Operationalizing Financial Covenants
This version: September 8, 2014

Dan Iancu, Nikos Trichakis, Gerry Tsoukalas*

Abstract
We study the interplay between financial covenants and the operational decisions of a retailer that obtains financing through a secured (asset-based) lending contract. While it is widely held that covenants serve to protect lenders, the ways in which a retailer adapts his operations in response have not been studied. We characterize the market conditions, involving demand distribution, growth potential, profit margin, and inventory depreciation rate, under which covenants are necessary, and argue that these are routinely met in practice. Furthermore, we show that covenants are not substitutable by other contractual terms, such as interest rates and loan limits, and provide operational insights for their optimal design. We discuss when covenants ensure that system-optimal decisions are taken in equilibrium, and show that additional operational flexibility can impact their effectiveness in a surprising, non-monotonic way.

1 Introduction
Asset-based lending (ABL) is a form of secured lending that allows firms to leverage their working assets as collateral, to obtain financing. The collateral typically consists of liquid assets, such as accounts receivable and inventory, which are expected to be monetized over time to generate funds for the loan repayment. This distinctive feature presents several challenges for both parties involved. On one hand, borrowing firms need to understand how the collateralization of their assets may impact their operations and investments. On the other hand, lenders need to develop a thorough understanding of firms’ business cycles and operations. Specifically, they must excel at two key functions. The first involves tracking the collateral’s value and the loan performance, by setting up proper monitoring and control processes. The second involves structuring the loan with adequate provisions that would grant them rights to take action, in case the firm becomes distressed or the collateral depreciates to a level that does not support the loan. Such provisions are typically implemented in the form of financial covenants, which are contractual terms stipulating...

*Iancu (daniancu@stanford.edu) is from Stanford GSB, Trichakis (ntrichakis@hbs.edu) is from Harvard Business School, Tsoukalas (gtsouk@wharton.upenn.edu) is from the Wharton School, University of Pennsylvania. Tsoukalas is thankful to the Fishman-Davidson Center for Service and Operations Management and to the Wharton School Dean’s Research Fund for grants that supported this work.
that borrowers meet particular financial metrics during the life of the loan. Failure to abide by these covenants transfers control rights to lenders, granting them the option to intervene and influence managerial decisions, e.g., by imposing changes in corporate policies, including new covenants, renegotiating claims, and even forcing the firm into bankruptcy, seizing and liquidating its assets to generate funds for the repayment of the entire loan (Hilson 2013, Tirole 2006).

The ABL market was almost inexistent in the early 1990s, and was considered a recourse of last resort, for distressed firms with poor or no credit history (Modansky and Massimino 2011). By 2009, ABL represented around 25% of the private credit extended to non-financial institutions, with over $480bn in loans outstanding (Alan and Gaur 2012, Foley et al. 2013). Today, ABL financing is provided by commercial banks, private equity, hedge funds etc., and is utilized by corporations of various sizes, including manufacturers, wholesalers, distributors and importers, in asset-intensive industries such as retail, oil and gas, transportation, and technology (see Stock (2010) and Schwimmer (2011), and white papers by Bank of America Merrill Lynch (BofA 2014) and GE Capital (GEC 1999)).

While recent changes in regulations and legislation in the U.S. have facilitated ABL adoption, several other factors make the practice particularly attractive for all parties involved. These are neatly summarized in the recent handbook issued by the Comptroller of the Currency of the U.S. Treasury (CH 2014). For borrowers:

“ABL provides important funding for companies in cyclical or seasonal industries by providing liquidity during slow sales periods and periods of inventory buildup. ABL provides rapidly growing companies the cash to fund growth or replenish internal capital used to fund growth by financing increases in receivables and inventory. ABL facilities are typically underwritten with a limited number of financial covenants [leading to] more flexibility”

The above quote highlights that the popularity of ABL among borrowers is related to the additional operational flexibility that it affords, by providing financing at key stages of the business cycle, and by including fewer financial covenants. The absence of such covenants is so important that it is, in fact, the main selling point in marketing campaigns run by lenders (see, e.g., BofA 2014, UPS 2013, WF 2011).

For lenders, the advantages are linked to the loans’ primary reliance on highly liquid and easily accessible collateral. Furthermore, the existence of a maturing appraisal and liquidation industry (e.g., Gordon Brothers, Hilco Global, ES Group, etc.) allows lenders to outsource the valuation and potential liquidation of the collateral, which are functions typically lying outside the scope of their core competencies. These features make ABL a “profitable, well-secured, and low-risk line of business if strong controls are established” (CH 2014).

---

1 Common financial covenants pertain to minimum requirements on cash-flow-to-debt ratios, net worth, EBITDA, etc. (Dalton 2014, Hilson 2013). We note that loan contracts typically include other forms of covenants (see Hilson (2013) for a detailed discussion); for our purposes, however, we only focus on financial covenants, and use the terms interchangeably (see discussion in Section 2.2 for more details).

2 Schwimmer (2011) lists several large companies that have entered into ABL contracts, including Georgia Gulf, Hertz, and Del Monte, and retailers like Sears, Neiman Marcus, Jo-Ann Stores, J Crew, and Liz Claiborne.

3 One such example is the introduction of the notion of a floating lien in the Uniform Commercial Code, which allows inventory to enter collateral when received by the debtor, and leave the collateral when sold (Hilson 2013).
However, the previous comments also hint at a few potential pitfalls of ABL. For instance, in retail, “many banks that have failed to properly structure loans to retailers, monitor the retailers performance, or monitor the value of the collateral have suffered significant credit losses” (CH 2014). While monitoring is achieved through inspections of the collateral, and by requiring retailers to submit frequent financial and operational statements, the mechanism by which lenders acquire control over the collateral is in fact precisely through covenants (Hilson 2013, Tirole 2006).

The above discussion highlights a critical tension around the use of covenants in ABL: on one hand, lenders advertise their absence and the accompanying operational flexibility for borrowers; on the other hand, their presence is essential, in order to mitigate risk.

In fact, the exact role of financial covenants in ABL agreements is not well understood. In the academic literature, their interplay with operational considerations has been largely ignored. In practice, some maintain that their inclusion reflects the lenders’ realization that it is insufficient to base the loan exclusively on the value of the collateral, without accounting for other financial aspects, such as cash flow (Hilson 2013). Others, however, explicitly question their use: “Some banks institute financial covenants to monitor retail borrowers, but the usefulness of financial covenants is debatable, given 1) the overwhelming reliance on collateral liquidity to repay the debt, and 2) a retailer’s tendency to experience seasonal losses” (CH 2014).

Given the intriguing tensions between covenants, operational flexibility and risk, several important questions emerge. From the firms’ perspective, how do (financial) covenants impact operational decisions and flexibility? How should firms adjust their operations in the presence of covenants? What levers can they use? From the lenders’ perspective, under what conditions are covenants necessary and why? How can lenders “operationalize covenants,” i.e., think about their optimal design, accounting for firm operations and business cycles? What is the interplay between covenants and other contractual terms, such as interest rates or loan limits?

To address these questions, we consider a game between a cash-constrained newsvendor (retailer) and a bank that operates in a perfectly competitive lending market, and can provide additional funds via an inventory-based term loan. In addition to interest rate and loan limit, the loan agreement carries a covenant on the retailer’s ongoing performance. To capture the role played by the covenant, we extend the classical newsvendor model by including an intermediate period, at which the players are able to review sales: if considered “weak,” the retailer has the option to liquidate any unsold inventory and exit the market early, or otherwise continue. At the same time, if the covenant is breached, the bank has the option of seizing control and forcing liquidation, to protect its investment. Both players are risk-neutral and have perfect information. We work through the game via backward induction: at the intermediate period, the retailer and bank decide on how to optimally manage their investment, i.e., by liquidating or continuing based on observed operational performance; at contract inception, the bank chooses the interest rate, loan limit, and covenant terms, and the retailer chooses his order quantity. We also consider alternative settings and extensions, under which the lending market is monopolistic, or partial liquidation of inventory (e.g., through “fire sales”) is possible, for both players.
We contribute to the existing literature in the following ways:

1. We study the problem of dynamically managing a leveraged inventory investment through liquidation decisions based on observed operational performance. Surprisingly, we find that threshold policies need not be optimal, and discuss how this might lead to counter-intuitive, or even perverse, behavior. We also argue how particular business cycle and market conditions, such as stronger growth potential, smaller margins, faster depreciation of products, lower initial equity or higher interest rates can all lead to increased propensity for such behavior.

2. We highlight that covenants emerge as critical terms in debt contracts. In particular, they are non-substitutable by interest rates or loan limits, and are essential even in the absence of information asymmetry, irrespective of whether the lending market is perfectly competitive or monopolistic. We argue that their primary role is to alleviate agency issues arising from the players’ distinct preferences concerning dynamic (operational) decisions. We characterize the market conditions that accentuate agency issues and covenant potency, and show that they are related to, but not identical with, conditions leading to bankruptcy risk.

3. We discuss how retailers and lenders can “operationalize” covenants, i.e., think about their effectiveness and design, in light of operational and market considerations. We identify conditions under which properly-designed covenants ensure that profit-maximizing decisions are taken in equilibrium, and show that additional operational flexibility can impact the effectiveness of covenants in a surprising, non-monotonic way.

We believe that our paper is the first in the operations literature to study dynamic investment management by strategic agents operating under covenants and/or liquidation decisions. While this problem has been addressed in the finance, economics and accounting literature, all papers we are aware of either omit the operational decisions or model them in a rather simplistic way. Our focus on operations allows us to elicit new insights, and provide theoretical justifications for several empirical observations in the literature.

Literature Review

Our work lies at the interface of operational and financial decision-making. Several recent papers have pioneered connections between these areas. Xu and Birge (2004), Dada and Hu (2008) and Boyabath and Toktay (2011) extend the newsvendor model to include financing constraints, and show how these can affect the optimal order quantity or capacity choice. Similarly, Birge and Yang (2013), Chod (2014) and Kouvelis and Zhao (2012) look at issues of supply chain coordination under leverage.\footnote{For work related to supply chain coordination and contracting (but without external financing considerations) see Cachon and Fisher (2000), Cachon (2003), Cachon and Lariviére (2005), Babich et al. (2012) and references therein.} The focus here is on short-term supplier financing in the form of trade credit or debt. We refer the reader to Kouvelis (2012) for a thorough review of this literature. In a similar spirit, Lai et al. (2009) study risk sharing between suppliers and retailers, focusing on the impact
of financial constraints imposed on the supplier. None of the papers above consider a dynamic setting, where covenants could be relevant.

Among the first to look at operations management with capital constraints in a dynamic setting is Porteus (1972), who deals with inventory policies, and the issue of maintaining a cash safety level. More recently, Archibald et al. (2002) study the optimal inventory policies of new start-up firms seeking to maximize their probability of survival. Babich and Sobel (2004) study the problem of maximizing initial public offering (IPO) cash flows, by formulating an optimal stopping problem. Babich (2010) studies a manufacturer’s problem of jointly deciding inventory orders and financial subsidies for risky suppliers. Gong et al. (2014) consider inventory control under leverage, and discuss conditions under which modified base-stock policies are optimal. Li et al. (2013) develop a multi-period model of a dividend-maximizing firm financed through short-term debt; their model includes a constraint bounding the loan amount, in order to prevent the firm from “looting the till,” i.e., using the loan to pay shareholders in anticipation of an imminent default. Besbes and Maglaras (2012) show that a non-leveraged retailer selling goods in a deterministic market should adjust pricing decisions to track the most stringent future financial constraint. Related to the interface literature, Dong and Tomlin (2012) show that the availability of insurance can impact inventory investment in non-monotonic ways. None of these papers focus on the role of financial covenants, nor on strategic interactions between the different parties involved.

Another related strand of the literature studies the interplay between operations and public capital markets. Alan et al. (2014) and Gaur et al. (2014) explain how operations metrics can relate to, and sometimes forecast publicly observed stock prices. Similarly, Schmidt et al. (2012) study how stocking decisions can serve as a signaling mechanism to public investors. Caldentey and Haugh (2009), Chod et al. (2010), Ding et al. (2007), Gaur and Seshadri (2005), Huchzermeier and Cohen (1996) and Kouvelis et al. (2013) demonstrate the benefits of hedging operations jointly with financial instruments. Brown et al. (2010) study optimal asset liquidation under distress. While these papers focus on capital markets, we study contract design.

Closer to our work, Buzacott and Zhang (2004) focus on asset-based lending, and formulate a static one-period newsvendor model with capital constraints, capturing the strategic interaction between a retailer and a monopolistic bank. They find that the interest and advance rates influence the retailer’s optimal order quantity, and are not necessarily substitutable. More recently, in the same area, Alan and Gaur (2012) study the role of asymmetric information, taxes and advance rate optimization in a static newsvendor model, and discuss their impact on the optimal capital structure of the firm. We focus attention on the role of covenants, in a dynamic setting, under both monopolistic and perfectly competitive lending markets, where both the lender and the newsvendor can influence the management of inventory through (partial) liquidation decisions.

On a broader level, our work is related to a large stream of accounting, finance and economics literature that deals with dynamic incentive problems in financial contracting. Covenants have been studied within this context, both empirically and theoretically (see Graham and Leary (2011) and Roberts and Sufi (2009b) for recent surveys). However, as argued in Cohen et al. (2012), “While
the literature studies the effects of covenant violations, it does not study how covenants affect the firm’s capital structure prior to violation.” In contrast, our work studies the ex-ante implications, showing that covenants can affect (inventory) investment in non-trivial ways, at various stages throughout the life of an ABL contract. Moreover, we depart from the literature by incorporating the operational perspective. Below, we discuss how these two features can lead to novel insights.

Empirical evidence brought forth in Dichev and Skinner (2002), Nini et al. (2009) and Bradley and Roberts (2004) suggests that: 1) covenants influence debt choices well before the trigger point; 2) covenants can lead to lower interest rates; 3) managers make accounting choices to reduce the likelihood of covenant violations, and 4) loans are more likely to include covenants when the borrower is small, has high growth opportunities or is highly levered. Our model can provide theoretical justifications for all these phenomena. In particular, it can be used to explain the first two in terms of the firm’s strategic operational response. It complements the third by showing that managers can pull operational levers, such as fire sales, to prevent covenant violations, and it confirms and augments the fourth, by explicitly showing the dependence on collateral liquidation values and profit margins.

While the theoretical literature in finance and economics has not focused specifically on asset-based lending, several papers are relevant to our work. The agency theory of covenants was developed in Jensen and Meckling (1976), Myers (1977) and Smith and Warner (1979). According to the theory, covenants serve to alleviate the agency conflict between bond and equity holders. Aghion and Bolton (1992) and Tirole (2006) show how this can be achieved through the (threat of) transfer of control rights. In line with this idea, Hart and Moore (1998) and DeMarzo and Fishman (2007) analyze the role played by the contract structure in preventing managers from diverting cash flows to shareholders. Similarly, Gárleanu and Zwiebel (2009) and Quadrini (2004) focus on issues of information asymmetry and/or debt renegotiation. Clementi and Hopenhayn (2006) study the optimal growth of a firm under borrowing constraints. Operational decisions in these papers are either absent, exogenous or modeled in a simplistic way. By incorporating more operational details, we elicit the dependence of covenant design and effectiveness on important market parameters, such as demand distribution, market growth, profit margins, collateral depreciation, etc.

2 Model

This section details the two-period model we develop to study contract design in ABL. We first focus on the base case of a retailer without financing, facing two selling periods and an intermediate decision to liquidate his inventory or continue operating. The model is then extended to the case of a cash-constrained retailer financed by a bank through an inventory-based term loan. In both cases, we start by presenting the model setup, including the action spaces of the players and sequence of events, and then provide a more detailed discussion of the assumptions and modeling choices.
2.1 Unleveraged Retailer

Model Setup

At time $t = 0$, a retailer endowed only with $x_0$ units of cash purchases $q$ units of a single product, at a unit cost $c > 0$. Full payment of $cq$ is due upon delivery of the purchased units, which occurs with zero lead time. The retailer’s cash is used to cover the entire payment, i.e., $x_0 \geq cq$.

The retailer faces two selling periods, over which units are sold at price $p = 1 > c$. At $t = 1$, the first period demand is realized and fulfilled to the largest extent possible using the starting inventory of $q$ units. Let $D_1$ denote the random demand in the first period, assumed to have non-negative support, with cumulative distribution function (c.d.f.) $F_1$ and probability density function (p.d.f.) $f_1$. A realization of $D_1$ is denoted $d_1$. Any unmet demand is assumed lost, with no penalty.

At this point, the retailer faces two options: he can either liquidate (e.g., salvage) his remaining inventory (if any) at unit price $s < c$ and exit the market immediately, or he can continue operating for a second selling period using his leftover inventory. The decision to liquidate or continue is based on the retailer’s information set at $t = 1$, which consists of the realized first period demand, $d_1$. Let $\ell_R : \mathbb{R} \times \mathbb{R} \rightarrow \{0, 1\}$ denote the retailer’s liquidation policy, with $\ell_R(q, d_1) = 1$ if and only if the retailer chooses to liquidate, when the initial inventory is $q$, and the first period demand is $d_1$.

If the retailer chooses to continue, at $t = 2$, the second period demand is realized and fulfilled, to the largest extent possible. We denote this demand by $D_2$, and allow its distribution to depend on $d_1$. We assume that unmet demand is lost without any penalty to the retailer, and any remaining inventory is disposed of, free of cost and without salvage value. The retailer then exits the market.

The retailer’s expected profits are given by

$$\pi_R(q) = \mathbb{E} \left[ \left( \min(D_1, q) + s(q - D_1)^+ \right) \mathbb{1}\{\ell_R(q, D_1) = 1\} \right] + \mathbb{E} \left[ \min(D_1 + D_2, q) \mathbb{1}\{\ell_R(q, D_1) = 0\} \right] - cq, \quad \text{for } x_0 \geq cq,$$

where $\mathbb{1}$ is an indicator function. To ease notation, we suppress the dependence of $\pi_R$ on the liquidation policy $\ell_R$. The first term denotes the retailer’s expected revenues by liquidating at $t = 1$, while the second term denotes his expected revenues by exiting the market at $t = 2$. The third term represents his purchasing costs. For simplicity, we assume a zero discount rate throughout the analysis.

Furthermore, we introduce the following assumption regarding the second period demand distribution, for the sake of tractability.

**Assumption 1.** The second period demand conditional on first period demand, i.e., $D_2 | \{D_1 = d\}$, follows a two-point distribution, taking values $(M - 1)d$ and 0 with equal probability, where $M > 1$.

Note that the two-point distribution introduces an adverse and a favorable scenario for the second period demand. Our choices of zero demand in the adverse scenario and equally likely scenarios allow us to ease exposition, without affecting the validity of the structural results. Furthermore, we also choose to parameterize $D_2$ via $M$, which determines the demand growth rate, or market
strength. In particular, when the first period demand is equal to \(d\), then the expected second period demand is equal to \(\frac{M-1}{2}d\). Thus, demand is projected to increase if \(M \geq 3\), and decrease otherwise. We refer to these particular cases as follows.

**Definition 1.** When \(M \geq 3\) (\(M < 3\)), we say that the market is non-shrinking (shrinking).

Under Assumption 1, the probability of selling a unit of inventory over the second selling period is at most \(1/2\). As such, the expected revenues from a unit of inventory in the second period are at most \(1/2\) as well, since the selling price is 1. On the other hand, liquidation yields revenues of \(s\) per unit at \(t = 1\). It is easy to see that, unless \(s < 1/2\), the retailer would never prefer to continue in the second period, but rather behave as a classical newsvendor who sells over one period and then liquidates (or salvages) his leftover inventory. To circumvent this degeneracy, we introduce the following assumption:

**Assumption 2.** The liquidation price satisfies \(s < 1/2\).

It is hardly surprising that an upper bound on the value of \(s\) is needed to eliminate trivial cases when liquidation is always preferred.

**Discussion**

Our model extends the classical newsvendor framework by providing a liquidation option to the retailer at an intermediate point. We assume that the entire leftover inventory at \(t = 1\), equal to \((q-d_1)^+\), can be liquidated at price \(s\). This is in accordance with the salvaging assumption in the newsvendor setup, whereby all remaining inventory is sold at a deterministic price that is known a priori, and is independent of the quantity and first period demand. This assumption, albeit simplistic,\(^5\) is standard in the operations literature, particularly within a newsvendor setting. In our model, it indeed adequately captures the mechanisms at play and becomes even more sensible in later parts of our model (e.g., when a bank is also endowed with an option to liquidate this inventory, see Section 2.2 for details). In practice, such liquidations are typically contracted to an external liquidation house, e.g., Gordon Brothers Group, Wells Fargo, etc. The liquidation house provides binding inventory appraisals upfront (i.e., in period \(t = 0\)), and, in case of liquidation, purchases the entire inventory at a fixed per-unit price, which reflects a pre-determined haircut on the original appraised value. Thus, any remaining liquidation risk beyond this haircut is essentially transferred from the retailer to the liquidation house (see Craig and Raman (2013) for more details).

The intermediate liquidation option can be valuable, since it provides an exit strategy for a retailer faced with a depreciating inventory value, from \(c\) at \(t = 0\), to \(s\) at \(t = 1\), and finally 0 at \(t = 2\).\(^6\) If inventory is losing value quickly and/or the forecasted second period demand is low, it may be preferable for the retailer to exit early.

\(^5\)Cachon and Kok (2007) recognize that the effective salvaging price depends on remaining inventory, and discuss appropriate ways for estimating this price. There is also an extensive literature in revenue management concerned with optimal markdown, the practice of dynamically adjusting prices for remaining inventory at the end of a selling season, for the purpose of freeing valuable shelf space (see, e.g., Talluri and Van Ryzin (2004) and Phillips (2005)).

\(^6\)More generally, even if the inventory value is not depreciating, the retailer is incurring operating costs over time, which reduce his valuation.
Note that the realized demand at $t = 1$ is informative for the retailer, as $D_1$ and $D_2$ are assumed to be correlated (see, e.g. Azoury 1985, Fisher and Raman 1996, Graves 1999, Lee et al. 2004), thereby increasing the exit option’s value. Note also that our assumption of a discrete second period demand distribution has been widely used in the literature in similar dynamic multi-period game-theoretic models (e.g., see Bolton and Dewatripont (2004) and references in our literature review in Section 1). As we argue later, despite its simplicity, the two-point distribution is sufficiently rich to capture the main trade-offs driving the dynamic decisions of the players.

We also tacitly assume that, in period $t = 1$, the retailer cannot replenish inventory, e.g., by reordering. This assumption makes the model relevant for products with short life cycle and/or long lead times, or for retailers operating in highly seasonal industries, where typical operating cycles involve an inventory accumulation stage, followed by a selling stage. This latter feature is particularly pertinent to ABL practice, as alluded to in the Introduction. We leave an extension that allows replenishments for future research.

Finally, we do not consider the possibility of a partial liquidation at this point. We revisit and relax this in Section 6.

Our model is rooted in reality, and is inspired by the decision process firms can go through when dealing with under-performing divisions, units or products. In the summer of 2011, facing stiff competition, lackluster sales, and weak demand, Hewlett Packard (HP) decided to discontinue its webOS TouchPad tablet devices. Sitting on sizable inventory, HP dropped prices overnight by over 75% for some of its products. The tablets were sold out over a two-day period in most locations during the first round of liquidation. During the second round, which occurred in December 2011, 8,000 units were sold out in just 30 minutes. At the end of the liquidation cycle, HP had effectively exited the tablet market. This example underlines the main trade-offs captured in our model, between depreciating inventory value (in HP’s case, in the form of technological obsoleteness) and future demand. The lack of a replenishment option is also consistent with our assumption, which involves short life-cycle products and longer lead-times.

2.2 Leveraged Retailer

We now deal with the case where the retailer’s capital is insufficient to cover the inventory cost $c q$. The model is extended to include a second player: the retailer $R$ (“he”) has access to a lending market, where a bank $B$ (“she”) can provide additional financing.

Model Setup

We study the game under perfect and symmetric information between the two players.

At $t = 0$, the retailer borrows an amount $w \triangleq c q - x_0$ from the bank, under an inventory-based loan contract with the following terms.

1. [Interest rate and repayment] Payment of principal plus interest charges (at rate $r$) is due at $t = 2$. For ease of exposition, we consider $R = 1 + r$ in the remainder of the paper, and refer
to it simply as the interest rate. The net amount due is then $R w$.

2. [Loan Limit] The credit to be extended, or principal, is subject to a loan limit of $\bar{w}$, i.e., $w \leq \bar{w}$, and is immediately available.

3. [Collateral] The bank obtains \textit{perfect security interest} in the entire inventory (to be) purchased by the retailer. In other words, the retailer’s inventory is used as collateral to secure the loan.

4. [Covenant] In period $t = 1$, the retailer is required to record a cash-flow-to-debt ratio higher than or equal to a threshold $\alpha$. Equivalently, at $t = 1$ his demand is required to be higher than a threshold, which we refer to as the \textit{covenant demand threshold}, denoted by $\delta \overset{\text{def}}{=} \alpha R w$.\footnote{Formally, note that the covenant is breached if and only if the cash from sales does not exceed a fraction $\alpha$ of the debt, i.e., $\min(q, d_1) < \alpha R w$, or equivalently $d_1 < \delta$ (the covenant is never breached upon stockout, i.e., if $d_1 \geq q$).}

Failure to abide by the covenant is considered an event of default, and gives the bank the right to request immediate repayment of all due principal plus interest, $R w$.

The bank chooses the contract terms, denoted by $\kappa \overset{\text{def}}{=} (R, \bar{w}, \delta)$. The retailer chooses his order quantity $q$, which he funds using his available equity $x_0$ and by borrowing $w$ from the bank under the provided inventory-based loan.\footnote{We implicitly assume that all available equity is used to purchase the inventory. In the absence of other considerations, this is without loss (Kouvelis and Zhao 2012). See Alan and Gaur (2012) for a discussion.}

At $t = 1$, demand $d_1$ is realized, revealed to both players and fulfilled, thus generating a cash flow of $\min(d_1, q)$ for the retailer. Similarly to the previous setup, the retailer has the option of continuing operating for a second selling period, or liquidating his leftover inventory at $s$ and exiting the market. If he chooses the latter option, he uses his generated cash flow and liquidation revenues, equal to $\min(d_1, q) + s(q - d_1)^+$, to repay his debt of $R w$. If the retailer chooses to continue operating, in accordance with the contractual agreement, the bank observes his performance in the first selling period and assesses his cash flow: if it is lower than $\delta$, i.e., $\min(d_1, q) < \delta$, the bank has the option of forcing liquidation in order to secure full payment of the debt. In case the bank exercises this option, she seizes remaining inventory and liquidates it at unit price $s$. All proceeds from sales and liquidation are first used towards debt servicing. If the bank is made whole, the remaining revenues are returned to the retailer. In accordance with our previous notation, let $\ell_B : \mathbb{R} \times \mathbb{R} \to \{0, 1\}$ denote the liquidation policy of the bank. Note that, unless the covenant is breached, the bank does not have any right to influence the liquidation/continuation decision.

When the retailer chooses to continue and the bank does not force liquidation, at $t = 2$, the second period demand $D_2$ is realized and filled to the largest extent possible. Any residual inventory at the end of period 2 is disposed of free of cost, and without salvage value. The retailer then uses the generated sales revenues from the two periods to repay his debt, and then exits the market.

In this setting, the retailer’s and the bank’s cash flows at the end of the game under liquidation (irrespective of the party choosing that option) and continuation are respectively given by

\[
\begin{align*}
X_{\mathcal{R}, \mathcal{L}}(q, D_1) &= (\min(D_1, q) + s(q - D_1)^+ - R w)^+ , \\
X_{\mathcal{R}, \mathcal{C}}(q, D_1, D_2) &= (\min(D_1 + D_2, q) - R w)^+ , \\
X_{\mathcal{B}, \mathcal{L}}(q, D_1) &= \min \{ R w , \min(D_1, q) + s(q - D_1)^+ \} , \\
X_{\mathcal{B}, \mathcal{C}}(q, D_1, D_2) &= \min \{ R w , \min(D_1 + D_2, q) \} .
\end{align*}
\]
For the retailer, both expressions have a floor at 0, reflecting his limited liability. The expected profits of the two players can then be compactly expressed as

\[ \pi_R(q, \kappa) = \mathbb{E} [X_{R,S}(q, D_1) \mathbb{1}\{S\} + X_{R,S}(q, D_1, D_2) \mathbb{1}\{\mathcal{C}\}] - x_0, \quad \text{for } 0 \leq cq - x_0 \leq \bar{w}, \]

\[ \pi_B(q, \kappa) = \mathbb{E} [X_{B,S}(q, D_1) \mathbb{1}\{S\} + X_{B,S}(q, D_1, D_2) \mathbb{1}\{\mathcal{C}\}] - w(q), \quad \text{for } 0 \leq cq - x_0 \leq \bar{w}, \]

where \( \mathcal{L} \overset{\text{def}}{=} \{\ell_R(q, D_1) = 1\} \cup \{d_1 < \delta\} \cap \{\ell_B(q, D_1) = 1\} \) is the liquidation event, and \( \mathcal{C} \overset{\text{def}}{=} \mathcal{L}^c \) is the continuation event.

At \( t = 0 \), the retailer chooses the order quantity \( q \) maximizing his expected profit, by solving:

\[
\begin{align*}
\text{maximize} \quad & \pi_R(q, \kappa) \\
\text{subject to} \quad & cq - x_0 \leq \bar{w} \\
& q \geq 0.
\end{align*}
\] (2)

The bank chooses the debt contract terms \( \kappa = (R, \bar{w}, \delta) \) so as to break even in expectation:

\[
\begin{align*}
\text{maximize} \quad & 0 \\
\text{subject to} \quad & \pi_B(q, \kappa) = 0 \\
& R \geq 1 \\
& \bar{w}, \delta \geq 0.
\end{align*}
\]

Put differently, the lending market is \textit{perfectly competitive}, a standard assumption in the literature.

For clarity, we refer to the game at \( t = 1 \) that determines whether \( \mathcal{S} \) or \( \mathcal{C} \) occurs as subgame \( \mathcal{S} \), depicted in Figure 1(b). The pure subgame perfect equilibrium actions of each player can be characterized via backward induction. In particular, the outcome of the subgame \( \mathcal{S} \) at \( t = 1 \) can be viewed as a Stackelberg game with the retailer leading by choosing \( \ell_R \) and the bank following by choosing \( \ell_B \). The sequence of all events is illustrated in Figure 1(a).
Discussion

We make several remarks about our modeling choices and implicit assumptions.

Covenant type. The covenant of a minimum cash-flow-to-debt ratio we study is among the most commonly used in practice (see Dichev and Skinner 2002, Roberts and Sufi 2009a). Other covenants include conditions imposed on net worth, the interest payments, the leverage ratio, the debt to equity ratio, the debt coverage ratio, the EBIT to interest ratio, etc. (for a detailed discussion, see Chapter 7 of Hilson (2013)). In our model, it turns out that all of the aforementioned covenants simply translate into a threshold requirement on first period demand.\(^9\) The choice of a particular type of covenant is therefore without loss of generality. As such, we henceforth base our analysis and discussion on the demand threshold \(\delta\). Finally, note that we also assume that in case of a covenant breach, the lender is entitled to early and immediate payment of the debt. This is routinely included as a clause in asset-based loan contracts, see Hilson (2013).

Information sets. We choose a framework with perfect, symmetric information and pure strategies, to model the interactions between the two players, a standard practice in the operations management literature, see e.g., Buzacott and Zhang (2004), Kouvelis and Zhao (2012), and Birge and Yang (2013). The assumption of symmetric information implies that both retailer and bank agree on the distribution of demand in the game and that the bank can costlessly monitor his cash flows. In practice, the amounts of information available to each player might be unequal and not of the same fidelity, resulting in some degree of asymmetry (Alan and Gaur 2012). However, our assumption remains reasonable, since lenders often require detailed demand forecasts from retailers, and require revenues from the sales of collateral to be directly remitted to lockbox accounts. Furthermore, lenders can seek to design contracts in a way that promotes truthful reporting (Bolton and Dewatripont 2004).

Bankruptcy costs and procedures. In the event of default, we only consider Chapter 7 bankruptcy (“liquidation”), but not Chapter 11 (“reorganization”), which would involve having to model the renegotiation process between the parties. Note also that, in view of our assumption of perfect and symmetric information, a renegotiation process would be superfluous (see Gărleanu and Zwiebel 2009). We also ignore bankruptcy costs. For a discussion on the effects of non-zero bankruptcy costs and reorganization, we refer the reader to Birge and Yang (2013) and Birge et al. (2014).

Symmetric liquidation costs. For the sake of simplicity, we ignore possible asymmetries in liquidation costs that may exist between the two players, and assume that either party would derive per unit revenues of \(s\) when opting for liquidation at \(t = 1\). This is consistent with our earlier explanation that, in practice, the liquidation process is typically outsourced to a third party, such as a liquidation house. In reality, liquidation prices may differ slightly, but not vastly so: if a bank forces liquidation, additional litigation and administrative costs might lead to a lower price; however, due to potential repeated business with a liquidation house and negotiation power, the

\(^9\)As an example, consider a covenant that requires the net worth to be higher than a threshold \(\tau\). For \(d_1 < q\), debt at \(t = 1\) is equal to \(Rw\), and equity is equal to \(d_1 + s(q - d_1)\). The covenant then equivalently requires the first period demand to be higher than a threshold, in particular \(d_1 > \frac{Rw - sq}{1 - s}\).
price might also be higher.

3 Liquidation Policies

To derive some intuition concerning the role of covenants, we first analyze the players’ optimal liquidation policies in subgame $S$, at the intermediate time $t = 1$.

3.1 Retailer’s Liquidation Policy

We first characterize the optimal liquidation policy of an unleveraged retailer, when his order quantity was $q$, and the first period demand realization was $D_1 = d$.

Lemma 1 (Unleveraged Retailer). The optimal liquidation policy for a retailer without leverage is a threshold one, i.e., $\ell_{\text{unlev}}^R(q, d) = 1\{d < d_{\text{unlev}}^R(q)\}$, where $d_{\text{unlev}}^R(q) \overset{\text{def}}{=} \frac{sq}{s^2 + 1}$.

Such a threshold policy is consistent with intuition and hardly surprising. Many studies dealing with dynamic investment management under debt either de facto assume or derive such policies as optimal (see, e.g., Babich and Sobel 2004, Gigler et al. 2009, Hart and Moore 1998, Swinney and Netessine 2009). Furthermore, as one might intuitively expect, the threshold $d_{\text{unlev}}^R$ is increasing in $q$ and $s$ and decreasing in $M$, confirming that the propensity to liquidate increases with investment size and liquidation value, but decreases with the second period market strength.

Under debt, the retailer’s liquidation decisions generally depart from the ones above. The optimal policy turns out to critically depend on whether the order quantity $q$ exceeds a particular threshold $q_D$, given by

$$q_D \overset{\text{def}}{=} \begin{cases} \frac{R_x}{R_c - \frac{2}{M-1+s}} & \text{if } M \geq \tilde{M} \text{ and } s < s_1^D, \\ \frac{R_x}{R_c - \frac{2}{M+s-1}} & \text{if } M < \tilde{M} \text{ and } s < s_2^D, \\ \infty & \text{otherwise}, \end{cases}$$

(3)

where $\tilde{M} \overset{\text{def}}{=} 2(1 - s)$, $s_1^D \overset{\text{def}}{=} \frac{R_c(M-1)}{2(1-R_c)}$, $s_2^D \overset{\text{def}}{=} \frac{R_c(M-1)}{M-R_c}$. We introduce the following definitions.

Definition 2. When $q > q_D$ ($q \leq q_D$), we say that the retailer is (not) sufficiently leveraged.

Definition 3. When $M < \tilde{M}$ ($M \geq \tilde{M}$), we say that the market is (not) rapidly shrinking.

The first moniker is intuitive, since larger $q$ is tantamount to a larger debt load. Note that whether this condition holds critically depends on market parameters. In particular, retailers are never sufficiently leveraged when inventory does not depreciate too rapidly (i.e., $s$ exceeds some critical value), and an important distinction is drawn depending on whether markets are “rapidly shrinking,” as captured in the second definition. Since $\tilde{M} < 2$, a rapidly shrinking market is reflective of a second period expected demand considerably below the first period realized demand, which is consistent with (and stronger than) our earlier condition for a “shrinking market,” $M < 3$.

With these definitions, we now characterize the liquidation policy of a leveraged retailer.
Lemma 2 (Leveraged Retailer). In equilibrium,

(a) a retailer that is not sufficiently leveraged follows the same liquidation policy as an unleveraged retailer, i.e., \( \ell_{\text{lev}}^\text{R}(q,d) = \ell_{\text{unlev}}^\text{R}(q,d) \).

(b) a retailer that is sufficiently leveraged and operates in a market that is not rapidly shrinking follows a threshold liquidation policy: \( \ell_{\text{lev}}^\text{R}(q,d) = 1\{d < d_{\text{lev}}^\text{R}(q)\} \).

(c) a retailer that is sufficiently leveraged and operates in a market that is rapidly shrinking follows a non-threshold policy:

\[
\ell_{\text{lev}}^\text{R}(q,d) = \begin{cases} 
\frac{Rw(q)}{M}, & \text{if } M = \tilde{M}, \\
\max \left( \frac{2sq - Rw}{M - 2(1-s)}, \frac{Rw}{M} \right), & \text{otherwise.}
\end{cases}
\]

where \( d_{\text{lev}}^\text{R}(q) \) is defined as follows:

\[
d_{\text{lev}}^\text{R}(q) = \begin{cases} 
\frac{Rw}{M}, & \text{if } M = \tilde{M}, \\
\max \left( \frac{2sq - Rw}{M - 2(1-s)}, \frac{Rw}{M} \right), & \text{otherwise.}
\end{cases}
\]

It is worth noting that a leveraged retailer follows the same policy in both the upper and the lower node of subgame \( S \), i.e., his policy is unaffected by a covenant breach. This is intuitive, since his own policy becomes irrelevant when the breach happens, so that the optimal policy in equilibrium is the same as when the covenant is not breached.

Lemma 2 is summarized in Figure 2. Note that a retailer’s behavior critically depends on whether he is sufficiently leveraged, and on whether the market is rapidly shrinking or not. As intuition would dictate, under small enough debt levels, a leveraged retailer acts as if the entire order were funded using his own equity (see Figure 2(a)). However, as leverage increases, the policy starts to deviate substantially.

Figure 2: Retailer optimal liquidation policy as a function of realized demand \( d_1 \). The shaded area denotes a preference for liquidation at \( t = 1 \). The policy above the horizontal axis corresponds to a leveraged retailer, when he is (a) not sufficiently leveraged, or sufficiently leveraged in a market that is (b) not rapidly shrinking or (c) rapidly shrinking. For comparison, an unleveraged retailer’s optimal policy is depicted below the axis.

When the market is not rapidly shrinking, a sufficiently leveraged retailer still follows a threshold policy, but starts liquidating less often than an unleveraged one, see Figure 2(b). In fact, the new threshold \( d_{\text{lev}}^\text{R}(q) \) is not only lower than \( d_{\text{unlev}}^\text{R}(q) \), but also increases in \( q \) at a lower rate, implying that the discrepancy in policies becomes even more pronounced as leverage increases.

Surprisingly, a sufficiently leveraged retailer operating in a rapidly shrinking market entirely departs from threshold policies, see Figure 2(c). To understand this behavior, note that market conditions in this case are particularly dire, as bleak second period prospects are compounded by a rapidly depreciating inventory value \( (s < s_2^2) \). At intermediate sales levels \( \left( \frac{Rw}{M} < d < d_{\text{lev}}^\text{R}(q) \right) \), the
latter effect takes precedence, as liquidating (a relatively large) inventory would mean immediate insolvency or extremely low profits for a leveraged retailer, while continuing could yield hope of high(er) profit if the high demand scenario materializes. As before, it is important to note that a leveraged retailer liquidates less often than an unleveraged one. However, the disagreement here occurs at intermediate sales levels, where an unleveraged retailer liquidates so as to recover a higher total asset value, while a leveraged retailer gambles on continuation. We note that non-threshold policies are not a by-product of our assumptions, and they persist under considerably more general settings – see the discussion in Section A.1 of the Appendix.

To better understand the circumstances under which leveraged retailers act like unleveraged ones, we consider the dependency of the threshold \( q_{D} \) on the market parameters. As noted before, in markets where inventory value does not depreciate rapidly (i.e., \( s \geq s_{1}^{D} \) if \( M \geq \bar{M} \), or \( s \geq s_{2}^{D} \) if \( M < \bar{M} \)), the threshold \( q_{D} \) is infinite, implying that retailers act as if they were unleveraged, no matter how high their actual debt may be. It can be checked that the minimal liquidation prices \( s_{1}^{D} \) and \( s_{2}^{D} \) governing this regime are both increasing in the demand growth rate \( M \), as well as in \( c \) and \( R \). Furthermore, when \( q_{D} \) is finite, it is always strictly decreasing in \( M, c \) and \( R \), and strictly increasing in \( x_{0} \). This implies that, ceteris paribus (in particular, for a fixed \( q \)), a leveraged retailer’s liquidation policy is more likely to start departing from an unleveraged policy as the market strength becomes more attractive, but also as the debt load becomes heavier, due to either less initial capital (smaller \( x_{0} \)), higher unit costs (larger \( c \)), or increased interest rates (larger \( R \)). Interestingly, this also suggests that, by charging larger interest rates \( R \), the bank may inadvertently induce retailers to act in a “riskier” way in managing the collateral, so that larger interest rates may act like an amplifier of risk and of the shareholder-debtholder conflict.\(^{10}\)

The critical threshold \( q_{D} \) can also be connected with the existence of bankruptcy risk in the debt agreement between the retailer and the bank, as summarized in the following result.

**Lemma 3** (Sufficient leverage and bankruptcy). When the optimal liquidation policy of either a leveraged or an unleveraged retailer is followed, \( q > q_{D} \) is

(i) a necessary and sufficient condition for bankruptcy risk in non-shrinking markets, and
(ii) a sufficient (but not necessary) condition for bankruptcy risk in shrinking markets.

The result implies that sufficiently leveraged retailers always induce bankruptcy risk, and, in fact, the former phenomenon is actually synonymous with the latter in non-shrinking markets. Since such markets may be quite natural in practice, this fact bears very relevant implications for our analysis, suggesting that risky lending agreements of the type examined here are likely to involve sufficiently leveraged retailers, whose liquidation policies depart from those of unleveraged ones. Interestingly, this is not necessarily the case in shrinking markets. Note that the distinction between the two regimes is given by our earlier definition of shrinking markets (i.e., \( M \geq 3 \)), instead of rapidly shrinking markets (i.e., \( M \geq \bar{M} \)).

\(^{10}\)In equilibrium, such parameter changes are likely to induce modifications in the retailer’s order quantity, \( q \), as well. Since a larger \( M \) typically leads to larger orders, the aforementioned effects would become even more pronounced. However, larger \( R \) or \( c \) typically lead to lower orders, hence diminishing the effects.
3.2 Bank’s Liquidation Policy

We next analyze the bank’s optimal liquidation policy in subgame S (see Figure 1(b)).

**Lemma 4** (Bank’s liquidation policy). *In equilibrium, the bank follows the threshold liquidation policy \( \ell_B(q, d_1) = 1 \{ d_1 < d_B(q) \} \), unless the collateral value depreciates very rapidly \( (s < s_B) \) and the retailer is highly leveraged \( (q > q_B) \), in which case she follows the non-threshold policy \( \ell_B(q, d_1) = 1 \{ d_1 \in [0, d_B^{\text{unlev}}(q)) \cup (Rw - 2qs, d_B(q)) \} \), where \( q_B \) is defined by:

\[
q_B \overset{\text{def}}{=} R \left( M - 1 + 2s \right)^{\frac{1}{2}} - 2Ms,
\]

and \( d_B(q) \) is defined by:

\[
d_B(q) = Rw.
\]

The result suggests that, barring a particular case, the bank follows a threshold liquidation policy, with a threshold \( d_B(q) \) that increases in \( q, R \) and \( c \), and decreases in \( x_0 \). Interestingly, the bank also departs from a threshold policy, under similar conditions as the retailer, i.e., low liquidation value and high leverage. Here, when the realized sales are low \( (d_1 < \frac{Rw - 2qs}{1-2s}) \), it can be checked that the retailer is bankrupt, and the bank is set to seize all his assets, either at \( t = 1 \) or at \( t = 2 \).

As such, the bank effectively becomes the owner and operator of the inventory, and prefers to act as an unleveraged retailer, liquidating below the threshold \( d_B^{\text{unlev}}(q) \), and continuing otherwise.

Not surprisingly, the market conditions conducive to this regime are very similar to those leading leveraged retailers to shift their liquidation policy away from unleveraged ones. More precisely, it can be checked that \( s_B \) (\( q_B \)) is strictly increasing (decreasing) in \( M, R \) and \( c \), and \( q_B \) is strictly increasing in \( x_0 \). As such, ceteris paribus, larger demand growth and higher leverage (due to higher interest rates, higher unit costs, or less initial capital) make the effect more likely to occur.

While in this paper we assumed perfect and symmetric information, it is nonetheless interesting to briefly consider a setting where the bank is unable to directly observe the retailer’s first period sales. In such a case, when optimal, the bank’s non-threshold liquidation policy might induce a retailer with sales above \( \frac{Rw - 2qs}{1-2s} \), but below \( d_B \), to underreport the sales, so as to avoid liquidation.\(^{11}\) This effect has already been documented in the context of debt-service renegotiation, where a borrower in default may overstate their debt service abatement in order to obtain better terms from the lender. Specifically, Bourgeon and Dionne (2013) find that asymmetric information about liquidation value might induce firms with high values to act as firms with lower values. In our case, sales underreporting would not be driven only by information asymmetry, but also by the players’ operational preferences. Similar to Bourgeon and Dionne (2013), our model also suggests that this behavior would be more likely in markets where liquidation values are low and leverage is high.

3.3 Liquidation Conflict And The Role of Covenants

To understand how tension between the players may arise, we now compare their optimal liquidation policies, and identify the circumstances under which they are in (dis)agreement. To this end, note that the retailer would never prefer liquidation if this action lead to insolvency. As such, whenever the retailer prefers liquidation, the bank is always made whole, and

\(^{11}\)It can be checked that such a retailer would prefer continuation to liquidation, and may thus choose to report sales strictly below \( \frac{Rw - 2qs}{1-2s} \), but above \( d_B^{\text{unlev}} \).
the two players are in agreement. When the retailer prefers to continue, however, it is possible that the bank might prefer liquidation. This prompts us to introduce the following definition.

**Definition 4.** We define the disagreement region $\mathbb{D}$ as the set of first period demand realizations for which the liquidation preferences of the two players are misaligned. More formally,

$$
\mathbb{D} \overset{\text{def}}{=} \left\{ d \geq 0 \mid X_{R,L}(q,d) < \mathbb{E}[X_{R,L}(q,d,D_2) \mid D_1 = d] \text{ and } X_{B,L}(q,d,D_2) > \mathbb{E}[X_{B,L}(q,d,D_2) \mid D_1 = d] \right\}.
$$

Whenever $\mathbb{D} \neq \emptyset$, we say that liquidation conflict exists between the two players.

Liquidation conflict here is a direct manifestation of agency issues, driven by the shareholder-debtholder conflict of interest (Jensen and Meckling 1976, Myers 1977, Smith and Warner 1979). Intuitively, by continuing, the retailer (equity holder) has limited downside and potentially large upside, due to his leverage. He is thus effectively transferring the underlying risk to the bank (debt holder). On the contrary, the bank may prefer liquidation, so as not to expose the collateral to further potential depreciation. Note that the existence and the extent of liquidation conflict critically depend on the retailer’s order quantity $q$ and market parameters. To this end, our next result precisely characterizes the disagreement region, showing that it is always a (possibly empty) interval of demand values, intrinsically related to whether a retailer is sufficiently leveraged.

**Lemma 5 (Liquidation conflict).** In equilibrium, liquidation conflict arises if and only if the retailer is sufficiently leveraged. More precisely,

$$
\mathbb{D} = \begin{cases} 
\emptyset, & \text{if } q \leq q_\mathbb{D}, \\
(d(q), \overline{d}(q)), & \text{otherwise},
\end{cases}
$$

where

$$
d(q) \overset{\text{def}}{=} \begin{cases} 
\max \left( \frac{d_{\mathbb{D}}^{\mathbb{K}}(q)}{1-2s}, \frac{Rw-2sq}{1-2s} \right), & \text{if } M \geq \tilde{M}, \\
\max \left( \frac{Rw}{M}, \frac{Rw-2sq}{1-2s} \right), & \text{otherwise},
\end{cases}
$$

and

$$
\overline{d}(q) \overset{\text{def}}{=} \begin{cases} 
d_B(q), & \text{if } M \geq \tilde{M}, \\
\min \left( d_B(q), d_{\mathbb{K}}^{\mathbb{D}} \right), & \text{otherwise},
\end{cases}
$$

Moreover, $\overline{d}(q) - d(q)$ is increasing in $q$.

The result shows that the two players are in complete agreement, i.e., $\mathbb{D} = \emptyset$, when the retailer is not sufficiently leveraged. Otherwise, liquidation conflict always arises, at intermediate levels of sales, $(d(q), \overline{d}(q))$. This is quite intuitive, since for low (high) enough sales, both players agree that the optimal action is to liquidate (continue). Furthermore, there is increasing conflict as leverage increases, and, ceteris paribus, conflict is more likely as the market strength $M$, the interest rate $R$ or the per-unit cost $c$ increase, or as the retailer’s initial capital $x_0$ decreases.

In view of our earlier results, it is interesting to note that liquidation conflict arises exactly when leveraged retailers use optimal liquidation policies that differ from those of unleveraged ones. Also, while liquidation conflict always arises in the presence of bankruptcy risk in non-shrinking

---

12 We note that agency issues may exist between the two players in other forms, as well, e.g., concerning the choice of initial order quantity $q$ (see, e.g., Buzacott and Zhang 2004). We use the term “liquidation conflict” to completely isolate the effect, and pinpoint that it is related to dynamic inventory decisions, i.e., liquidation policies.
markets, that is not necessarily the case if the market is shrinking: strictly higher leverage may be required to generate liquidation conflict, than to result in bankruptcy risk (see Lemma 3).

These results also shed light on why financial covenants may be useful for lenders. When there is disagreement concerning the management of collateral, a well chosen covenant has the potential of offering the lender protection through the transfer of control rights. Specifically, assume that the retailer orders a quantity \( q \) that would result in liquidation disagreement, i.e., \( \mathbb{D} \neq \emptyset \). Then, if the covenant demand threshold \( \delta \) were chosen such that \( \delta \in \mathbb{D} \), the covenant would give the bank the right to force liquidation at \( t = 1 \) whenever \( d_1 \in (d(q), \delta) \), an action that would be optimal for her, but not for the retailer. As such, a well designed covenant would reduce the risk transferred by the retailer to the bank through his operational decisions, i.e., inventory management.

The use of contract terms to alleviate shareholder-debtholder conflicts of interest has been recognized in the operations management literature before. In a static (one-period) model, Buzacott and Zhang (2004) argue that highly leveraged retailers tend to over-invest in inventory, and discuss how loan limits can be used to reduce the risk transferred to the bank. In their model, loan limits are a control for the lender that is enforceable only ex-ante, at period \( t = 0 \). Our results suggest that retailers under high leverage can still exhibit similar behavior, not just at the inception of the debt agreement, but during the entire life of the contract. In this sense, covenants emerge as a suitable control mechanism for mitigating agency issues in a dynamic setting. Furthermore, as our findings in the next sections will emphasize, the two mechanisms are not substitutable, so that, even if the parties are in agreement about the investment size \( q \) at time \( t = 0 \) or the bank does impose a loan limit, that does not prevent the retailer from transferring risk to the bank through the liquidation decision at period \( t = 1 \).

4 Equilibrium Analysis

Thus far, our analysis has focused on the players' liquidation policies in subgame \( S \). We now switch our focus to the decisions at \( t = 0 \) and the equilibrium of the overall game.

As discussed in Section 2.2, the bank in a perfectly competitive lending market chooses her contract terms \( \kappa \) at \( t = 0 \) so as to only break even, i.e., for any order quantity \( q \) desired by the retailer, \( R, \tilde{w},^13 \) and \( \delta \) are chosen so that \( \pi_B(q, \kappa) = 0 \). As such, a profit-maximizing retailer should, in principle, seek decisions that maximize the total expected profit of the channel,\(^14 \) since the latter is always identical to his own in this setting. In fact, these channel-optimal decisions can be readily characterized, as they exactly correspond to the actions of an unleveraged retailer with ample capital: the order quantity, denoted by \( q^{\text{unlev}} \), would maximize the profit expression in (1), and the liquidation policy would be exactly \( \ell_{R}^{\text{unlev}} \), described in Lemma 1. For the rest of the analysis, we therefore assume that \( x_0 < cq^{\text{unlev}} \), since otherwise the bank becomes irrelevant.

\(^{13}\)In fact, the loan limit is a redundant parameter in this setting, since \( R \) itself is sufficient to ensure that the bank breaks even (see, e.g., Xu and Birge 2004).

\(^{14}\)In keeping with classical terminology used in supply chain management, we use the term channel to refer to the system consisting of all the players, i.e., the retailer and the bank.
In this context, from a leveraged retailer’s perspective, the inclusion of a covenant in the debt agreement would have ambivalent effects, implying less operational flexibility concerning the liquidation policy, but also a potentially lower interest rate from the lender (reflective of the additional protection afforded by covenants). Thus, the players’ equilibrium decisions will critically reflect this trade-off between flexibility and interest rates. Our next result characterizes the precise conditions under which, in equilibrium, covenants are necessary in lending agreements.

**Theorem 1.** In equilibrium, a covenant is necessarily included if and only if one of the following equivalent conditions holds:

(a) there is liquidation conflict,

(b) the retailer’s initial capital $x_0$ is below a particular threshold $\tilde{x}_0$, which depends only on the market parameters $(M, c, s, \text{and } F_1)$.

Furthermore, bankruptcy risk is

(i) a necessary and sufficient condition for covenants to be included in a non-shrinking market,

(ii) a necessary (but not sufficient) condition for covenants to be included in a shrinking market.

The result suggests that, under liquidation conflict, retailers who are sufficiently leveraged (i.e., have sufficiently low initial capital) would always prefer covenants in their debt agreements. They would thus always relinquish operational flexibility, in exchange for better interest rates.

According to Theorem 1, covenants are tantamount to bankruptcy in non-shrinking markets. Given that, in practice, bankruptcy risk persists in a vast majority of debt agreements, this suggests that covenants should also be ubiquitous. This is consistent with empirical findings: Bradley and Roberts (2004) find a positive relation between the inclusion of covenants and bankruptcy risk (as measured through credit spreads), and Roberts and Sufi (2009a) find that 97% of all loans contain at least one financial covenant. It is also aligned with insights in the finance literature, which often informally equate\(^\text{15}\) the presence of covenants to bankruptcy risk (Myers 1977). In addition, our result also highlights a distinction between bankruptcy and covenants in shrinking markets, where risky debt agreements without covenants may be possible. To the best of our knowledge, this insight is new, and is afforded exclusively by the more detailed operational model.

In a different sense, our findings also suggest that, when adjusted for bankruptcy risk, the prevalence of covenants is likely to be higher in non-shrinking markets than in shrinking ones. This may seem surprising at first sight, as collateral/inventory in a non-shrinking market is worth more, and hence might be deemed as more “secure” by lenders. However, the insight is reversed when one recognizes that the primary role of covenants is mitigating liquidation conflict, which is more likely under non-shrinking markets, where (sufficiently) leveraged retailers behave in riskier ways. Along the same train of thought, it is also natural to expect that retailers with lower initial capital are more likely to be faced with covenants, as our theorem confirms. All these findings are well aligned with empirical results in corporate finance. In particular, Bradley and Roberts (2004) find

\(^{15}\)For instance, in a summary of his insights, Myers (1977) states that “[...], a firm with risky debt outstanding, and which acts in its stockholders’ interest, will follow a different decision rule than one which can issue risk-free debt or which issues no debt at all.”
that firms that are smaller or have fewer tangible assets (i.e., small $x_0$), and firms with greater growth opportunities face more covenants in their debt agreements.

### 4.1 Equilibrium With(out) Covenants

To better understand the underlying trade-offs and the role of covenants in this setting, it will be helpful to first study a hypothetical game in which no covenants are allowed in the debt agreement. Recall from our earlier discussion that ordering $q^{\text{unlev}}$ and liquidating according to $\ell^{\text{unlev}}$ would be profit-maximizing actions for the retailer (and the channel). Consider now a retailer who desires to order $q^{\text{unlev}}$, but has insufficient capital. Since the debt agreement signed at $t = 0$ cannot explicitly bind him to a particular liquidation policy at $t = 1$, once the debt is in place, the (now leveraged) retailer would actually follow the policy $\ell^{\text{lev}}$, which may generally differ from $\ell^{\text{unlev}}$, see Lemma 2.

Since the bank rationally anticipates this behavior, she would charge a higher interest rate. In the resulting equilibrium of subgame $S$, $\ell^{\text{lev}}$ would then be the policy followed, resulting in a profit loss for the channel, and thus the retailer himself. Furthermore, under these circumstances, it is even unclear whether $q^{\text{unlev}}$ would be the equilibrium order quantity for the retailer.

The following result formalizes the discussion above, by characterizing the equilibrium in this hypothetical game without covenants.

#### Theorem 2.
When no covenants are allowed, in equilibrium,

(i) if $x_0 \geq \bar{x}_0$, the channel-optimal actions are followed, i.e., the order quantity is $q^* = q^{\text{unlev}}$, and the liquidation policy is $\ell^* = \ell^{\text{unlev}}$.

(ii) if $x_0 < \bar{x}_0$, the retailer’s order quantity is smaller than the channel-optimal, i.e., $q^* < q^{\text{unlev}}$.

The result highlights that, without covenants, retailers with low initial capital will underinvest, resulting in a loss of profit compared to the channel optimum. In this context, the inclusion of a covenant that would more closely align the equilibrium policy at $t = 1$ with $\ell^{\text{unlev}}$ would increase the retailer/channel profit, by partially alleviating the liquidation conflict. To confirm this intuition, we now analyze the equilibrium under our original setting, where covenants can be included in the debt agreement, and show that the covenant is, in fact, able to restore channel optimality.

#### Theorem 3.
When covenants are allowed, the channel-optimal actions are followed in equilibrium, i.e., the order quantity is $q^* = q^{\text{unlev}}$ and the liquidation policy is $\ell^* = \ell^{\text{unlev}}$, independent of the retailer’s initial capital $x_0$.

When contrasted with our earlier findings in Theorem 2, this result critically highlights the role of covenants, and their impact on operational decisions. Specifically, covenants act not only as a protection mechanism for the bank, but also as a value-creating mechanism for the retailer, allowing him to maximally exploit the potential of a business opportunity, irrespective of his initial capital.

---

16This is consistent with typical assumptions in the literature on incomplete contracts (see, e.g. Aghion and Bolton 1992, Hart and Moore 1998, Roberts and Sufi 2009b), as well as with observed practice. Contracts that seek to prescribe actions or payments for every possible contingency would be overly complex, and would also not be enforceable ex-post in a court (Gärleanu and Zwiebel 2009, Hilson 2013, Tirole 2006).
Covenants emerge as a non-substitutable contract term under perfectly competitive lending markets. Moreover, we find they are particularly effective for retailers with limited initial capital, thus providing theoretical backing for empirical findings in Bradley and Roberts (2004).

As a side remark, we note that the ability of the covenant to restore channel optimality does rely on some specific details of our present setting. For instance, it is no longer true when the lender behaves in a monopolistic way (see Section 5) or if retailers have increased operational flexibility. In the latter case, additional covenants may be needed to alleviate liquidation conflict (see Section 6).

We conclude our discussion by characterizing the optimal covenant level, and highlighting its dependence on market parameters.

**Theorem 4.** In equilibrium, the optimal covenant demand threshold is given by \( \delta^* = \frac{2s q^{unlev}}{M+2s-1} \). Moreover, \( \delta^* \) is increasing in \( s \), and decreasing in \( c \).

These comparative statics might initially seem surprising. Ceteris paribus, one might expect markets with higher liquidation values for the collateral (i.e., larger \( s \)) or lower production costs (i.e., lower \( c \)) to be more secure, and to thus warrant less protection, in the form of lower covenant demand thresholds. However, as discussed in Section 3, these regimes are also exactly the ones making retailers sufficiently leveraged, hence leading to more liquidation conflict, and thus the need for stricter covenants.

Finally, we note that \( \delta^* \) has a non-monotonic behavior in \( M \), since \( q^{unlev} \) typically increases in the demand growth rate \( M \). Intuitively, higher \( M \) would translate into better second period prospects, hence a lower liquidation threshold. At the same time, however, it also leads to more intense agency issues, due to increased liquidation conflict, as well as higher leverage (through larger order quantities).

## 5 Covenants Under Monopolistic Lending

We now consider extensions to our model and alternative settings. We start by studying the game under the assumption of a monopolistic lending market, which may be relevant in situations where the collateral is highly specialized, and only a confined number of lenders have the adequate expertise, i.e., as far as valuation, assessment and potential liquidation are concerned. Furthermore, several papers in the operations management literature have considered this alternative setting as well, e.g., Buzacott and Zhang (2004), Dada and Hu (2008), Alan and Gaur (2012), and Boyabatlı and Toktay (2011).

Under a monopolistic lending market, both players now choose their actions so as to maximize their expected profits. Thus, the retailer chooses \( q \) again by solving (2). Let \( q^*(\kappa) \) be his optimal order quantity, assumed to be unique to avoid unnecessary technical complications.

In this setting, we consider a Stackelberg game, with the bank leading by choosing the contract...
terms so as to maximize her profit, anticipating the retailer’s optimal response:

\[
\begin{align*}
\text{maximize} & \quad \pi_B(q^*(\kappa), \kappa) \\
\text{subject to} & \quad R \geq 1 \\
& \quad w, \delta \geq 0.
\end{align*}
\]

The sequence of events under this alternative game is illustrated in Figure 3.

![Figure 3: Game under a monopolistic lending market.](image)

The bank’s choice of contract terms is now affected by particular trade-offs involving the retailer’s response. To start, it can be checked that, all else being equal, as the bank increases the covenant demand threshold \(\delta\), her expected profit is increasing, i.e., \(\frac{\partial \pi_R}{\partial \delta} \geq 0\), while the retailer’s expected profit is decreasing, i.e., \(\frac{\partial \pi_R}{\partial \delta} \leq 0\). The latter fact might then induce the retailer to adjust his order quantity in order to mitigate his losses, a strategic response that may reduce the bank’s profit, as well. Furthermore, the interaction of the covenant with other contract terms is potentially unclear, as well. One may suspect that the protection afforded by a covenant might, in fact, be equivalent to an interest rate increase, as far the lender’s expected profit is concerned. On the other hand, as our previous discussion in Section 3.2 suggested, covenants allow lenders to dynamically react as new information is revealed, unlike interest rates.

In light of these issues, it remains unclear under what circumstances (if any) covenants actually persist in equilibrium under a monopolistic lending market. Our next result addresses this question.

**Theorem 5.** In equilibrium,\(^{17}\) under a monopolistic lending market, a covenant is necessarily included if and only if there is liquidation conflict.

Furthermore, bankruptcy risk is

(i) a sufficient condition for covenants to be included in non-shrinking markets, and
(ii) neither sufficient, nor necessary for covenants to be included in shrinking markets.

Our results mirror the findings under a perfectly competitive lending market. Interestingly, we find that, under liquidation conflict, including covenants is always optimal for the lender, despite the strategic response of the retailer. Furthermore, covenants emerge as necessary terms in lending

\(^{17}\)We assume that \(\frac{\partial^2 \pi_R}{\partial q^2} |_{q^*} < 0\) holds. This technical condition is rather harmless in practice. It is only slightly stronger than requiring \(q^*\) to be a strict local optimum of \(\pi_R\). The assumption is unnecessary if \(\bar{w}\) is binding.
agreements, and cannot be substituted, in general, by adjustments to the interest rate and/or the loan limit. Note that this is in contrast with the relationship between the latter two terms: it is known that, in the absence of other considerations, loan limits are substitutable by interest rates under monopolistic lending markets (see, e.g., Buzacott and Zhang 2004, Dada and Hu 2008).

Compared to our results under perfect competition, we find that for non-shrinking markets, bankruptcy is no longer a necessary condition for covenants to exist, but is still sufficient. Loosely speaking, covenants are no longer synonymous with bankruptcy risk, but are just implied by it. The reason behind this discrepancy is that, by using covenants, a monopolist bank may be able to completely eliminate bankruptcy risk in some cases. In shrinking markets, bankruptcy risk is no longer indicative for the presence of covenants. Specifically, it is possible that contracts include covenants that in fact eliminate bankruptcy risk, as we remarked above, or that there is bankruptcy risk that does not necessarily result in liquidation conflict, and thus makes covenants superfluous. It is interesting to note that Theorem 5, interpreted in a different light, also suggests that covenants would be more prevalent under monopolistic settings compared to perfect competition (after adjusting for bankruptcy risk).

We conclude our study of the monopolistic setting by characterizing the dependence of the equilibrium covenant demand threshold $\delta^*$ on market parameters. Note that under a monopolistic setting, channel-optimal actions will not be taken in general, due to double marginalization. Furthermore, the expected profits $\pi_R$, $\pi_B$ are typically multi-modal, making it intractable to fully characterize the equilibrium solution in closed form, even for simple demand distributions, e.g., uniform. For that reason, we conduct numerical experiments.

We present comparative statics for the optimal covenant demand threshold $\delta^*$ obtained in equilibrium under a monopolistic lending market. Figure 4(a) shows a heatmap of $\delta^*$ for different values of the purchase cost $c$ and the liquidation value $s$ (for a fixed $x_0 = 2$ and $M = 3$), while Figure 4(b) shows a heatmap of $\delta^*$ for different values of demand growth rate $M$ and initial equity $x_0$ (for fixed $c = 0.6$ and $s = 0.3$). Red (blue) indicates higher (lower) optimal covenant demand thresholds in equilibrium. Unsurprisingly, both figures show non-monotonic relationships between the parameter values and the covenant threshold. Nonetheless, one can identify the following trends. According to Figure 4, lower liquidation values, higher purchasing costs, lower equity, and higher demand growth rate all tend to lead to higher demand thresholds.

To conclude this section, we note that one of the most important implications of our findings is that covenants emerge once again as critical contractual terms in debt agreements, non-substitutable by interest rates or loan limits. Since covenants have direct implications on the borrower’s operational decisions and flexibility, we believe that they should be an integral part of any operations models that include long-term debt financing.
6 Operational Flexibility with Partial Liquidation

The models we considered so far only allowed the two players (retailer and lender) a choice between full liquidation or continuation at the intermediate period, $t = 1$. Full liquidations correspond to discontinuation of divisions, products, or going out of business sales (Craig and Raman 2013). In practice, however, retailers may have additional flexibility, and be able to conduct partial liquidations, e.g., by performing temporary promotional or clearance sales (Talluri and Van Ryzin 2004), or by closing down only a number of (underperforming) stores. Similarly, upon a covenant violation and the transfer of control rights, lenders may force retailers to only partially liquidate, thereby increasing cash holdings and being able to continue with a leaner business.

In this section, we investigate the impact of this additional flexibility on the operational and lending decisions, and on the role played by covenants. To facilitate the comparison, we preserve all other assumptions made so far. That is, partial liquidation still generates per-unit revenues of $s$, independent of the quantity liquidated. We furthermore implicitly assume that, after a partial liquidation, the retailer can continue selling at the full price $p = 1$ in the second period, and the demand distribution is unaffected by the liquidation price $s$ or the amount of liquidated inventory.\footnote{For instance, this assumption would apply when liquidating geographically segregated stores or units of the same business, or returning products to the manufacturer at less than the wholesale price. For other examples, see Talluri and Van Ryzin (2004).}

We preserve the same notation throughout, and occasionally include the superscript $\dagger$ to denote corresponding quantities or policies under the partial liquidation model. For conciseness, we also omit including some of their explicit formulas; we refer the reader to the Appendix and the proofs for details.

Let $\ell_P : \mathbb{R} \times \mathbb{R} \to [0, 1]$ denote the optimal standalone policy for player $P \in \{R, B\}$, in case (s)he were solely responsible for making the liquidation decision. That is, $\ell_P(q, d_1)$ denotes the...
fraction of leftover inventory in period \( t = 1 \) that player \( P \) would prefer to liquidate, when the starting inventory was \( q \), and the realized first period demand was \( d_1 \). For instance, under the retailer’s policy, \((q - d_1) + \ell^\text{R}_\text{†}(q,d_1)\) units would be liquidated at time \( t = 1 \), resulting in a starting inventory level of \((q - d_1) + (1 - \ell^\text{R}_\text{†}(q,d_1))\) for the second selling period. Note that this is essentially the same model as in Section 2, except that both policies \( \ell^\text{P}_\text{†}, P \in \{R, B\} \), can now take fractional values between 0 and 1. Additionally, note that the policy \( \ell^\text{R}_\text{†} \) would also be the equilibrium policy of the retailer when no covenant is in place. The following result characterizes the policies \( \ell^\text{P}_\text{†}, P \in \{R, B\} \), highlighting the region of demand realizations where disagreement may arise.

**Lemma 6 (Standalone liquidation policies).** (a) The players’ optimal standalone policies at \( t = 1 \), in case they were solely responsible for the liquidation decision, are

\[
\ell^\text{R}_\text{†}(q,d) = 1 - \frac{\min\{(M - 1)d, q - d\}}{q - d},
\]

\[
\ell^\text{B}_\text{†}(q,d) = \max\left\{ \ell^\text{R}_\text{†}(q,d), 1 - \frac{\max\{d_{\text{AI}}(q) - d, 1 - s(d - d_{\text{AI}}(q))\}}{q - d} \right\},
\]

where \( d_{\text{AI}}(q) \equiv \frac{R_w - sq}{1 - s} \).

(b) The optimal policies of the two players are in disagreement, i.e., \( \ell^\text{B}_\text{†}(q,d) > \ell^\text{R}_\text{†}(q,d) \), if demand falls in the interval \( \mathbb{D}^\dagger \), which is nonempty if and only if \( q > q^\dagger_{\text{D}} \).

Note that the policies critically depend on the demand level \( d_{\text{AI}} \), which can be interpreted as a threshold for *accounting insolvency*. That is, if \( d_1 \leq d_{\text{AI}}(q) \), and all the remaining inventory were liquidated, the retailer’s net worth, of \( d_1 + s(q - d_1) \), would be lower than his liabilities, \( R_w \).

Comparing the players’ standalone policies yields similar qualitative insights as our earlier analysis. In particular, the bank takes a more conservative approach than the retailer, preferring higher liquidation fractions \( (\ell^\text{B}_\text{†}(q,d) \geq \ell^\text{R}_\text{†}(q,d)) \) so as not to expose the collateral to further potential depreciation. Disagreement arises only for high enough order quantities \( (q > q^\dagger_{\text{D}}) \), i.e., for high leverage, and this becomes increasingly likely in markets exhibiting faster depreciation, larger growth opportunities, smaller profit margins, more capital-constrained retailers, or larger interest rates. In such cases, there is a *disagreement interval* \( \mathbb{D}^\dagger \) of demand realizations, over which the retailer prefers to liquidate a strictly smaller quantity than the bank. Here, by carrying a higher stock into the second period, the retailer takes a riskier bet, and essentially transfers risk to the bank, as in our previous model.

It is interesting to compare the disagreement intervals under the two settings.

**Corollary 1 (Comparison of disagreement intervals).** For \( q \leq q^\dagger_{\text{D}}, \emptyset = \mathbb{D}^\dagger \subset \mathbb{D} \). For \( q > \frac{R_w}{R_c - \max\{1, M, s\}} > q^\dagger_{\text{D}}, \mathbb{D} \subset \mathbb{D}^\dagger \).

Corollary 1 highlights the impact of the additional operational flexibility afforded by partial liquidation on the game between the retailer and lender, and the critical dependency on leverage. Ceteris paribus, a retailer with low leverage \( (q_D < q \leq q^\dagger_{\text{D}}) \) would use the additional flexibility in
a way that serves both players well, as the disagreement interval vanishes. This is consistent with Chod and Zhou (2014), who find that flexibility can partially mitigate the shareholder-debtholder agency conflict. However, our results also show that more operational flexibility would result in more acute disagreement when the retailer is highly leveraged \((q > q^\dagger)\), since the agents’ liquidation policies would differ over a wider range of possible demand realizations. In this sense, this nuances the findings of Chod and Zhou (2014), by eliciting the dependency on leverage.

As with our original model, the existence of a disagreement interval, and thus liquidation conflict, implies that a covenant can be potentially beneficial for the bank. In particular, consider again a cash-flow-to-debt covenant, imposing a minimum cash flow threshold \(\delta\). The cash flow of the retailer at \(t = 1\) is given by his sales and liquidation revenues. In case the retailer follows \(\ell^\dagger_{\text{R}}\), then the covenant is violated if

\[
\text{retailer’s cash flow} = \min\{d_1, q\} + s(q - d_1)^+ + \ell^\dagger_{\text{R}} < \delta.
\]

Interestingly, the retailer now has the operational flexibility to affect and possibly avert a covenant violation through his actions, i.e., his liquidation policy, unlike the model we have considered thus far. An adjustment to his (standalone) policy \(\ell^\dagger_{\text{R}}\) to this end could only be beneficial when following the policy would lead to disagreement in liquidation and a covenant violation. We next identify the interval of demand realizations over which this would be the case.

**Lemma 7 (Covenant violation).** When the retailer follows his standalone liquidation policy \(\ell^\dagger_{\text{R}}\), there is disagreement in liquidation and the covenant is violated if and only if demand falls in the interval \(D^\dagger_{\text{CV}} \subset D^\dagger\).

The interval \(D^\dagger_{\text{CV}}\) precisely captures the demand realizations over which the covenant would effectively offer protection to the lender.\(^{19}\) If the first period demand \(d_1\) falls in that interval though, the retailer could anticipate that following his standalone liquidation policy \(\ell^\dagger_{\text{R}}\) would lead to a violation of the covenant, which, in turn, would result in a transfer of control rights, and a forced, higher liquidation fraction \(\ell^\dagger_{\text{B}}\). As a result, the retailer’s optimal liquidation policy in equilibrium could differ from \(\ell^\dagger_{\text{R}}\), as formalized in the next result.

**Lemma 8 (Equilibrium liquidation policies).** In equilibrium, the retailer’s liquidation policy \(\ell^\star_{\text{R}}\) deviates from his standalone optimal policy \(\ell^\dagger_{\text{R}}\) if and only if, under the latter, there is liquidation disagreement and the covenant is violated, i.e., \(d \in D^\dagger_{\text{CV}}\). In particular,

\[
\ell^\star_{\text{R}}(q, d) = \begin{cases} 
\ell^\dagger_{\text{R}}(q, d) & \text{if } d \notin D^\dagger_{\text{CV}}, \\
\min\{\ell^\dagger_{\text{B}}(q, d), \ell_C(q, d)\} & \text{otherwise},
\end{cases}
\]

where \(\ell_C(q, d) \overset{\text{def}}{=} \min\{z \geq 0 \mid d + s(q - d)^+ z \geq \delta\}\) is the minimum required liquidation fraction so that the retailer’s cash position at \(t = 1\) exactly covers the covenant.

\(^{19}\)In the previous full liquidation model, that interval was \((d(q), \delta)\).
Moreover, for $d \in D_{CV}^\dagger$ and $d > d_{A1}(q)$,

$$\ell_R^\dagger(q, d) < \ell_R^*(q, d) = \ell_C(q, d) < \ell_B^\dagger(q, d),$$

and the covenant does not get violated.

The result confirms that, in equilibrium, a retailer faced with a covenant violation and liquidation conflict would always deviate from his standalone policy $\ell_R^\dagger$, by liquidating a higher fraction of his inventory. In fact, unless he is in accounting insolvency, he would always liquidate just enough inventory to cover the covenant. This would always result in less inventory being liquidated than what the lender would have preferred upon the transfer of control rights. A retailer in accounting insolvency would still behave in the same way, unless the lender preferred him to liquidate a smaller fraction than $\ell_C(q, d)$.

From a practical standpoint, the insights from Lemma 8 suggest that highly leveraged retailers are likely to pull any additional operational levers to prevent covenant violations. From the lender’s perspective, this essentially means that additional operational flexibility diminishes the effectiveness of covenants. This result complements the debt-covenant hypothesis developed in the empirical accounting literature, which argues that managers pull accounting levers to avoid a costly covenant breach. Here, we show that managers can also pull operational levers to achieve the same result. These effects could potentially be empirically observed. For instance, Dichev and Skinner (2002) find that there is an unusually large number of firms with metrics that are right at the covenant breach level. Similarly, one could look at whether fire sales or aggressive markdowns and promotions for retailers financed via ABL are related to the proximity of the covenant breaching point.

In a different (but related) sense, this result is also aligned with Besbes and Maglaras (2012), who argue that non-leveraged retailers operating in a deterministic market, and faced with (exogenous) financial milestones should always adjust prices dynamically to track the most stringent future constraint. Our result confirms and extends this intuition to a stochastic setting, by arguing that, when feasible, it is optimal for leveraged, strategic retailers to conduct fire sales in order to exactly meet a financial covenant.

The results above also suggest that, in the presence of additional operational flexibility, the choice of financial covenant(s) might become relevant for the lender. To see this, consider a case where the bank imposed a covenant involving a valuation of the collateral, e.g., minimum net worth. The evaluation of such a covenant would be agnostic to the retailer’s liquidation actions. In fact, when used in conjunction, a cash-flow-to-debt ratio and a net worth covenant in this setting would allow the lender to distinguish “healthy sales” at full price from “fire sales.” Consequently, lenders may prefer to enact (additional) financial covenants in accordance with (additional) operational flexibility. This result provides theoretical backing for the intuition that lenders often include additional covenants prohibiting the sale of a large percentage of the collateral assets (Tirole 2006).
7 Conclusions

This paper takes an operational view of optimal contracting in asset-based lending. We consider a setting in which a borrowing firm (retailer) can decide to collateralize its working assets (inventory) to obtain financing. The retailer manages inventory through two different decisions: 1) by choosing the order quantity at contract inception, and 2) by (partially) liquidating inventory, after observing initial sales. The lender designs the contract terms, in light of the firm’s operational capabilities.

Endogenizing the inventory decisions allows us to connect a retailer’s operational practices with the design of financial covenants. We argue how agency issues can arise purely through operational frictions, without the need to consider any information or collateral valuation asymmetry. We show that the liquidation decisions of each player can depart from simple threshold policies, leading to perverse reporting incentives for retailers. Furthermore, we also characterize how specific market conditions, such as large market growth, fast depreciation of products, low margins, low equity or larger interest rates, can lead to increased levels of leverage and increased conflict with lenders.

We argue that covenants serve as a key mechanism for alleviating agency issues in both competitive and monopolistic markets, and characterize conditions routinely met in practice under which they emerge in equilibrium. We show that covenants are not substitutable by other contractual terms, such as interest rates or loan limits. As such, their design should be carefully crafted, taking into account the retailer’s operational response. In particular, properly-designed covenants enjoy synergies with interest rates, and can strictly increase channel profits. We also find that 1) absence of covenants can lead to under-investment, 2) higher firm operational flexibility can surprisingly lead to either increased or decreased agency issues, and 3) retailers can employ partial inventory liquidations (product “fire sales”) to avoid a covenant breach, thus diluting its effectiveness.
Notation and Definitions

\[ p = 1, c, s \] selling, purchasing and liquidating per unit prices
\[ D_t, F_t, f_t \] demand value, c.d.f., p.d.f. at time \( t \)
\[ M \] demand growth rate, or market strength parameter
\[ \mathcal{R}, \mathcal{B} \] retailer, bank
\[ x_0, q \] retailer’s initial capital, order quantity
\[ w, R, \bar{w}, \delta \] debt principal, interest rate, loan limit, covenant demand threshold
\[ \kappa \] debt contract terms
\[ \ell_{\mathcal{P}} \] liquidation policy of player \( \mathcal{P} \)
\[ \mathcal{L}, \mathcal{C} \] liquidation, continuation events
\[ d_{\mathcal{P}} \] liquidation demand threshold for player \( \mathcal{P} \)
\[ \mathfrak{D} \] demand disagreement region
\[ X_{\mathcal{P}, \mathcal{E}} \] cash flows of player \( \mathcal{P} \) under event \( \mathcal{E} \)
\[ \pi_{\mathcal{P}} \] expected profits of player \( \mathcal{P} \)
\( \text{(un)lev} \) superscript referring to an (un)leveraged \( \mathcal{R} \)
\( \ast \) superscript denoting quantity in equilibrium
\( \dagger \) superscript denoting quantity for model with partial liquidation

(non-)shrinking market if \( M < 3 \) \((M \geq 3)\)
rapidly shrinking market if \( M < \bar{M} = 2(1 - s) \)
sufficiently leveraged \( \mathcal{R} \) if \( q > q_{\mathfrak{D}} \)

References


BoF. 2014. Frequently asked questions about asset-based lending.


Hilson, JF. 2013. *Asset-based lending: a practical guide to secured financing*. Practicing Law Institute, NY.


31


Appendices

A Robustness of Non-threshold Liquidation Policies

In this section, we confirm that our insights and results pertaining to non-threshold policies are robust, and persist under more general demand distributions, as well as in equilibrium.

A.1 General Demand Distribution

Our main treatment concerning the players’ liquidation policies was conducted under Assumption 1, namely that the second period demand was following a discrete, two-point conditional distribution. In this context, we showed that, under debt, any of the players may find it optimal to follow a non-threshold liquidation policy (see Lemma 2 and Lemma 4). We show that these insights continue to hold for much more general demand distributions.

Theorem 6. Suppose that, conditional on the first period demand, the second period demand distribution has a density that is a Polya frequency function of order 2 (PF2). Then, the retailer may follow a non-threshold liquidation policy. In particular, his liquidation policy has at most two switching points, i.e., there exist (not necessarily identical) \( \xi_1, \xi_2 \in [0, q] \) with \( \xi_1 \leq \xi_2 \), such that

\[
\ell^{lev}_{R}(q, d) = \mathbb{1}\{d \in (\xi_1, \xi_2)\}.
\]

Moreover, the bank also follows a liquidation policy with at most two switching points, and the result is true for any second period demand distribution.

We note that PF2 distributions have been studied extensively in operations management (see, e.g., Porteus (2002) for an overview). It is known that a distribution is PF2 if and only if it is log-concave, i.e., the logarithm of its density function \( f \) is concave. Many of the common distributions are PF2, for instance the exponential, the reflected exponential, the uniform, the Erlang, the normal, and all truncations, translations and convolutions of such distributions (Porteus 2002). In this sense, our assumption on the demand is not very limiting, and is well aligned with standard assumptions made in operations management.

The result confirms that two types of liquidation policies are possible. A player either follows a threshold policy, i.e., liquidating below a demand threshold and continuing above (where the threshold can also be 0 or \( q \)), or follows a non-threshold policy, preferring continuation below a threshold \( \xi_1 \) and above a threshold \( \xi_2 > \xi_1 \), and preferring liquidation between \( \xi_1 \) and \( \xi_2 \). These are exactly the two possible patterns encountered in our base model (see Lemma 2 and Lemma 4), and, as such, our insights concerning the implication of non-threshold liquidation policies persist.

A.2 Non-threshold Policies in Equilibrium

We now confirm that, when the lender is monopolistic, non-threshold policies can persist in equilibrium, for either of the players. We provide both sufficient conditions, as well as particular instances.
Proposition 1. (a) [Retailer] Suppose that \( s < \frac{x_0}{M-c} \), \( M < \bar{M} \), and \( D_1 \geq \frac{x_0}{c(M-1+s)} > 0 \) almost surely. Then, the retailer always follows a non-threshold liquidation policy in equilibrium. One such instance is given by \( s = \frac{1}{16}, c = \frac{1}{3}, x_0 = 1 \) and \( M = \frac{3}{2} \).

(b) [Bank] Suppose that \( s < \frac{(M-1)c}{2(M-c)} \) and \( D_1 \geq \frac{(M-1+2s)x_0}{c(M-1+2s)-2Ms} > 0 \) almost surely. Then, the bank always follows a non-threshold liquidation policy in equilibrium. One such instance is given by \( s = \frac{1}{16}, c = \frac{1}{4}, x_0 = 1 \) and \( M = 2 \).

B Proofs

To simplify notation slightly, we sometimes omit showing the explicit dependency of some quantities on \( q, d \) and/or \( \delta \), e.g., debt, revenues, profits, etc.

B.1 Liquidation Policies

We use the following notation for the two possible values of the critical quantity \( q_\delta \), see (3),

\[
q_1^{1/2} = \frac{R x_0}{Rc - \frac{2s}{M-1+2s}}, \quad q_2^{1/2} = \frac{R x_0}{Rc - \frac{sM}{M+s-1}}.
\]

Proof of Lemma 1. Consider a fixed order quantity \( q \leq \frac{q_0}{c} \). We seek the retailer’s liquidation policy as a function of the first period demand realization, \( D_1 = d \). If \( d \geq q \), the retailer is out of stock, and is indifferent between continuation and liquidation. As such, we consider \( d < q \).

The revenues from a liquidation action are given by \( x_{R\cdot \nu}(q,d) = X_{R\cdot \nu}(q,d) = (1-s)d + sq \).

In view of Assumption 1, the expected revenues from continuation are given by

\[
x_{R\cdot \nu}(q,d) = E[\min(d + D_2, q) | D_1 = d] = \begin{cases} \frac{M+1}{2}d, & \text{if } d \leq \frac{q}{M} \\ \frac{q+d}{2}, & \text{if } \frac{q}{M} < d \leq q. \end{cases}
\]

Note that \( x_{R\cdot \nu}(q, \cdot) \) is an affine function of \( d \) for \( d \in [0, q] \), with slope \( 1-s \). Also, \( x_{R\cdot \nu}(q, \cdot) \) is continuous and piece-wise affine in \( d \), with slope \( \frac{M+1}{2} \) for \( d \in [0, \frac{q}{M}] \), and slope \( \frac{1}{2} \) for \( d \in (\frac{q}{M}, q] \).

Furthermore, \( x_{R\cdot \nu}(q,0) = sq > x_{R\cdot \nu}(q,0) = 0 \), and \( x_{R\cdot \nu}(q,q) = x_{R\cdot \nu}(q,q) \). As such, since \( s < \frac{1}{2} \) (by Assumption 2), we immediately have that: \( x_{R\cdot \nu}(q,d) > x_{R\cdot \nu}(q,d), \forall d \in (\frac{q}{M}, q) \); \( x_{R\cdot \nu}(q,d) > x_{R\cdot \nu}(q,d), \forall d \in (0, \xi) \); and \( x_{R\cdot \nu}(q,d) < x_{R\cdot \nu}(q,d), \forall d \in (\xi, \frac{q}{M}] \), where \( \xi = \frac{(q-d)}{2} \) is the solution to the equation \( \frac{M+1}{2}d = (1-s)d + sq \). It can be readily checked that this exactly corresponds to the liquidation policy \( L_{R\cdot \nu}(q,d) \) described in the lemma. \( \square \)

Proof of Lemma 2. Consider a fixed order quantity \( q > \frac{q_0}{c} \), resulting in a debt \( Rw(q) = cq - x_0 \).

We seek to characterize the retailer’s liquidation policy as a function of the realized demand \( D_1 = d \), at the given order quantity \( q \).

We first argue that the policy does not depend on whether the covenant is breached, i.e., it is identical in the upper and lower nodes of subgame \( S \) at \( t = 1 \) (see Figure 1(b)). In the lower node
of the subgame, when the covenant is not breached, the retailer is the sole player responsible for the liquidation/continuation decision, and obtains his optimal policy by solving the problem:

$$\max_{\ell \in \{0, 1\}} \ell \cdot X_{R, L}(q, d) + (1 - \ell) \cdot \mathbb{E}[X_{R, C}(q, d, D_2) \mid D_1 = d].$$

We claim that this problem also yields an optimal policy in the upper node of the subgame, i.e., when there is a covenant breach. Here, the retailer solves the following problem:

$$\max_{\ell \in \{0, 1\}} \left[\ell + (1 - \ell)\ell_B^*\right] \cdot X_{R, L}(q, d) + (1 - \ell)(1 - \ell_B^*) \cdot \mathbb{E}[X_{R, C}(q, d, D_2) \mid D_1 = d],$$

where $\ell_B^*$ denotes the bank’s optimal response. Note that, if $\ell_B^* = 1$, the retailer’s profits are independent of his decision. When $\ell_B^* = 0$, the two problems above are equivalent.

The retailer’s revenues from liquidation as a function of $D_1 = d$ are

$$x_{R, L}(d) = X_{R, L}(q, d) = \begin{cases} 0, & \text{if } d \leq d_{AI} \\
(1 - s)d + sq - Rw, & \text{if } d_{AI} < d \leq q \\
q - Rw, & \text{if } q < d. \end{cases}$$

Here, $d_{AI} \overset{\text{def}}{=} \frac{Rw - sq}{1 - s}$ denotes the demand level for “accounting insolvency,” i.e., the demand such that, by liquidating all remaining inventory, the retailer is exactly able to replay the entire debt $Rw$. As $d$ raises just above $d_{AI}$, the retailer’s revenues increase linearly, with a slope of $1 - s$, and then saturate upon stock-out, at $d = q$. Similarly, his expected revenues from continuation are

$$x_{R, C}(d) = \mathbb{E}[X_{R, C}(q, d, D_2) \mid D_1 = d] = \mathbb{E}[\min(d + D_2, q) - Rw^+] \mid D_1 = d].$$

In view of Assumption 1, this expression simplifies to the following cases:

If $Rw < \frac{q}{M}$,

$$x_{R, C}(d) = \begin{cases} 0, & \text{if } d \leq \frac{Rw}{M} \\
\frac{M - R}{M}d - Rw, & \text{if } \frac{Rw}{M} < d \leq Rw \\
R - \frac{q - Rw}{M}, & \text{if } \frac{q}{M} < d \leq q \end{cases}$$

If $Rw \geq \frac{q}{M}$,

$$x_{R, C}(d) = \begin{cases} 0, & \text{if } d \leq \frac{Rw}{M} \\
\frac{M - R}{M}d - Rw, & \text{if } \frac{Rw}{M} < d \leq \frac{q}{M} \\
\frac{q - R}{M} - Rw, & \text{if } \frac{q}{M} < d \leq Rw \\
q - Rw, & \text{if } q < d. \end{cases}$$

By continuing, the retailer’s revenues are exactly zero if $d$ falls below $\frac{Rw}{M}$, and then increase in a piece-wise linear fashion, initially with a slope of $\frac{M}{2}$, before saturating upon stock-out, at $d = q$.

The retailer’s liquidation decision exactly entails comparing $x_{R, L}(d)$ with $x_{R, C}(d)$, and liquidating (continuing) if the former (latter) is strictly larger. Several cases emerge, depending on whether $d_{AI} \geq \frac{Rw}{M}$ and $1 - s \geq \frac{M}{2}$. These are treated in Propositions 2 and 3, and are summarized
below. From Proposition 2, if \( M \geq \tilde{M} \), an optimal liquidation policy for the retailer is:

\[
\ell^\text{lev}_R(q, d) = \begin{cases} 
1\{d < d^\text{lev}_R(q)\}, & \text{if } s < s^1_D \text{ and } q > q^1_D, \\
1\{d < d^\text{unlev}_R(q)\}, & \text{otherwise.}
\end{cases}
\]  

(7)

Furthermore, \( \ell^\text{lev}(q, d) \leq \ell(q, d) \) always holds. From Proposition 3, if \( M < \tilde{M} \), an optimal liquidation policy is:

\[
\ell^\text{lev}_R(q, d) = \begin{cases} 
1\{d \in \left[0, \frac{Rw(q)}{M}\right] \cup (d^\text{lev}_R(q), d^\text{unlev}_R(q))\}, & \text{if } s < s^2_D \text{ and } q > q^2_D, \\
1\{d < d^\text{unlev}_R(q)\}, & \text{otherwise.}
\end{cases}
\]

(8)

Furthermore, \( \ell^\text{lev}(q, d) \leq \ell^\text{unlev}(q, d) \) holds, unless \( s < \min(s^1_D, s^2_D) \) and \( q > \max(q^1_D, q^2_D) \).

By defining \( q_D \) as in the statement of the lemma, it can be readily verified that these policies exactly reduce to the desired ones.

\[\square\]

**Proof of Lemma 3.** We first show that for \( M \geq \tilde{M}, q > q_D \) holds if and only if \( d^\text{unlev}_R < Rw \). To see this, note that

\[
Rw - d^\text{unlev}_R = \frac{(s^1_D - s)q}{2(1 - Rc)(M - 1 + 2s)} - Rx_0 = \frac{(s^1_D - s)(q - q^1_D)}{2(1 - Rc)(M - 1 + 2s)}.
\]

(9)

If \( q > q_D \) holds, that implies that \( q_D = q^1_D < q \), as well as \( s < s^1_D \). Combining these with (9), we get that \( d^\text{unlev}_R < Rw \). On the other hand, if \( s \geq s^1_D \), then \( q \leq q_D = \infty \) holds, and by (9) we also have \( d^\text{unlev}_R > Rw \). If \( s < s^1_D \) and \( q \leq q_D = q^1_D \) holds, we also have \( d^\text{unlev}_R \geq Rw \). Hence, \( d^\text{unlev}_R < Rw \) implies \( q > q_D \).

We now prove the claim of the Lemma in case \( d^\text{lev}_R \) is followed. The proof in case \( d^\text{unlev}_R \) is followed is similar and is omitted. Consider the case when \( q > q_D \). Then, for \( M \geq \tilde{M} \), we argued that \( d^\text{unlev}_R < Rw \). However, \( \ell^\text{lev}_R(q, d^\text{unlev}_R) = 0 \), i.e., if the first period demand is equal to \( d^\text{unlev}_R \), there is continuation which would then lead to bankruptcy if the second period demand is zero. For \( M < \tilde{M} \), we have that \( \ell^\text{lev}_R(q, \frac{Rw}{M}) = 0 \), which again would lead to the same result. As such, \( q > q_D \) is a sufficient condition for bankruptcy risk.

To show that \( q > q_D \) is also necessary for bankruptcy risk when \( M \geq 3 \), let us suppose, for the sake of contradiction, that \( q \leq q_D \) and that there is bankruptcy risk. First note that

\[
(1 - s)d_{A1} = Rw - sq < Rw - d^\text{unlev}_R \leq 0,
\]

where the first inequality holds since \( M \geq 3 \), and the second since \( M \geq \tilde{M} \) and \( q \leq q_D \). Since \( d_{A1} \) is negative, there is no bankruptcy risk at \( t = 1 \). Also, continuation occurs only if \( d \geq d^\text{unlev}_R \) at \( t = 1 \) and cannot thus lead to bankruptcy.

Finally, to show that \( q > q_D \) is not necessary for bankruptcy risk when \( M < 3 \), consider the case when \( q = 27, c = 0.5, s = \frac{7}{16}, R = \frac{13}{8}, M = \frac{5}{4} \) and \( x_0 = 1 \). Then, we get \( s^1_D = \frac{13}{24} > s \),
There is bankruptcy risk at $t = 1$, since $0 < d_{A1} < d_{A1}^{unlev}$. 

**Proof of Lemma 4.** Consider a fixed order quantity $q > \frac{Rw}{c}$, resulting in a debt $Rw(q)$, and seek to characterize the banks’s liquidation policy as a function of the realized demand $D_t = d$, at the given order quantity $q$. In view of Assumption 1, the bank’s expected revenues from liquidation $(x_{B,Φ})$ and continuation $(x_{B,Ψ})$ as a function of $D_t = d$ are respectively given by:

$$x_{B,Φ}(d) = X_{B,Φ}(q, d) = \begin{cases} (1 - s)d + sq, & \text{if } d \leq d_{A1} \\ Rw, & \text{if } d_{A1} < d, \end{cases} \quad (10a)$$

$$x_{B,Ψ}(d) = \mathbb{E}[X_{B,Ψ}(q, D_1, D_2 | D_1 = d)] = \begin{cases} \frac{(M + 1)d}{2}, & \text{if } d \leq \frac{Rw}{M} \\ \frac{d + Rw}{2}, & \text{if } \frac{Rw}{M} < d \leq Rw \\ Rw, & \text{if } Rw < d, \end{cases} \quad (10b)$$

where $d_{A1} \overset{\text{def}}{=} \frac{Rw - sq}{1-s}$. Note that the bank is indifferent if $d \geq Rw$, since, in that case, the first period sales alone are sufficient to cover the entire debt. As such, we focus the discussion of the liquidation decision to cases where $d < Rw$.

It can be readily checked that $x_{B,Φ}$ and $x_{B,Ψ}$ are both continuous and concave in $d$, with $x_{B,Φ}(0) > x_{B,Ψ}(0)$ and $x_{B,Φ}(d) > x_{B,Ψ}(d), \forall d \in (d_{A1}, Rw)$. As such, the bank strictly prefers liquidation for “sufficiently low” or “sufficiently high” demands (i.e., larger than $d_{A1}$). In fact, by comparing (10a) and (10b), it can be seen that two cases can arise:

**Case 1:** If $x_{B,Φ}(\frac{Rw}{M}) \geq x_{B,Ψ}(\frac{Rw}{M})$, then $x_{B,Φ}(d) \geq x_{B,Ψ}(d), \forall d \in [0, Rw)$, so that the optimal liquidation policy for the bank is $\ell_B(q, d) = 1\{d < d_B(q)\}$.

**Case 2:** If $x_{B,Φ}(\frac{Rw}{M}) < x_{B,Ψ}(\frac{Rw}{M})$, then there exist two demand levels $\xi_1 \in (0, \frac{Rw}{M}]$ and $\xi_2 \in [\frac{Rw}{M}, d_{A1})$, such that $\ell_B(q, d) = 1\{d \in (0, \xi_1) \cup (\xi_2, Rw]\}$. Here, $\xi_1 = \frac{sq}{M - s} = d_{A1}^{unlev}(q)$ is the solution of the equation $(1 - s)d + sq = \frac{(M + 1)d}{2}$, and $\xi_2 = \frac{Rw - 2gs}{1-2s}$ is the solution to the equation $(1 - s)d + sq = \frac{d + Rw}{2}$.

Finally, note that $x_{B,Φ}(\frac{Rw}{M}) < x_{B,Ψ}(\frac{Rw}{M})$ holds if and only if $(\frac{M - 1}{2} + s)Rw_0 < \left[\frac{M - 1}{2} + s\right]Rc - Ms$. Since the left term is always strictly positive, the inequality either never holds (when the term multiplying $q$ is non-positive, which is equivalent to $s \geq s_B$), or results in a valid lower bound on $q$, equal to $q_B$ (and valid only for $s < s_B$). As such, **Case 2** arises if and only if $s < s_B$ and $q > q_B$, which completes the proof. 

**Proof of Lemma 5.** The proof follows readily by unifying the cases from Lemma 2 and Lemma 4. We omit it for space considerations. 

**B.1.1 Auxiliary Results**

**Proposition 2.** For $M \geq \tilde{M} \overset{\text{def}}{=} 2(1 - s)$, an optimal liquidation policy for the retailer is given by (7). Furthermore, $\ell_{lev}(q, d) \leq \ell_{unlev}(q, d)$, for all $q$ and $d$. 

37
Proof. We follow the same notation as in the proof of Lemma 2. Several cases emerge.

Case 1: \( \frac{R_w}{M} < d_A \) or \( \{ \frac{R_w}{M} \leq d_A \) and \( M < \bar{M} \)\}. By comparing (4) with (5) and (6), it can be readily checked that \( x_{R,w}(d) > x_{R,w}(d), \forall d \in \left( \frac{R_w}{M}, \frac{q}{M} \right], \) since \( x_{R,w} \) has slopes \( \frac{M}{2} \) (or \( \frac{M+1}{2} \)), which are larger than \( 1-s \) (strictly larger when \( M < \bar{M} \), and at least as large, but with \( x_{R,w}(d_A) > x_{R,w}(d_A) \) when \( M = \bar{M} \)). Combining with Proposition 4, we can see that the optimal liquidation policy for the retailer becomes \( \ell^R_{\text{lev}}(d) = \{ d < \frac{R_w}{M} \} \). To check that this corresponds to (7), note from the definition of \( d^R_{\text{lev}} \) and (11d) that in Case 1 we have \( d^R_{\text{lev}}(q) = \frac{R_w}{M} \). Furthermore, (11b) and (11c) imply that \( s < s^1_D \) and \( q > q^1_D \) hold, which concludes the proof of the case.

Case 2: \( M = 2(1-s) \) and \( \frac{R_w}{M} = d_A \). This is a degenerate case, where \( x_{R,w}(d) = x_{R,w}(d), \forall d \in \left[ \frac{R_w}{M}, \min(Rw, \frac{q}{M}) \right], \) and \( x_{R,w}(d) > x_{R,w}(d), \forall d \in \left( \min(Rw, \frac{q}{M}), q \right) \). As such, any policy of the form \( \ell^R_{\text{lev}}(d) = \{ d < \xi \} \), for some \( \xi \in \left[ \frac{R_w}{M}, \min(Rw, \frac{q}{M}) \right] \) is optimal. Without loss, we can take \( \xi = \frac{R_w}{M} \), but also \( \xi = R_w \). Note that, in this case, \( d^R_{\text{lev}}(q) = \frac{R_w(q)}{M} \), and (11h) implies \( d^R_{\text{lev}}(q) = Rw(q). \) Therefore, both branches in (7) yield optimal policies.

Case 3: \( \frac{R_w}{M} > d_A \). It can be readily seen that \( x_{R,w}(d) > x_{R,w}(d), \forall d \in \left( d_A, \frac{R_w}{M} \right] \). Furthermore, by Proposition 4, \( x_{R,w}(d) < x_{R,w}(d), \forall d \in \left( \frac{q}{M}, q \right] \). Therefore, \( x_{R,w} \) and \( x_{R,w} \) must intersect for some \( d \in \left( \frac{R_w}{M}, \frac{q}{M} \right] \). We distinguish two sub-cases.

Case 3-A: \( R_w \geq \frac{q}{M} \). By (6), \( x_{R,w} \) and \( x_{R,w} \) are both affine functions on \( \left( \frac{R_w}{M}, \frac{q}{M} \right] \), with slopes \( 1-s \) and \( \frac{M}{2} \), respectively. We must also have \( M < \bar{M} \) (if \( M = \bar{M} \), the two lines would be parallel and there could be no intersection). In this case, the policy is given by:

\[
x_{R,w}(d) > x_{R,w}(d), \forall d \in \left[ \frac{R_w}{M}, \xi \right] \quad \text{and} \quad x_{R,w}(d) < x_{R,w}(d), \forall d \in \left( \xi, \frac{q}{M} \right],
\]

where \( \xi = \frac{2s-q-Rw}{M-d} \) is the solution to the equation \( \frac{Md-Rw}{2} = (1-s)d + sq - Rw \).

To see that this exactly corresponds to (7), note first that, by (11d), \( \xi = d^R_{\text{lev}}(q), \) which proves that, in Case 3-A, \( \ell^R_{\text{lev}}(q, d) = \{ d < d^R_{\text{lev}}(q) \}. \) Then, (11c) implies that \( \{ s < s^1_D \) and \( q > q^1_D \} \) must hold, which concludes the proof.

Case 3-B: \( R_w < \frac{q}{M} \). In this case, (5) implies that the (unique) demand level \( d \) where \( x_{R,w}(d) = x_{R,w}(d) \) can occur for either \( d \in \left( \frac{R_w}{M}, Rw \right] \) or \( d \in \left[ Rw, \frac{q}{M} \right] \). The former (latter) occurs if and only if \( x_{R,w}(Rw) \) (larger) occurs if and only if \( x_{R,w}(Rw) \) is larger (smaller) than \( x_{R,w}(Rw) \).

- If \( x_{R,w}(Rw) > x_{R,w}(Rw), \) the demand level satisfying \( x_{R,w}(d) = x_{R,w}(d) \) is the solution to the equation \( \frac{Md-Rw}{2} = (1-s)d + sq - Rw, \) identical to Case 3-A. As such, we again have that \( \ell^R_{\text{lev}}(q, d) = \{ d < d^R_{\text{lev}}(q) \}. \)
- If \( x_{R,w}(Rw) \leq x_{R,w}(Rw), \) the demand level satisfying \( x_{R,w}(d) = x_{R,w}(d) \) is the solution to the equation \( \frac{Md+1}{2} = (1-s)d + sq, \) which is exactly \( \frac{sq}{M+2+s} = d^R_{\text{lev}}. \) As such, we conclude that \( \ell^R_{\text{lev}}(q, d) = \{ d < d^R_{\text{lev}}(q) \}. \)

To see that this corresponds to (7), note that (11b) implies that \( x_{R,w}(Rw) > x_{R,w}(Rw) \) holds if and only if \( \{ s < s^1_D \) and \( q > q^1_D \} \), which is exactly what is required for (7).
Finally, to see that \( \ell_{\text{lev}}(q, d) \leq \ell_{\text{unlev}}(q, d) \) holds for all \( q \) and \( d \), note that (11f) and (11g) imply that \( d_{\text{unlev}}(q) < d_{\text{lev}}(q) \) can only hold if \( s < s_0^d \) and \( q > q_0^d \). As such, (7) directly leads to the desired conclusion.

**Proposition 3.** For \( M < \tilde{M} \overset{\text{def}}{=} 2(1 - s) \), an optimal liquidation policy for the retailer is given by (8). Furthermore, the following modified policy \( \hat{\ell}_{\text{lev}}(q, d) \) is also optimal for the retailer, and satisfies \( \hat{\ell}_{\text{lev}}(q, d) \leq \ell_{\text{unlev}}(q, d) \) for any \( q \) and \( d \):

\[
\hat{\ell}_{\text{lev}}(q, d) = \begin{cases} 
0 & \text{if } \{ M < \tilde{M} \text{ and } s < \min(s_0^d, s_0^q) \} \text{ and } \{ q > \max(q_0^d, q_0^q) \}
\end{cases} \ell_{\text{lev}}(q, d) \text{ otherwise.}
\]

**Proof.** We follow the same notation as in the proof of Lemma 2. Several cases emerge.

**Case 1:** \( \frac{Rw}{M} \leq d_{\text{AI}} \). In this case, we have that \( x_{R,\ell}(d_{\text{AI}}) \geq x_{R,\ell}(d_{\text{AI}}) = 0 \), and (11a) also implies that \( \{ s < s_0^d \} \) and \( q \geq q_0^d \), i.e., we are in the first case of (8). Two sub-cases emerge.

**Case 1-A:** \( Rw \geq \frac{q}{M} \). In this case, (6) and Proposition 4 imply that \( x_{R,\ell}(d) > x_{R,\ell}(d) \), \( \forall d \in (\frac{Rw}{M}, q) \). This is equivalent to the optimal liquidation policy \( \ell_{\text{lev}}(q, d) = 1 \{ d < \frac{Rw(q)}{M} \} \). To see that this is equivalent to (8), note that **Case 1-A** and (11c) imply that \( s < s_0^d \) and \( q > q_0^d \), which, by (11f), implies that \( d_{\text{unlev}}(q) \leq d_{\text{lev}}(q) \), so that \( (d_{\text{lev}}(q), d_{\text{unlev}}(q)) = \emptyset \).

**Case 1-B:** \( Rw < \frac{q}{M} \). Here, we have \( x_{R,\ell}(d) > x_{R,\ell}(d) \), \( \forall d \in (\frac{Rw}{M}, d_{\text{AI}}) \cup (\frac{q}{M}, q) \). In the interval \( (d_{\text{AI}}, \frac{q}{M}) \), \( x_{R,\ell} \) is piecewise-affine, with two pieces, and \( x_{R,\ell} \) is affine. Two possibilities emerge:

- If \( x_{R,\ell}(Rw) \geq x_{R,\ell}(Rw) \), then \( x_{R,\ell}(d) \geq x_{R,\ell}(d) \), \( \forall d \in (\frac{Rw}{M}, q) \), and \( x_{R,\ell} \) and \( x_{R,\ell} \) have at most one point of tangency in \( (d_{\text{AI}}, \frac{q}{M}) \), so the liquidation policy becomes \( \ell_{\text{lev}}(q, d) = 1 \{ d < \frac{Rw(q)}{M} \} \). To see that this corresponds to (8), note that (11b) implies that \( \{ s < s_0^d \} \) and \( q \geq q_0^d \), which, by (11f), implies that \( (d_{\text{lev}}(q), d_{\text{unlev}}(q)) = \emptyset \).

- If \( x_{R,\ell}(Rw) < x_{R,\ell}(Rw) \), then \( x_{R,\ell} \) and \( x_{R,\ell} \) have two intersection points. One such point is given by the solution to the equation \( \frac{Md - Rw}{2} = (1 - s)d + sq - rw \), i.e., \( \frac{2sq - Rw}{M - 2(1 - s)} \). By (11e), this is exactly \( d_{\text{lev}}(q) \). The other such point is the solution to the equation \( \frac{(M + 1)d}{2} = (1 - s)d + sq \), which is exactly \( \frac{sq}{d + s} = d_{\text{unlev}}(q) \). By (11b) and (11f), we also have \( d_{\text{unlev}}(q) > d_{\text{lev}}(q) \), so that the liquidation policy is \( \ell_{\text{lev}}(q, d) = 1 \{ d \in [0, \frac{Rw(q)}{M}] \cup (d_{\text{lev}}(q), d_{\text{unlev}}(q)) \} \), which exactly corresponds to (8).

**Case 2:** \( \frac{Rw}{M} > d_{\text{AI}} \). In this case, we claim that \( Rw < \frac{q}{M} \). To see this, note that \( Rw \geq \frac{q}{M} \) and \( x_{R,\ell}(\frac{Rw}{M}) \geq x_{R,\ell}(\frac{Rw}{M}) = 0 \) would imply, through (6), that \( x_{R,\ell}(\frac{Rw}{M}) > x_{R,\ell}(\frac{Rw}{M}) \), since the slope of the liquidation profits \( (1 - s) \) is greater than the slope of the continuation profits \( \frac{M}{2} \). The latter would be in direct contradiction with Proposition 4. Since \( Rw < \frac{q}{M} \), a similar reasoning to the one above applied to the profits in (5) shows that \( x_{R,\ell}(Rw) > x_{R,\ell}(Rw) \), so that

\[
x_{R,\ell}(d) > x_{R,\ell}(d), \forall d \in (0, \xi) \quad \text{and} \quad x_{R,\ell}(d) < x_{R,\ell}(d), \forall d \in (\xi, \frac{q}{M});
\]
where $\xi$ is the solution to the equation $\frac{(M+1)d}{2} = (1-s)d + sq$, which is exactly $\frac{sq}{d-s} = d^\text{unlev}_R$. As such, $d^\text{lev}_R(q,d) = 1\{d < d^\text{unlev}_R(q)\}$. To see that this exactly corresponds to (8), note that $\frac{Rw}{M} > d_A$ and (11a) imply that $\{s < s^2_D$ and $q > q^2_D\}$ cannot hold.

To see that the modified policy $\hat{d}^\text{lev}_R(q,d)$ satisfies $\hat{d}^\text{lev}_R(q,d) \leq d^\text{unlev}_R(q,d)$, note from (8) that the only case where $d^\text{lev}_R(q,d) \leq d^\text{unlev}_R(q,d)$ might not hold is when $s > s^2_D$, so that the retailer is actually indifferent between liquidation and continuation, so that the policy $\hat{d}^\text{lev}_R(q,d)$ is also optimal. \hfill $\square$

**Proposition 4.** $x_{R,\psi}(d) > x_{R,\varphi}(d), \forall d \in \left[\frac{q}{M}, q\right], \text{i.e., } d^\text{lev}_R(q,d) = d^\text{unlev}_R(q,d) = 0, \forall d \in \left[\frac{q}{M}, q\right]$.

**Proof.** The result trivially holds if $\frac{q}{M} < d_A$, so we only consider the reverse case. We distinguish two cases, depending on whether $Rw \geq \frac{q}{M}$.

If $Rw < \frac{q}{M}$, note from (5) that $x_{R,\psi}$ and $x_{R,\varphi}$ are affine in $d \in \left[\frac{q}{M}, q\right]$, and $x_{R,\varphi}(q) = x_{R,\varphi}(q)$.

Showing the main result is thus equivalent to arguing $x_{R,\psi}(\frac{q}{M}) > x_{R,\varphi}(\frac{q}{M})$, which holds, since:

$$x_{R,\psi}(\frac{q}{M}) - x_{R,\varphi}(\frac{q}{M}) = \frac{q(M-1)(1-2s)}{2M} > 0, \text{ since } s < \frac{1}{2} \text{ and } M > 1.$$ 

If $Rw \geq \frac{q}{M}$, note from (6) that $x_{R,\psi}$ is constant for $d \in \left[\frac{q}{M}, Rw\right]$, and affine, with slope $\frac{1}{2}$ for $d \in (Rw,q]$. Since $x_{R,\varphi}$ is affine for $d \in (\frac{q}{M}, q]$, with slope $0 \leq s < \frac{1}{2}$, and $x_{R,\varphi}(q) = x_{R,\varphi}(q)$, to argue the result, it suffices to show that $x_{R,\psi}(Rw) > x_{R,\varphi}(Rw)$, which holds, since

$$x_{R,\psi}(Rw) - x_{R,\varphi}(Rw) = (q - Rw)\left(\frac{1}{2} - s\right) > 0, \text{ since } q > Rw \text{ and } s < \frac{1}{2}. \hfill \square$$

**Proposition 5.** We have: \cite{20}

$$\frac{Rw}{M} \leq d_A \iff \{s < s^2_D \text{ and } q \geq q^2_D\} \quad (11a)$$

If $Rw < \frac{q}{M}$, then $x_{R,\psi}(Rw) \geq x_{R,\varphi}(Rw) \iff \{s < s^1_D \text{ and } q \geq q^1_D\}$ \quad (11b)

$$Rw \geq \frac{q}{M} \iff \{s < s^1_D \text{ and } q > q^1_D\} \quad (11c)$$

If $M > 2(1-s)$, then $d^\text{lev}_R(q) = \frac{Rw(q)}{M} \iff \frac{Rw}{M} \leq d_A$ \quad (11d)

If $M < 2(1-s)$, then $d^\text{lev}_R(q) = \frac{Rw(q)}{M} \iff \frac{Rw}{M} \geq d_A$ \quad (11e)

If $M < 2(1-s)$, $d^\text{unlev}_R(q) \leq \frac{2sq - Rw}{M - 2(1-s)} \iff \{s < s^1_D \text{ and } q \geq q^1_D\}$ \quad (11f)

$$d^\text{unlev}_R(q) \leq \frac{Rw}{M} \iff \{s < s^1_D \text{ and } q > q^1_D\} \quad (11g)$$

$$\{M = 2(1-s) \text{ and } \frac{Rw}{M} = d_A\} \Rightarrow d^\text{unlev}_R(q) = Rw(q). \quad (11h)$$

\cite{20}In order to save space, since we require both the strict and non-strict versions of the inequalities, we use a compact notation that shows in parenthesis the alternate versions.
Proof. Throughout this proof, recall that $0 < Rc \leq 1$, $M > 1$, and $s < \frac{1}{2}$.

To prove (11a), note that $\frac{Rw}{M} \leq d_{A1}$ holds if and only if:

$$\frac{x_0 R (M + s - 1)}{M (1 - s)} \leq \left( \frac{cR - s}{1 - s} - \frac{cR}{M} \right) q.$$  

Since the left term is always strictly positive, the inequality either never holds (when the term multiplying $q$ is non-positive, which is equivalent to $s \geq s^2_D$), or results in a valid lower bound on $q$ (equal to $q^2_D$, and valid only for $s < s^2_D$). The strict version of the inequality follows similarly.

To prove (11b), note that, by (5), $x_{R, e}(Rw) \geq x_{R, e}(Rw)$ holds if and only if:

$$\frac{M - 1 + 2s}{2} Rx_0 \leq \left( \frac{M - 1 + 2s}{2} Rc - s \right) q.$$  

(12)

Since the left term is always strictly positive, the inequality either never holds (when the term multiplying $q$ is non-positive, which is equivalent to $s \geq s^1_D$), or results in a valid lower bound on $q$ (equal to $q^1_D$, and valid only for $s < s^1_D$). The strict version of the inequality follows similarly.

To prove (11c), note first that $Rw \geq \frac{q}{M}$ is equivalent to

$$(MRc - 1)q \geq MRx_0 \iff \left\{ MRc > 1 \quad \text{and} \quad q \geq \frac{MRx_0}{MRc - 1} \right\}.$$  

(13)

We first show that $MRc > 1$ implies $s < s^1_D$. To this end, note that the latter is equivalent to $Rc > \frac{2s}{M + 2s - 1}$. But $MRc > 1$ implies $Rc > \frac{1}{M}$, and $\frac{1}{M} > \frac{2s}{M + 2s - 1} \iff M - 1 > 2s(M - 1) \iff s < \frac{1}{2}$, which is always true. Therefore, $s < s^1_D$, which implies, by (12), that $q^1_D > 0$. To see that $q > q^1_D$, note that (13) also requires $q > \frac{MRx_0}{MRc - 1}$, and we always have:

$$\frac{MRx_0}{MRc - 1} > \frac{Rx_0}{Rc - \frac{2s}{M - 1 + 2s}} \iff M - 1 + 2s > 2Ms \iff 1 > 2s.$$

To prove (11d) and (11e), note that $d^R_{rev} = \frac{Rw}{M}$ holds if and only if $\frac{Rw}{M} \geq \frac{2q - Rw}{M - 2 + 2s}$. The latter inequality is exactly equivalent to

$$(M + s - 1)Rx_0 \leq \left[ M(Rc - s) - Rc(1 - s) \right] q \quad \text{if} \quad M > 2(1 - s),$$

$$(M + s - 1)Rx_0 \geq \left[ M(Rc - s) - Rc(1 - s) \right] q \quad \text{if} \quad M < 2(1 - s).$$

The former condition is exactly equivalent to condition (13), i.e., to $\frac{Rw}{M} \leq d_{A1}$. Similarly, the latter is equivalent to $\frac{Rw}{M} \geq d_{A1}$.

To prove (11f), note that the relation is equivalent, under $M < 2(s - 1)$, to $Rx_0 (M + 2s - 1) \leq [(M - 1) Rc - 2s(1 - Rc)] q$, which is exactly (12). By the same arguments as in (11b), this is equivalent to $\{ s < s^1_D \}$ and $q \geq q^1_D$.

To prove (11g), note that the condition is equivalent to $Rx_0 (M + 2s - 1) \leq [(M + 2s - 1) Rc - 2sM] q$. Comparing this with (12), it can be seen that the left-hand-sides are identical, while the term multiplying $q$ here is smaller than in (12). As such, if the relation above holds, it must be that $s < s^1_D$ and $q > q^1_D$. 

41
To prove (11h), note that $d_R^{\text{unlev}}(q) - Rw$ becomes equal in this case to:

$$\frac{2sq}{M + 2s - 1} - Rw = sq\left[\frac{2}{M + 2s - 1} - \frac{M}{M + s - 1}\right] = 0,$$

where the first equality follows by expressing $Rw$ from the identity $Rw = \frac{Rw - sq}{1 - s}$, and the second follows by using $M = 2(1 - s)$.

\[\square\]

### B.2 Perfect Competition

For this section, recall that the loan limit $\bar{w}$ is a superfluous contract parameter. To simplify the exposition, we also introduce some notation. Let $R : \mathbb{R} \rightarrow [1, \infty)$ and $\delta : \mathbb{R} \rightarrow \mathbb{R}^+$ be functions that map an order quantity to an interest rate and a covenant demand threshold requirement. In this section, we denote a contract under these terms with $\kappa = (R, \delta)$, and the set of all such competitively priced loans with $K$.

Consider also the following three particular competitively priced contracts of interest:

(a) $\kappa_0 = (R_0, \delta_0)$, with $\delta_0(q) \equiv 0$;

(b) $\kappa_1 = (R_1, \delta_1)$, with $\delta_1(q) \equiv \frac{2sq}{M + 2s - 1}$;

(c) $(R_{\text{unlev}}, 0)$, offered to a retailer following the channel-optimal actions, i.e., ordering $q_{\text{unlev}}$, and liquidating according to $\ell_{\text{unlev}}$. Let $q_{\text{unlev}}$ be as in Lemma 2, calculated using $R_{\text{unlev}}$.

Finally, we derive the first order optimality conditions (FOC) that are necessary and sufficient for the optimal order quantity of an unleveraged retailer $q_{\text{unlev}}$ (also, channel-optimal). Note that, using Lemma 1, we can express (1) as

$$\pi_R(q) = \int_0^{q_{\text{unlev}}^{\text{unlev}}} (sq + (1 - s)u)f_1(u)du + \frac{1}{2} \int_{q_{\text{unlev}}^{\text{unlev}}}^{q} (M + 1)uf_1(u)du$$

$$+ \frac{1}{2} \int_{q}^{q_{\text{unlev}}^{\text{unlev}}} (q + u)f_1(u)du + \int_{q}^{\infty} qf_1(u)du - cq, \quad \forall q \geq 0.$$

Thus, the FOC is: $1 + sF_1(d_R^{\text{unlev}}) = \frac{1}{2} F_1(q/M) + \frac{1}{2} F_1(q) + c$.

**Proof of Theorem 1.** To prove (a), note that Proposition 6 implies that, in equilibrium, the order quantity is always $q_{\text{unlev}}$, and the liquidation policy followed is $\ell_{\text{unlev}}$. By definition (see Lemma 5), liquidation conflict exists if and only if $q_{\text{unlev}} > q_{\text{unlev}}^{\text{unlev}}$. The proof is complete by invoking again Proposition 6.

To prove (b), in view of Proposition 6, it is sufficient to show that if $q_{\text{unlev}} > q_{\text{D}}^{\text{unlev}}$ holds for some $\bar{x}_0$, it also holds for any $x_0 < \bar{x}_0$. Since $q_{\text{unlev}}$ does not depend on $x_0$, it then suffices to show that $q_{\text{D}}^{\text{unlev}}$ is increasing in $x_0$. Note that

$$\text{sgn} \left(\frac{dq_{\text{unlev}}^{\text{unlev}}}{dx_0}\right) = \text{sgn} \left[\frac{d}{dx_0} \left( \frac{R_{\text{unlev}}x_0}{R_{\text{unlev}}^{\text{unlev}} - \theta} \right) \right] = \text{sgn} \left[ R_{\text{unlev}}^{\text{unlev}}(R_{\text{unlev}}^{\text{unlev}} - \theta) - x_0\theta \frac{\partial R_{\text{unlev}}^{\text{unlev}}}{\partial x_0} \right],$$

where $\theta$ is independent of $x_0$ (see (3)). Thus, since $R_{\text{unlev}}^{\text{unlev}} - \theta > 0$, and $R_{\text{unlev}}^{\text{unlev}}$ is decreasing in $x_0$,
by Proposition 7, the proof is complete.

The connection with bankruptcy follows from the results of Proposition 6 and Lemma 3.

**Proof of Theorem 2.** (i) If $x_0 \geq \tilde{x}_0$, by Theorem 1(b) we have that $q_{\text{unlev}} \leq q_{\text{lev}}$. By Proposition 6, the retailer prefers $\kappa_0$, which does not have covenants, orders $q_{\text{unlev}}$ and follows $\ell_{\text{unlev}}$.

(ii) Suppose now that $x_0 < \tilde{x}_0$. Let $g(q, x_0)$ denote the threshold $q_D$ evaluated under the interest rate $R_0(q)$ for a retailer with initial capital of $x_0$, i.e.,

$$g(q, x_0) \overset{\text{def}}{=} \frac{R_0(q)x_0}{R_0(q)c - \theta},$$

where $\theta$ is defined as in the proof of Theorem 1. Arguing in precisely the same way as in (14), since $R_0(q)$ and $R_{\text{unlev}}$ have the same monotonicity in $x_0$, by Proposition 7, we have that $g(q_{\text{unlev}}, x_0)$ is increasing in $x_0$. Based on Theorem 1(b) and Proposition 6, $g(q_{\text{unlev}}, \tilde{x}_0) = q_{\text{unlev}}$. By the monotonicity of $g(q_{\text{unlev}}, x_0)$ with respect to $x_0$, we then get that

$$q_{\text{unlev}} = g(q_{\text{unlev}}, \tilde{x}_0) > g(q_{\text{unlev}}, x_0). \quad (15)$$

Suppose now that in equilibrium the retailer orders $q^*$ and follows $\ell_{\text{unlev}}$. Then, by Lemma 2,

$$q^* \leq g(q^*, x_0).$$

Note also that $g(q, x_0)$ is decreasing in $q$. To see this,

$$\frac{\partial g(q, x_0)}{\partial q} = \frac{\partial g(q, x_0)}{\partial R_0} \frac{\partial R_0}{\partial w} \frac{\partial w}{\partial q} < 0,$$

since the first multiplier above is negative, the second is positive by Proposition 7, and the third is equal to $c$. If $q^* \geq q_{\text{unlev}}$, we get $q^* \geq q_{\text{unlev}} > g(q_{\text{unlev}}, x_0) > g(q^*, x_0)$, a contradiction. Hence, $q^* < q_{\text{unlev}}$.

Finally, if the retailer follows $\ell_{\text{lev}} < \ell_{\text{unlev}}$ in equilibrium, then, since his expected profits are equal to the channel’s when following $\ell_{\text{lev}}$, $q^*$ solves the FOC $sF_1(d_{\text{R}}(q)) + 1 = \frac{1}{2}F_1(q) + \frac{1}{2}F_1(\frac{q}{M}) + c$, where $d_{\text{R}} = d_{\text{lev}}$. Since $q_{\text{unlev}}$ solves the same equation, but for $d_{\text{R}} = d_{\text{lev}} > d_{\text{lev}}$, and $F_1$, $d_{\text{unlev}}$ and $d_{\text{lev}}$ are all increasing in $q$, we conclude that $q^* < q_{\text{unlev}}$.

**Proof of Theorem 3.** The proof follows from Proposition 6.

**Proof of Theorem 4.** By Proposition 6, when covenants are present, contract $\kappa_1$ is optimal and thus the corresponding covenant demand threshold is exactly equal to $\delta^* = \frac{2s_0_{\text{unlev}}}{M + 2s - 1}$.
To prove the comparative statics, it can be readily checked that:

$$\text{sgn} \left( \frac{\partial \delta^*}{\partial s} \right) = \text{sgn} \left[ (M-1)2q_{\text{unlev}} + 2s(M+2s-1) \frac{\partial q_{\text{unlev}}}{\partial s} \right] \geq 0$$

$$\text{sgn} \left( \frac{\partial \delta^*}{\partial c} \right) = \text{sgn} \left( \frac{\partial q_{\text{unlev}}}{\partial c} \right).$$

Recall that $q_{\text{unlev}}$ is the maximizer of the profit $\pi_R$ given by (1). It can be readily checked that:

$$\frac{\partial^2 \pi_R}{\partial q \partial s} = F_1 \left( \frac{2sq}{M+2s-1} \right) + sf_1 \left( \frac{2sq}{M+2s-1} \right) \frac{\partial q_{\text{unlev}}^R}{\partial s} \geq 0,$$

since $\frac{\partial q_{\text{unlev}}^R}{\partial s} = \frac{2q(M-1)}{(M+2s-1)^2} \geq 0$. As such, $\pi_R$ is supermodular in $(q,s)$ on the lattice $\mathbb{R}^2$, so that $q_{\text{unlev}}$ is increasing in $s$ (see, e.g., Topkis 1998). Similarly, it can be checked that $\frac{\partial q_{\text{unlev}}^R}{\partial c} \leq 0$, so that $q_{\text{unlev}}$ is decreasing in $c$.

**B.2.1 Auxiliary Results**

**Proposition 6.** In equilibrium, the retailer prefers $\kappa_0$ ($\kappa_1$) to any other $\kappa \in K$ if $q_{\text{unlev}} \leq q_{\text{unlev}}^D$ ($q_{\text{unlev}} > q_{\text{unlev}}^D$). Moreover, in equilibrium, he orders $q_{\text{unlev}}$, and liquidates according to $\ell_{\text{unlev}}^R$.

**Proof.** Let $\pi_P(q; (R, \delta))$ be the expected profits of agent $P \in \{R, B\}$ in equilibrium when the order quantity is $q$ and the contract offered has an interest rate $R$ and a covenant of $\delta$. Then, under perfect competition $\pi_B(q; (R_i(q), \delta_i(q))) = 0$, for all $q \geq 0, i \in \{0, 1\}$.

Let $x(q; \ell)$ be the net expected revenues at the end of the game, generated by sales and possible liquidation of the inventory, under an order quantity $q$ and a liquidation policy $\ell$. We can express the retailer’s profits as the net expected revenues minus inventory purchasing costs minus the bank’s profits, that is, for any $\kappa = (R, \delta) \in K$

$$\pi_R(q; (R(q), \delta(q))) = x(q; \ell_{\kappa,q}) - cq - \pi_B(q; (R(q), \delta(q))) = x(q; \ell_{\kappa,q}) - cq,$$

where $\ell_{\kappa,q}$ is the liquidation policy followed in equilibrium under quantity $q$ and contract $\kappa$.

Let $\pi_{R^*}^\kappa$ be the retailer’s optimal expected profits under contract $\kappa \in K$. Then,

$$\pi_{R^*}^\kappa = \max_q \pi_R(q; (R(q), \delta(q))) = \max_q \{ x(q; \ell_{\kappa,q}) - cq \}.$$

**Case 1:** $q_{\text{unlev}} \leq q_{\text{unlev}}^D$. We argue that $R_0(q_{\text{unlev}}) = R_{\text{unlev}}$. To see this, note that if the retailer wishes to order $q_{\text{unlev}}$ and the bank offers the contract $(R_{\text{unlev}}, 0)$, then, since $q_{\text{unlev}} \leq q_{\text{unlev}}^D$, the
retailer will follow $\ell^\text{unlev}_R$, according to Lemma 2. Consequently, we also have $\ell_{\kappa_0, q_{\text{unlev}}} = \ell^\text{unlev}_R$, and

$$
\pi^\kappa_{\kappa_0*} = \max_q \{ x(q; \ell_{\kappa_0,q}) - cq \}
$$

$$
\geq x(q_{\text{unlev}}; \ell_{\kappa_0,q_{\text{unlev}}}) - cq_{\text{unlev}}
$$

$$
= x(q_{\text{unlev}}; \ell^\text{unlev}_R) - cq_{\text{unlev}}
$$

$$
\geq \sup_{\kappa \in \mathbb{K}} \max_q \{ x(q; \ell_{\kappa,q}) - cq \}
$$

$$
= \sup_{\kappa \in \mathbb{K}} \pi^\kappa_{\kappa_0*} \geq \pi^\kappa_{\kappa_0*},
$$

where the second inequality follows from the optimality of ordering $q_{\text{unlev}}$ and following $\ell^\text{unlev}_R$ for the unleveraged case and the third inequality from the fact that $\kappa_0 \in \mathbb{K}$.

**Case 2:** $q^\text{unlev} > q^\text{D}_{\text{unlev}}$. Consider contract $\kappa_1$. Then, by the choice of $\delta_1(q)$, we have that $\delta_1(q) = d^\text{unlev}_R(q)$. According to Lemma 4, the bank will prefer liquidation for any $d_1 < d^\text{unlev}_R(q)$. Moreover, since $\ell^\text{lev}_R \leq \ell^\text{unlev}_R$ (see Lemma 2), we have that $\ell_{\kappa_1,q} = \ell^\text{unlev}_R$. Hence,

$$
\pi^\kappa_{\kappa_1*} = \max_q \{ x(q; \ell_{\kappa_1,q}) - cq \}
$$

$$
= \max_q \{ x(q; \ell^\text{unlev}_R) - cq \}
$$

$$
= x(q_{\text{unlev}}; \ell^\text{unlev}_R) - cq_{\text{unlev}}
$$

$$
\geq \sup_{\kappa \in \mathbb{K}} \pi^\kappa_{\kappa_1*} \geq \pi^\kappa_{\kappa_0*}.
$$

**Proposition 7.** For any fixed $q$, $R_0(q)$ is decreasing in $x_0$. Similarly, $R^\text{unlev}$ is decreasing in $x_0$.

**Proof.** We provide a unifying proof for both quantities. Let $R$ below denote either $R_0(q)$ or $R^\text{unlev}$.

Let $\zeta$ denote the probability of bankruptcy, and $x_{br}$ denote the expected channel revenues conditional on bankruptcy. By definition, $x_{br} < Rw$. The bank’s expected profit can be written as

$$
\pi_B = Rw(1 - \zeta) + x_{br}\zeta - w \equiv 0.
$$

As a side remark, this implicit equation yields the $R$ that should be charged by the bank. Using the implicit function theorem, this then yields:

$$
\frac{R(1 - \zeta) - 1}{\leq 0} - \left( Rw - x_{br} \right) \frac{\partial \zeta}{\partial w} > 0 \quad \text{and} \quad +w(1 - \zeta) \frac{\partial R}{\partial w} = 0.
$$

Above, the first term is negative due to the identity $\pi_B \equiv 0$ above, and $\frac{\partial \zeta}{\partial w} \geq 0$ since, ceteris paribus, the probability of bankruptcy increases in the size of the principal. Since $\frac{\partial R}{\partial x_0} = -\frac{\partial R}{\partial w}$, the proof is complete. \qed

45
B.3 Monopoly

For the purposes of this section, it will help to think about the covenant in terms of the parameter \( \alpha \), instead of the demand threshold \( \delta \). Recall that these parameters were equivalent (see Section 2.2).

We first introduce some simplifying notation. Let \((R^*, \bar{w}^*, \alpha^*)\) denote the optimal contract offered by the bank in equilibrium, and let \( q^*(\alpha) \) denote the retailer’s optimal response to a contract \((R^*, \bar{w}^*, \alpha)\), i.e., where only the covenant threshold is changing, and the interest rate and loan limit are fixed to their optimal values. For simplicity, also let \( q^*(\alpha^*) = q^{**} \).

Several quantities of interest in the proof depend on the interest rate \( R \) – for instance, the threshold \( q_D \) defined in (3), and the values \( d(q) \) and \( \bar{d}(q) \), defined in Lemma 5. To avoid introducing unnecessary notation, for all the proofs in the rest of the section, we use the implicit understanding that any such quantity is calculated under \( R = R^* \), i.e., using the optimal interest rate for the bank in equilibrium. This should not create any confusion, since all the arguments presented will not rely on changing \( R \). For simplicity, we also introduce \( \alpha_{min} \) and \( \alpha_{max} \).

We let \( \Delta X_R(q, d) \) denote the expected revenue difference between liquidation and continuation of player \( P \in \{R, B\} \), when the bank contract is \((R^*, \bar{w}^*, \alpha)\), and the retailer orders a generic \( q \). If \( q \) results in liquidation conflict, note that

\[
\Delta X_R(q, d(q)) = 0, \tag{16}
\]

by the definition of \( d(q) \) (see Lemma 5).

**Proof of Theorem 5.** When there is no liquidation conflict, by Lemma 5, the disagreement region is empty, and the equilibrium liquidation policy followed is \( \ell^{lev}_{R}(q^{**}, d) \), according to Lemma 2. As such, a contract \((R^*, \bar{w}^*, 0)\) would result in exactly the same liquidation policy in equilibrium, and would also yield \( q^*(0) = q^{**} \). Without loss, then, no covenant is needed in equilibrium.

For the converse, it suffices to show that, when there is liquidation conflict, \( \alpha^* > 0 \). For the purposes of deriving a contradiction, assume that \( \alpha^* = 0 \). Consider the case when the loan limit \( \bar{w}^* \) is binding, i.e., \( \frac{\partial x}{\partial \bar{w}} q^*(0) > 0 \). Then, for sufficiently small \( \epsilon > 0 \), we would have \( q^*(\epsilon) = q^*(0) \), so the bank can strictly increase her profits by offering the contract \((R^*, \bar{w}^*, \epsilon)\), a contradiction.

We now consider the case where the loan limit is non-binding, and as such, \( q^*(0) \) is given by the first-order optimality condition (FOC) of \( \pi_R \).

We first argue for cases where the bank follows a threshold policy (see Lemma 4). By the implicit function theorem applied to the derivative of the FOC with respect to \( \alpha \), we get that

\[
\frac{\partial^2 \pi_R}{\partial q \partial \alpha} \bigg|_{q^*(\alpha)} + \frac{\partial^2 \pi_R}{\partial q^2} \bigg|_{q^*(\alpha)} \frac{dq^*(\alpha)}{d\alpha} = 0. \tag{17}
\]

We now evaluate the terms above at \((q^*(\alpha_{min}), \alpha_{min})\). Recall that \( q^*(\alpha_{min}) = q^*(0) \). By our standing assumption,

\[
\frac{\partial^2 \pi_R}{\partial q^2} \bigg|_{q^*(\alpha_{min})} = \frac{\partial^2 \pi_R}{\partial q^2} \bigg|_{q^*(0)} < 0. \tag{18}
\]
To evaluate the mixed derivative above, we express the retailer’s expected profits, for \( q > q_B \) and \( \alpha_{min} \leq \alpha \leq \alpha_{max} \), as
\[
\pi_R(q, \alpha) = \pi_R(q, 0) + \int_{d(q)}^\delta \Delta X_R(q, u) f_1(u) du,
\]
where \( \delta \) is the covenant demand threshold. To see this, note that in the disagreement region where the covenant is breached, \((d(q), \delta)\), the retailer would have otherwise continued for \( \alpha = 0 \), whereas he faces liquidation for \( \alpha \). We then have
\[
\frac{\partial^2 \pi_R}{\partial q \partial \alpha} \bigg|_{q^*(\alpha_{min})} = \left[ \left( \frac{\partial}{\partial q} \Delta X_R(q, \delta) \right) f_1(\delta) \frac{\partial \delta}{\partial \alpha} \right] q^*(\alpha_{min}) + \left[ \Delta X_R(q, \delta) \frac{\partial}{\partial q} \left( f_1(\delta) \frac{\partial \delta}{\partial \alpha} \right) \right] q^*(\alpha_{min}).
\]
Note that \( \delta \) evaluated at \((q^*(\alpha_{min}), \alpha_{min})\) is equal to \( d(q^*(\alpha_{min})) \). Therefore, by equation (16), the second term above is zero.

To determine the sign of the first term, note that \( \delta \) is increasing in \( \alpha \), and \( f_1 \) is positive. Thus, at the point \((q^*(\alpha_{min}), \alpha_{min})\), the sign of \( \frac{\partial^2 \pi_R}{\partial q \partial \alpha} \) is the same as the sign of \( \frac{\partial \Delta X_R(q, \delta)}{\partial q} \). To evaluate the latter, we apply the implicit function theorem to equation (16) to obtain:
\[
\frac{\partial \Delta X_R}{\partial q} \bigg|_{q^*(\alpha_{min})} = -\frac{\partial d(q)}{\partial q} \bigg|_{q^*(\alpha_{min})} \cdot \frac{\partial \Delta X_R}{\partial d} \bigg|_{q^*(\alpha_{min})}.
\]
Since \( d \) is increasing in \( q \) (see Lemma 5), the sign of the first multiplier in the right-hand side is positive. By Proposition 8, the sign of the second multiplier in the right-hand side is negative. We conclude that the sign of \( \frac{\partial^2 \pi_R}{\partial q \partial \alpha} \) at \((q^*(\alpha_{min}), \alpha_{min})\) is positive.

By combining the above fact with (17) and (18), we get that
\[
\frac{dq^*(\alpha)}{d\alpha} \bigg|_{\alpha_{min}} \geq 0. \tag{19}
\]

We now focus on the bank’s expected profits. For \( q > q_B \) and \( \alpha_{min} \leq \alpha \leq \alpha_{max} \), we can express them as
\[
\pi_B(q, \alpha) = \pi_B(q, 0) + \int_{d(q)}^\delta \Delta X_B(q, u) f_1(u) du.
\]
Thus,
\[
\frac{\partial \pi_B(q, \alpha)}{\partial \alpha} = R^*(cq - x_0) \Delta X_B(q, \delta) f_1(\delta).
\]
By the implicit function theorem we also get
\[
\frac{d\pi_B(q^*(\alpha), \alpha)}{d\alpha} = \frac{\partial \pi_B(q, \alpha)}{\partial \alpha} \bigg|_{q^*(\alpha)} + \frac{dq^*(\alpha)}{d\alpha} \frac{\partial \pi_B(q, \alpha)}{\partial q} \bigg|_{q^*(\alpha)}.
\]
Thus, by combining the two equations above, we get
\[
\frac{d\pi_B(q^*(\alpha), \alpha)}{d\alpha} \bigg|_{\alpha_{min}} = R^*(cq^*(\alpha) - x_0) \Delta X_B(q^*(\alpha), \delta_{min}) f_1(\delta_{min}) + \frac{dq^*(\alpha)}{d\alpha} \bigg|_{\alpha_{min}} \frac{\partial \pi_B(q, \alpha)}{\partial q} \bigg|_{q^*(\alpha)} > 0,
\]
Note that the first term above is strictly positive due to the existence of liquidation conflict (also
see Proof of Lemma 2). The second term is positive by (19). To argue that the third term is positive, assume for the sake of contradiction that it is negative. Then, by imposing a loan limit that is equal to \( \bar{w} = q^*(0) - \epsilon \), for small enough \( \epsilon > 0 \), the bank’s profit would strictly increase, thus contradicting optimality of the contract \((R^*, \bar{w}^*, 0)\). This show that the assumed contract \((R^*, \bar{w}^*, 0)\) is suboptimal for the bank, since she can increase her profits by including a covenant.

We now argue for the case when the bank does not follow a threshold policy, i.e., when \( q^*(0) > q_B \), \( s < s_B \). By using our results from Lemma 2 and Lemma 4, note that for \( d < d_{\text{unlev}}^R \), it is optimal for the retailer to either liquidate or continue, and for the bank to liquidate. Thus, in equilibrium, \( \delta^* \geq d_{\text{unlev}}^R > 0 \), which implies that a covenant is needed.

Finally, as in Lemma 3, one can show that when the optimal liquidation policy of the bank is followed, sufficient leverage (or, equivalently, liquidation conflict) is a necessary condition for bankruptcy risk in non-shrinking markets, and neither necessary, nor sufficient in shrinking markets.

\[ \square \]

B.3.1 Auxiliary Results

**Proposition 8.** When there is liquidation conflict and the bank is following a threshold policy, \( \frac{\partial \Delta X_R}{\partial d} \) evaluated at \((q^*(\alpha_{\text{min}}), d(q^*(\alpha_{\text{min}})))\) is negative.

**Proof.** For simplicity of notation, we denote \( q^*(\alpha_{\text{min}}) \) with a generic \( q \). We distinguish two cases. For \( M \geq \tilde{M} \), since the bank follows a non-threshold liquidation policy, \( d(q) = d_{\text{lev}}^R(q) \) (see Lemma 5).

Since the retailer always prefers continuation for \( d \geq q_M \) (see Proposition 4), and prefers liquidation for \( d < d_{\text{unlev}}^R(q) \), we must have \( d(q) < \frac{q}{M} \). Therefore, \( x_{\text{R}, q}(q, d) = \frac{1}{2}(Md - wR) \). By combining this with the liquidation payoffs given by (4), we have that

\[
\Delta X_R(q, d) = \begin{cases} 
-\frac{1}{2}(Md - wR), & \text{if } d < d_{\text{AI}} \\
 sq + (1 - s)d - wR - \frac{1}{2}(Md - wR), & \text{otherwise}
\end{cases}
\]

As such, \( \frac{\partial \Delta X_R}{\partial d} \) is equal to \( -\frac{M}{2} \) or \( \frac{M - M}{2} \), and is therefore negative.

For \( M < \tilde{M} \), \( d(q) = \frac{wR}{M} < \frac{q}{M} \). As such, the liquidation and continuation profits are exactly the same as above. Note, however, that \( d(q) < d_{\text{AI}} \), by (11e) in Lemma 5. Thus, \( \frac{\partial \Delta X_R}{\partial d} \) is equal to \( -\frac{M}{2} \), and the proof is complete. \[ \square \]

B.4 Partial Liquidation

**Proof of Lemma 6.** Assume that \( d < q \) is the first period demand; for \( d \geq q \) the retailer stocks out and the players’ policies are trivial. Let \( y \leq q - d \) be the carrying inventory after performing the partial liquidation. Since the second period demand is at most \( (M - 1)d \), the agents will never prefer to carry more inventory than that, i.e., \( y \leq (M - 1)d \). Consequently, the players decide on
by solving
\[
\begin{align*}
\text{maximize} & \quad x_P(y) \\
\text{subject to} & \quad y \leq q - d \\
& \quad y \leq (M - 1)d \\
& \quad y \geq 0,
\end{align*}
\]
where \(x_P(y)\) are player’s \(P \in \{R, B\}\) expected revenues by liquidating a fraction \(1 - \frac{y}{q - d}\) of the inventory and carrying \(y\) into the second period.

(a) We first deal with the retailer’s problem. The retailer’s expected revenues are
\[
x_R(y) = \mathbb{E} \left[ (d + \min(y, D_2) + s(q - d - y) - Rw)^+ | D_1 = d \right] = \frac{1}{2} (d + s(q - d - y) - Rw)^+ + \frac{1}{2} (d + y + s(q - d - y) - Rw)^+.
\]
It is easy to check that \(x_R\) is convex increasing in \(y\) over the feasibility set of Problem 20. Hence, the retailer’s optimal carrying inventory level is \(y^*_R = \min((M - 1)d, q - d)\) and \(\ell^\dagger_R(q, d) = 1 - \frac{y^*_R}{q - d}\).

We now deal with the bank’s problem. Her expected revenues are
\[
x_B(y) = \mathbb{E} \left[ \min (d + \min(y, D_2) + s(q - d - y), Rw) | D_1 = d \right] = \frac{1}{2} \min(d + s(q - d - y), Rw) + \frac{1}{2} \min(d + y + s(q - d - y), Rw).
\]
Note that \(x_B\) is concave in \(y\). Moreover, for \(y\) such that \(d + y + s(q - d - y) \leq Rw\), or equivalently \(y \leq d_{AI} - d\), \(x_B\) is strictly increasing in \(y\) if and only if \(d + s(q - d - y) \leq Rw\), or equivalently \(y \geq \frac{1 - s}{s} (d - d_{AI})\), and increasing (in fact, constant) otherwise. Hence, \(x_B\) is increasing up to \(\max(d_{AI} - d, \frac{1 - s}{s} (d - d_{AI}))\) and non-increasing beyond that point. Thus, the bank’s optimal carrying inventory level is
\[
y^*_B \in \min \left\{ (M - 1)d, q - d, \max \left\{ d_{AI} - d, \frac{1 - s}{s} (d - d_{AI}) \right\} \right\}
\]
and her optimal policy is
\[
\ell^\dagger_B(q, d) = 1 - \frac{y^*_B}{q - d} = 1 - \frac{\min \left\{ y^*_R, \max \left\{ d_{AI} - d, \frac{1 - s}{s} (d - d_{AI}) \right\} \right\}}{q - d}\]
\[
= \max \left\{ \ell^\dagger_R(q, d), 1 - \frac{\max \left\{ d_{AI} - d, \frac{1 - s}{s} (d - d_{AI}) \right\}}{q - d} \right\}.
\]

(b) We show that the optimal policies of the two players are in disagreement if and only if \(q > q^\dagger_D\)
and \( d \in \mathbb{D}^\dagger \equiv (\mathcal{g}^\dagger(q), \mathcal{d}^\dagger(q)) \), where \( q_\mathbb{D}^\dagger \equiv \frac{Rx_0}{Rc - \min\{sM, s\}} \).

\[
\mathcal{g}^\dagger(q) \equiv \begin{cases} \frac{d_{AI}(q)}{M} & \text{if } s \leq \frac{M}{M}, \\ \max\left\{ \frac{d_{AI}(q)}{M}, \frac{(1-s)d_{AI}(q)}{1-sM} \right\} & \text{otherwise,} \end{cases} \quad \mathcal{d}^\dagger(q) \equiv \begin{cases} \min\left\{ Rw, \frac{(1-s)d_{AI}(q)}{1-sM} \right\} & \text{if } s < \frac{M}{M}, \\ Rw & \text{otherwise.} \end{cases}
\]

We have \( l_R^\dagger(q, d) > l_R^\dagger(q, d) \) if and only if \( y_R^* > y_R^* \), i.e.,

\[
\max\left\{ d_{AI} - d, \frac{1-s}{s} (d - d_{AI}) \right\} < \min\{(M - 1)d, q - d\}.
\]

Depending on whether \( d < d_{AI} \) or not, we can then identify the following sets,

\[
A_1 \equiv \{ d : d < d_{AI}, d_{AI} - d < (M - 1)d, d_{AI} - d < q - d \}, \\
A_2 \equiv \{ d : d \geq d_{AI}, \frac{1-s}{s} (d - d_{AI}) < (M - 1)d, \frac{1-s}{s} (d - d_{AI}) < q - d \},
\]

such that \( \mathbb{D}^\dagger = \{ d \geq 0 : d \in A_1 \cup A_2 \} \).

Note that we can rewrite \( A_1 = \left( \frac{d_{AI}}{M}, d_{AI} \right) \), and as such \( A_1 = \emptyset \) if and only if 

\[
d_{AI} \leq 0 \iff q \leq \frac{Rx_0}{Rc - s} \equiv q_\mathbb{D}^\dagger.
\]

We now deal with different cases:

**Case 1.** \( 1 - sM < 0 \). In this case, \( A_2 = \left\{ d : d \geq d_{AI}, \frac{(1-s)d_{AI}}{1-sM} < d < Rw \right\} \). We have that \( A_2 = \emptyset \) if and only if 

\[
\frac{(1-s)d_{AI}}{1-sM} \geq Rw \iff q \leq \frac{Rx_0}{Rc - M} \equiv q_\mathbb{D}^\dagger.
\]

In this case, \( q^2 = q_\mathbb{D}^\dagger \). Note also that \( q^2 < q_\mathbb{D}^\dagger \), since \( RMc > Ms > 1 \), using that \( R > 1 \) and \( c > s \). Then, for \( q \leq q^2 < q_\mathbb{D}^\dagger \), \( A_1 = A_2 = \emptyset \) and thus \( \mathbb{D}^\dagger = \emptyset \).

For \( q^2 < q \leq q_\mathbb{D}^\dagger \), \( d_{AI} \leq 0 \) and thus \( A_1 = \emptyset \). Moreover, \( \frac{d_{AI}}{M} \leq 0 \leq \frac{(1-s)d_{AI}}{1-sM} \). Combining these facts, we get that 

\[
\mathbb{D}^\dagger = A_2 = \left( \max\left\{ \frac{d_{AI}}{M}, \frac{(1-s)d_{AI}}{1-sM} \right\}, Rw \right).
\]

For \( q > q_\mathbb{D}^\dagger \), \( d_{AI} > 0 \) and thus \( \frac{d_{AI}}{M} > 0 \geq \frac{(1-s)d_{AI}}{1-sM} \). Thus, we again get 

\[
\mathbb{D}^\dagger = A_1 \cup A_2 = \left( \frac{d_{AI}}{M}, d_{AI} \right) \cup [d_{AI}, Rw] = \left( \max\left\{ \frac{d_{AI}}{M}, \frac{(1-s)d_{AI}}{1-sM} \right\}, Rw \right).
\]

**Case 2.** \( 1 - sM > 0 \). In this case, \( A_2 = \left\{ d : d \geq d_{AI}, d < \frac{(1-s)d_{AI}}{1-sM}, d < Rw \right\} \). For \( q \leq q_\mathbb{D}^\dagger \), then, \( d_{AI} \leq 0 \) and trivially \( \mathbb{D}^\dagger = A_1 = A_2 = \emptyset \).

For \( q > q_\mathbb{D}^\dagger \), \( d_{AI} > 0 \), and since \( 1 - s > 1 - sM \), we also have that \( d_{AI} < \frac{(1-s)d_{AI}}{1-sM} \). By combining
Thus, \(D^\dagger = A_1 \cup A_2 = \left( \frac{d_{M1}}{M}, d_{A1} \right) \cup \left[ d_{A1}, \max \left\{ \frac{(1-s)d_{A1}}{1-sM}, Rw \right\} \right] = \left( \frac{d_{A1}}{M}, \max \left\{ \frac{(1-s)d_{A1}}{1-sM}, Rw \right\} \right). \)

Case 3. \(1 - sM = 0.\) In this case, \(A_2 = \{ d : d \geq d_{A1}, d < Rw, d_{A1} \leq 0 \}.\) For \(q \leq q^\dagger = \frac{q}{d}, d_{A1} \leq 0\) and trivially \(D^\dagger = A_1 = A_2 = \emptyset.\) For \(q > q^\dagger, d_{A1} > 0\) and

\[D^\dagger = A_1 \cup A_2 = \left( \frac{d_{A1}}{M}, d_{A1} \right) \cup [d_{A1}, Rw) = \left( \frac{d_{A1}}{M}, Rw \right). \]

**Proof of Lemma 7.** Recall that for \(q \leq q^\dagger, D^\dagger = \emptyset.\) Hence, we only consider the case when \(q > q^\dagger.\) We show that \(D_{CV}^\dagger = \emptyset\) for \(\delta \leq \delta^\dagger_d,\) and \(D_{CV}^\dagger = (d_{CV}^\dagger, d_{CV}^\dagger)\) otherwise, where \(\delta^\dagger_d \equiv \min \{sq + \left( \frac{1}{M} - s \right)d_{A1}d(q), \frac{q}{M} \},\) and

\[d_{CV}^\dagger \equiv \begin{cases} \frac{d_{A1}(q)}{M} & \text{if } s \leq \frac{1}{M} \\ \max \left\{ \frac{d_{A1}(q)}{M}, \delta - sq \right\} & \text{otherwise}, \end{cases} \quad d_{CV}^\dagger \equiv \begin{cases} \min \left\{ \delta, \frac{\delta - sq}{1-sM} \right\} & \text{if } s < \frac{1}{M} \\ \delta & \text{otherwise.} \end{cases} \]

Let \(x_1\) denote the retailer’s cash flow at \(t = 1\) given that he follows \(\ell^\dagger_R.\) In particular, for \(d \in D^\dagger,\)

\[x_1 = d + s(q - d - y^*_R) = (1-s)d + sq - s \max \{(M-1)d, q - d\} = d + sq - s \max \{Md, q\}.\]

Thus, \(D_{CV}^\dagger = \{ d \in D^\dagger : x_1 < \delta \}.\) Using the expression above for \(x_1,\) we can express \(D_{CV}^\dagger\) as the union of two sets that capture the cases of \(Md < q\) and \(Md \geq q,\) i.e., \(D_{CV}^\dagger = B_1 \cup B_2,\) where \(B_1 = \{ d \in D^\dagger : Md < q \}, (1-sM)d < \delta - sq\},\) and \(B_2 = \{ d \in D^\dagger : Md \geq q, d < \delta \}.\) We next deal with different cases, as in the Proof of Lemma 6, and consider without loss \(\delta \leq Rw.\)

Case 1. \(1 - sM < 0.\) Note that in this case, we have the following inequalities:

\[Rw - q \geq (Rc - \frac{1}{M}) q - Rx_0 = \left( Rc - \frac{1}{M} \right)(q - q^\dagger_d) > 0, \quad \frac{q}{M} - \frac{Rw - sq}{1-sM} = \frac{1}{sM-1} \left( Rw - \frac{q}{M} \right) > 0. \]

By using Lemma 6 to substitute for \(D^\dagger\) and the above inequalities, we get

\[B_1 = \left( \max \left\{ \frac{d_{M1}}{M}, \delta - sq \right\}, \frac{q}{M} \right), \quad B_2 = \left[ \frac{q}{M}, \delta \right]. \]

For \(\delta \leq \frac{q}{M},\) we have \(B_1 = B_2 = D_{CV}^\dagger = \emptyset\) and for \(\delta > \frac{q}{M}, D_{CV}^\dagger = \left( \max \left\{ \frac{d_{M1}}{M}, \delta - sq \right\}, \delta \right).\) Note also that \(\delta^\dagger_d = \frac{q}{M} = sq + \left( \frac{1}{M} - s \right) q \leq sq + \left( \frac{1}{M} - s \right) d_{A1},\) since \(d_{A1} \leq q\) and \(1 - sM < 0.\)

Case 2. \(1 - sM > 0.\) Note that in this case, \(\delta^\dagger_d = sq + \left( \frac{1}{M} - s \right) d_{A1} \leq sq + \left( \frac{1}{M} - s \right) q = \frac{q}{M}. \)
If \( \frac{q}{M} \geq Rw \), then \( B_2 = \emptyset \). We also have that

\[
\frac{\delta - sq}{1-sM} = \frac{\delta - sq - sM\delta + sM\delta}{1-sM} \leq \delta + \frac{sM(Rw - \frac{q}{M})}{1-sM} \leq \delta \leq 1,
\]

which implies that \( B_1 = D_{CV}^\dagger = \left( \frac{d_{AI}}{M}, \min \left\{ \delta, \frac{\delta - sq}{1-sM} \right\} \right) \). Also, \( D_{C}^\dagger = \emptyset \) if and only if \( \frac{d_{AI}}{M} \geq \frac{\delta - sq}{1-sM} \Leftrightarrow \delta \leq \delta_{min}^\dagger \). If \( \frac{q}{M} < Rw \), then \( B_1 = D_{CV}^\dagger = \emptyset \).

For \( \delta \leq \delta_{min}^\dagger \), we get \( B_1 = B_2 = D_{CV}^\dagger = \emptyset \). For \( \delta_{min} < \delta < \frac{q}{M} \), it is easy to see that \( B_2 = \emptyset \) and

\[
\frac{\delta - sq}{1-sM} = \frac{\delta - sq - sM\delta + sM\delta}{1-sM} = \delta + \frac{sM(\delta - \frac{q}{M})}{1-sM} \leq \delta < \frac{q}{M},
\]

and hence \( B_1 = D_{CV}^\dagger = \left( \frac{d_{AI}}{M}, \min \left\{ \delta, \frac{\delta - sq}{1-sM} \right\} \right) \). For \( \delta \geq \frac{q}{M} \),

\[
\delta \leq \delta + \frac{sM(\delta - \frac{q}{M})}{1-sM} = \frac{\delta - sq}{1-sM} \leq Rw - sq = (1-s)\frac{d_{AI}}{M}.
\]

Thus, \( B_1 = \left( \frac{d_{AI}}{M}, \frac{q}{M} \right) \), \( B_2 = \left[ \frac{q}{M}, \delta \right) \) and \( D_{CV}^\dagger = B_1 \cup B_2 = \left( \frac{d_{AI}}{M}, \min \left\{ \delta, \frac{\delta - sq}{1-sM} \right\} \right) \).

**Case 3.** \( 1 - sM = 0 \). As in Case 1, we get that \( Rw - \frac{q}{M} > 0 \). Thus, we get

\[
B_1 = \left\{ d : \frac{d_{AI}}{M} < d < \frac{q}{M}, \delta > sq \right\}, \quad B_2 = \left[ \frac{q}{M}, \delta \right).
\]

Also, for \( \delta \leq \delta_{min}^\dagger = \frac{q}{M} = sq = sq + \frac{1}{M-s} d_{AI} \), we get \( B_1 = B_2 = D_{CV}^\dagger = \emptyset \). For \( \delta > \delta_{min}^\dagger \), \( D_{CV}^\dagger = \left( \frac{d_{AI}}{M}, \delta \right) \).

**Proof of Lemma 8.** We use the notation and results in the proofs of Lemmata 6 and 7. In case \( q \leq q_D^\dagger \), or \( \delta < \delta_{min}^\dagger \) or \( d \in D_{CV}^\dagger \), there is no disagreement or covenant violation and the retailer follows \( \ell_R^\star \). Suppose now that \( q > q_D^\dagger \), \( \delta \in [\delta_{min}^\dagger, Rw] \), without loss, and \( d \in D_{CV}^\dagger \).

Let \( y_C, y^\star \) be the carrying inventory levels that correspond to the policies \( \ell_C \) and \( \ell^\star \), respectively, i.e., \( y_C = 1 - (q - d)\ell_C(q, d) \) and \( y^\star = 1 - (q - d)\ell^\star(q, d) \).

For \( d \in D_{CV}^\dagger \), we have \( y_R^\dagger < y_C \) and \( y_R^\dagger < y_B^\dagger \), i.e., when the retailer follows his optimal policy, the covenant is breached and there is disagreement.

Let \( x_{R,\ell}(y) \) be the retailer’s revenues if he chooses his carrying inventory to be \( y \) for the second period, in the presence of the covenant. We deal with two different cases:

**Case 1.** \( y_C > y_B^\dagger \). We then have that for any carrying inventory level above \( y_C \), the covenant is breached and the bank, who would then have the decision rights, would set the carrying inventory level to \( y_B^\dagger \). For any carrying inventory level below \( y_C \), the covenant is not breached. That is, \( x_{R,\ell}(y) = x_R(y_B^\dagger) \), if \( y > y_C \), and \( x_{R,\ell}(y) = x_R(y) \) otherwise. Since \( x_R \) is increasing, we get that...
$y^* = y_C = \max\{y_C, y_B^*\}$ and thus $\ell^*(q, d) = \min\{\ell_B^\uparrow(q, d), \ell_C(q, d)\}$.

**Case 2.** $y_C \leq y_B^*$. We then have that for any carrying inventory level above $y_B^*$, the covenant is breached and the bank, who would then have the decision rights, would set the carrying inventory level to $y_B^*$. For any carrying inventory level below $y_B^*$, even if the covenant is breached, the bank does not impose a further liquidation, since $x_B$ is concave. Hence, $x_{R_{y^*}}(y) = x_R(y_B^*)$, if $y > y_B^*$, and $x_{R_{y^*}}(y) = x_R(y)$ otherwise. Arguing as above, we get that $y^* = y_B^* = \max\{y_C, y_B^*\}$ and thus $\ell^*(q, d) = \min\{\ell_B^\uparrow(q, d), \ell_C(q, d)\}$.

For $\delta < Rw$ and $d > d_{AI}$, it suffices to show that $\ell_C(q, d) < \ell_B^\uparrow(q, d)$. But, since $y_R^* < y_B^*$ and $d > d_{AI}$, we get that $y_B^* = \max\{d_{AI} - d, \frac{1-s}{s}(d-d_{AI})\} = \frac{1-s}{s}(d-d_{AI})$. Then, for $d < q$,

$$d + s(q-d)\ell_B^\uparrow(q, d) = d + s(q-d) \left(1 - \frac{y_B^*}{q-d}\right) = d + s(q-d) \left(1 - \frac{(1-s)(d-d_{AI})}{s(q-d)}\right) = sq + (1-s)d_{AI} = Rw > \delta,$$

which completes the proof, since $\ell_C(q, d) = \min\{z \geq 0 \mid d + s(q-d)^+z \geq \delta\}$.

\[\square\]

### B.5 Robustness of Non-Threshold Policies

**Proof of Theorem 6.** Since $q$ is fixed, we no longer explicitly show the dependency of the various functions on $q$. When the first period demand is $D_1 = d$, the retailer’s expected liquidation and continuation payoffs are respectively given by:

$$x_{R_{y^*}}(d) = X_{R_{y^*}}(q, d) = \begin{cases} 0, & \text{if } d \leq d_{AI} \\ (1-s)d + sq - Rw, & \text{if } d_{AI} < d \leq q \\ q - Rw, & \text{if } q < d. \end{cases}$$

$$x_{R_{y^*}}(d) = \mathbb{E}_{D_2|D_1=d}[X_{R_{y^*}}(q, d, D_2)] = \mathbb{E}_{D_2|D_1=d}[\min(d + D_2, q) - Rw^+] \downarrow,$$

Consider the function $g(y) \overset{\text{def}}{=} (\min(y, q) - Rw)^+$. Since $Rw \leq q$, note that $g$ is identically zero for $y \leq Rw$, then increases linearly, with a slope of 1, and then becomes constant, equal to $q - Rw$, for $y \geq q$. In particular, $g(y)$ is convex-concave, and increasing. Since $x_{R_{y^*}}(d) = \mathbb{E}_g(d + D_2)$, and $D_2$ has a distribution that is Polya frequency of order 2, the function $x_{R_{y^*}}(d)$ will also be convex-concave and increasing in $d$ (Porteus 2002).

Consider the two functions $x_{R_{y^*}}(d)$ and $x_{R_{y^*}}(d)$. Both are positive and increasing on $[0, q)$, and satisfy $x_{R_{y^*}}(d) = x_{R_{y^*}}(d) = q - Rw$, $\forall d \geq q$. It can be readily checked that these functions can have at most two intersection points on $(0, q)$, based on the convex-concave structure of $x_{R_{y^*}}$. Furthermore, if there exists $0 < \xi < q$ such that $x_{R_{y^*}}(\xi^-) > x_{R_{y^*}}(\xi^-)$ and $x_{R_{y^*}}(\xi^+) < x_{R_{y^*}}(\xi^+)$, then $x_{R_{y^*}}(d) \overset{\text{def}}{=} x_{R_{y^*}}(d), \forall d \in (\xi, q)$. The conclusion of the theorem concerning the retailer follows.

\[\text{Following standard terminology, we call a function } g : \mathbb{R} \rightarrow \mathbb{R} \text{ convex-concave if it is convex on } (-\infty, a) \text{ and convex on } (a, \infty), \text{ for some } a \in \mathbb{R} (\text{see, e.g., Porteus (2002)).}\]
The proof for the bank follows in a similar fashion, by recognizing that the expected profits from liquidation and continuation, i.e.,

\[ x_{B, \mathcal{X}}(d) = X_{B, \mathcal{X}}(d) = \min \{ Rw(q), \min(d, q) + s(q - d) + \} , \]

\[ x_{B, \mathcal{E}}(q, d) = \mathbb{E}[X_{B, \mathcal{E}}(q, d, D_2)] = \mathbb{E}[\min \{ Rw(q), \min(d + D_2, q) \} ] , \]

are concave increasing on \( d \in (0, Rw) \), and can have at most two intersection points on this interval. Note that the PF2 requirement is not needed, since the bank’s profits from continuation are always concave increasing, for any second period demand distribution.

\[ \square \]

**Proof of Proposition 1.** (a) Note first that, since \( \mathcal{D}_1 > \frac{x_0}{c - \frac{x_0}{M + s - 1}} \geq \frac{x_0}{c} \) holds almost surely, the retailer will prefer to be leveraged, and to order a quantity \( q > \frac{x_0}{c - \frac{x_0}{M + s - 1}} \), for any interest rate \( R \leq \frac{1}{c} \). In particular, since the equilibrium interest rate always satisfies this inequality, we have \( q^* > \frac{x_0}{c - \frac{x_0}{M + s - 1}} \).

Recall that \( s^2_D = \frac{Rc(M-1)}{M-Rc} \), \( q^2_D = \frac{Rc x_0}{c - \frac{x_0}{M + s - 1}} \). As such, the conditions in the proposition ensure that \( s < s^2_D \) and \( q^2_D > 0 \) holds at \( R \geq 1 \). Since \( s^2_D \) is increasing in \( R \), and \( q^2_D \) is decreasing in \( R \), we immediately see that \( s < s^2_D \) and \( q^* > q^2_D \) must hold in equilibrium. Since \( M < \tilde{M} \), this implies that the retailer will always be sufficiently leveraged in equilibrium, and will follow non-threshold policies (see Lemma 2).

(b) As above, we must have that, in equilibrium, the retailer prefers to be leveraged, and \( q^* > \frac{(M-1+2s)x_0}{(M-1+2s)-2Ms} \geq \frac{x_0}{c} > 0 \).

Recall that \( s_B = \frac{(M-1)cR}{2(M-cR)} \) and \( q_B = \frac{R(M-1+2s)x_0}{cR(M-1+2s)-2Ms} \). The conditions in the proposition ensure that \( s < s_B \) and \( q^* > q_B \) holds for \( R = 1 \). Since \( s_B \) is increasing in \( R \), and \( q_B \) is decreasing in \( R \), we have that \( s < s_B \) and \( q^* > q_B \) must also hold for the optimal interest rate, proving that the bank will always follow non-threshold policies in equilibrium (see Lemma 4). \( \square \)