Supplemental Materials

Durable Products, Time Inconsistency, and Lock-in

Proof of Lemma 3.1

We apply the standard mechanism design results based on Maskin and Riley (1984)'s derivation for the continuous distribution model to derive the optimal quantity, price pair. To facilitate exposition, and because the pricing policy is time invariant, we drop the subscript $t$ on the variables $q(a)$ and $t(a)$. Denoting the information rent received by a consumer of type $a$ by $V(a) = U(a, q(a)) - t(a)$, we can write the objective function of the consumables pricing problem ($π_{NLcons} (δ, Q)$) for $Q = 1$ as follows:

$$\max_{q(a)} \int_{1-δ}^{1+δ} \left( U(a,q(a)) - V(a) \right) f(a) da = \max_{q(a)} \int_{1-δ}^{1+δ} \left( U(a,q(a)) - \int_{1-δ}^{a} q(τ) dτ \right) f(a) da \quad (A.1)$$

where the expression on the right-hand side comes from substituting the incentive compatibility constraint (3.3). We can now use an integration by parts to express the above objective function as:

$$\max_{q(a)} \int_{1-δ}^{1+δ} \left( U(a,q(a)) - \frac{1 - F(a)}{f(a)} q(a) \right) f(a) da \quad (A.2)$$

recognizing that $U(a,1,q(a)) = q(a)(2a - q(a))/2$ and that $f(a) = \frac{1}{2\delta}$ and $F(a) = \frac{a+δ-1}{2\delta}$ for a $U(1-δ,1+δ)$ distribution, we can use the first-order condition to maximize pointwise to obtain, $q(a)^* = \max \{0, 2a - (1 + δ)\}$. Because the lowest valuation consumer is type $a = 1 - δ$, it is easy to confirm that all consumer types $a \in (1 - δ, 1 + δ)$ will be allocated positive quantities so long as $δ < 1/2$. For these low values of $δ$, we obtain the total price paid by a consumer of type $a$ as:

$$t(a) = U(a,q(a)) - \int_{1-δ}^{a} (2τ - (1 + δ)) dτ = \frac{4a(1+δ) - 2a^2 - 3(2 - δ)δ - 1}{2} \quad (A.3)$$

When $δ \geq \frac{1}{3}$, then only those consumers of type $a > \frac{1+δ}{2}$ receive $q(a) > 0$. For these consumers $a \in (\frac{1+δ}{2}, 1 + δ)$, the total price paid can be obtained as:

$$t(a) = U(a,q(a)) - \int_{(1+δ)/2}^{a} (2τ - (1 + δ)) dτ = \frac{8a(1+δ) - 4a^2 - 3(1 + δ)^2}{4} \quad (A.4)$$

The expressions for the profit associated with $π_{NL} (δ)$ can be obtained by integrating (A.3) or (A.4) over the appropriate range of $a$ depending on whether $δ \geq \frac{1}{3}$ to obtain $π_{t, NLcons} (δ, 1)$ and then using the definition $π_{NL} (δ) = (1 + ρ) π_{t, NLcons} (δ, 1)$.■

Proof of Lemma 3.2 From the expression for profits under future price commitment that is shown
in (3.9) it is obvious that the trade-off between the durables quantity and the consumables price is the same in both periods. So that we can restrict attention to solutions in which \( Q_1 = Q_2 = Q \) and \( p_1 = p_2 = p \) for some \( Q \) and \( p \). By taking \( \beta = 0 \) and substituting (3.7) and (3.8) into (3.9), we can write the DGM’s profits as the following function of \( Q \) and \( p \).

\[
\pi^{cc}(Q, p, Q, p) = \frac{1 + \beta}{2} Q \left( 2pQ - p^2 + (1 - 2Q) \delta \right)^2
\]

It can be confirmed that there are four pairs of values for \( Q \) and \( p \) that satisfy the first-order conditions for this profit function. However, the determinant of the Hessian matrix is positive, i.e. the second-order conditions are satisfied, only for \( Q = \frac{1 + \delta}{\delta} \) and \( p = \frac{1 + \delta}{2} \). Thus, so long as \( \delta \geq \frac{1}{4} \), so that \( \frac{1 + \delta}{\delta} \leq 1 \), this will be the optimal solution. When \( \delta < \frac{1}{4} \), then the optimal solution will be along the boundary with \( Q = 1 \). Since \( \pi^{cc}(Q, p, Q, p) \) is concave in \( p \), if follows that the optimal value for \( p \) satisfies the first order conditions. Thus, \( p = \delta Q \).

**Proof of Lemma 3.3**

By substituting (3.7) into (3.10), the second period profits can be expressed as:

\[
\pi_2(Q, p_2) = \begin{cases} 
Q(1 - p_2 + \delta(1 - Q))p_2 & \text{for } p_2 \leq (1 - \beta)a_m \\
\frac{(p_2 - (1 - \beta)(1 + \delta))(p_2(1 - 2\beta) - (1 - \beta)(1 + \delta))}{4(1 - \beta)^2 \delta}p_2 & \text{otherwise}
\end{cases}
\]

(A.5)

The lower branch is unimodal in \( p_2 \) with a maximum at \( p_2^l = \frac{(1 - \beta)(1 + \delta)}{2(1 - \beta) + \sqrt{1 - 2\beta + 4\beta^2}} \), to see this, we look at the first derivative. Ignoring the constant term, \( \left( 4(1 - \beta)^2 \delta \right)^{-1} \), this derivative can be expressed as:

\[
p_2^2(3 - 6\beta) - 4p_2(1 - \beta)^2(1 + \delta) + (1 - \beta)^2(1 + \delta)^2
\]

The first derivative is continuous and the roots of the above expression are: \( i_1 = \frac{(1 - \beta)(1 + \delta)}{2(1 - \beta) + \sqrt{1 - 2\beta + 4\beta^2}} \) and \( i_2 = \frac{(1 - \beta)(1 + \delta)}{2(1 - \beta) - \sqrt{1 - 2\beta + 4\beta^2}} \). The root \( i_2 \) is either negative or greater than \( (1 - \beta)(1 + \delta) \). Note that \( (1 - \beta)(1 + \delta) \) is the maximum price that the manufacturer can charge for the consumable so as to induce purchase of her consumable. Therefore \( i_2 \) is not a feasible price for the consumable. Given that \( i_2 < 0 < i_1 \) or \( 0 < i_1 < i_2 \) we can see that the first derivative is positive when \( p \in (0, i_1) \) and negative for \( p \in (i_1, (1 - \beta)(1 + \delta)] \). Therefore the lower branch of (A.5) is unimodal at \( i_1 \). Further \( i_1 = p_2^l > (1 - \beta)a_m \) if and only if \( Q > \hat{Q} = \frac{\delta(1 + \delta)}{4\delta - 1 + \sqrt{1 - 2\beta + 4\beta^2}} \).

It follows that if \( Q \leq \hat{Q} \), the optimal price will be \( p_2^l \leq (1 - \beta)a_m \). It is easy to confirm that the upper branch of (A.5) is concave in \( p_2 \), and that its first order condition is satisfied at:

\[
p_2^u = \frac{(1 - \delta)(1 - Q)}{2}
\]

It follows that the optimal price is: \( \text{Min} \{(1 - \beta)a_m, p_2^u\} \).

**Proof of Corollary 3.4**
The partial derivatives of $\tilde{Q}$ with respect to $\beta$ and $\delta$ are as follows:

\[
\frac{\partial \tilde{Q}}{\partial \beta} = \frac{(1 + \delta)(-1 + \beta + \sqrt{1 - 2\beta + 4\beta^2})}{\delta (1 - 4\beta + \sqrt{1 - 2\beta + 4\beta^2})^2 \sqrt{1 - 2\beta + 4\beta^2}}
\]

\[
\frac{\partial \tilde{Q}}{\partial \delta} = -\frac{\beta}{\delta^2 (1 - 4\beta + \sqrt{1 - 2\beta + 4\beta^2})}
\]

By differentiating twice with respect to $\beta$, we can confirm that the term $-1 + \beta + \sqrt{1 - 2\beta + 4\beta^2}$ is convex, and that it is minimized at $\beta = 0$. Evaluating this term for $\beta = 0$, confirms that $-1 + \beta + \sqrt{1 - 2\beta + 4\beta^2} \geq 0$. It follows immediately that $\frac{\partial Q}{\partial \delta} < 0$. Since $-1 + 2\beta + 4\beta^2 > 0$ for all $\beta \in [0, 1]$, we also have that $\frac{\partial Q}{\partial \beta} < 0$.

We can see that $(1 - \beta)a_m$ is non-increasing in $\beta$. Therefore the Min $\{(1 - \beta)a_m, \frac{(1 + \delta (1 - Q))}{2}\}$ is non-increasing in $\beta$. Similarly by evaluating the expression $\frac{\partial p_2^*}{\partial \beta}$ we obtain the following expression:

\[
-\frac{3\beta (1 + \delta)}{2(1 - \beta) + \sqrt{1 - 2\beta + 4\beta^2})^2 \sqrt{1 - 2\beta + 4\beta^2}
\]

which is clearly non-positive for all $\beta \in [0, 1]$.

To show how $p_2^*$ responds to changes in $\delta$, we first observe that both $p_2^* = \frac{(1 - \beta)(1 + \delta)}{2(1 - \beta) + \sqrt{1 - 2\beta + 4\beta^2}}$ and $p_2^* = \frac{(1 + \delta (1 - Q))}{2}$ are increasing in $\delta$, while $(1 - \beta)a_m = (1 - \beta)(1 + \delta(1 - 2Q))$ is increasing (decreasing) in $\delta$ for $Q < \frac{1}{2}$ ($Q > \frac{1}{2}$). Thus, from the definition of $p_2^*$ in Lemma 3.3, we can see that it is decreasing in $\delta$ if and only if $(1 - \beta)a_m < p_2^*, Q > \frac{1}{2}$, and $Q < \tilde{Q}$. The result follows from the fact that $(1 - \beta)a_m > p_2^*$ if and only if $Q < \frac{(2\beta - 1)(1 + \delta)}{4\beta - 1}\delta$.  

**Proof of Lemma 3.5**

It is easy to confirm that $\beta^*$ is value of $\beta$ for which $(1 - \beta)(1 + \delta(1 - 2Q)) = p^c$ when $Q = Q^c$. It remains to be shown that, if the DGM sets $Q = Q^c$, then she will find it optimal to set $p_2$ to $(1 - \beta^*)(1 + \delta(1 - 2Q^c)) = p^c$. From Lemma 3.3, we know that $p_2^*$ depends upon whether $Q \leq \tilde{Q} = \frac{\beta(1 + \delta)}{\delta(4\beta - 1 + \sqrt{1 - 2\beta + 4\beta^2})}$. Specifically, we know that, for $Q \leq \bar{Q}$, we will have $p_2^* = \text{Min} \left\{ (1 - \beta)(1 + \delta(1 - 2Q)), \frac{1 + \delta (1 - Q)}{2} \right\}$. Because $\beta^* > \frac{1}{2}$ for all $\delta$, it follows that, for $\beta = \beta^*$, we will have $p_2^* = (1 - \beta^*)(1 + \delta(1 - 2Q^c)) = p^c$ so long as $Q^c \leq \bar{Q}$. To see why this is the case, we observe that:

\[
(1 - \beta^*)(1 + \delta(1 - 2Q)) < \frac{1}{2}(1 + \delta(1 - 2Q)) \leq \frac{1 + \delta(1 - Q)}{2}
\]

It remains to be shown that $Q^c \leq \bar{Q}$ when $\beta = \beta^*$. To do this, we note that $\beta^* > \frac{1}{2}$ for all $\delta$. Consequently, by replacing $\sqrt{1 - 2\beta + 4\beta^2}$ with $4\beta^2$ in the denominator of $\bar{Q}$, we can obtain the
Differentiating with respect to $Q$: $Q' = \frac{\beta(1+\delta)}{\delta(\delta p - 1)}$ where $Q' < \bar{Q}$. Thus, it suffices to show that $Q^{cc} \leq \bar{Q}'$. For $\delta \leq \frac{1}{4}$, we have $Q^{cc} = 1$. By substituting $\beta^* = \frac{3-2\delta}{\delta + 3\delta}$ into $Q' = \frac{\beta(1+\delta)}{\delta(\delta p - 1)}$, we have $\bar{Q}' = \frac{3+\delta-2\delta^2}{15\delta-15\delta}$, which is decreasing in $\delta$ for $\delta \in [0, 3-\sqrt{6})$, and is strictly greater than $Q^{cc} = 1$ for $\delta = \frac{1}{4} < 3 - \sqrt{6}$. Thus, $\bar{Q}' < \bar{Q}$ for $\delta \leq \frac{1}{4}$.

For $\delta \geq \frac{1}{4}$, we have $Q^{cc} = \frac{1+\delta}{5\delta}$. By substituting $\beta^* = \frac{2}{3}$ into $Q' = \frac{\beta(1+\delta)}{\delta(\delta p - 1)}$, we have $\bar{Q}' = \frac{2(1+\delta)}{9\delta}$, which is obviously larger than $Q^{cc} = \frac{1+\delta}{5\delta}$. □

**Proof of Lemma 3.6** For the case in which $\beta = 1$, the DGM’s profit is $\pi_1(1, Q, 0)$ as shown in (3.12). Differentiating with respect to $Q$, we have:

$$
\frac{d\pi_1(1, Q, 0)}{dQ} = \frac{1 + \rho}{2} (1 + \delta(1 - 2Q)) (1 + \delta (1 - 6Q))
$$

(A.6)

which has two roots: $j_1 = \frac{1+\delta}{6\delta}$ and $j_2 = \frac{1+\delta}{2\delta}$. $j_2$ is clearly greater than 1 for all $\delta < 1$. It can be easily verified that $\frac{d\pi_1(1, Q, 0)}{dQ} > 0$ for $Q \in [0, j_1)$ and $\frac{d\pi_1(1, Q, 0)}{dQ} < 0$ for $Q \in (j_1, 1]$. Therefore $\pi_1(1, Q, 0)$ is unimodal in $Q$, with a maximum at $j_1$. Further, $j_1 \leq 1$ only if $\delta \geq \frac{1}{5}$. Therefore the optimal quantity is given by

$$
Q^{b1} = \begin{cases} 
1 & \text{for } \delta \leq 1/5 \\
\frac{1+\delta}{6\delta} & \text{for } \delta \geq 1/5
\end{cases}
$$

The expressions for the optimal profits can be obtained by substituting $Q^{b1}$ into (3.12). □

**Proof of Lemma 3.7** When $\beta = 0$ the DGM’s profits in period 1 can be obtained by substituting which can be obtained by substituting (3.14) into (3.11) to obtain:

$$
\pi_1(0, Q, p_1) = \begin{cases} 
\frac{Q}{2} \left(-p_1^2 + 2p_1Q\delta + (1 + (1 - 2Q)\delta)^2\right) & \text{for } Q \leq (1 + \delta)/3\delta \\
+ \frac{Q^2}{8} \left(3 + \delta \left(11Q^2\delta - 10Q(1 + \delta) + 3(2 + \delta)\right)\right) & \text{for } Q \leq (1 + \delta)/3\delta \\
\frac{1}{5\delta} \left(27Q\delta \left(-p_1^2 + 2p_1Q\delta + (1 + (1 - 2Q)\delta)^2\right) + 2(1 + \delta)^3\rho\right) & \text{otherwise}
\end{cases}
$$

(A.7)

It is easy to confirm that $\frac{d^2\pi_1(0, Q, p_1)}{dp_1^2} < 0$ so that the DGM’s profits are concave in $p_1$. The FOC for both the upper and lower branches of (A.7) is $p_1 = Q\delta$, so that the conditionally optimal price for any $Q$ is $p_1^{co}(Q) = \min \{Q\delta, a_m\}$, which ensures that the marginal consumer is at least indifferent to using the product. We now note that, $p_1^{co}(Q) = Q\delta$ if and only if $Q \leq \frac{1+\delta}{5\delta}$. Thus, when we substitute $p_1^{co}(Q) = Q\delta$ and $p_1^{co}(Q) = a_m$ into the upper and lower branches of (A.7) and differentiate with respect to $Q$, we obtain:
\[
\frac{d\pi_1(0, Q, Q\delta)}{dQ} = \begin{cases} 
\frac{1}{8}(1 + (1 - 3Q)\delta)(4 + 3\rho + \delta(4 + 3\rho - Q(20 + 11\rho))) & \text{for } Q \leq (1 + \delta)/3\delta \\
2Q\delta(1 + \delta(1 - 3Q)) & \text{otherwise}
\end{cases}
\]

(A.8)

It is easy to confirm that the lower branch is negative for all \(Q > \frac{1 + \delta}{3\delta}\), so we will never set the durables quantity above this threshold. The lower branch of (A.8) has two roots, \(k_1 = \frac{(1 + \delta)(4 + 3\rho)}{\delta(20 + 11\rho)}\) and \(k_2 = \frac{1 + \delta}{3\delta}\). It can be confirmed that \(k_1 < k_2\) and that \(\frac{d\pi_1(0, Q, Q\delta)}{dQ} > 0\) for \(Q < k_1\) and negative beyond \(k_1\). Therefore the optimal quantity of durables can be represented as \(Q^{b_0} = \min\{k_1, 1\}\), where \(k_1 < 1\) only if \(\delta > \frac{4 + 3\rho}{8(2 + \rho)}\). By substituting \(Q^{b_0}\) and \(p_{1}^c(Q^{b_0})\) into (A.8), we can confirm that the optimal profits for \(\beta = 0\) are as stated in the Lemma.

\[\textbf{Proof of Corollary 3.8}\]
Note that \(\frac{4 + 3\rho}{8(2 + \rho)} > \frac{1}{5}\) for all \(\rho \in [0, 1]\). For \(\delta < \frac{1}{5}\), \(Q^{b_0} = 1 = Q^{b_1}\). When \(\delta \in \left[\frac{1}{5}, \frac{4 + 3\rho}{8(2 + \rho)}\right]\) then \(Q^{b_0} = 1 > \frac{(1 + \delta)/6\delta}{Q^{b_1}}\). For \(\delta > \frac{4 + 3\rho}{8(2 + \rho)}\), \(Q^{b_0} = \frac{(1 + \delta)(4 + 3\rho)}{\delta(20 + 11\rho)} > \frac{(1 + \delta)/6\delta}{Q^{b_1}}\) so long as \(\frac{4 + 3\rho}{20 + 11\rho} > \frac{1}{5}\) which is always true.

\[\textbf{Proof of Proposition 3.9}\]
For \(\delta < \frac{1}{5}\) the difference between the DGM's profit with free access to consumables (\(\beta = 1\)) and with lock-in (\(\beta = 0\)) can be expressed as:

\[\Delta_1(\delta) = \pi_1^{b_1} - \pi_1^{b_0} = \frac{(1 + \rho)(1 - \delta)^2 - 4 + 3\rho - 4(1 - \delta)(2 + \rho)}{2} \frac{8}{8} \]

The difference \(\Delta_1(0) = \frac{\rho}{8} > 0\). Solving for \(\Delta_1(\delta) = 0\) gives us a threshold \(\frac{1}{2}\left(-\rho + \sqrt{\rho(1 + \rho)}\right)\) above which \(\Delta_1 < 0\). Note that \(\frac{1}{2}\left(-\rho + \sqrt{\rho(1 + \rho)}\right) < \frac{1}{5}\) only if \(\rho < \frac{4}{5}\). Otherwise for \(\rho \geq \frac{4}{5}\), the DGM prefers \(\beta = 1\) for all \(\delta \leq \frac{1}{5}\).

For \(\delta \in \left[\frac{1}{5}, \frac{4 + 3\rho}{8(2 + \rho)}\right]\), the relevant difference to evaluate is:

\[\Delta_2(\delta) = \pi_1^{b_1} - \pi_1^{b_0} = \frac{(1 + \rho)(1 + \delta)^3 - 4 + 3\rho - 4(1 - \delta)(2 + \rho)}{27\delta} \frac{8}{8} \]

Clearly the difference \(\Delta_2(\delta) > 0\) for \(\delta\) below \(\frac{8(1 + \rho)}{52 + 25\rho}\), and it is easy to confirm that \(\frac{8(1 + \rho)}{52 + 25\rho} < \frac{4 + 3\rho}{8(2 + \rho)}\).

However \(\frac{8(1 + \rho)}{52 + 25\rho} > \frac{1}{5}\) only if \(\rho > \frac{4}{5}\). Otherwise, for \(\rho \leq \frac{4}{5}\), the DGM prefers \(\beta = 0\) for all \(\delta \in \left[\frac{1}{5}, \frac{4 + 3\rho}{8(2 + \rho)}\right]\). Finally, for \(\delta > \frac{4 + 3\rho}{8(2 + \rho)}\), the relevant difference to evaluate is:

\[\Delta_3(\delta) = \pi_1^{b_1} - \pi_1^{b_0} = \frac{(1 + \rho)(1 + \delta)^3 - (1 + \delta)^3(2 + \rho)(4 + 3\rho)^2}{27\delta} \frac{2\delta(20 + 11\rho)^2}{2\delta(20 + 11\rho)^2} \]

Clearly the difference \(\Delta_3(\delta) > 0\) for all \(\delta \in \left[\frac{1}{5}, \frac{4 + 3\rho}{8(2 + \rho)}\right]\).
Proof of Lemma 3.10 By substituting (3.7) and (3.8) into (3.16), the DGM’s problem in period 2 reduces to maximizing the following:

\[
\Pi_2(Q_{t-1}, Q_t, p_t) = \begin{cases} 
(1 - p_2 + \delta(1 - Q_t))p_t + \frac{(Q_t - Q_{t-1})}{2} (a_m(Q_t) - p_t)^2 & \text{for } p_t \leq (1 - \beta)a_m(Q_t) \\
\frac{(p_t - (1 - \beta)(1 + \delta))(p_t(1 - 2\beta) - (1 - \beta)(1 + \delta)}{4(1 - \beta)^2 \delta} p_t + \frac{(Q_t - Q_{t-1})}{2} a_m(Q_t)^2 & \text{otherwise}
\end{cases}
\]

It is easy to see that the upper branch of (A.9) is concave in \(p_t\), for any \(0 \leq Q_1 \leq Q_2 \leq 1\). As established in the proof of Lemma 3.3, the lower branch is unimodal in \(p_t\). Therefore the FOC to the upper and lower branches are:

\[
FOC^u_p = \frac{Q_{t-1} + (Q_{t-1} - 2Q_{t-1}Q_t + Q_t^2)\delta}{Q_{t-1} + Q_t} \quad \text{and} \quad FOC^l_p = \frac{(1 - \beta)(1 + \delta)}{2(1 - \beta) + \sqrt{1 - 2\beta + 4\beta^2}}
\]

respectively. From the limit on the prices of \(p_t \leq (1 - \beta)a_m\) on the upper branch and \(p_t \geq (1 - \beta)a_m\) on the lower branch the conditionally optimal prices given \(Q_1\) and \(Q_2\) and either the constraint that \(p_t \leq (1 - \beta)a_m(Q_t)\) or \(p_t \geq (1 - \beta)a_m(Q_t)\):

\[
p_t(Q_1, Q_2) = \text{Min}\{FOC^u_p, (1 - \beta)a_m\} \quad \text{and} \quad p_h(Q_1, Q_2) = \text{Max}\{FOC^l_p, (1 - \beta)a_m\}
\]

respectively, and it is easy to see that the manufacturer chooses the optimal price of consumable \(p^*_2(Q_1, Q_2)\) from \(p_t\) and \(p_h\), whichever results in a higher profit.

By substituting the optimal second period quantity of durables, \(Q^*_2\) and \(p^*_2\) into (3.17), the manufacturer’s problem in period 1 reduces to maximizing the following profit function:

\[
\Pi_1(\beta, Q_1, p_1) = \begin{cases} 
(1 - p_1 + \delta(1 - Q_1))p_1 + \frac{Q_1}{2} (a_m(Q_1) - p_1)^2 & \text{for } p_1 \leq (1 - \beta)a_m(Q_1) \\
\frac{(p_1 - (1 - \beta)(1 + \delta))(p_1(1 - 2\beta) - (1 - \beta)(1 + \delta)}{4(1 - \beta)^2 \delta} p_1 + \frac{Q_1\beta^2}{2} a_m(Q_1)^2 + \rho \Pi_2(Q_1, Q_2^{**}, p_2^{**}) & \text{otherwise}
\end{cases}
\]

Note that \(Q^{**}\) and \(p^{**}\) are a function of \(Q_1\) but independent of \(p_1\). Given this, for the purposes of identifying the value of \(p_1\) that maximizes (A.10) conditional upon \(Q_1\), we can ignore the term, \(\rho \Pi_2(Q_1, Q_2^{**}, p_2^{**})\), that appears in both the upper and lower branches. After ignoring these terms, we can compare (A.10) to (A.9) and it is easy to see that the value of \(p_1\) that maximizes (A.9) for \(Q_{t-1} = 0\) and \(Q_t = Q_1\) also maximizes (A.10) for a given value of \(Q_1\).
Proof of Lemma 3.11 By substituting \( \delta = 1 \) and \( \beta = 1 \) into (3.8), we can see that the implicit rental price, \( r (1, Q, p) = 2 (1 - Q)^2 \), regardless of the consumables price \( p \). In addition, the DGM will make no revenue from consumables sales, so that her profit in period 2 will be the following function of \( Q_2 \) and \( Q_1 \):

\[
\Pi_2 (Q_1, Q_2, p) = (Q_2 - Q_1) 2 (1 - Q)^2
\]

The first derivative of this function has two roots, one at \( Q_2 = \frac{1}{3} (1 + 2Q_1) \), and the other at \( Q_2 = 1 \). However, the second derivative is negative only at \( Q_2 = \frac{1}{3} (1 + 2Q_1) \). It follows that the conditionally optimal total quantity of durables in period 2 is \( Q_2^{\beta_1} (Q_1) = \frac{1}{3} (1 + 2Q_1) \) for all \( Q_1 \in [0, 1] \). For period 1, we can substitute \( r (1, Q, p) = 2 (1 - Q)^2 \) into (3.17) and ignore the term for consumables income to obtain the following profit function:

\[
\Pi_1 (1, Q_1, p) = Q_1 (2 (1 - Q_1)^2 + 2 \left( 1 - \frac{1 + 2Q_1}{3} \right)^2) + \left( \frac{1 + 2Q_1}{3} - Q_1 \right) 2 \left( 1 - \frac{1 + 2Q_1}{3} \right)^2
\]

\[
= \frac{2}{27} (1 - Q_1)^2 (4\rho + Q_1 (27 + 8\rho))
\]

The first derivative of this profit function has two roots, one at \( Q_1 = \frac{9}{27 + 8\rho} \) and the other at \( Q_1 = 1 \). However, the second derivative is negative only at the first root. Thus, \( Q_1 = \frac{9}{27 + 8\rho} \) represents a local maximum while \( Q_1 = 1 \) is a local minimum. The result follows from substituting \( Q_1^{\beta_1} = \frac{9}{27 + 8\rho} \) into \( Q_2^{\beta_1} (Q_1) \) and into \( \Pi_1 (1, Q_1, p) \).

Proof of Lemma 3.12 a) For every possible \( Q_1 \), the conditionally optimal second period consumables price will be either \( p_2^h (Q_1, Q_2) \) or \( p_2^l (Q_1, Q_2) \). If the price is \( p_2^h (Q_1, Q_2) \), then the implicit rental price is zero and the consumables income is the only source of income. Thus, conditional upon \( p_2 \geq 2(1 - Q_2) \), the second period profit function can be written as:

\[
\Pi_2^{p_2^h} (Q_1, Q_2, p_2) = p_2 y (Q_2, p_2) = p_2 \left( \frac{p_2 - 2}{4} \right)^2
\]

where the latter expression is obtained by substituting the lower branch of (3.7) for \( y(Q, p) \) and subsequently substituting \( \delta = 1 \) and \( \beta = 0 \). The first derivative of this expression has a single root in the interval \((0, 2)\) that is located at \( p_2 = \frac{2}{3} \). The second derivative is negative at \( p_2 = \frac{2}{3} \), confirming it as a local maximum. From the right hand side of (A.11) it is obvious that \( Q_2 \) plays no role in the maximization of consumables income, other than by limiting the number of consumers who have access to the durable, i.e. through the constraint \( p_2 \geq 2(1 - Q_2) \). To prevent this constraint from interfering, we need \( Q_2 \geq \frac{2}{3} \). It follows that conditional upon setting \( p_2 \geq 2(1 - Q_2) \), we will set \( Q_2^{p_2^h} = Min \{ Q_1, \frac{2}{3} \} \) and \( p_2^{p_2^h} = \frac{2}{3} \). By substituting into (A.11), we can see that the conditionally optimal profits from doing this are \( \Pi_2^{p_2^h} (Q_1, Q_2^{p_2^h}) = \frac{8}{27} \).

If the consumables price is set to \( p_2^l (Q_1, Q_2) \), then by substituting \( \delta = 1 \) and \( \beta = 0 \) into (3.8), the implicit rental price can be written as \( r (0, Q, p) = \left[ \frac{1}{2} (2 - 2Q - 2p)^2 \right]^{+} \), where \( [x]^+ = Max \{x, 0\} \).
For \( p_l( Q_1, Q_2) = \frac{2Q_1 - 2Q_1^2 + Q_2^2}{Q_1 + Q_2} < 2(1 - Q_2) \), we can substitute into (3.16) and re-arrange to obtain the following expression for the second period profits as a function of \( Q_1 \) and \( Q_2 \):

\[
\Pi_2^{p_l} (Q_1, Q_2, p_2) = \frac{Q_2^2 (4 + 4Q_1 (1 - Q_2) - Q_2 (8 - 5Q_2))}{2 (Q_1 + Q_2)} \tag{A.12}
\]

The first derivative of this expression with respect to \( Q_2 \) has two roots that lie in the interval \((0, 1)\). These two roots are \( Q_2^{p_1} (Q_1) = \frac{1}{3} (1 - 2Q_1 + \sqrt{1 + 8Q_1 (2 + 3Q_1)}) \), and \( Q_2^{p_2} (Q_1) = \frac{2}{3} \). However, by evaluating \( \frac{d\Pi_2^{p_l}}{dQ_2} \) at \( Q_2^{p_1} \) and at \( Q_2^{p_2} \), it can be confirmed that \( Q_2^{p_1} \) is a local maximum if and only if \( Q_1 < \frac{1}{3} \), and that \( Q_2^{p_2} \) is a local maximum if and only if \( Q_1 > \frac{1}{3} \). It can also be confirmed that \( p_l( Q_1, Q_2) < 2(1 - Q_2) \) for \( Q_2 = Q_2^{p_1} (Q_1) \) if and only if \( Q_1 \leq \frac{1}{3} \), while \( p_l( Q_1, Q_2) = 2(1 - Q_2) \) for \( Q_2 = Q_2^{p_2} (Q_1) \) for any value of \( Q_1 \). Thus, for any \( Q_1 \geq \frac{1}{3} \), it follows that the conditionally optimal second period response is to set \( Q_2 = Q_2^{p_2} (Q_1) = \frac{2}{3} \) and \( p_2 = p_l( Q_1, \frac{2}{3} ) = p_2^{p_h} = \frac{2}{3} \).

It remains to be shown that \( \Pi_2^{p_l} (Q_1, Q_2^{p_1}, p_l( Q_1, Q_2^{p_1}) ) \geq \Pi_2^{p_h} (Q_1, Q_2^{p_h}) = \frac{8}{27} \) for \( Q_1 \leq \frac{1}{3} \). To do this, let us first show that \( \Pi_2^{p_l} (Q_1, Q_2^{p_1}, p_l( Q_1, Q_2^{p_1}) ) \) is decreasing in \( Q_1 \) for \( Q_1 \in [0, \frac{1}{3}] \) and that \( \Pi_2^{p_l} (Q_1, Q_2^{p_1}, p_l( Q_1, Q_2^{p_1}) ) = \frac{8}{27} \) at the point \( Q_1 = \frac{1}{3} \). The first derivative of \( \Pi_2^{p_l} (Q_1, Q_2^{p_1}, p_l( Q_1, Q_2^{p_1}) ) \) with respect to \( Q_1 \) can be written as:

\[
\frac{d\Pi_2^{p_l}}{dQ_1} (Q_1, Q_2^{p_1}, p_l( Q_1, Q_2^{p_1}) ) = \frac{8}{27} \left( 12 + 36Q_1 \right) \sqrt{1 + 8Q_1 (2 + 3Q_1)} - 13 - Q_1 (118 + 177Q_1) \right)
\]

This derivative is negative at \( Q_1 = 0 \) and has only one root at \( Q_1 = \frac{1}{3} \). Thus, it is decreasing over the interval, \( Q_1 \in [0, \frac{1}{3}] \). Finally, when we substitute \( Q_2^{p_1} (Q_1) \) and \( Q_1 = \frac{1}{3} \) into (A.12), it is easy to confirm that \( \Pi_2^{p_l} (Q_1, Q_2^{p_1}, p_l( Q_1, Q_2^{p_1}) ) = \frac{8}{27} \).

b) Given the conditionally optimal responses in period 2, the profit function in period 1 depends upon \( Q_1 \).

\[
\Pi_1 (0, Q_1, p_1) = \begin{cases} 
\Pi_1^{QL} (Q_1, p_1) & \text{for } Q_1 \leq \frac{1}{3} \\
\Pi_1^{QH} (Q_1, p_1) & \text{for } Q_1 \leq \frac{1}{3}
\end{cases}
\]

where:

\[
\Pi_1^{QL} (Q_1, p_1) = Q_1 \left( r (0, Q_1, p_1) + \rho r (0, Q_2^{p_1} (Q_1), p_l( Q_1, Q_2^{p_1}) ) \right) + p_1 y(Q_1, p_1) + \rho \Pi_2^{p_l} (Q_1, Q_2^{p_1}, p_l( Q_1, Q_2^{p_1}) ) \tag{A.13}
\]

\[
\Pi_1^{QH} (Q_1, p_1) = Q_1 \left( r (0, Q_1, p_1) + \rho r (0, Q_2^{p_1} (Q_1), p_l( Q_1, Q_2^{p_1}) ) \right) + p_1 y(Q_1, p_1) + \rho \Pi_2^{p_h} (Q_1, Q_2^{p_h}, p_2^{p_h}) \tag{A.14}
\]
For $Q_1 \leq \frac{1}{3}$, the first period profit function is obtained by substituting $Q_2^{p_1}(Q_1)$ and $p_1(Q_1, Q_2^{p_1}(Q_1))$ for $Q_2^{**}$ and $p_2^{**}$ into the right-hand side of (3.17), while for $Q_1 \geq \frac{1}{3}$, we substitute $Q_2^{p_h}$ and $p_2^{p_h}$.

It is easy to confirm that for both forms of first period profit function are concave in $p_1$ and are maximized at $p_1 = Q_1$. Thus, we can substitute $p_1 = Q_1$ into $\Pi_1^{QL}(Q_1, p_1)$ and $\Pi_1^{QH}(Q_1, p_1)$ so that both forms of profit function are now functions of one variable only. It can be confirmed that $\frac{d\Pi_1^{QL}(Q_1, Q_1)}{dQ_1} > 0$ for all $Q_1 \in [0, \frac{1}{3}]$. Thus, we can restrict our attention to $Q_1 \geq \frac{1}{3}$. The first derivative of $\Pi_1^{QH}(Q_1, Q_1)$ has two roots, one at $Q_1 = \frac{2}{3}$ and another at $Q_1 = \frac{2}{3}$. However, the second derivative of $\Pi_1^{QH}(Q_1, Q_1)$ is negative at the $Q_1 = \frac{2}{3}$, while it is positive at $Q_1 = \frac{2}{3}$. Thus, the optimal quantity for period 1 must be at the point $Q_1^{QH} = \frac{2}{3}$. By substituting this into $Q_2^{p_1}(Q_1)$ and $p_1^{p_1}(Q_1)$ from part a), we can confirm the equilibrium quantity and price in the second period, and by substituting into (A.14), we can confirm that total profit is $\Pi_1^{QH} = \frac{8}{27} + \frac{8\rho}{27}$. 

**Proof of Proposition 3.14** From Lemma 3.12, we have that $Q_1^{p_1} = 2/5 < Q_2^{p_1} = 2/3$. From Lemma 3.7, we have that $Q_1^{p_0} = \frac{2(4+3\rho)}{20+11\rho}$ for $\delta = 1$. This is increasing in $\rho$ and is equal to $Q_1^{p_0} = \frac{2}{5}$ for $\rho = 0$, and is equal to $Q_1^{p_0} = \frac{14}{27}$ for $\rho = 0$. Thus, for all $\rho \in (0, 1]$, it follows that $Q_1^{p_0} < Q_1^{QH} < Q_2^{p_0}$.

From Lemma 3.12, we have that $p_1^{p_0} = \frac{2}{5}$ and that $p_2^{p_0} = \frac{2}{3}$. From Lemma 3.7, we have that $p_1^{p_0} = Q_1^{p_0} = \frac{2(4+3\rho)}{20+11\rho}$ for $\delta = 1$. As argued above, $\frac{2(4+3\rho)}{20+11\rho}$ is increasing in $\rho$ and is equal to $\frac{2}{5}$ when $\rho = 0$. It follows that $p_1^{p_0} < p_1^{p_0}$ for all $\rho \in (0, 1]$. By substituting $Q_1^{p_0} = \frac{2(4+3\rho)}{20+11\rho}$ into the upper branch of (3.13), we have that $p_2^{p_0} = \frac{16+8\rho}{20+11\rho}$. Since $\frac{16+8\rho}{20+11\rho}$ is decreasing in $\rho$ and is equal to $\frac{24}{37} > \frac{2}{5} = p_2^{p_0}$ when $\rho = 1$, it follows that $p_2^{p_0} < p_2^{p_0}$ for all $\rho \in (0, 1]$.

Finally, from Lemma 3.12, we have that $\Pi_1^{p_0} = \frac{8}{27} + \frac{8\rho}{27}$, and from Lemma 3.7, we have that $\Pi_1^{p_1} = \frac{8(2\rho)(4+3\rho)^2}{2(20+11\rho)^2}$. To see that $\Pi_1^{p_0} > \Pi_1^{p_0}$, it suffices to show that:

$$(27 + 25\rho) \frac{2(20 + 11\rho)^2}{27(2 + \rho)(4 + 3\rho)^2} > 25 \cdot 27(2 + \rho)(4 + 3\rho)^2$$

By collecting terms and cancelling, this inequality holds so long as $\rho(560 + (184 - 25\rho)\rho) > 0$. Thus it is true for all $\rho \in (0, 1]$. 