Abstract. Motivated by the growing popularity of online penny or pay-to-bid auctions we consider a setting where consumers have access to two sales channels: a fixed list price and an all-pay auction. In the fixed list price channel consumers can purchase the item at $p_1$. In the all-pay auction consumers place a bid $b$, which they forfeit regardless of the outcome, and the highest bid wins the item (and pays $b$). Consumers that lose out on the auction can choose to return to the fixed list price channel and buy the item at $p_1$, or simply accept their loss and do nothing further. Further motivated by buy-it-now options available on most penny auction sites, we consider a modification to the all-pay auction where consumers can use their sunk bid $b$ as a credit and buy the item directly for $p_2$; i.e. pay an additional $p_2 - b$ and receive the item. We characterize a symmetric equilibrium in the bidding/buying strategy and derive optimal prices $p_1$ and $p_2$ in order for the seller to maximize expected revenue. We consider both the situation when there is only one seller operating both channels, and when there are two sellers operating the two channels.

1. Introduction

Auctions as a formal sales mechanism date back to antiquity, and have been used extensively in both business-to-consumer and business-to-business markets. With the advent of the internet, auctions expanded to the consumer-to-consumer markets, and although the popularity of online auctions has stagnated some in recent years, their presence seems fortified by the online auction behemoth eBay. The latest internet based auction phenomena which has been growing in popularity are the so called penny or pay-to-bid auctions. Examples of these include, the former swoopo.com, dealdash.com, quibids.com, and beezid.com. There are three features that separate these from the more common online auction sites and in particular eBay. First, unlike eBay, which only hosts the auctions for sellers, the penny auction sites are the seller of the items auctioned off, which usually are brand new and predominately consumer electronics. Second, unlike the hard ending rule of eBay, where an auction ends at a pre-specified date and time, the penny auctions employ a sort of going, going,...gone ending rule by extending the auction clock for each incoming bid in the final moments. Third, the penny auction format can in effect can be labeled as an all-pay auction, with the basic premise as follows. Bidders first buy a pack of 'bids' for a fixed price, e.g. a pack of 100 ‘bids’ for $60. These ‘bids’ can then be used to nominally raise the price in an auction. That is, each time a bidder places a ‘bid’ the auction price increases by some small amount, e.g. $.01; hence the name penny auction. In other words, for this example, to raise the price by $.01 costs $.60. The bidder who places the last bid in an auction wins the item and pays the final auction
price (in addition to the incurred sunk cost of the ‘bids’ he or she placed). More explanation of all-pay auctions are provided in Section 2.

The attraction for bidders to participate in these types of auctions, is the chance of winning an item at a highly discounted price. From the online penny auctioneers’ perspective, the assumption or hope is that in general an auction will attract enough bidders who place enough ‘bids’ such that the revenue from all ‘bids’ exceed the cost of the item. The typical anecdotal stories regarding penny auction sites are that you can, for instance, win a brand new iPhone for $50, while the penny auction site collects a total revenue of $2,000. These type of stories have generated lots of negative press in both academia, newspapers and blogs, and basically labeled penny auctions as online gambling (Robinson, Giebelhausen, and Cotte 2011; USA Today 2011; NY Times 2009; Gimein 2009; Guina 2009). Even a class action lawsuit has been filed against quibids.com, claiming it is more akin to a gambling website than to an auction website (BusinessWire 2010; White, Mandel, and Kierstead 2010). Since online gambling is illegal in the US, the general strategy from the penny auction sites has been to position themselves as ‘entertainment shopping’ channels.

In this paper, we show that penny auction sites do not necessarily have to be classified as gambling, and provide a rational Nash equilibrium bidding strategy. In fact our results show that if bidders were to follow the proposed bidding strategy then their expected surplus is positive, while the expected surplus for the penny auction site is negative. We focus on how a retailer operating a fixed list price sales channel should set prices given the presence of an all-pay auction channel. To analyze the dynamics we consider two cases. In the first case, we assume there is only one seller operating both the fixed list price channel and the all-pay auction channel. In the second case, we assume there are two sellers; one operating the fixed list price channel and one operating the all-pay auction channel. In the latter scenario we assume the auctioneer buys the item from the retailer. This mimics the original business model of, for instance, swoopo.com, who would only order and buy the item at retail once an auction was concluded. Since then, many penny auction sites do buy wholesale before starting an auction. Motivated by another recent development that most, if not all, penny auction sites provide, we consider an extension where bidders who lose an auction may use their sunk bid as a credit and buy the item directly from the auctioneer. In other words, most penny auction sites provide a sort of buy-it-now feature, which we incorporate.
The main contributions of the paper are as follows. First, we provide an equilibrium bidding strategy in the private valuation framework for a single all-pay auction when bidders have an option to buy the item directly at a fixed price. The extension to include an external option for the bidders and not analyze the auction in isolation is rare and, we hope, serves the call of Rothkopf and Harstad (1994) and Haruvy et al. (2008). Second, we discuss how the retailer and auctioneer should set their respective fixed prices in order to maximize revenue. The call for more consideration on the business and operational aspects of selling using auctions has been made, for instance, by Pinker et al. (2003) and Shen and Su (2007). Some examples of related papers that also incorporate the operational aspects of selling through dual channels include Etzion et al. (2006) and Caldentey and Vulcano (2007). Although our paper is related and complements their insights, there are some key differences. First, while they consider a dual channel setting with a fixed list price and a multi-unit uniform price auction (where the winning bidders all pay the price of the highest non-winning bid), we consider a setting where the competing auction channel is a single-unit all-pay auction. Second, they consider a Poisson arrival process of bidders, while we restrict the discussion to a fixed number of bidders. Consequently the focus of our discussion is quite different from theirs.

The remaining paper is organized as follows. In Section 2 we briefly discuss some basic auction theory results and literature regarding online penny auctions. In Section 3 we discuss the model setup and bidding behavior of the consumers. We first consider the setting when consumers can either get the item through a fixed list-price channel or an all-pay auction channel. After that we consider the setting when consumers who lose out in the auction may use their sunk bid towards credit for purchasing the item at a second fixed price. In Section 4 we discuss the seller’s price optimization problem. We first consider a system with only one seller operating both the fixed list-price channel and the all-pay auction channel. After that we discuss the setting when there are two sellers; one operating the fixed list-price and one operating the all-pay auction channel. In Section 4.3 we illustrate how the retail seller’s and all-pay auctioneer’s expected profit change if bidders underestimate the number of bidders. In Section 5 we provide some concluding remarks and discuss potential future research topics. All proofs can be found in the Appendix.

2. Auction Theory and Literature Review

The standard private valuation single item auction model typically centers on a fixed number of bidders $N$, each with i.i.d. valuation $V$ drawn from distribution $F_V(v) = \Pr\{V \leq v\}$. Each bidder knows only their own (realized) valuation $v$ and the distribution $F_V(v)$ (but not the realized
valuations of the other bidders). Although the auctions may either be open or sealed-bid, we focus on the sealed-bid versions. In sealed-bid auctions, each bidder places their bid in a sealed envelope, when all bids have been submitted the seller (or auctioneer) opens the bids and announces, according to the specified format, the winner and the amount the winner has to pay. Typically two auction formats are considered: (1) First-Price Auction - where the highest bidder wins and pays the amount of his or her bid, (2) Second-Price Auction - where the highest bidder wins but only pays the amount of the second highest bid. A third and a bit more unusual format is the so called All-Pay Auction, where everyone pays the amount they bid but still only the highest bidder wins the item. For all three auction formats, a bidder’s decision regarding whether to bid and if so how much to bid can be illustrated by a decision tree; see the decision tree in panel A of Figure 1.

In the first-price and second-price auction formats $y_b = 0$, while for the all-pay auction $y_b = -b$.

In the first-price and all-pay auction formats $x_b = b$, while for the second-price auction $x_b$ is the amount of the second highest bid. The probability of winning $\rho_b$, depends on the bid $b$.

In the Economics literature and in particular the Auction Theory literature, the primary focus is on bidding strategies of a symmetric equilibrium. These are characterized by: (1) the bid amount $b$ is increasing in the valuation $v$, thus ensuring that the bidder with the highest valuation wins, and (2) no individual bidder has an incentive to deviate from the bidding strategy given that the other $N-1$ bidders adhere to it. The classic and well-established Nash equilibrium bidding strategies are: (1) in a first-price auction, $b = \mathbb{E}[V^{(1)}|V^{(1)} < v]$, (2) in a second-price auction, $b = v$, and (3) in an all-pay auction, $b = \Pr\{V^{(1)} < v\} \times \mathbb{E}[V^{(1)}|V^{(1)} < v]$; where $V^{(1)}$ is the highest valuation from the other $N-1$ bidders. In other words, in a second-price auction you should truthfully bid your valuation, in a first-price auction you should bid the expected value of the second highest valuation given that it is below your own valuation, and in the all-pay auction you bid a fraction (namely the probability that the second highest valuation is below yours) of the second highest valuation given that it is below your valuation. An interesting difference between the three bidding strategies is that, unlike the first-price and second-price auctions where the bid $b$ is non-decreasing in the number of bidders $N$, for the all-pay auction the bid $b$ is not monotone in $N$. That is, it may be the case that for bidders with ‘high’ valuation as the number of bidders $N$ increase the bid $b$ decreases, while for bidders with ‘low’ valuations the bid $b$ increases. More details regarding this and in particular the impact on the auctioneers expected profit in presented in Section 4.3.

Another remarkable and rather un-intuitive result regarding the three auction formats, is that the
expected surplus for the bidder and the expected revenue for the seller is the same for all three auction formats. Furthermore, the probability of winning is the same in each auction format, namely $\rho_b = \Pr\{V_1 < v\}$. See Krishna (2002) for more background.

Traditionally the theory and results of all-pay auctions have been applied to model lobbying and selection of research and development projects; see, for example, Baye et al. (1993), and Anderson et al. (1998). For an experimental study that compares the results from Anderson et al. (1998) with empirical results, see Gneezy and Smorodinsky (2006). More recently the all-pay auction literature has focussed on the emerging penny auction business model. Three recent working papers
that model the bidding dynamics of online penny auctions include Platt et al. (2010), Augenblick (2010) and Hinnosaar (2010). Independently but based on a similar underlying model framework the three papers derive equilibrium results for the auction end time. More specifically they model a single penny auction with a fixed number of bidders in a common valuation framework. Using discrete time periods they all characterize a bidding strategy based on the probability of placing a bid in a period. In addition, they all provide extensive empirical analysis on 10,000s of auctions. Our paper and model approach differs dramatically from theirs. First, we consider a private valuation framework regarding bidders’ valuation. Second, we do not analyze the auction in isolation but include that consumers have a choice to also buy the item directly from a fixed list price channel. In an extension to our framework we also incorporate that non-winning consumers can use their sunk bid towards a credit and buy the item directly from the all-pay auctioneer. Third, we assume the auction is standard and that bidders only place one bid. An auction is standard if the bidder who bids the most is guaranteed to win the auction (Krishna 2002, pg.29).

This last point is an important technical detail to note. Strictly speaking the current format of most, if not all, online penny auctions is that they are not standard since the winner is the bidder who places the last bid. It is the uncertainty of still losing despite having spent the most that gives rise to the gambling attribute of the penny auction sites. On the other hand, one might conjecture that in practice the main bidding activity and in particular counter bidding takes place close to the end of the auction, and that consequently it is the bidder who places the most bids that does indeed win. Furthermore, most penny auction sites tend to provide a service where bidders enter the maximum amount they are willing to spend and then the auction sites monitor and bid as minimally necessary on behalf of the bidder. This service can be related to what usually is refereed to as the proxy bidding of eBay. Some examples include, the former bid butler on swoopo.com, BidBuddy on dealdash.com, and beezid sniper on beezid.com. Although these services do not guarantee the penny auctions to be standard, it would seem that in reality they would be. However, whether penny auctions are de facto standard is an empirical question that we do not seek to answer in this paper.

3. Bidding and Buying Framework

We consider a stylized setting where consumers only have access to two sales channels: a store selling at a fixed list price, and an online all-pay auction site. In the fixed list price channel the consumers can buy the item for $p_1$. In the all-pay auction consumers can place a bid $b$ (which
they pay regardless of the outcome), and if their bid is the highest then they win the item, but if their bid is not the highest then they can choose to either simply accept the loss of \( b \) and not do anything or return to the store and buy the item. We assume consumers arrive with a realized valuation \( v \) and that they know there are \( N \geq 1 \) other consumers whose valuation is drawn i.i.d. from \( F_V(v) = \Pr\{V \leq v\} \), with support on \([v, \tau]\).\(^1\) We are interested in characterizing a symmetric equilibrium in the bidding/buying strategy. Therefore, for the remainder of the paper, we define \( \rho_v \) as the probability a consumer with valuation \( v \) has the highest valuation and hence wins the auction, \( \rho_v = \Pr\{V(1) < v\} = (F_V(v))^N \); where \( V(1) \) is the highest order statistics of the other \( N \) bidders’ valuation. We consider two forms of list item purchasing following a failed bid attempt. First, a standard model where the consumer may purchase the item at \( p_1 \) from the retail seller. Second, a modified version where consumers can use their failed bid value as a credit towards purchasing the item at \( p_2 \) from the auctioneer, or at \( p_1 \) from the retail seller but without any credit from the failed bid.

### 3.1. All-Pay Auction with Retail Option.

In the first setting we assume consumers decide between purchasing an item at \( p_1 \), placing a bid \( b \), or do nothing. The consumers who place a bid \( b \) but fail to win the item may either choose to purchase the item at \( p_1 \) or simply do nothing further (see panel (B) of Figure 1 for model illustration). Logically, consumers will never bid more than \( p_1 \) regardless of their valuation \( v \). Furthermore, a consumer with valuation \( v > p_1 \) will never just forfeit their bid if they lose the auction, i.e. if they lose the auction then they would always be better off buying the item at \( p_1 \). Consequently, it is not immediate that the standard bidding strategy results from Auction Theory are applicable. On the contrary, we propose that they bid a given fraction of \( p_1 \). Namely, a fraction representing the probability that their valuation is the highest as well as a constant representing the expected surplus of the consumer with \( v = p_1 \). For consumers with \( v \leq p_1 \), it will never be optimal to buy the item at \( p_1 \); neither from the onset or after having lost an auction. It therefore follows that for consumers with \( v \leq p_1 \) they are faced with the traditional auction theory framework and hence should bid accordingly. We formally summarize this result in the following Lemma.

**Lemma 3.1.** If there are \( N + 1 \) consumers and a consumer with valuation \( v \) acts as follows,

1. if \( v > p_1 \), then bid \( b = \rho_v p_1 - u_{p_1} \), and if lose the auction then buy at \( p_1 \)
2. if \( v \leq p_1 \), then bid \( b = \rho_v E[V(1)|V(1) < v] \), and if lose the auction then do nothing

\(^1\)Note that we assume there are \( N + 1 \) total bidders. We do this to simplify the notation in the ensuing analysis.
where \( \rho_v = (F_{V_1}(v))^{N} \), \( V_1 \) is the highest order statistics of the other \( N \) consumers’ valuation, and \( u_{p_1} \) is the expected surplus for the consumer with valuation \( v = p_1 \) (i.e. \( u_{p_1} = \rho_{p_1}(p_1 - E[V_1 | V_1 < p_1]) \)), then the strategy is a Nash equilibrium.

Given the above bidding strategy the expected surplus for a consumer with valuation \( v > p_1 \) is,

\[
\phi_v = \rho_v (v - \rho_v p_1 + u_{p_1}) + (1 - \rho_v) (v - p_1 - \rho_v p_1 + u_{p_1}) \\
= v - p_1 + u_{p_1}
\]

and for a consumer with valuation \( v \leq p_1 \) it is,

\[
\phi_v = \rho_v (v - \rho_v E[V_1 | V_1 < v]) - (1 - \rho_v) (\rho_v E[V_1 | V_1 < v])
\]

\[
= \rho_v (v - E[V_1 | V_1 < v])
\]

In other words, for all consumers the expected surplus is higher with the added all-pay auction channel than if there was only a fixed list price store. The consumers with low valuation derive some surplus from having the chance of winning an item that they cannot afford, while the high valuation consumers receive the added expected surplus \( u_{p_1} \). As consumers with \( v > p_1 \) receive the added benefit and \( u_{p_1} \) they do not have an incentive to bid as if their valuation were below \( p_1 \), i.e. ensures they do not deviate from the equilibrium bidding strategy.

3.2. Modified All-Pay Auction with Retail Option. We now extend the above setting and allow losing bidders the option of using their sunk bid \( b \) towards a credit and buy the item for \( p_2 \). That is, if they lose the auction then they can choose to pay an additional \( p_2 - b \) and receive the item, or as above either simply accept the loss of \( b \) and not do anything or return to the store and buy the item at \( p_1 \) (but without any credit from the sunk cost bid). See panel (C) in Figure 1 for an illustration of the decision tree. The motivation for this extension comes from the buy-it-now features most penny auction sites offer. With this feature a consumer can essentially at any point during the auction choose to buy the item at a fixed price using their sunk bids. It should be noted though, that this service is not always offered on all online penny auctions and that the credit amount does not always correspond to the amount spent on the bids. Furthermore and quite naturally, there usually is a premium on the price that the penny auction sites offer.

With this extension there are a few comments to make. First, assuming the auctioneer buys the item at \( p_1 \) and the number of consumers is fixed, it is only of interest to consider prices such that \( p_1 < p_2 \). An anecdotal example and motivation includes a set of auctions we observed on
swoopo.ca for Sony Playstation 3 (250GB) consoles. In June, 2010, Swoopo listed the value (price) as $389.00, although a quick search at that time revealed that, for instance, BestBuy.com was selling the same item at $349.99. A second comment is that, as above no consumer would ever bid more than \( p_1 \) regardless of their valuation \( v \). Third, consumers with valuation \( v > p_1 \) will still not just forfeit their bid if they lose the auction. Now, however, they have an option of either buying it at \( p_2 \) or \( p_1 \). Fourth, although consumers with \( v < p_1 \), will never buy the item at \( p_1 \) they might, upon losing the auction, have an incentive to buy the item at \( p_2 \). This would happen if their bid is more than \( p_2 - v \), in which case rather than take the loss \(-b\) they take the loss \( v - p_2 \). In order to avoid this issue and simplify the analysis, we impose that \( p_2 \geq p_1 + \rho p_1 E[V(1)|V(1) < p_1] \). The reason for this will be clear in the ensuing analysis. Given this assumption there exists a threshold \( v^+ = F^{-1}(([p_2 - p_1 + u_{p_1}]/p_1)^\frac{1}{N}) \), such that if (1) \( v \leq v^+ \), then the consumer should act according to Lemma 3.1, while if (2) \( v > v^+ \) then the consumer should first bid \( b = (p_1 - (1 - \rho v)p_2 - u_{p_1})/\rho v \), and if unsuccessful buy at \( p_2 \); with \( u_{p_1} \) as defined in Lemma 3.1. We summarize this in the following Lemma.

**Lemma 3.2.** If there are \( N + 1 \) consumers, \( p_2 \geq p_1 + \rho p_1 E[V(1)|V(1) < p_1] \), and a consumer with valuation \( v \) acts as follows,

1. if \( v > v^+ \), then bid \( b = (p_1 - (1 - \rho v)p_2 - u_{p_1})/\rho v \), and if lose the auction then buy at \( p_2 \)
2. if \( v \leq v^+ \), then act according to Lemma 3.1

where \( v^+ \equiv \max\{p_1, F^{-1}(\min\{1, ([p_2 - p_1 + u_{p_1}]/p_1)^\frac{1}{N}\})\} \), \( \rho v = (F_V(v))^N \), \( V(1) \) is the highest order statistics of the other \( N \) consumers’ valuation, and \( u_{p_1} = \rho p_1 (p_1 - E[V(1)|V(1) < p_1]) \), then the strategy is a Nash equilibrium.

With the added option the expected surplus for a consumer with valuation \( v > v^+ \) is,

\[
\phi_v = \rho v (v - (p_1 - (1 - \rho v)p_2 - u_{p_1})/\rho v) + (1 - \rho v)(v - p_2)
\]

\[\text{(3.3)}\]

while for the consumers with \( v \leq v^+ \), the expected surplus is the same as before. In other words, although adding \( p_2 \) segments the consumers with \( v > p_1 \) into two different bidding strategies, their expected surplus does not change. Note that the threshold \( v^+ \) could depending on the particular values of \( p_1 \) and \( p_2 \) collapse to either \( p_1 \) or \( \overline{v} \). For instance, if \( p_2 \) is too high then \( v^+ \) collapses to \( \overline{v} \), resulting in that there is only one type of high valuation consumers. An upper bound on \( p_2 \) in order to ensure the segmentation is provided in the following corollary.

**Corollary 3.3.** If \( p_2 \geq 2p_1 \) then no one will buy the item at \( p_2 \).
In other words, if $p_2$ is twice as high as $p_1$ then no one will buy it at $p_2$, and there is no point for the auctioneer in even offering the price $p_2$. It should be noted that the upper bound in Corollary 3.3 is not the least upper bound for the result to hold. Since the consumers with $v > p_1$ are given the added surplus of $u_{p_1}$ the supremum for the result to hold is $2p_1 - u_{p_1}$. The infimum for the lower bounds on $p_2$ such that the threshold $v^+$ collapses to $p_1$ is the lower bound given in Lemma 3.2. That is, if $p_2 = p_1 + \rho_{p_1} E[V(1)|V(1) < p_1]$ then $v^+ = p_1$. We summarize the consequence of this in the following corollary.

**Corollary 3.4.** If $p_2 = p_1 + \rho_{p_1} E[V(1)|V(1) < p_1]$ then no one will buy the item at the store for the fixed list price $p_1$.

### 3.3. Example with Valuations Uniformly Distributed

We illustrate the bidding strategies from Lemmas 3.1 and 3.2 with some numerical examples. Assume there are five consumers whose valuation are uniformly distributed on $[0, 1]$. We consider three sets of prices: (A) $p_1 = .35$, $p_2 = .75$, (B) $p_1 = .50$, $p_2 = .55$, (C) $p_1 = .70$, $p_2 = .90$. For the three cases we have the following thresholds: (A) $v^+ = 1.0$, (B) $v^+ = 0.58$, (C) $v^+ = 0.76$. Figure 2 displays the bidding strategy (left) and expected surplus (right) as a function of a consumer’s valuation for each set of prices. It should be noted that although the expected surplus seems to be zero for consumers with valuation less .40, it is in fact positive. In the first case there are two types of consumers: (1) consumers with $v \leq p_1$ who bid $b_v = v \frac{N}{N+1} v$, and if they lose do nothing, (2) consumers with $p_1 < v \leq v^+ = \overline{v}$ who bid $b_v = v^N p_1 - \frac{p^{N+1}}{N+1}$, and if they lose buy the item for $p_1$. In other words, the first case is equivalent to $p_2$ not being offered. In the latter two cases there are three types of consumers: (1) consumers for which $v \leq p_1$ who bid $b_v = v^N \frac{N}{N+1} v$, and if they lose do nothing, (2) consumers with $p_1 < v \leq v^+$ who bid $b_v = v^N p_1 - \frac{p^{N+1}}{N+1}$, and if they lose buy the item for $p_1$, and (3) consumers with $v^+ < v \leq \overline{v}$ who bid $b_v = \frac{p_1 - (1-v^N)p_2 - \frac{p^{N+1}}{N+1}}{v^N}$ and if they lose buy the item at $p_2$.

### 4. Seller’s Optimization Model

Given the bidding strategy from Section 3 we now consider the retail seller’s and auctioneer’s problem in setting optimal prices $p_1$ and $p_2$ to maximize expected revenue. We first consider a system where one seller operates both the fixed list price store and the all-pay auction. In Section 4.2 below we extend the setting and consider a system with two sellers: a retail seller and an all-pay auctioneer.
Figure 2. Numerical illustration of bidding for five consumers with $V \sim U[0,1]$. 

(A) $p_1 = .35, p_2 = .75, v^+ = 1.0$

(B) $p_1 = .50, p_2 = .55, v^+ = .58$

(C) $p_1 = .70, p_2 = .90, v^+ = .76$
4.1. **Single Seller.** If there is only one seller then the seller receives bids \( b_v \) from consumers with valuations below \( p_1 \), either \( b_v \) or \( b_v + p_1 \) for those with valuations between \( p_1 \) and \( v^+ \) depending on whether the consumer wins or loses the auction, and similarly \( b_v \) or \( p_2 \) from winners and losers with valuations above \( v^+ \). Therefore the seller’s expected profit (per consumer) \( \pi \) is expressed as,

\[
\pi = \int_0^{p_1} b_v f(v)dv + \int_{v^+}^p [\rho_v b_v + (1 - \rho_v)(p_1 + b_v)] f(v)dv + \int_{v^+}^\infty [\rho_v b_v + (1 - \rho_v)p_2] f(v)dv
\]

(4.1) = \int_0^{p_1} b_v f(v)dv + \int_{p_1}^{v^+} [b_v + (1 - \rho_v)p_1] f(v)dv + \int_{v^+}^\infty [\rho_v b_v + (1 - \rho_v)p_2] f(v)dv

with \( b_v \) and \( v^+ \) as defined in Lemmas 3.1 and 3.2. A result that may seem a bit counter-intuitive is that the seller’s expected profit \( \pi \) is independent of \( p_2 \). We summarize this result in the following proposition.

**Proposition 4.1.** If there is one seller operating both the fixed list price store and the online all-pay auction channel, then the expected profit per consumer is independent of \( p_2 \) and becomes,

\[
\pi = \int_0^{p_1} \rho_v E[V(1)|V(1) < v] f(v)dv + (p_1(1 - \rho_p) + \rho_p E[V(1)|V(1) < p_1])(1 - F_V(p_1))
\]

(4.2) = \int_0^{p_1} \rho_v E[V(1)|V(1) < v] f(v)dv + \rho_p E[V(1)|V(1) < p_1](1 - F_V(p_1))

where \( \rho_v = (F_V(v))^N \).

In other words, although \( p_2 \) may segment the consumers into different bidding strategy segments, it does not effect the seller’s expected profit. From equation (4.2) it is not immediately clear if the seller is better or worse off with the added all-pay auction channel. Although the seller incurs an additional expected revenue of \( \int_0^{p_1} \rho_v E[V(1)|V(1) < v] f(v)dv \) and \( \rho_p E[V(1)|V(1) < p_1](1 - F_V(p_1)) \), this comes at the expense of a lesser revenue \( p_1(1 - \rho_p)(1 - F_V(p_1)) \) as compared with only operating a fixed list price channel with expected revenue of \( p_1(1 - F_V(p_1)) \). To further explore this trade-off we next discuss the case when consumers’ valuations are uniformly distributed.

4.1.1. **Optimal Prices for Valuations Uniformly Distributed.** We can explicitly solve the seller’s expected revenue (4.2) if we assume a functional form for the valuation uncertainty. In the following example we continue our earlier assumption that consumers’ valuations are uniformly distributed on \([0, 1] \), in which case equation (4.2) becomes,

\[
\pi = \int_0^{p_1} v^N \frac{N}{N + 1} f(v)dv + (p_1(1 - p_1^N) + p_1^N \frac{N}{N + 1} p_1)(1 - p_1)
\]

(4.3) = \int_0^{p_1} v^N \frac{N}{N + 1} f(v)dv + \frac{2}{N + 2} - \frac{N + 1}{N + 2}

Graph A in Figure 3 display equation (4.3) for the case with three and five consumers. On the horizontal axis is \( p_1 \) and on the vertical axis is \( \pi \). The dashed line is the case when there are three
Proposition 4.2. If there is only one seller operating both the fixed list price and the online all-pay auction channel, and consumers with valuation drawn from \( U[0,1] \) follow the strategies given in Lemmas 3.1 and 3.2, then the optimal price that maximizes expected profit is \( p^*_1 = 1/2 \).

Another thing to note in Graph A of Figure 3 is that for ‘high’ prices it seems that the seller is better off with the added channel, while for ‘low’ prices the seller is better off with just the fixed list price sales channel. Comparing the expected revenue equation (4.3) with the expected revenue for a seller with only a fixed list price channel (\( \pi_M \)), we can derive a threshold in \( p_1 \) for when the seller is better off with the added all-pay auction channel. This result is summarized in the following result.
Proposition 4.3. If there are $N+1$ consumers with valuation drawn from $U[0,1]$ who follow the strategies given in Lemmas 3.1 and 3.2, and $p_1 \geq \frac{N+2}{2(N+1)}$, then the seller’s expected revenue (per consumer) is higher with the added all-pay auction than with just a fixed list price sales channel.

We see that the threshold provided in Proposition 4.3 asymptotically approaches $\frac{1}{2}$ as $N$ increases. So while the addition of an all-pay auction in the presence of optimally set prices decreases seller revenue (as it provides a consumer surplus) this loss decreases with increased number of consumers and is actually a gain if the firm has sub-optimally set prices too high.

4.2. Retail Seller vs Online All-Pay Auctioneer. We now extend the above setting and consider a system with two sellers: a seller operating a fixed list price store and a seller operating an online all-pay auction. To make the revenue comparison consistent with the setting with a single operator, we assume the all-pay auctioneer purchases the goods auctioned at $p_1$ from the retail seller. That is, we are interested in deriving expressions for the retail seller’s expected profit $\pi_r$ and the all-pay auction seller’s expected profit $\pi_{ap}$, such that $\pi = \pi_r + \pi_{ap}$. Recall that both the winner of the all-pay auction as well as the losers with $v > v^+$ receives an item from the all-pay auction channel. With these changes the expected profit (per consumer) for the retail seller becomes,

\begin{align}
\pi_r &= \int_{v}^{p_1} \rho_v p_1 f(v) dv + p_1 (1 - F_V(p_1)) \tag{4.4}
\end{align}

Note that the second part of the profit function is the expected profit for a retail seller operating in a setting without the all-pay auction channel. In other words, compared to the setting when there is only a retailer, the retail seller is better off with the added all-pay auction channel. This is of course intuitive since we have assumed the all-pay auctioneer buys the items from the retail seller at $p_1$, and hence there is some positive expected profit coming from the scenarios when a consumer with $v \leq p_1$ wins the auction.

The expected profit (per consumer) for the all-pay auctioneer is,

\begin{align}
\pi_{ap} &= \int_{v}^{p_1} (b_v - \rho_v p_1) f(v) dv + \int_{v^+}^{p_1} (b_v - \rho_v p_1) f(v) dv + \int_{v^+}^{p_2} (\rho_v b_v - p_1 + (1 - \rho_v)p_2) f(v) dv \\
&= \int_{v}^{p_1} \rho_v (E[V(1)|V(1) < v] - p_1) f(v) dv - u_{p_1} (1 - F_V(p_1)) \tag{4.5}
\end{align}

Since $E[V(1)|V(1) < v] \leq p_1$ for all $v \leq p_1$, and $u_{p_1} \geq 0$ it follow that the all-pay auctioneer’s expected profit is always negative, which we formally summarize in the following result.
Proposition 4.4. If there are \( N + 1 \) consumers who act according to Lemmas 3.1 and 3.2, and the all-pay auctioneer buys the items from the retail seller at \( p_1 \), then the all-pay online auction seller’s expected profit is always negative.

To gain further insight into the retailer’s and all-pay auctioneer’s expected profit we next discuss the case with structural assumptions on the consumers’ valuation. More specifically we follow the previous examples and assume valuations are drawn from \( U[0, 1] \).

4.2.1. Optimal Prices for Valuations Uniformly Distributed - Dual Sellers. When consumers valuation is drawn from \( U[0, 1] \), then the retailer’s expected profit (4.4) becomes,

\[
\pi_r = \int_0^{p_1} v^N p_1 f(v) dv + p_1 (1 - p_1)
\]

\[
= \frac{p_1^{N+2}}{N+1} + p_1 (1 - p_1)
\]

(4.6)

While the all-pay auctioneer’s expected profit (4.5) becomes,

\[
\pi_{ap} = \int_0^{p_1} v^N \left( \frac{N}{N+1} v - p_1 \right) f(v) dv - u p_1 (1 - p_1)
\]

\[
= \frac{p_1^{N+1}}{N+1} \left( \frac{N p_1}{N+2} - 1 \right)
\]

(4.7)

As we observed above, we see that the retail seller’s expected profit is always positive, while the all-pay auctioneer’s expected profit is always negative. The retail seller would naturally want to set the optimal price \( p_1^* \) that maximizes (4.6). However, unlike the scenario when the retail seller also employs the all-pay auction channel, the optimal price is not necessarily \( 1/2 \). In fact, there is no closed form solution for (4.6) since the function depends on the number of consumers through the \( N + 2 \) exponent.

Numerical illustrations of the expected profit functions (4.6) and (4.7) are provided in Figure 4. On the horizontal axis is the fixed list price as set by the retail seller, while on the vertical axis is the expected profit per consumer for the retail seller (top lines) and all-pay auctioneer (bottom lines). The dashed, dotted and solid line represent the case with three, five, and nine consumer respectively. As expected we see that the all-pay auctioneer always incurs an expected loss, and that the loss is increasing in the fixed list price \( p_1 \). In addition, we see that as expected the retail seller’s expected profit is always positive, and that depending on the number of consumers the optimal price \( p_1^* \) changes. Note too, that the sum of the retailer’s and all-pay auctioneer’s expected profit is equal to the expected profit for the case with a single seller operating both channels.
4.3. Over-Bidding with Valuations Uniform Distributed Valuations. Our earlier results indicate that if the posted prices are set optimally then the operation of an all-pay auction results in a loss to the auctioneer or at best case scenario no incremental gain to a seller operating both a fixed list price channel and all-pay auction channel. In this section we show how the all-pay auction can increase system revenue if consumers mildly over-bid; for illustrative purposes we examine the case where bidders underestimate the number of consumers participating in the auction. As mentioned in Section 2 a difference between the all-pay auction and the more common auction formats of first-price and second-price auctions, is that the bidding strategy is not monotone in the number of bidders. That is, for the traditional all-pay auction framework and the Nash equilibrium bidding strategy mentioned in Section 2, the bid amount $b_v$ may increase or decrease as the number of bidders $N$ increase (depending on the bidder’s valuation $v$). This is in contrast to the two Nash equilibrium bidding strategies of the first-price and second-price auctions that were mentioned in Section 2, where in a first-price auction $b_v$ is non-decreasing in $N$, while for second-price auction
$b_v$ is constant in $N$. This rather peculiar feature of the all-pay auction also holds in our framework with the added retail channel.

To illustrate consider the numerical example provided in Figure 5. The figure represents the second numerical example discussed in Section 3.3 with the added scenarios when the five consumers underestimate the number of competing consumers by $m$, $m = 0, 1, 2, 3$. In other words, when $m = 0$ they do not underestimate the number of consumers, when $m = 1$ they all think there are only four consumers in total when in fact there are five, and so on. Similar to Figure 2 the left graph represent the bidding strategy while the right graph represent the expected surplus. However, in this case when $m > 0$, the right graph represent the perceived expected surplus. Note that the five consumers by underestimating the number of competing consumers all overestimate their chance of having the highest valuation, and hence overestimate $\rho_v$. There are a few interesting observations to make. First, we note that as $m$ increases for consumers with ‘low’ valuation the bid amount increases, while for consumers with ‘high’ valuation the bid amount decreases. Second, the threshold $v^+$ is decreasing in $m$, and for $m > 0$ we have $v^+ = p_1$. Note though, that what appears as a discontinuity in the bidding strategy at $p_1$, is strictly due to the ‘resolution’ of the graph. The bidding strategy is in fact a smooth continuous curve. Third, we see that the (perceived) expected surplus is increasing in $m$. This is of course intuitive as with fewer consumers the expected gain should be higher (this also holds in the traditional first- and second-price auction formats).

Next we analyze the impact of over-bidding on the seller’s and auctioneer’s expected revenue. We adjust equilibrium bids from Lemma 3.1 to reflect bidding assuming all consumers believe there are only $N - m$ other consumers (versus $N$). As earlier, initially we focus on the setting with a single operator of both fixed price selling and the auction, followed by discussion of the auction being operated by second party.

The seller providing both fixed price selling and an all-pay auction with consumers underestimating the number of competing bidders by $m$ has expected profit (per consumer) of,

$$\tilde{\pi} = \int_{v}^{p_1} \tilde{\rho}_v E[\tilde{V}_1|\tilde{V}_1 < v] f(v) dv + \int_{p_1}^{\tilde{\rho}_v} (\tilde{\rho}_v p_1 - \tilde{u}_p + (1 - \tilde{\rho}_v) p_1) f(v) dv$$

where $\tilde{\rho}_v = (F V(v))^{N-m}$, $E[\tilde{V}_1|\tilde{V}_1 < v]$, and $\tilde{u}_{p_1} = \tilde{\rho}_p (p_1 - E[\tilde{V}_1|\tilde{V}_1 < p_1])$. 

(4.8)
To gain some further insight and compare with our earlier results we assume consumer valuations are drawn from $U[0,1]$. In this case (4.8) becomes,

$$\tilde{\pi} = \int p_1 v^{N-m} \frac{N-m}{N-m+1}vdv + \int p_1 v^{N-m} p_1 - p_{1}^{N-m+1} \frac{1}{N-m+1} + (1-v^N)p_1dv$$

(4.9)

$$= p_1 \left( \frac{(N+1)^2 - Nm}{(N-m+1)(N+1)} - p_1 \right) + p_1^{N+2} \left( \frac{p_1^{-m}(N-m)}{(N-m+1)(N-m+2)} + \frac{1}{N+1} \right) - \frac{p_1^{N-m+1}}{N-m+1}$$

We see that when $m = 0$ equation (4.9) becomes (4.3). Graph B in Figure 3 display two illustrations of equation (4.9). The dashed line is when there are three consumers who all underestimate the number of consumers by 1, i.e. $N = 2, m = 1$. The dotted line is when there are five consumers who all underestimate the number of consumers by one, i.e. $N = 4, m = 1$. The solid line is the benchmark expected profit for a seller selling only using a fixed list price. We see that even by underestimating the number of consumers by only one the seller is better off. It is also interesting to note that over-bidding may increase the system revenues beyond the optimal posted price selling without the all-pay auction. Another comment is that the same analysis for the scenario with a modified all-pay auction channel (i.e. with the added option to buy at $p_2$) is possible, but then the resulting equation would be a bit more difficult to compare with our earlier results. In particular, equation (4.9) would contain $p_2$. 

Figure 5. Numerical illustration of over-bidding for five bidders with $V \sim U[0,1]$. 

\begin{align*}
    p_1 &= .50, p_2 = .55, v^+ = .58, m = 0, 1, 2, 3
\end{align*}
Table 1. Numerical illustration of the expected profit per consumer for the all-pay auctioneer with consumers overbidding due to underestimating the number of consumers by \( m \); assuming \( V \sim U[0, 1] \). The last column represents the optimal price \( p_1^* \) set by the retailer.

Next we consider over-bidding for the setting with one retail seller and one all-pay auctioneer. The first thing to note is that the retail seller is not affected by consumers underestimating the number of competing consumers, with the retail seller still receiving a surplus (above posted price selling) with the addition of the auction by a second party. That is, \( \pi_r \) in equation (4.4) is independent of the consumers’ bidding strategy. On the other hand, the all-pay auctioneer’s expected revenue is of course affected and (4.5) becomes,

\[
\tilde{\pi}_{ap} = \int_{b_0}^{p_1} (\tilde{b}_v - \rho_v p_1) f(v) dv + \int_{p_1}^{\tilde{b}_v} (\tilde{b}_v - \rho_v p_1) f(v) dv
\]

where as earlier \( \tilde{\rho}_v = (F_V(v))^{N-m}, \ E(\tilde{V}(1)|\tilde{V}(1) < v] \), and \( \tilde{u}_{p_1} = \tilde{\rho}_v (p_1 - E(\tilde{V}(1)|\tilde{V}(1) < p_1)) \), results from consumers solving for equilibrium but underestimating the number of consumers by \( m \). To compare and contrast the implications of underestimating the number of bidders by our earlier results, we assume consumer valuations are drawn from \( U[0, 1] \), in which case (4.10) becomes,

\[
\tilde{\pi}_{ap} = \int_{0}^{p_1} v^{N-m} \frac{N-m}{N-m+1} v - v^{N} p_1 dv + \int_{p_1}^{1} v^{N-m} p_1 - p_1^{N-m+1} \frac{1}{N-m+1} - v^{N} p_1 dv
\]

We see that when \( m = 0 \) equation (4.11) becomes (4.7). Note too that while \( \pi_{ap} \) is always negative, we see that \( \tilde{\pi}_{ap} \) could be positive. We illustrate this last point in Table 1 with some numerical examples when the retail seller sets an optimal price \( p_1^* \) based on solving for (4.6). There are a few observations to make. First, if there are only two or three consumers then the retail seller should set \( p_1 = 1.0 \). Though this may seem a bit counter-intuitive, it is results from the auctioneer
buying the product from the retail seller. Consequently, the retail seller is always guaranteed one sale. Second, when there is no overbidding and \( m = 0 \), then the all-pay auctioneer always incurs an expected loss. Third, if there are enough consumers such that the retail seller has an incentive to set a price less than 1.0, then the all-pay auctioneer will incur a positive expected gain as soon as the consumers underestimate the number of consumers.

5. Conclusion

In this paper we have attempted to clarify some issues around all-pay auctions and the recent proliferation of penny auction websites. To this end we have illustrated equilibrium bidding behavior when consumers have access to regular sales channels as well as all-pay auctions. Furthermore, given the proposed bid behavior we derive optimal prices for list price selling. Anecdotally our models are consistent with current practice, as most penny auction websites are supplied by items readily available from third parties. Our developments show that retailers and consumers benefit from the creation of all-pay auctions with these benefits coming at a cost to the all-pay auctioneer. This result comes in stark contrast to the much negative connotation that has grown in the wake of the proliferation of penny auction websites. Due to the unusual and unintuitive format of all-pay auctions it is perhaps not surprising that penny auction websites are being labeled as online gambling. We hope our paper can clarify some of the misunderstanding and provide consumers with better insight in how to regard and bid in all-pay type auctions.

Another issue to note regarding the negative feedback is that it does not seem reasonable to expect that penny auction websites can be based on business models where consumers derive negative (expected) surplus. In the long-run that simply would not seem sustainable, and to quote Myerson (1991, p.5),

‘If a theory [business model] predicts that some individuals will be systematically fooled or led into making costly mistakes, then this theory [business model] will tend to lose its validity when these individuals learn (from experience or from a published version of the theory [business model] itself) to better understand this situation.’

From this quote and the results presented in this paper it is noteworthy to observe that the first entrant and dominant player swoopo.com filed for bankruptcy in March, 2011 (Oswold 2011). As illustrated by our efforts all-pay auctions operated in the face of rational consumers generate a loss. Initially as has been empirically shown all-pay auctions may generate profits, but these profits
result from consumers overbidding for items. We have illustrated that this overbidding may result from rational behavior simply with small underestimates of the number of other bidders.

A final comment to make is that online penny auctions are still in their infancy, and it is not unlikely that additional applications of all-pay auctions will develop. Similar to the early dot-com period when there were many different auction formats, e.g. first vs. second price, fixed vs. variable end-time, etc., it would not seem unreasonable to expect firms experimenting with other versions of all-pay auctions. Furthermore, similar to how second-price auctions are used in, for instance, online ad-space bidding, other online applications of all-pay auctions are likely to arise. We hope our models and results can provide a fruitful basis for researchers and practitioners alike.

References


Proof Lemmas 3.1 and 3.2 - Let \( b_v \) be the bid and \( u_v \) be the expected surplus for a consumer with valuation \( v \) who bids according to Lemmas 3.1 and 3.2. For a consumer with \( v \leq p_1 \) we know from Auction Theory that he has no incentive to place a bid \( b_{v'} \) for \( v' \leq p_1 \) and \( v' \neq v \) (Krishna 2002,
which is either

\[ u_{p1} < \rho_{v'}(p_1 - b_{v'}) + (1 - \rho_{v'}) \max\{p_1 - p_2, -b_{v'}\} \]

which is either

\[ u_{p1} < \rho_{v'}(p_1 - b_{v'}) - (1 - \rho_{v'})b_{v'} \]

or

\[ u_{p1} < p_1 - \rho_{v'}p_1 - b_{v'} \]

which both are false as \( b_{v'} = \max\{\rho_{v'}p_1 - u_{p1}, (p_1 - (1 - \rho_{v'})p_2 - u_{p1})/\rho_{v'}\} \) for \( v' > p_1 \).

Next we show that a consumer with \( v > p_1 \) has no incentive to deviate from the bidding strategy in Lemmas 3.1 and 3.2. We first show that a consumer with \( v > p_1 \) has no incentive to bid \( b_{v'} \) for \( v' \leq p_1 \). Proof by contradiction. Since \( b_{v'} + p_1 \leq p_2 \) for \( v' \leq p_1 \), show the following is false,

\[ u_v < \rho_{v'}(v - b_{v'}) + (1 - \rho_{v'})(v - p_1 - b_{v'}) \]

which is either

\[ \rho_v(v - b_v) + (1 - \rho_v)(v - p_1 - b_v) < \rho_{v'}(v - b_{v'}) + (1 - \rho_{v'})(v - p_1 - b_{v'}) \]

\[ -b_v - (1 - \rho_v)p_1 < -b_{v'} - (1 - \rho_{v'})p_1 \]

\[ b_v + (1 - \rho_v)p_1 > b_{v'} + (1 - \rho_{v'})p_1 \]

\[ \rho_v(p_1 - u_{p1} + (1 - \rho_v)p_1 > \rho_{v'}E[V_{1(1)}|V_{1(1)} < v'] + (1 - \rho_{v'})p_1 \]

\[ -\rho_{p1}(p_1 - E[V_{1(1)}|V_{1(1)} < p_1]) > \rho_{v'}(E[V_{1(1)}|V_{1(1)} < v'] - p_1) \]

\[ \rho_{p1}(p_1 - E[V_{1(1)}|V_{1(1)} < p_1]) < \rho_{v'}(p_1 - E[V_{1(1)}|V_{1(1)} < v']) \]
or

\[
\rho_v(v - b_v) + (1 - \rho_v)(v - p_2) < \rho_v'(v - b_{v'}) + (1 - \rho_v')(v - p_1 - b_{v'}) \\
-\rho_v b_v - (1 - \rho_v)p_2 < -b_{v'} - (1 - \rho_{v'})p_1 \\
\rho_v b_v + (1 - \rho_v)p_2 > b_{v'} + (1 - \rho_{v'})p_1 \\
\rho_v(p_1 - (1 - \rho_v)p_2 - u_{p_1})/\rho_v + (1 - \rho_v)p_2 > \rho_v'E[V(1)|V(1) < v'] + (1 - \rho_{v'})p_1 \\
\quad -u_{p_1} > \rho_v'(E[V(1)|V(1) < v'] - p_1) \\
\rho_{p_1}(p_1 - E[V(1)|V(1) < p_1]) < \rho_v'(p_1 - E[V(1)|V(1) < v'])
\]

where the last condition for both cases is false as the left hand side is maximized at \( v' = p_1 \).

Next show that a consumer with \( p_1 < v \leq v^+ \) has no incentive to bid \( b_{v'} \) for \( v' > v^+ \). Proof by contradiction. Let \( v' > v^+ \), then show the following is false,

\[
u_v < \rho_v'(v - b_{v'}) + (1 - \rho_{v'}) \max\{v - p_2, v - p_1 - b_{v'}\}
\]

which is either

\[
\rho_v(v - b_v) + (1 - \rho_v)(v - p_1 - b) < \rho_v'(v - b_{v'}) + (1 - \rho_{v'})(v - p_2) \\
v - b_v - (1 - \rho_v)p_1 < v - \rho_v b_{v'} - (1 - \rho_{v'})p_2 \\
-\rho_v p_1 + u_{p_1} - (1 - \rho_v)p_1 < -\rho_v'(p_1 - (1 - \rho_{v'})p_2 - u_{p_1})/\rho_{v'} - (1 - \rho_{v'})p_2 \\
u_{p_1} < u_{p_1}
\]

or

\[
\rho_v(v - b_v) + (1 - \rho_v)(v - p_1 - b) < \rho_v'(v - b_{v'}) + (1 - \rho_{v'})(v - p_1 - b_{v'}) \\
v - b_v - (1 - \rho_v)p_1 < v - b_{v'} - (1 - \rho_{v'})p_1 \\
\quad b_{v'} - b_v - (\rho_{v'} - \rho_v)p_1 < 0 \\
(p_1 - (1 - \rho_{v'})p_2 - u_{p_1})/\rho_{v'} - \rho_v p_1 + u_{p_1} - (\rho_{v'} - \rho_v)p_1 < 0 \\
(1 - \rho_{v'})(1 + \rho_{v'})p_1 - (1 - \rho_{v'})p_2 - (1 - \rho_{v'})u_{p_1} < 0 \\
(1 + \rho_{v'})p_1 - p_2 - u_{p_1} < 0 \\
p_1 - p_2 - u_{p_1} < -\rho_{v'}p_1 \\
(p_2 - p_1 + u_{p_1})/p_1 > \rho_{v'}
\]
where the last condition in the first case is clearly false, and the last condition in the second case is also false since \((p_2 - p_1 + u_{p_1})/p_1 < \rho_{v'}\) for all \(v' > v^+\) by definition of \(v^+\).

Remaining to show that \(v > v^+\) has no incentive to bid \(b_{v'}\) for \(p_1 < v' \leq v^+\). Proof by contradiction. Let \(p_1 < v' \leq v^+\), then show the following is false,

\[
u_v < \rho_{v'}(v - b_{v'}) + (1 - \rho_{v'})(v - p_2, v - p_1 - b_{v'})\]

which is either

\[
\rho_v(v - b_v) + (1 - \rho_v)(v - p_2) < \rho_{v'}(v - b_{v'}) + (1 - \rho_{v'})(v - p_2) \\
v - \rho_v b_v - (1 - \rho_v)p_2 < v - \rho_{v'}b_{v'} - (1 - \rho_{v'})p_2 \\
-\rho_v(p_1 - (1 - \rho_v)p_2 - u_{p_1})/\rho_v - (1 - \rho_v)p_2 < -\rho_{v'}(\rho_v p_1 - u_{p_1}) - (1 - \rho_{v'})p_2 \\
-p_1 + u_{p_1} < -\rho_{v'}(\rho_v p_1 - u_{p_1}) - (1 - \rho_{v'})p_2 \\
0 < (1 - \rho_{v'})(1 + \rho_{v'})p_1 - (1 - \rho_{v'})u_{p_1} - (1 - \rho_{v'})p_2 \\
0 < (1 + \rho_{v'})p_1 - u_{p_1} - p_2 \\
-\rho_{v'}p_1 < p_1 - u_{p_1} - p_2 \\
\rho_{v'} > (p_2 - p_1 + u_{p_1})/p_1
\]

or

\[
\rho_v(v - b_v) + (1 - \rho_v)(v - p_2) < \rho_{v'}(v - b_{v'}) + (1 - \rho_{v'})(v - p_1 - b_{v'}) \\
v - \rho_v b_v - (1 - \rho_v)p_2 < v - b_{v'} - (1 - \rho_{v'})p_1 \\
-(p_1 - (1 - \rho_v)p_2 - u_{p_1}) - (1 - \rho_v)p_2 < -\rho_{v'}p_1 + u_{p_1} - (1 - \rho_{v'})p_1 \\
u_{p_1} < u_{p_1}
\]

which are both false for the same reason as before, i.e. \((p_2 - p_1 + u_{p_1})/p_1 > \rho_{v'}\) for all \(p_1 < v' \leq v^+\) by definition of \(v^+\). QED

**Proof Corollary 3.3** - If \(p_2 \geq 2p_1\) then \((p_2 - p_1 + u_{p_1})/p_1 > 1\), and hence \(v^+ = 7\). QED

**Proof Corollary 3.4** - If \(p_2 = p_1 + \rho_{p_1}E[V(1)|V(1) < p_1]\) then \((p_2 - p_1 + u_{p_1})/p_1 = \rho_{p_1}\), and hence \(v^+ = p_1\). QED
Proof of Proposition 4.1 - Replacing \( b_v \) with their respective bidding strategies as specified by Lemmas 3.1 and 3.2, equation (4.1) becomes,

\[
\pi = \int_v^{p_1} \rho_v E[V(1) | V(1) < v] f(v) dv + \int_{p_1}^{v^+} \left[ \rho_v p_1 - u_{p1} + (1 - \rho_v) p_1 \right] f(v) dv
\]

\[
+ \int_{v^+}^{v} \left[ \rho_v (p_1 - (1 - \rho_v) p_2 - u_{p1}) / \rho_v + (1 - \rho_v) p_2 \right] f(v) dv
\]

which after simplification becomes,

\[
\pi = \int_v^{p_1} \rho_v E[V(1) | V(1) < v] f(v) dv + \int_{p_1}^{\nu} (p_1 - u_{p1}) f(v) dv
\]

with \( u_{p1} = \rho_{p1} (p_1 - E[V(1) | V(1) < p_1]) \) the result holds. QED

Proof of Proposition 4.2 - The optimal price \( p_1^* \) can be found by taking the derivative of equation (4.3) with respect to \( p_1 \), i.e. by setting \( \frac{\partial \pi}{\partial p_1} = 0 \),

\[
\frac{\partial \pi}{\partial p_1} = 1 - 2p_1 + 2p_1^{N+1} - p_1^N = 0
\]

\[
\Rightarrow (1 - 2p_1) - p_1^N(1 - 2p_1) = 0
\]

\[
\Rightarrow (1 - 2p_1)(1 - p_1^N) = 0
\]

\[
\Rightarrow p_1^* = \frac{1}{2}
\]

QED

Proof of Proposition 4.3 -

\[
p_1 \geq \frac{N + 2}{2(N + 1)}
\]

\[
\Rightarrow 2p_1(N + 1) - (N + 2) \geq 0
\]

\[
\Rightarrow \frac{2p_1(N + 1) - (N + 2)}{(N + 2)(N + 1)} \geq 0
\]

\[
\Rightarrow p_1^{N+1} \left( \frac{2p_1}{N + 2} - \frac{1}{N + 1} \right) \geq 0
\]

\[
\Rightarrow p_1^{N+2} \frac{2}{N + 2} - p_1^{N+1} \frac{1}{N + 1} \geq 0
\]

\[
\Rightarrow p_1(1 - p_1) + p_1^{N+2} \frac{2}{N + 2} - p_1^{N+1} \frac{1}{N + 1} \geq p_1(1 - p_1)
\]

QED
Proof of Proposition 4.4 - If $v \leq p_1$ then $E[V_{(1)}|V_{(1)} < v] \leq p_1$. QED

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