Selling with Binding Reservations
in the Presence of Strategic Consumers

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Abstract

We analyze a revenue management problem in which a seller endowed with an initial inventory
operates a *selling with binding reservations* scheme. Upon arrival, each consumer, trying to
maximize his own utility, must decide either to buy at the full price and get the item immediately,
or to place a non-withdrawable reservation at a discount price and wait until the end of the sales
season where the leftover units are allocated according to first-come first-serve priority. We study
structural properties of this noncooperative game, and prove the existence of a Bayesian-Nash
equilibrium. The equilibrium is of the threshold type, meaning that a consumer will place a
reservation if and only if his valuation is below a function of his arrival time. The computation
of this consumer’s strategy is intensive, and provably convergent under specific conditions. To
overcome this limitation, we formulate an asymptotic version of the problem that leads to a simple
closed-form solution, which is then used as an approximate equilibrium for the original problem.
Our computations show that this heuristic is accurate for medium to large-size problems.

Finally, through an extensive numerical study, we find that the proposed mechanism with
reservations almost consistently delivers higher revenue than the standard markdown with a
preannounced fixed-discount. The benefit is more emphasized when 1) The ratio between the
initial number of units and the expected demand is moderate to large, or 2) the heterogeneity of
the consumers’ valuations is moderate to high. In our numerical experiments, the revenue gap
can reach a level up to 5%, which is quite significant for retail businesses that typically operate
with narrow margins.

Key words: Revenue management, dynamic pricing, markdowns, strategic consumer behavior,
asymptotic analysis, Bayesian-Nash equilibrium.

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1 Introduction

The use of markdowns for selling a limited supply of perishable assets is an extended practice in retailing, spanning industries from entertainment (e.g., TKTS sells discount Broadway show tickets in New York at half-price) to computer games (where old versions are deeply discounted when new releases are launched). In the apparel industry, a recent study suggests that around 50% of the inventory of a typical retailer is sold at discount prices (e.g., see Hardman et al. (2007)). Here, as well as in other comparable retail operations with high gross margins and low net profits, small changes in revenue can have a big impact on the financial performance.\(^1\) Indeed, scientific pricing is considered the fastest and most cost-effective way to increase profits (see Phillips (2005), Section 1.2). Analysts and vendors tout a 1% to 3% boost in overall sales, and in some cases a 10% rise in gross margins for companies that employ price-optimization technology (e.g., see Sullivan (2005)).

In this paper, we touch upon markdown pricing. This practice is suitable for settings where typically price insensitive customers arrive early in the sales season, and price sensitive customers arrive late, as it is the case for apparel, high-tech, and perishable-foods retailing, and concert and sport events.\(^2\) Its extended implementation has also raised some concerns among retailers, since consumers have been trained to strategize over the timing of their purchases and buy on sale. This higher market sophistication requires a refinement of the usual markdown practice that preserves the advantage of price discrimination but mitigates the downside of the consumers’ strategic wait. One possible approach is the deliberate introduction of scarcity in the market. A well-known case is Zara, a Spanish apparel chain that sells 85% of its inventory at the regular price. Zara sets low inventory levels for its fashion-sensitive products to induce consumers to buy early rather than wait for sales. The focus of our paper is the analysis of an alternative refinement of the markdown practice: The use of binding reservations under first-come-first-served, or first-in-first-out (FIFO) rationing rule.

\(^1\)According to a report from Reuters dated February 5th, 2008, the average gross margin across 63 apparel retailers based in U.S. was 37.8%, and the net profit was 6.6%. By the tighter economic times of April 2009, the average net profit across 119 “Retail-Apparel & Accessories” companies was just 0.62%.

\(^2\)Following Desiraju and Shugan (1999)’s classification, these settings are labeled class B retailing services.
1.1 Motivation

We study a mechanism where consumers can place non-withdrawable reservations for the leftover inventory that the retailer will clear at the end of the selling season. To our knowledge, the current use of this type of mechanism is very limited in business practice. One of the few examples that we are aware of is the “plunging price” that was used online by Sam’s Club (www.samsclub.com) for few years in the early 2000s to clear excess inventory. It consisted of a preannounced (price, time) schedule, where each consumer visiting the website could buy at the prevailing price or place a non-withdrawable reservation through a credit card stand-by transaction at a specific future (price, time) combination. The reservation would be fulfilled provided there is a unit in stock by that time. Our model is a two-stage version of that (price, time) schedule.

Despite the discontinuity of the “plunging price” we still think that web-based B2C channels could become appropriate platforms to implement binding reservations. In fact, after the exponential growth of online auctions as an electronic retail selling mechanism (eBay being the canonical example), there has been a recent trend to meld them with more conventional fixed-price settings (like for example, by designing auctions with “buy-now” options), or to even favor the use of plain fixed-prices. An explanation for this shift is the proliferation of pricing information online that has made it easier for consumers to bargain-hunt and lessened the need to risk overbidding in an auction (e.g., see Holahan (2008) and Flynn (2008)). While seeking ways of providing posted-price alternatives for consumers, online sellers are certainly still interested in mechanisms that allow price discrimination. The one that we explore in this paper follows this direction.

Binding reservations could also be applied in conventional bricks-and-mortar retailers through the installation of on-site kiosks so that consumers could choose a product using a GUI and then swipe a credit card to put the purchase on-hold. The reservation would become a stand-by transaction, reflecting the commitment to honor it provided a unit is available at the end of the horizon, and the reservation time grants fulfillment under FIFO priority. By implementing this mechanism, the retailer would avoid the display of merchandise at low prices during a clearance season (which

\[3\text{By April 2009, a binding reservation practice has been implemented by a commercial U.S. airline: The “AirTran U StandBy Ticket” program (http://www.airtranu.com), where college students can get non-confirmed, deeply discounted tickets at the airports but could eventually need to wait at the gate for hours.}\]
typically decreases the productivity of the shelf space), speed-up the introduction of new products, and reduce the holding cost incurred over the old merchandise. A downside could be given by the fact that availability of the reservation option might deter slightly compulsive consumers from buying immediately.

From the consumers' perspective, our reservation mechanism is convenient because they would not need to revisit the store looking for a bargain. In addition, the FIFO priority rule delivers a sense of fairness (since earlier reservations are honored first). Different from auctions, whose allocations are theoretically founded on valuation-based priorities, the fairness here is rooted on time-based priorities which are observable first-hand. Moreover, our mechanism is easy to implement for the retailer, and easy to explain to the consumers. It does not involve any a priori fee, and reduces consumers’ search cost. Therefore, in principle, it exhibits desirable properties that combined with positive revenue performance, would appeal to retailers and would encourage them to implement this business model in practice.

1.2 Overview of main results

In our model for studying this reservation mechanism, a seller endowed with inventory of a particular product faces a nonhomogeneous Poisson arrival stream of consumers. The sales horizon is of length $T$. The seller announces the inventory level $Q_0$ put up for sale, the regular price $p_h$, and the clearance price $p_l$, with $p_l \leq p_h$. Consumers with single-unit demand have a private value for the product, independently drawn from a continuous distribution. Based on a discounted utility function, they must decide whether to buy at the full price $p_h$, or place a non-withdrawable reservation and wait for the clearance season, where the leftover units (if any) will be allocated. Even though our approach allows for more general allocation rules, we concentrate the discussion on FIFO, where the $Q_T$ leftover units are allocated to the earliest reservations placed.

Given this non-withdrawable reservation setting, how should strategic, utility-maximizing consumers behave? Certainly, consumers with valuations between $p_l$ and $p_h$ should place a reservation to get a non-negative utility, but those consumers with valuation above $p_h$ face a tradeoff between getting a unit now at no risk by paying a high price, or placing a reservation with the hope of getting
a unit at a low price later. Of course, the chance of getting a unit will depend on the purchasing behavior of other consumers. We prove that a symmetric purchasing equilibrium strategy for this noncooperative game exists, and that it is characterized by a threshold function in the space (time, valuation): For a consumer arriving at time \( t \) with valuation \( v \), given all publicly known parameters of the problem, there is a threshold \( H(t) \) such that if \( v > H(t) \) he will purchase right away at price \( p_h \). Otherwise, he will place a reservation. This purchasing strategy can be computed using an iterative algorithm, provably convergent under specific conditions. Unfortunately, this procedure is computationally intensive. To overcome this limitation, we formulate an asymptotic version of the problem, in which the demand rate and the initial inventory grow proportionally large. We get a simple closed form expression for the equilibrium strategy in this limiting regime, which is then used as an approximate solution for the original problem. Numerical computations show that this heuristic is very accurate for moderate- to large-size problems.

Then, we analyze the seller’s revenue optimization problem. We plug the consumer’s asymptotic purchasing strategy into it to compute optimal values of the parameters \( Q_0, T, p_h, \) and \( p_l \) (or a subset of them) that the seller must announce in order to maximize expected revenues. We can assert then that the asymptotic solution culminates in a simple threshold function that prescribes how consumers should behave and that allows the seller to design the reservation mechanism efficiently.

We also study a benchmark setting that models the usual markdown practice implemented by retailers; namely, the random allocation (RA) rule. Under RA, the sales horizon of length \( T \) is split into two periods with respective preannounced prices \( p_h \) and \( p_l \). Consumers arriving during the first period can either buy at \( p_h \), or come back at the beginning of the clearance season and get a unit at \( p_l \). During the clearance period, other consumers come to purchase the remaining units at \( p_l \). We characterize a symmetric equilibrium for this game, and its corresponding limiting regime. Through an extensive numerical study, we compare the consumer behavior and revenue performance of FIFO and RA with the objective of identifying cases where the former is more profitable for the retailer. We find that FIFO almost consistently beats RA. The discount factors of the seller and the consumers play a critical role in this assessment. If the seller is more patient than the consumers, then the most beneficial cases over the usual markdown practice occur when: 1) The ratio between
the initial number of units and the expected demand is moderate to large, or 2) the heterogeneity of
the consumers’ valuations is moderate to high. In our numerical experiments, the revenue gap over
the usual markdown practice can reach a level of 5%.

Next, we check the impact that different market compositions (in terms of the mix of strategic
and impulsive consumers) could have on the seller’s revenue performance. We notice that FIFO
consistently achieves the highest revenues. Finally, we study the effect of FIFO and RA rationing
rules on the aggregate consumer surplus, and verify that FIFO delivers a higher one, mainly because
it tends to sell more units.

1.3 Organization

The remainder of the paper is organized as follows: We review the related literature in the next
section. In Section 3, we introduce the model. The strategic behavior of consumers when facing
the purchasing decision (i.e., “buy now” versus “place a reservation”) is analyzed in Section 4.
In Section 4.2 we describe the benchmark, regular markdown setting, and analyze the associated
strategic consumer behavior. The development of the asymptotic analysis for the mechanisms under
consideration is included in Section 5, and the seller’s revenue optimization problem is analyzed
numerically in Section 6. Our conclusions are summarized in Section 7. All the proofs are included
in the appendix.

2 Literature review

Recently, there has been a growing interest within the revenue management (RM) literature in
capturing the intertemporal strategic behavior of consumers, and in developing ways to mitigate
the adverse impact of this phenomenon on retailers’ revenue performance. An excellent source that
compiles the existing knowledge on this topic is the book chapter by Aviv et al. (2009). A short list
of the proposed mechanism includes capacity rationing (e.g., Liu and van Ryzin (2008), Su (2007),
Zhang and Cooper (2008)), making price and capacity commitments (e.g., Elmaghraby et al. (2008),
Aviv and Pazgal (2008), and Su and Zhang (2008)), using internal price matching policies (e.g.,
Levin et al. (2007)), and limiting inventory information (e.g., Yin et al. (2007)).
Clearly, the strategic behavior of consumers has challenged the pricing strategies of firms and inspired the search of innovative mechanisms. Pricing with the added service of taking consumers’ reservations is inscribed within this trend, and our paper contributes to it. In an earlier work, although in a setting where consumers’ valuations and prices are likely to increase, Png (1989) showed that reservations are indeed very effective when risk-averse customers are uncertain about their own (future) valuations, and available supply is limited. In his two-period model, if a customer decides to place a reservation in period 1, then he will exercise it in period 2 if the revealed valuation is high enough.4 A related pricing mechanism is given by what is named contingent pricing in the marketing literature, an agreement where a product is sold to a customer at a low price if the seller does not succeed in obtaining a higher price during a specified period. If a higher price is obtained during the arranged time period, the original sale does not take place, and the first potential buyer receives a previously agreed-upon compensation. The paper by Biyalogorsky and Gerstner (2004) analyzes this problem between a consumer and a seller, for the single-unit supply case. The spirit of these models is to take advantage of an increasing pattern of prices, while securing advance sales at low prices. More recently, inspired by the restaurant business, Alexandrov and Lariviere (2007) study the impact of accepting withdrawable reservations in firms’ sales by altering consumer behavior, and argue that under unknown consumer market size, competition among firms makes reservations more attractive when enough customers consider dining at either restaurant. Çıl and Lariviere (2007) study the problem faced by a service provider (e.g., a restauranteur) facing two customer segments (reservations and walk-ins), and who has to decide how much capacity to make available to the reservation segment. They show that when the reservation segment is more valuable, the firms may still opt to decline some reservation requests; while if the reverse is true, the firm may still decide not to save capacity for walk-ins.

A paper closer to ours is the one by Elmaghraby et al. (2006). They also study a problem where a (homogeneous) Poisson arriving stream of rational customers can choose either to buy during the selling season at a high price, or to place a reservation at a pre-announced low price and wait until the end of the season to get the product (subject to availability). They analyze the

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4Png (1989) provides some rationality for why airlines take reservations free of charge, and are willing to overbook. These reservations are withdrawable though.
seller’s expected payoff under this reservation regime, and compare it with the no-reservation regime. They show that the seller prefers the reservation regime when there is a significant proportion of high valuation consumers. Our work differs from theirs along several important modeling features. First, Elmaghraby et al. (2006) consider a single-unit supply case, while we consider a multi-unit supply to be depleted during the sales horizon. This extension scales-up the complexity of the analysis. Second, Elmaghraby et al. (2006) consider time-homogeneous valuations, while in our case consumers’ valuations are time-nonhomogeneous and are discounted over time. Therefore, in our case, when a consumer places a reservation, he does not only run the stockout risk, but also his payoff is discounted. This observation has a clear impact on the equilibrium behavior: While in their case the equilibrium is defined by a single threshold time value, in our multiplicative utility function case, the equilibrium is defined by a continuous threshold function in the space (time, valuation). Third, Elmaghraby et al. (2006) assume that consumers are split into a finite number of segments, where segment \( i \) is defined by a constant valuation \( v_i \). We assume that consumers are heterogeneous (in the sense that their valuations can differ), and that these valuations are private information taken from time-nonhomogeneous, continuous probability distributions.

Another closely related paper is the aforementioned Aviv and Pazgal (2008). Our RA model follows from the setting introduced by these authors. They study two classes of pricing strategies for a single price-drop event: inventory-level-dependent and announced, fixed-discounts. Through an extensive numerical study, they show that the latter could revenue-wise dominate the former by up to 8%, and also estimate that the benefit of capturing explicitly strategic consumer behavior (versus ignoring it when customers are indeed strategic) could reach up to 20%. Aviv and Pazgal (2008) do not study the effect of different rationing rules though. We take their best strategy (i.e., announced-fixed discounts) and use it as a benchmark for our proposed FIFO reservation mechanism.

In addition, there are a couple of important technical differences between the two papers: First, we consider time-variant consumers’ valuations; and second, even though Aviv and Pazgal (2008) pose necessary conditions that a Bayesian-Nash equilibrium must satisfy, they do not demonstrate its existence.\(^5\) A major contribution of our piece of research in this regard is that by using the technical

\(^5\)Aviv and Pazgal (2008), in Theorem 2 (page 348), formulate the threshold function \( \theta \) (i.e., our function \( H \)) that would be the Bayesian-Nash equilibrium of the game under their announced fixed-discount pricing strategy. The proof of existence would require showing the convergence of the successive application of their equations (7) and (8), but they
machinery developed here, which is close to the one presented in Caldentey and Vulcano (2007), we prove the existence (and sufficient conditions for uniqueness) of an equilibrium under both FIFO and the random rationing rules, hence also reinforcing the support for Aviv and Pazgal (2008)'s results.

The paper by Yin et al. (2007) is also related to ours. There, the authors present a model where consumers arriving during a selling horizon must decide to either buy immediately at a high price $p_h$, or wait for a low price $p_l$ that will be offered at the end of the season. Both prices are fixed and preannounced, and the leftover units are allocated randomly among the consumers who decided to wait. There are major differences with respect to our model though: First, and most importantly, the focus of their paper is different from ours: they study the impact of two different inventory display format, and verify that by displaying one unit at a time (as opposed to all the available inventory), the seller is able to introduce a sense of scarcity in the market and achieve higher profits. We explore the use of binding reservations as a way to increase the retailer revenues. Second (time-homogeneous valuations) and third (two customer segments with fixed valuations $v_i$) are similar to the setting in Elmaghraby et al. (2006). One advantage of Yin et al. (2007) is that consumers have real time information of the inventory level $Q_t$. In our stochastic model, customers just know if the item is in-stock or sold-out, but the asymptotic analysis gives a first-order approximation for the real-time value $Q_t$ that consumers are able to infer. Indeed, a distinguishing characteristic of our piece of research is the asymptotic analysis of the game, which provides simple and well-behaved heuristics for the rationing allocation rules that we study. Our approach in this regard follows the asymptotic analysis of Maglaras and Meissner (2006).

3 Model description

A retailer (seller) is endowed with an initial inventory $\bar{Q}$ of a homogeneous product. The inventory needs to be depleted over a selling season of length $T$. Following a RM approach, we assume that the inventory is not replenished. The seller can ration the inventory by choosing a quantity $Q_0 \leq \bar{Q}$ to put up for sale. The remaining quantity $\bar{Q} - Q_0$ is discarded at no extra cost or salvage value.

...do not prove such result. A related comment acknowledging this limitation appears at the end of their Section 4.3, where they claim that the proof of convergence of their iterative algorithm “presents an even harder theoretical challenge”. ...
The seller announces a regular unit price $p_h$ to be posted during the interval $[0, T]$, and a discounted price $p_l \leq p_h$ that will be realized at the end of the horizon. At time $t$, she also announces the time left for the season end, $T - t$, and the inventory $Q_0$ initially available for sale. We assume that there is no real time update for the current inventory level $Q_t$ during $(0, T]$, but upon each arrival the seller discloses whether the product is available or not (i.e., she reveals if $Q_t > 0$ or not).

During the selling process, the seller depletes inventory and hence, both for the seller and for any arriving consumer during $[0, T]$, the effective number of units to be left at time $T$ is described by a random variable $Q_T$, with support $\{0, \ldots, Q_0\}$. The clearance is modeled as an instantaneous event that occurs right after time $T$, and where transactions at price $p_l$ are due to consumer reservations placed during the regular season. If the number of reservations placed is less than or equal to the leftover inventory level, then all reservations are fulfilled. Otherwise, units are allocated following a rationing rule set by the seller at the beginning of the season. In principle, we allow any time-based rationing rule that defines a total order among reservations w.p.1 (details provided in Section 4), but focus our attention on a particular case: the FIFO rationing rule.

On the demand side, the description of customer arrivals is similar to the one in Bitran and Mondschein (1997). Consumers have single unit requests, and visit the store or website following a nonhomogeneous Poisson process with intensity $\lambda(t)$. They are characterized by two quantities: (i) their arrival time, and (ii) their private valuation for the product. For notational convenience, we denote the private valuation of a consumer arriving at time $t$ by $v_t$. Observe that this notation is well-defined since, w.p.1, the Poisson process has at most one arrival at any given time. The cumulative probability distribution $F$ of the random variable $v_t$ is allowed to be time dependent, to account for potential changes in product’s popularity throughout the selling season or for the dynamics of consumers’ preferences (e.g., a valuation for a winter coat for a consumer in New York maybe low in September, high in November, and then lower in February, near the end of the season). Let $F(v, t)$ have the common support $V \triangleq [0, \bar{v}] \times [0, T]$, with $\bar{v} > p_h$. We assume that $F(v, t)$ is differentiable in $v$ for all $t$ and admits a density function $f(v, t)$. We also require the mild technical condition that $F$ is $K_F$-Lipschitz continuous in the first argument, for some constant $K_F$, for all $t$.\(^6\)

\[^6\]A scalar function $F$ is $K_F$-Lipschitz continuous in $x$ if for all $x_1, x_2$, $|F(x_1, t) - F(x_2, t)| \leq K_F|x_1 - x_2|$, \(\forall t\). A sufficient condition to verify Lipschitz continuity of $F$ is continuous differentiability, i.e., the fact that $\frac{\partial}{\partial v} F(v, t) \equiv f(v, t)$.
Both $\lambda$ and $F$ are common knowledge. Without loss of generality, we assume from now on that $\bar{v} = 1$, that is, we scale all prices in this economy by $\bar{v}$.

When visiting the store, consumers must choose either to buy the product at the current price $p_h$ or to reserve it for the later price $p_l$ (and run the risk of not getting it), with the objective of maximizing their own surplus. We assume that they are sensitive to delay, and denote by $u(t, \tau, v - p)$ a quasilinear discounted utility function of a consumer arriving at time $t$ with valuation $v$ who eventually gets at time $\tau$ a unit of product at price $p$ (paid at the moment of getting it). In particular, we consider an exponentially discounted utility function of the form:

$$u(t, \tau, v - p) = (v - p) \exp(-w(\tau - t)), \quad (1)$$

where $w$ is a fixed, nonnegative constant shared by all consumers that captures their disutility for waiting.\(^7\)

We assume that a consumer arriving at $t \in [0, T)$ bases his purchasing decision on his private valuation, his knowledge of the arrival rate $\lambda(t)$ and the distribution of valuations $F$, the initial inventory $Q_0$, the remaining season time $T - t$, the announced prices, and the rationing rule for reservations. Pictorially, the consumers’ type space (arrival time, valuation) is divided into four regions, as shown in Figure 1. Consumers with valuation below $p_l$ quit without making any transaction. For those with valuation between $p_l$ and $p_h$ the only profitable option is to place a reservation. Consumers with valuation above $p_h$ act strategically according to a threshold function $H(\cdot)$, such that a $v_t$-consumer reserves an item only if his valuation verifies $p_h < v_t \leq H(t)$. Those with valuation $v_t > H(t)$ are the buy-now consumers. We will characterize such $H(\cdot)$ in Section 4.

Figure 1 also provides a good insight for the behavior of strategic consumers under FIFO. Among them, those with valuation slightly above $p_h$ would tend to place reservations, since the tiny difference with respect to $p_h$ makes the instantaneous payoff small (and hence, the wait profitable). Those with very high valuations would tend to buy now in order to avoid the risk of wait. For those with

\(^7\)Note that the consumer’s utility function is of the intertemporal type (e.g. see Mas-Colell et al. (1995), Chapter 20). We assume the exponential decay due to technical convenience, since it guarantees nonnegativity for all $t$ and $\tau$, but our main theoretical results are not tight to this functional form of utility, as long as it remains increasing in $v - p$ and decreasing in delay $\tau - t$. 

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moderately high valuations, there could be two reasons for buying now: Certainly, one reason is to
arrive late, when the chances of getting a unit from the FIFO rule are low. This is reflected by the
decaying shape of \( H(\cdot) \) towards the end of the horizon. But also, depending on the parameters of
the problem, \( H(\cdot) \) could be increasing at the beginning of the horizon (as is the case here), capturing
the fact that the utility discount faced by early arrivals makes the wait less appealing.

The retailer’s problem is to design the selling season by setting the values for \( T, Q_0, p_h \) and \( p_l \)
in order to maximize her expected revenue, which is also exponentially discounted over time. In this
regard, the game between the seller and the consumers is of the Stackelberg-type, with the seller
being the leader, and the consumers being the followers.

4 Strategic consumer’s purchasing behavior under reservations

In this section we study consumers’ purchasing decision. We focus on the strategic consumers, i.e.,
those with valuation \( v \geq p_h \). We assume that the seller has already announced the parameters of
the selling mechanism \((Q_0, T, p_h, \text{ and } p_l)\) and the rationing rule. The seller’s problem of optimally
designing the mechanism is postponed to Section 6.

For ease of the exposition, we rescale consumers’ valuations (and the corresponding probability
distributions), price \( p_h \) and arrival rate \( \lambda(t) \) under the normalization \( p_l = 0 \). That is, based on the
original value $p_l$, we set:

$$v \leftarrow \frac{v - p_l}{1 - p_l}, \quad p_h \leftarrow \frac{p_h - p_l}{1 - p_l}, \quad F(v, t) \leftarrow \frac{F(v, t) - F(p_l, t)}{1 - F(p_l, t)}, \quad \lambda(t) \leftarrow \lambda(t) (1 - F(p_l, t)).$$ (2)

Then, we set $p_l \leftarrow 0$. Note that under this scaling the range of valuations remains $[0, 1]$. If the consumers’ valuations are time heterogeneous, then this scaling results in a time-dependent arrival rate. We define the maximum (scaled) arrival intensity $\bar{\lambda} \triangleq \max_{t \in [0, T]} \lambda(t)$.

We can characterize the decision of a consumer arriving at time $t$ with private valuation $v_t$ by a threshold function $H(t)$ such that the consumer will place a reservation if and only if $v_t \leq H(t)$. The fact that we can represent the purchasing strategy for all $t$-consumers by a single threshold $H(t)$ is a consequence of the monotonicity of the utility function in the instantaneous payoff $v_t - p$. In other words, if it is optimal for a $v_t$-consumer to wait $T - t$ time units for the clearance season then it is also optimal to wait for any other consumer arriving at $t$ with valuation lower than $v_t$.

Two assumptions are used in this representation of the purchasing strategy. First, note that this characterization is based on the notion of a symmetric equilibrium in which all consumers use the same threshold function $H(t)$. In addition, we are also assuming that a consumer arriving at time $t$ is incapable to observe the number of reservations placed and units left in the system. That is, we are assuming that the only information that a consumer uses to decide whether or not to place a reservation – besides $\lambda$, $T$, $Q_0$, $F$, $p_h$, and the rationing rule – is his arrival time and private valuation.

We will denote by $\mathcal{H}$ the strategy space. In order to keep our formulation reasonably simple, we assume that $\mathcal{H} \subseteq \mathcal{D}$, the set of piecewise continuous functions with right and left limits, which is broad enough to include most strategies that are practically relevant. We will show that for a large class of rationing rules the set $\mathcal{D}$ is larger than necessary in the sense that in equilibrium any symmetric purchasing strategy $H \in \mathcal{H}$ is actually continuous. Note that by our scaling based on $\bar{v} = 1$ and $p_l = 0$, the elements of $\mathcal{H}$ are functions with domain $[0, T]$ taking values in $[0, 1]$. Furthermore, for any $H \in \mathcal{H}$, we must have $H(t) \geq p_h$. This reflects the fact illustrated in Figure 1 that any consumer with valuation less than $p_h$ cannot afford to buy the product during the regular season; the reservation is his only potentially profitable decision, no matter his arrival time. In summary, we define the set of potential purchasing strategies as the set of functions $\mathcal{H} = \left\{ H \in \mathcal{D}, H : [0, T] \rightarrow [0, 1], \text{ such that } H(t) \geq p_h \text{ for all } t \right\}$. 

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We use a two-step approach to characterize a symmetric purchasing equilibrium (PE) \( H \in H \). First, we look at a consumer’s best-response purchasing strategy assuming that other consumers use a fixed strategy \( H \in H \). We will denote by \( \mathcal{R}(H) \in H \) this best-response purchasing strategy and refer to \( \mathcal{R} \) as the best-response mapping on \( H \). Second, we impose the equilibrium condition \( \mathcal{R}(H^*) = H^* \).

Suppose a consumer –that we refer as consumer \( \tau \)– arrives at time \( \tau \) with private valuation \( v_\tau > p_h \) and suppose that every other consumer is using the purchasing strategy \( H \). The relevant case for a consumer is when there are still units available upon his arrival (i.e., \( Q_\tau > 0 \)). If consumer \( \tau \) decides to buy a unit at \( p_h \), then his expected utility would be: \( u(\tau, \tau, v_\tau - p_h) \). On the other hand, if he decides to reserve, then his expected payoff would be: \( u(\tau, T, v_\tau - p_l) \mathbb{P}(\text{reserving at } \tau \text{ and getting an item}|Q_\tau > 0) \). Thus, a rational consumer \( \tau \) places a reservation if and only if

\[
u(\tau, T, v_\tau - p_l) \mathbb{P}(\text{reserve and get an item}|Q_\tau > 0, H) \geq u(\tau, \tau, v_\tau - p_h),
\]
or equivalently, if and only if

\[
u(\tau, T, v_\tau - p_l) \mathbb{P}(\text{reserve and get an item}|Q_\tau > 0, H) \geq u(\tau, \tau, v_\tau - p_h) P(H), \tag{3}
\]

where \( \mathbb{P}(\text{reserve and get an item}|Q_\tau > 0, H) \) is the (unconditional) probability of getting an item through a reservation placed at time \( \tau \). We still need to explicitly characterize this reservation constraint in terms of the function \( H \), which will depend (through \( \mathbb{P}(H) \)) on the time-based rationing rule chosen.

The rationing rule is an ordering relationship governing priorities among placed reservations. It can be defined as a function \( \xi(\tau) : [0, T] \to [0, 1] \) which assigns a time-based priority to each reservation so that if \( \xi(\tau_1) > \xi(\tau_2) \), then a customer arriving at time \( \tau_1 \) has higher priority than one arriving at \( \tau_2 \) in case both place reservations. The priority ordering is assumed to be strict, in the sense that w.p.1, for arrival times \( \tau_1 \neq \tau_2 \), either \( \xi(\tau_1) > \xi(\tau_2) \) or \( \xi(\tau_1) < \xi(\tau_2) \). For example, FIFO rationing rule, i.e., “earlier reservations first”, is a strict ordering relationship and would correspond to a strictly decreasing function \( \xi(\cdot) \). Assume \( \xi \in \Xi \triangleq \{ \xi \in \mathcal{D} : [0, T] \to [0, 1], \text{ s.t. number of local extrema is finite} \} \). Note that \( \xi(\cdot) \) is well defined for every \( \tau \in [0, T] \), which guarantees that any possible reservation has an assigned priority. See Section A2 of the appendix for illustrations of different priority rules.
Given \( H \in \mathcal{H} \) and \( \xi \in \Xi \), next quantities are important in the derivation of \( \Pi_H(\tau) \). Define for all \( \tau \in [0, T] \):

- Average number of consumers that “buy now” during \([0, \tau]\),
  \[
  \Lambda_{HB}(\tau) \triangleq \int_0^{\tau} \lambda(t) \bar{F}(H(t), t) dt.
  \]  

- Average number of consumers whose reservations have a higher priority than the reservations placed at time \( \tau \),
  \[
  \Lambda_{HR}(\tau) \triangleq \int_0^{T} 1\{\xi(t) > \xi(\tau)\} \lambda(t) F(H(t), t) dt
  \]  

Since \( \mathcal{H} \subset \mathcal{D} \) and \( \xi \in \Xi \), both \( \Lambda_{HB}(\tau) \) and \( \Lambda_{HR}(\tau) \) are well defined functions continuous in \( \tau \). Consider a customer who arrived at time \( \tau \) and under strategy \( H \) placed a reservation with priority \( \xi(\tau) \). Following our reasoning, the probability of the customer getting an item depends on priorities of other customers’ reservations.

### 4.1 Strict priority rationing rules and the FIFO case

Let us denote by \( N(x) \) a Poisson random variable with mean \( x \). If, w.p.1, no two reservations can have the same priority, each customer can get an item through the reservation channel if the seller has available units after serving: i) all buy-now customers, and ii) all reservations with higher priorities. These quantities are independent, Poisson random variables \( N(\Lambda_{HB}(T)) \) and \( N(\Lambda_{HR}(\tau)) \), respectively, and the probability of getting an item through a reservation placed at time \( \tau \) is given by:

\[
\Pi_H(\tau) = \mathbb{P}(N(\Lambda_{HB}(T)) + \Lambda_{HR}(\tau)) \leq Q_0 - 1)
\]  

Next lemma shows the differentiability of \( \Pi_H(\tau) \), which becomes important in our proof of existence of an equilibrium purchasing strategy.

**Lemma 1** For any rationing rule defined by \( \xi(\cdot) \in \Xi \), and for any strategy profile \( H \in \mathcal{H} \), \( \Pi_H(\tau) \) is differentiable and \( |\Pi'_H(\tau)| \leq K_\Pi < \infty \), for all \( \tau \).
Given $H \in \mathcal{H}$, we compute the best-response strategy $R(H)$ for consumer $\tau$ by looking at the threshold function that is consistent with (3). For the exponentially discounted utility function defined in equation (1), and according to condition (3), a consumer places a reservation when

$$\frac{v_\tau}{v_\tau - p_h} \geq \frac{\exp(w(T - \tau))\mathbb{P}(Q_\tau > 0|H)}{\Pi_H(\tau)} \triangleq g_H(\tau).$$

Next lemma characterizes an important feature of the elements of $\mathcal{H}$, namely, there is always a range of consumers with valuations above $p_h$ that prefer to reserve an item, irrespective of their arrival times:

**Lemma 2** For all $H \in \mathcal{H}$ there is a valuation $v_H > p_h$ such that $R(H)(\tau) \geq v_H$ for all $\tau \in [0, T]$. The infimum of $v_H$ over $H$ satisfies $\tilde{v} = \inf_{H \in \mathcal{H}} \{v_H\} \geq \frac{p_h\tilde{g}}{\tilde{g} - 1}$, where $\tilde{g} = \exp(wT)/\mathbb{P}(N(\tilde{\lambda}T) \leq Q_0 - 1)$.

The following proposition characterizes $R(H)$. It is a simple consequence of the continuity of $g_H(\tau)$, and the shape of the LHS in (7): It is monotonically decreasing, with minimum $1/(1 - p_h)$ achieved when $v_\tau = 1$.

**Proposition 1** A consumer places a reservation (i.e., condition (7) is satisfied) when

$$v_\tau \leq \frac{p_h g_H(\tau)}{g_H(\tau) - 1} \text{ for } g_H(\tau) > \frac{1}{1 - p_h}. \quad (8)$$

Thus, a consumer arriving at time $\tau$ with valuation $v_\tau$ places a reservation if and only if $v_\tau \leq R(H)(\tau)$, where

$$R(H)(\tau) \triangleq \begin{cases} 
1 & \text{if } g_H(\tau) \leq \frac{1}{1 - p_h} \\
\frac{p_h g_H(\tau)}{g_H(\tau) - 1} & \text{if } g_H(\tau) > \frac{1}{1 - p_h}.
\end{cases} \quad (9)$$

This best-response mapping $R(H)(\tau)$ is continuous in $\tau$.

Because the best-response strategy $R(H)(\tau)$ is continuous in $[0, T]$, it follows that $R$ effectively maps $\mathcal{H}$ into $\mathcal{H}$. Furthermore, since a PE is characterized by the fixed-point condition $R(H^*) = H^*$, we conclude that a symmetric purchasing equilibrium of this game $H^* \in \mathcal{H}$ is in fact continuous. Our next result shows that the best-response strategies are $K$-Lipschitz continuous functions in $[0, T]$, for an appropriate constant $K > 0$. This additional property of the purchasing strategies becomes relevant in our proof of existence of an equilibrium.
**Proposition 2**  For all $H \in \mathcal{H}$, there is a positive constant $K$ (independent of $H$) such that the best-response strategy $R(H)(\tau)$ is a $K$-Lipschitz continuous function.

A standard way to prove the existence of a symmetric equilibrium $H(\tau)$ is to verify that the set of purchasing strategies $\mathcal{H}$ has the fixed-point property (i.e., every continuous mapping $R : \mathcal{H} \to \mathcal{H}$ has a fixed-point; see Cheney (2001), Section 7.1 for details) and that the best-response mapping $R$ is continuous in $\mathcal{H}$. We will take this approach here and formulate sufficient conditions on the rationing rule under which the existence of equilibrium is guaranteed.

**Theorem 1**  For any rationing rule defined by $\xi(\cdot) \in \Xi$, and for any strategy profile $H \in \mathcal{H}$, the set of strategies $\mathcal{H}$ equipped with the uniform norm $\|X\| = \sup_{0 \leq \tau \leq T} |X(\tau)|$ in $[0, T]$ exhibits the fixed-point property. For all $H, \tilde{H} \in \mathcal{H}$, the mapping $R$ satisfies:

$$\|R(H) - R(\tilde{H})\| \leq \frac{3(1 - p_h)^2 \exp(wT) \beta(Q_0 - 1)}{p_h \mathbb{P}(N(\lambda T) \leq Q_0 - 1)^2} \|H - \tilde{H}\|,$$

where $\beta(Q_0 - 1) = \mathbb{P}(N(Q_0 - 1) = Q_0 - 1)$. In addition, if

$$\frac{3(1 - p_h)^2 \exp(wT) \beta(Q_0 - 1) \bar{\lambda} K_F T}{p_h \mathbb{P}(N(\lambda T) \leq Q_0 - 1)^2} < 1,$$

then $R$ is a contraction. In this case, the fixed-point $R(H^*) = H^*$ is guaranteed to be unique in $\mathcal{H}$ and can be found through the iteration $H^{(n+1)} = R(H^{(n)})$ starting at an arbitrary $H^{(0)} \in \mathcal{H}$.

The proof of Theorem 1 relies on the differentiability of $\Pi_H(\cdot)$. The intuition for this condition comes from the fixed-point theorem, that requires the best response to be a continuous mapping of the set of continuous functions into itself. A discontinuity or undefined derivative of $\Pi_H(\cdot)$ would break the continuity of the best response mapping. As Lemma 1 states, all rationing rules governed by the set of functions $\Xi$ result in a differentiable $\Pi_H(\cdot)$.

We refer the reader to Figure 2 (left) for an example of equilibrium purchasing strategy under FIFO. Indeed, we will focus our discussion on the FIFO rationing rule to run comparisons with the usual markdown practice. Next section is devoted to characterize consumers’ behavior under the latter.
4.2 Benchmark setting: Clearance season with random allocation

Our benchmark model generalizes the announced fixed-discount setting presented in (Aviv and Pazgal 2008, Section 5) to the time-nonhomogeneous valuation case. According to the numerical experiments there, this policy dominates the inventory-level-dependent pricing scheme in the presence of forward-looking consumers, and constitutes a challenging one to beat. Aviv and Pazgal (2008) explain this dominance by arguing that a credible pre-commitment to a fixed-discount level removes the hope of consumers on deep discounts during the clearance season.

The sales horizon of length $T$ is split into two parts: $[0, T_S]$, and $[T_S, T]$. During the first part of the season, a price $p_h$ is charged, and from time $T_S$ onwards, $p_l$ is charged. All the other basic details that apply to our previous model also apply here (i.e., Poisson arrivals at scaled rate $\lambda(t)$ during the whole horizon $[0, T]$, time-dependent valuations following distribution $F$, utility function as in (1)).

The strategic behavior occurs during the first period, when arriving consumers must decide either to purchase at the current price $p_h$, or come back at time $T_S$ and buy at price $p_l$ (subject to product availability). At time $T_S$, leftover units are allocated randomly among consumers who revisit the store. Consumers keep coming at rate $\lambda(t)$ during $[T_S, T]$. However, there is no gaming behavior of these late buyers; they just take the leftover inventory (if any).

We call this model the random allocation (RA) case, and think of the strategic wait consumers as if they were placing reservations that have the same priority and will be honored randomly at time $T_S$ according to a discrete uniform distribution. Specifically, the probability that a strategic consumer get a reserved item is

$$c(H) \triangleq \min \left\{ \frac{Q_{T_S}}{\# \text{ of consumers that reserved an item}}, 1 \right\} = \Pi_H(\tau),$$

where the number of consumers who reserve an item is a Poisson random variable with mean

$$\Lambda^{RA}_{HR}(T_S) = \int_0^{T_S} \lambda(t)F(H(t), t)dt. \quad (10)$$

Note that these expressions do not depend on a consumer arrival time $\tau$, i.e. the probability $c(H)$ of getting a reserved item is the same for any arrival $\tau \in [0, T_S]$. Then, for the exponential utility function (1), and according to condition (3), a consumer places a reservation when

$$\frac{v_{\tau}}{v_{\tau} - p_h} \geq \frac{\exp(w(T_S - \tau))P(Q_{\tau} > 0|H)}{c(H)} \triangleq g_{RA}^{HR}(\tau), \quad \tau \in [0, T_S], \quad (11)$$
where
\[
  c(H) = \sum_{k=1}^{Q_0} \mathbb{P}(Q_{T_S} = k|H) \left[ \sum_{n=0}^{k} \mathbb{P}(N(\Lambda_{H}^{RA}(T_S)) = n) + \sum_{n=k+1}^{\infty} \frac{k}{n} \mathbb{P}(N(\Lambda_{H}^{RA}(T_S)) = n) \right].
\] (12)

Note that \( c(H) \) is guaranteed to be strictly positive.\(^8\)

A consumer arriving at time \( \tau \) with valuation \( v_\tau \) places a reservation if and only if
\[
  v_\tau \leq R(H)(\tau),
\]
where
\[
  R(H)(\tau) \equiv \begin{cases} 
  1 & \text{if } g_{H}^{RA}(\tau) \leq \frac{1}{1-p_h} \\
  \frac{p_h g_{H}^{RA}(\tau)}{g_{H}^{RA}(\tau) - 1} & \text{if } g_{H}^{RA}(\tau) > \frac{1}{1-p_h}.
\end{cases}
\]

We can now establish results analogous to the ones for the strict ordering rationing rules. To avoid redundancy here, we present these results in Section A1.2.1 in the e-companion of the paper. A summary of them follows:

- There is an infimum \( \bar{v} > p_h \) such that \( R(H)(\tau) \geq \bar{v} \), for all \( H \in \mathcal{H} \) (see Lemma A3).
- For all \( H \in \mathcal{H} \), \( R(H)(\tau) \) is \( K \)-Lipschitz continuous (see Proposition A1).
- The set of strategies \( \mathcal{H} \) equipped with the uniform norm \( \|X\| = \sup_{0 \leq \tau \leq T_S} \{ |X(\tau)| \} \) in \([0, T_S]\) exhibits the fixed-point property (see Theorem A1).

Like in the case of rationing rules based on strict priority orderings, in this benchmark scenario the probability \( c(H) \) (formerly \( \Pi_H(\tau) \)) is also differentiable, which is critical for the verification of the conditions of the fixed-point theorem. Figure 2 (right) illustrates an example of the RA rationing allocation rule for the same parameters as in the FIFO case on the left. We can verify there that \( H(\tau) > p_h \), for all \( \tau \in [0, T_S] \).

Unfortunately, the exact analysis for all rationing rules under consideration is not exhaustive in the sense that there are instances of the problems for which we do not have a guaranteed method

\(^8\)Note that our formulation is slightly different from the one in Aviv and Pazgal (2008) (see equation (2) in page 344 and Theorem 2 in page 348 there). They use the unconditional probability of getting a unit via the random allocation rule (captured by the indicator \( \mathbb{I}(A|Q_T) \)), but do not account for the fact that at the moment \( t \) of making the decision there are units available; i.e., \( Q_t > 0 \). However, if \( Q_0 \) is relatively large, then \( \mathbb{P}(Q_t > 0) \approx 1 \), and hence both formulations are indeed equivalent.
for computing the equilibrium. Nevertheless, in the ones that we have tested, we have always been able to find a PE using the basic iteration \( H^{(n+1)} = R(H^{(n)}) \).

5 Asymptotic analysis of the game

In order to circumvent the computational burden and lack of general convergence guarantee of the iterative procedure in Theorem 1, we present here a fluid model derived by replacing the stochastic demand by a continuous flow, with intensity set at the arrival rate. The analytical derivation relies on a limit of the model under a Strong-Law-Of-Large-Numbers type of scaling when we let the demand rate \( \lambda(t), t \in [0, T] \), and the initial number of units \( Q_0 \) grow proportionally large. To this end, consider a sequence of instances of the problem indexed by \( n \) so that \( \lambda^{(n)}(t) \triangleq n\lambda(t) \) and \( Q_0^{(n)} \triangleq nQ_0 \) are the corresponding demand rate and initial inventory level for instance \( n \), respectively, and let \( n \) increase to infinity. All other parameters are kept independent of \( n \).

For each instance \( n \) of the problem, we let \( \rho^{(n)} \triangleq Q_0^{(n)}/(\int_0^T \lambda^{(n)}(t)dt) \). Then, \( \lim_{n \to \infty} \rho^{(n)} = \rho \), for \( \rho \triangleq Q_0/(\int_0^T \lambda(t)dt) \). We refer to \( \rho \) as the supply-demand ratio; it represents the average number of customers that try to buy the item per unit time.

\[ H^{(n+1)} = \alpha H^{(n)} + (1 - \alpha)R(H^{(n)}), \]

where \( \alpha \in [0, 1) \) is empirically selected.
of units available per arriving consumer, and denote by $Q^{(n)}_\tau$ the random number of units available at time $\tau$ for instance $n$ of the sequence of problems. In what follows, we analyze the limiting regime for the FIFO and RA rationing rules.

5.1 FIFO rationing rule

We first consider the FIFO rationing rule, and use the related notation introduced in Section 4. The following result characterizes the asymptotic regime.

**Theorem 2** Suppose that the purchasing strategy $H(\tau)$ is given. Then, in the limit as $n \to \infty$:

(i) The following convergence results hold almost surely (a.s.), and uniformly in $\tau$:

\[
N(\Lambda^{(n)}_{HR}(\tau))/n \to \Lambda_{HR}(\tau), \quad Q^{(n)}_\tau/n \to Q_\tau \triangleq (Q_0 - \Lambda_{HR}(\tau))^+, \quad \text{and} \quad N(\Lambda^{(n)}_{HB}(\tau))/n \to \Lambda_{HB}(\tau).
\]

(ii) The probability $\Pi^{(n)}_H(\tau) \triangleq \mathbb{P}(N(\Lambda^{(n)}_{HR}(\tau)) \leq Q^{(n)}_\tau - 1)$ converges to the two-point distribution:

\[
\Pi^{\infty}_H(\tau) = \begin{cases} 
1 & \text{if } \Lambda_{HR}(\tau) \leq Q_T \\
0 & \text{if } \Lambda_{HR}(\tau) > Q_T,
\end{cases}
\]

for $Q_T \triangleq (Q_0 - \Lambda_{HB}(T))^+$.

In order to find an equilibrium strategy $H^*$ for this limiting regime, we have to impose the condition $R(H^*) = H^*$. That would require finding the corresponding leftover quantity $Q_T$ that will be cleared through the reservations. We need to distinguish three cases:

i) **Limited supply.** Suppose that the initial supply of units is **limited** in the sense that $Q_0 \leq \int_0^T \lambda(t) \bar{F}(p_h, t) dt$ (i.e., $\rho \leq \frac{\int_0^T \lambda(t) F(p_h, t) dt}{\int_0^T \lambda(t) dt}$). Consider an arriving consumer $v_\tau$, and suppose that all other consumers choose the strategy $H^*(t) = p_h, \forall t \in [0, T]$. Given that the supply scarcity ensures no leftover inventory at time $T$, the arriving player could follow the strategy $H^*(\tau) = p_h$.

We note that $H^*(\tau) = p_h$ is not the only PE in this case. In fact, let $\tau^*$ be a solution to $Q_0 = \int_0^{\tau^*} \lambda(t) \bar{F}(p_h, t) dt$. Note that $\tau^*$ is guaranteed to exist on the interval $[0, T]$ for the limited supply case. Then, any $H$ of the form

\[
H(\tau) = \begin{cases} 
p_h & \text{if } \tau \leq \tau^* \\
s(\tau) \in [0, 1] & \text{if } \tau > \tau^*,
\end{cases}
\]

(13)
for any arbitrary function \( s(\tau) \in [0, 1] \), is indeed an equilibrium, because for such an \( H \), all \( Q_0 \) units will be depleted by time \( \tau^* \) (i.e., any consumer arriving after \( \tau^* \) will never get a unit, and so he becomes indifferent between placing or not a reservation).

ii) **Intermediate supply.** Suppose that initial supply is *intermediate* in the sense that \( \int_0^T \lambda(t) \bar{F}(p_h, t) \, dt < Q_0 < \int_0^T \lambda(t) \, dt \) (i.e., \( \frac{\int_0^T \lambda(t) \bar{F}(p_h, t) \, dt}{\int_0^T \lambda(t) \, dt} < \rho < 1 \)). In this case, \( QT > 0 \) and some consumers with valuation less than \( p_h \) get units by placing reservations. Indeed, the consumers who obtain units through the buy-now or reservation channels are the ones with valuation higher than \( p_h \) plus the early arrivals up to time \( \tau^* \), which is the solution to

\[
\int_{\tau^*}^T \lambda(t) \bar{F}(p_h, t) \, dt + \int_0^{\tau^*} \lambda(t) \, dt = Q_0. \tag{14}
\]

Note that \( \tau^* \) is guaranteed to exist in the interval \([0, T]\). From \( \tau^* \) onwards, only those with valuation \( v > p_h \) get units from the buy-now channel. The early arriving consumers, with \( \tau \in [0, \tau^*] \), must decide which channel to purchase from, and therefore need to solve the limiting version of equation (7):

\[
\frac{v}{v^* - p_h} \geq \frac{\exp(w(T - \tau)) \mathbb{P}(Q_0 > 0|H)}{\Pi_H^\infty(\tau)},
\]

where according to Theorem 2 and the selection of \( \tau^* \), both probabilities in the RHS are one. We conclude that in this intermediate case the unique PE \( H^*(\tau) \) is given by:

\[
H^*(\tau) = \begin{cases} \min \left\{ \frac{p_h \exp(w(T - \tau))}{\exp(w(T - \tau)) - 1}, 1 \right\} & \text{if } \tau \in [0, \tau^*] \\ p_h & \text{if } \tau \in (\tau^*, T], \end{cases}
\]

for \( \tau^* \) defined by (14).

iii) **Abundant supply.** Suppose that the initial supply is *abundant*, i.e., \( Q_0 \geq \int_0^T \lambda(t) \, dt \) (i.e., \( \rho \geq 1 \)). In this case all reservations will be satisfied with probability one, hence, every consumer will get an item from the channel he chooses w.p.1. The unique optimal strategy is given by

\[
H^*(\tau) = \min \left\{ \frac{p_h \exp(w(T - \tau))}{\exp(w(T - \tau)) - 1}, 1 \right\}. \tag{15}
\]

The result can be viewed as a particular case of the intermediate supply case where the strategy

\[
H^*(\tau) = p_h
\]

is never realized, because \( \tau^* \geq T \).

\(^{10}\)For the time-homogeneous valuation case where \( F(v, t) = F(v), \forall t \), equation (14) admits a simple closed form solution: \( \tau^* = \frac{Q_0 - \lambda T \bar{F}(p_h)}{\lambda F(p_h)} \).
With a slight abuse of notation, let $H^*(Q_0, \rho)$ be the optimal purchasing strategy if the seller offers $Q_0$ units and the supply-demand ratio is equal to $\rho$. In Figure 3 we compare the optimal asymptotic participation strategy $H^*(\infty, 0.7)$ (computed using Theorem 2) with several optimal purchasing strategies computed numerically using the iteration in Theorem 1. This matches case (ii) above. By setting $T = 1$, starting from $Q_0 = 7$ and $\lambda = 10$, we test the accuracy of the asymptotic approximation for systems with scale factors $n = 1, 2, 5, 10, \text{and} 20$, where a system of size $n$ is defined by $\lambda(n) = n \lambda$, and $Q_0(n) = n Q_0$. The error (last column in the table) is calculated as the fraction of consumers that make a suboptimal decision under the approximated PE, calculated as if all consumers play $H(\infty, \rho)$, with respect to the decision they would have made under the exact PE. We can observe that even for a moderate number of units, the approximation is satisfactorily accurate, and given the simplicity of the calculation, the approximated PE becomes a promising strategy to be pursued in practice.

Asymptotic approximation for the FIFO rationing rule

<table>
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<th>$Q_0$</th>
<th>Exact PE</th>
<th>Approx. PE</th>
<th>Error</th>
</tr>
</thead>
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<tr>
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<td>Reservations</td>
<td>Buy-nows</td>
<td>Reservations</td>
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<tr>
<td>140</td>
<td>134.60</td>
<td>65.40</td>
<td>131.26</td>
</tr>
</tbody>
</table>

Figure 3: Performance of the asymptotic approximation for the FIFO rationing rule case, for time-homogeneous valuations $\text{Unif}[0, 1]$, $T = w = 1$, $p_h = 0.5$, and $\rho = 0.7$. In this case, $\tau^* = 0.4$.

An important observation is that even though the exact PE is $K$-Lipschitz continuous, the asymptotic purchasing strategy $H^*(\infty, \rho)$ is not even continuous in general, with $\tau^*$ being the discontinuity point. Indeed, Theorem 2 guarantees weak convergence of the number of units left $Q_T$, and of the probability of getting a unit through the reservation channel, but it does not claim path-wise convergence of the trajectories $H^*(Q_0, \rho)$.

\[^{11}\text{The discontinuity point is represented by a vertical line for the plotting of } H^*(\infty, 0.7) \text{ in Figure 3.}\]
5.2 RA rationing rule

Let us consider the purchasing strategy under the RA rationing rule in the asymptotic regime. Recall that under the RA rule, the probability of getting an item is equal for all consumers who place a reservation.

**Theorem 3** Suppose that the purchasing strategy $H(\tau)$ is given. Then, in the limit as $n \to \infty$, the probability of getting an item after placing a reservation converges to

$$c^\infty(H) \triangleq \min \left\{ \frac{(Q_0 - \Lambda_{RB}(T_S))^{+}}{\Lambda_{RR}(T_S)}, 1 \right\},$$

where $\Lambda_{RB}(T_S) = \int_0^{T_S} \lambda(t)\bar{F}(H(t), t)\, dt$ and $\Lambda_{RR}(T_S)$ is defined in (10).

Again, we consider three different supply cases, although for the ease of exposition we reverse the order here:

i) **Abundant supply** (i.e., $Q_0 \geq \int_0^{T_S} \lambda(t)\, dt$): Here $c^\infty(H) = 1$, and the optimal strategy is given by (15).

ii) **Intermediate supply** (i.e., $\int_0^{T_S} \lambda(t)\bar{F}(p_h, t)\, dt < Q_0 < \int_0^{T_S} \lambda(t)\, dt$): We can rewrite (16) as

$$c^\infty(H) = 1 - \frac{(1 - \rho) \int_0^{T_S} \lambda(t)\, dt}{\Lambda_{RR}(T_S)},$$

where $0 < c^\infty(H) < 1$. The condition for reservation is given by (11), where now $\mathbb{P}(Q_\tau > 0 | H) = 1$ for all $\tau \in [0, T_S]$. Therefore, the optimal purchasing strategy is defined by the following threshold:

$$H^*(\tau) = \min \left\{ \frac{p_h \exp(w(T_S - \tau))}{\exp(w(T_S - \tau)) - c^\infty(H^*)}, 1 \right\}. \quad (18)$$

By substituting (18) into the denominator of (17), we get the equivalent condition

$$c^\infty(H^*) = 1 - \frac{(1 - \rho) \int_0^{T_S} \lambda(t)\, dt}{\int_0^{T_S} \lambda(t)F\left(\min \left\{ \frac{p_h \exp(w(T_S - \tau))}{\exp(w(T_S - \tau)) - c^\infty(H^*)}, 1 \right\}, t\right)\, dt}.$$  \quad (19)

Equation (19) is a fixed-point equation in $c^\infty(H^*)$. Note that given $c^\infty(H^*)$, $H^*(\tau)$ is uniquely defined by (18). Next proposition shows that this fixed-point equation always has a solution for the intermediate supply case:
**Proposition 3** If \( \int_0^{T_S} \lambda(t) \tilde{F}(p_h, t) \, dt < Q_0 < \int_0^{T_S} \lambda(t) \, dt \), then equation (19) has a solution for \( c^\infty(H) \) in the interval (0, 1), and therefore an equilibrium always exists.

The uniqueness of the solution to (19) cannot be guaranteed though. For example, if \( \lambda \) follows a time-homogeneous Beta distribution with parameters (0.4, 0.4), \( p_l = 0, p_h = 0.45, \rho = 0.55 \), and \( w = 0 \) (i.e., consumers are patient), equation (19) has two solutions leading to different purchasing strategies: One solution corresponds to \( H_1(\tau) = 0.5 \), where \( c^\infty(H_1) = 0.1 \), and where some consumers choose to buy immediately. The other solution corresponds to the strategy \( H_2(\tau) = 1 \), for all \( \tau \in [0, T_S] \) (i.e., all consumers choose to reserve), and \( c^\infty(H_2) = \rho \). Next proposition generalizes the case of \( H_2(\tau) = 1 \):

**Proposition 4** For the intermediate supply case, if \( \rho \geq \exp(w T_S)(1 - p_h) \), there is an equilibrium strategy where \( H^*(\tau) = 1 \), \( \forall \tau \in [0, T_S] \), and \( c^\infty(H^*) = \rho \).

Note also that equilibrium \( H_2(\tau) = 1 \) in the aforementioned example Pareto-dominates equilibrium \( H_1(\tau) \). To check this, consider the following two cases:

- Take a consumer with valuation \( v_l \in [0.5, 1] \) (i.e., a valuation between \( H_1(\tau) \) and \( H_2(\tau) \)).

  The buy now utility of this customer under \( H_1(\tau) \) is \( u(\tau, \tau, v_r - p_h) \geq u(\tau, T_S, v_r - p_l) \times c^\infty(H_1) \). His reservation utility for playing \( H_2(\tau) \) is \( u(\tau, T_S, v_r - p_l) \times c^\infty(H_2) \geq u(\tau, \tau, v_r - p_h) \). Therefore, he will prefer to play \( H_2(\tau) \).

- Take a consumer with valuation \( v_l \in [0.45, 0.50] \) (i.e., a valuation between \( p_h \) and \( H_1(\tau) \)).

  The reservation utility for this customer under \( H_1(\tau) \) is \( u(\tau, T_S, v_r - p_l) \times c^\infty(H_1) \geq u(\tau, \tau, v_r - p_h) \). His reservation utility under \( H_2(\tau) \) is \( u(\tau, T_S, v_r - p_l) \times c^\infty(H_2) > u(\tau, T_S, v_r - p_l) \times c^\infty(H_1) \), because \( c^\infty(H_2) > c^\infty(H_1) \).

This analysis could be generalized to more than two equilibria to prove that the Pareto-dominant equilibrium is the one with highest value \( c^\infty(H) \) among the solutions to (19).

iii) **Limited supply** (i.e., \( Q_0 \leq \int_0^{T_S} \lambda(t) \tilde{F}(p_h, t) \, dt \)): Multiple equilibria are also possible in this case. The purchasing strategy \( H^*(\tau) = p_h \) for all \( \tau \in [0, T_S] \), and more generally, any strategy given by (13) is an equilibrium. The probability of getting an item through a reservation is \( c^\infty(H) = 0 \) in this case.
In addition, there could be another type of equilibria in which consumers can get an item through a reservation with \( c^\infty(H) > 0 \). In this scenario, an equilibrium is given by (18), where \( c^\infty(H) \) is defined by (19). If a solution exists, it is not necessarily unique, as can be seen on Figure 4. In fact, the existence of a solution to (19) is not even guaranteed in this case. For example, this happens under the same parameters as in Figure 4, but with \( w = 1 \). Like in the intermediate supply case, if there are multiple equilibria, it could be verified that the one with highest value of \( c^\infty(H) \) is Pareto-dominant.

Asymptotic approximation for the RA rationing rule

![Diagram](image)

**Figure 4**: Two equilibrium purchasing strategies for the RA rationing rule in the asymptotic regime with limited supply and time-homogeneous (scaled) valuations Beta\((4, 4)\), \( T_S = 1 \), \( w = 0.25 \), \( p_h = 0.45 \), and \( \rho = 0.5 < \bar{F}(p_h) \approx 0.61 \). The probabilities of getting a reserved item are \( c(H_1) \approx 0.25 \) and \( c(H_2) \approx 0.47 \), respectively.

<table>
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<th>( Q_0 )</th>
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<td>32.95</td>
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**Figure 5**: Asymptotic approximation for the RA rationing rule for an intermediate supply case with time-homogeneous valuations Unif\([0, 1]\), \( T_S = w = 1 \), \( p_h = 0.5 \), and \( \rho = 0.7 \). The limiting probability of getting a reserved item is \( c^\infty(H) = 0.64 \).

We also compare here the optimal asymptotic purchasing strategy \( H^*(\infty, 0.7) \) (computed from (19) and (18)) with several optimal purchasing strategies computed numerically using the iteration in The-
Theorem A1. The solution to (19) is unique in this case. Starting from $Q_0 = 7$, we test the accuracy of system scale factors $n = 1, 2, 5, 10,$ and 20. Figure 5 shows the performance of the exact PEs with respect to the approximated PEs. Note that the error of the approximation again decreases as the system size increases.

To finalize this section, recall that one of the strong assumptions of our original (non-asymptotic) model for the rationing regimes under consideration is that the seller does not update the information about the remaining number of units $Q_\tau$ during the sales horizon; there is just information about the availability (or not) of the item. On the firm’s side, since the game is defined at time 0, and there is a precommitment of its rules, knowing the exact value of $Q_\tau$ in real-time is somehow irrelevant for her. On the consumers’ side, in an asymptotic sense, the inventory level $Q_\tau$ can be inferred in real time. Thus, our proposed fluid solution for these partial information settings is also an equilibrium for the game in which the seller provides full information about the inventory level over time. Hence, although restrictive, our model can be viewed as an asymptotic first-order approximation of the more complex game with real time information about $Q_\tau$.

6 Revenue optimization problem and numerical results

Following the usual approach for analyzing Stackelberg form games, we have assumed so far that the parameters that describe the business environment are fixed, and studied how strategic consumers behave in equilibrium. Our primary focus in this section is the seller’s revenue optimization problem under the FIFO rationing rule; that is, we determine which parameters the leader has to announce in practice. The optimal revenue performance under the RA rule is our benchmark measure. Second, we estimate the value of the market composition information for the retailer, that is, we compute how the optimal revenue changes if a fraction of consumers behaves non-strategically (i.e., myopically). Finally, we study the consumer surplus achieved under FIFO and RA.

In this section we assume that consumers’ arrival rate and valuations are time-homogeneous to parallelize the setting of Aviv and Pazgal (2008), and reduce the notation $F(\cdot, t)$ to $F(\cdot)$. Throughout all studies in this section we use the fluid equilibrium in the optimization problem. This is justified by the convergence results of Section 5 and the accuracy of the approximation.
6.1 Revenue performance

The first step in formulating the seller’s problem is to scale back the parameters of the model (i.e. take the inverse of the transformation (2)), and write the optimization problem in terms of the original parameters:

\[
V(\bar{Q}) = \max_{T, Q_0, p_l, p_h} \left\{ p_h \lambda \int_0^T e^{-\alpha t} \mathbb{1}\{Q_t > 0\} \hat{F}(H(t)) dt + p_l e^{-\alpha T} \min\{(Q_0 - \Lambda_{H_R}(T))^+, \Lambda_{H_R}(T)\}, \right. \\
\left. \quad \text{subject to } p_l \leq p_h, Q_0 \leq \bar{Q} \right\},
\]

(20)

Note that we are assuming a discount factor \(\alpha\) for the seller, that could be different from the \(w\) of the consumers.

The optimal revenue under our RA benchmark is the solution to the maximization problem:

\[
V_{RA}(\bar{Q}) = \max_{T, Q_0, p_l, p_h} \left\{ p_h \lambda \int_0^T e^{-\alpha t} \mathbb{1}\{Q_t > 0\} \hat{F}(H(t)) dt + p_l e^{-\alpha T} \min\{(Q_0 - \Lambda_{H_R}(T))^+, \Lambda_{RA}(T_S)\} + V_C, \right. \\
\left. \quad \text{subject to } p_l \leq p_h, Q_0 \leq \bar{Q} \right\},
\]

(21)

where \(V_C\) is the revenue collected during the clearance season, i.e.,

\[
V_C \triangleq \frac{1}{\alpha} p_l \hat{F}(p_l) \lambda \left( \exp(-\alpha T_S) - \exp(-\alpha \min(T, \tau^*)) \right), \text{ for } \alpha > 0,
\]

(22)

and where \(\tau^* \triangleq T_S + (Q_0 - \lambda \hat{F}(p_l) T_S)^+ / (\lambda \hat{F}(p_l))\) stands for the purchasing time of the “last” available (infinitesimal) unit. Define \(s\) as the proportion of time during the sales season where the full price is used, i.e., \(s \triangleq T_S / T\), and denote this rationing rule as RA-\(s\). In case of multiple Bayesian-Nash equilibria, we use the Pareto-dominant one in our reports.

Unfortunately, no simple analytical solutions to (20) and (21) exist, and therefore we have to study their properties numerically. We consider an illustrative base case with \(Q_0 = 500, \lambda = 1,000, T = 1, \alpha = 0.5, w = 2\), and a symmetric valuation distribution \(F \triangleq \text{Beta}(b, b)\), with shape parameter \(b = 1\), so that \(F = \text{Unif}[0, 1]\).\(^{12}\) We assume that the maximum available inventory is \(\bar{Q} = 1,000\).

From here, we explore the effect of each parameter on the expected revenue performance.

In what follows, when we refer to RA-\(s\), we will assume \(s < 1\). In Figure 6, we plot the revenue gaps obtained under FIFO with respect to RA-\(s\), with \(s = 0.7, 0.8, 0.9\), and also with respect to RA-\(1\).

\(^{12}\)Recall that the general Beta\((a, b)\) distribution has bounded support \([0, 1]\) and is symmetric when both shape parameters have the same value, i.e. when \(a = b\).
1. In general, the difference between revenues under FIFO and RA-s is higher when the clearance season is relatively longer (i.e., when \( s \) is smaller).

![Figure 6: Relative revenue increase under FIFO, compared to RA-s, as a function of a) Seller’s discount factor \( \alpha \), b) Shape parameter \( b \) of the Beta(\( b \), \( b \)) distribution, c) Initial inventory \( Q_0 \), d) Selling season duration \( T \). Default values of parameters are: \( Q_0 = 500 \), \( T = 1 \), \( \lambda = 1000 \), \( \alpha = 0.5 \), \( w = 2 \), and \( b = 1 \).](image)

Discount factors play a critical role for achieving price discrimination and extracting higher revenues, as illustrated in Figure 6, and further explored in Figure 7. Indeed, the prevalence of FIFO over RA-s is even more emphasized when \( \alpha < 1.5 \) (i.e., \( \alpha < 0.75w \)), since the seller can take advantage of the consumers’ impatience. In other words, when \( \alpha < 1.5 \), since the seller is more patient than the consumers, she can price more aggressively at the high price level. This is consistent with the findings of Aviv and Pazgal (2008), Section 7.3, and it is also aligned with traditional results of the economics literature.\(^{13}\) The benefit with respect to RA-0.7 could reach a level of 5%. For \( \alpha \approx w \), FIFO could even be beaten by RA-s, but by no more than 0.10%. According to Figure 7, the magnitude of the discount under FIFO is generally small to medium (here, up to 25%).

\(^{13}\)The classic work of Coase (1972) argues that a monopolist facing rational, patient customers who anticipate future lower prices is forced to price at the marginal cost and earn zero profit. Discount factors play a critical role in this assertion; if the seller is more patient than the customers, she may take advantage of price-discrimination (see von der Fehr and Kuhn (1995)).
the reported experiments, all prices remain in the range [0.47, 0.66]. We observe that even though the deeper discounts are applied on the highest full prices, consumers are induced to buy earlier, which in turn translates into a higher number of buy-nows under FIFO. Figure 7 also shows that for the random allocation schemes RA-0.7 and RA-0.9, given the length of the clearance season, more transactions occur at the low price, due to both reservations placed by early arrivals and purchases by late arrivals.

Figure 7: Composition of revenues: In the left column, percentage price decrease, and in the right column, fraction of transactions at the low price, as a function of a) $\alpha$, and b) $Q_0$. Default values of parameters: $Q_0 = 500, T = 1, \lambda = 1000, \alpha = 0.5, w = 2, b = 1$.

Another factor studied in Figure 6 is the heterogeneity of the consumers’ valuations. Clearly, a seller applying FIFO can benefit more when facing highly to moderately heterogeneous consumers (described by a small value of the shape parameter $b$ of the Beta$(b, b)$ distribution). Full prices are roughly similar under all policies, but FIFO discounts slightly more deeply than RA-$s$, and is able to sell more units for small values of $b$.

In Figure 6 we also study the impact of the supply-demand parameters, i.e., initial inventory $Q_0$. 
and selling season duration $T$. For $\alpha < w$, FIFO dominates RA-s when $Q_0$ is moderate or large. The revenue gap could go up to 3.5%. For moderate $Q_0$, FIFO offers a slightly higher full price, and discounts more aggressively than RA-s. The threat of scarcity in the future triggers more buy-nows for FIFO, and hence higher revenues. For a large $Q_0$, FIFO still dominates RA-s. The difference in revenues for the excess supply scenarios is due to the different saturation points: FIFO is able to sell 575 units, versus 562 and 540 of RA-0.9 and RA-0.7 respectively. Note that the discontinuity in revenues of Figure 6 coincide with the price discount drop in Figure 7 (left), and with the upper bound for the number of units to be sold at the low price in Figure 7 (right).

FIFO dominates RA-s when $T$ is moderate. The pricing policies are similar at the full price, but FIFO is more aggressive towards discounting because the threat of scarcity induces consumers to purchase earlier. For a small $T$, FIFO dominates RA-s by inducing more transactions at the full price. The revenue gap could go up to 3.3%. To avoid redundancy, we do not show graphs of revenue results when varying $\lambda$; they mirror those with respect to $Q_0$. For small and moderate values of $\lambda$, FIFO dominates RA-s.

In summary, revenue-wise, the most beneficial cases in favor of FIFO over RA-s occur when the seller is more patient than the consumers, and in particular when: 1) the supply-demand ratio $\rho = Q_0/(\lambda T)$ is moderate to large, and/or 2) the dispersion of the consumers’ valuations is moderate to high. In our experiments, the revenue advantage of FIFO with respect to the standard clearance practice can reach a level of up to 5%.

### 6.2 Value of the market composition information

Consider the mixed market case, where there are two different types of consumers. Here, the total arrival rate $\lambda(t)$ is split between a fraction $\gamma, 0 \leq \gamma \leq 1$, of myopic consumers, and a fraction $1 - \gamma$ of strategic consumers. We begin assuming that $\gamma$ is common knowledge.

The myopic, impulsive consumers behave according to the simple strategy “buy now if own valuation is higher than $p_h$, and reserve otherwise”. The strategic ones choose the channel that maximizes their expected utility. Both types of consumers participate in the clearing of excess inventory at the end of the selling season. Section A3 of the appendix provides the details of the
Interestingly, the stochastic, exact consumer strategy under FIFO rationing rule is sensitive to the parameter $\gamma$, but its asymptotic counterpart is invariant with respect to it. In other words, under the asymptotic FIFO regime, forward-looking consumers can ignore the fraction of myopic consumers in order to compute the (optimal) equilibrium strategy. This somewhat surprising result is anchored in the following two features: 1) Strategic consumers can assess the time of the last marginal arrival who will get a unit (i.e., the value of $\tau^*$ defined in (14)), which does not depend on $\gamma$, and hence they can also assess the limiting probability of getting a unit through each of the channels (which according to Theorem 2 is just 1 or 0); and 2) Given the setting of the game ($\lambda, T, Q_0$, and $w$) the other factors that define consumers' utility at the moment of making the purchasing decision are $v_\tau$ and $\tau$, which do not depend on $\gamma$ either.

For the RA rule, the factor $\gamma$ is included in both the exact and asymptotic strategies; the reason being that in this case, a consumer is uncertain about the fact of getting an item through the reservation channel in both regimes.

In Figure 8 (left), we first analyze the effect of FIFO and RA rationing rules on the seller’s revenue under this mixed market framework. We see that FIFO consistently achieves the best revenue performance, just matched by RA-1 when more than 40% of the consumers are myopic.

Second, we study the scenario when the seller incorrectly assumes that all consumers are myopic when in fact just a fraction $\gamma$ of them indeed are. In Figure 8 (right) we plot the revenue gap between the seller optimizing $p_l$ and $p_h$ under the correct proportion of myopic consumers (i.e., knowing the value of $\gamma$) and the seller optimizing under the wrong assumption that everybody is myopic. It is clear that the negative impact on revenue performance for all the rationing rules under consideration is decreasing in the proportion of myopic consumers. The four policies studied have a comparable sensitiveness to market composition information, specially when more than half of the consumers are myopic. Ignoring market composition could decrease the revenue potential by more than 5.5%, and is even more substantial when consumers are more heterogeneous (e.g., when $F=\text{Beta}(0.2, 0.2)$, the suboptimality gap of FIFO could reach 38%).
6.3 Consumer surplus

In what follows, the notation encompasses both FIFO and RA cases, where for the former $T_S = T$. Given prices $p_l$ and $p_h$, the total surplus obtained by buy-now consumers is

$$S_B = \lambda \int_0^{T_S} \int_{H(t)}^1 (v - p_h) \mathbb{1}\{Q_t > 0\} f(v) dv dt + S_C, \quad (23)$$

where $S_C$ is the surplus obtained during a clearance season, i.e.,

$$S_C = \lambda \int_{T_S}^{\min\{\tau^*, T\}} \int_{p_l}^1 (v - p_l) f(v) dv dt = \min \left\{ \frac{(Q_0 - \lambda F(p_l) T_S)^+}{F(p_l)}, \lambda (T - T_S) \right\} \int_{p_l}^1 (v - p_l) f(v) dv. \quad (24)$$

The total surplus obtained through the reservation channel is

$$S_R = \lambda \int_0^{T_S} \int_{p_l}^{H(t)} (v - p_l) \exp(w(t - T_S)) \Pi_H^\infty(t) f(v) dv dt,$$

where $\Pi_H^\infty(t)$ stands for the limiting probability of getting an item through the reservation channel. The exact expressions depend on the rationing rule and supply-demand ratio and were given in Section 5.

Figure 9 plots the total surplus obtained by consumers through the buy-now and reservation channels under FIFO and RA-0.7, 0.9, and 1. We notice that the consumer surplus under FIFO is just slightly less than under RA-1. Furthermore RA-s, with $s = 0.7, 0.9$, deliver noticeably lower consumer surplus, unless the amount of inventory put up for sale is high. This difference would have even been more emphasized had we included a consumer search cost in the utility function for the RA case (recall that under this mechanism, consumers typically need to revisit the store).
Figure 9: Total aggregated consumer surplus as a function of: a) Seller’s discount factor $\alpha$, b) Shape parameter $b$ of the Beta($b, b$) distribution, c) Initial inventory $Q_0$, d) Selling season duration $T$. Default values of parameters are: $Q_0 = 500, T = 1, \lambda = 1000, \alpha = 0.5, w = 2,$ and $b = 1$.

The higher degree of price discrimination of FIFO leads not only to a higher seller’s revenue but also to a higher aggregated consumer surplus as compared to RA-s, and almost matches the one under the RA-1 rule. This is also aligned with the fact that FIFO tends to sell more units.

7 Conclusions

In this paper, we develop a stylized model where a seller facing an arrival stream of strategic consumers operates a selling with binding reservations scheme. Upon arrival, each consumer, trying to maximize his own surplus, must decide either to purchase at a high price and get the item at no risk, or to place a reservation at a discount price and wait until the end of the sales season when the leftover units are allocated according to first-in, first-out (FIFO) priority. As a benchmark, we use the two-period, preannounced fixed-discount model introduced by Aviv and Pazgal (2008), that we call random allocation (RA). Consumers arriving early in the sales horizon choose between buying now or waiting for the beginning of the clearance season, where leftover units are allocated randomly.
among the consumers who took the gamble. More consumers come during the clearance season, and deplete the remaining inventory (if any).

For both settings, the seller announces the price path at the beginning of the horizon. Consumers are strategic in the sense that they are forward-looking: They know that price will decline over time, but they are also aware of the risk of later non-availability. Their private information consists of the arrival time and the valuation for one of the units being offered. Using a time-sensitive utility function, we show that their purchasing equilibrium strategy is of the threshold type, that is, a consumer will place a reservation if and only if his own valuation is lower than a function of his arrival time. Of course, for consumers with values between the full price and the discount price, the optimal strategy is to always place a reservation. Interestingly, we find that there is also a range of values above the full price for which it is always optimal to place a reservation regardless of the arrival time.

At a theoretical level, we prove that a symmetric purchasing equilibrium always exists for a broad class of rationing rules that include FIFO, and for RA, and use a contraction algorithm in a function space to find it. The procedure is computationally intensive and provably convergent under specific conditions. In general though, the exact algorithm does not have a convergence guarantee. To overcome these limitations, we develop an asymptotic, fluid-type approximation for the two settings, where the initial number of units and the demand rate grow proportionally large. For each setting, in the limit, the purchasing behavior converges weakly to an equilibrium that we are able to characterize in closed form. Using numerical experiments, we also verify that this limiting solution is a good approximation even for moderate-size problems.

Due to the simplicity and accuracy of the asymptotic analysis, we solve the seller’s revenue optimization problem under this limiting regime. We observe that the FIFO reservation rule almost consistently achieves the best performance. A critical condition to provide some extra revenue gains is that the discount factor of the seller must be less than the one of the consumers. Compared to RA, the most beneficial cases revenue-wise occur when: 1) There is a moderate to high number of units with respect to the expected demand, and/or 2) the dispersion of the consumers’ valuations is moderate to high. The relative benefit of FIFO versus the standard markdown practice is even more
emphasized when the clearance season is longer. Our numerical experiments show that the revenue gap can exceed 5%. Moreover, its advantage is still preserved when the market is composed by both myopic and strategic consumers.

All in all, given the narrow margins of retail operations, the additional revenues achievable under FIFO can have a profound impact on these businesses’ profitability. With its appealing features for the consumers, it makes the use of non-withdrawable reservations with FIFO rationing a promising mechanism to be further explored and implemented in practice.

References


A1 Proofs

A1.1 Preliminary results

We compile a few results that will be used in the main proofs.

**Lemma A1** The following results hold:

(i) Let $N(x)$ be a Poisson random variable with mean $x > 0$. For a nonnegative integer $n$,

$$\frac{d}{dx} P(N(x) \leq n) = -P(N(x) = n).$$

(ii) Let $N(x)$ be a Poisson random variable with mean $x > 0$. Let $a$ and $b$ be two nonnegative constants, and let $N \geq 1$ be an integer. Then, there exists $0 \leq \beta(N) \leq 1$ such that

$$|P(N(b) \leq n) - P(N(a) \leq n)| \leq \beta(n) |b - a|.$$  

In particular, $\beta(n) \triangleq P(N(n) = n)$.

(iii) For any constants $a, b, \text{ and } c$,

$$|\max\{a, c\} - \max\{b, c\}| \leq |a - b|.$$

**Proof:** Results (i) and (ii) are proved in Caldentey and Vulcano (2007) (see Lemmas A1 and A3 in the corresponding Online Appendix). The third result is easy to verify. □

The next lemma introduces some useful bounds for future reference:

**Lemma A2** The following bounds hold for $\tau \in [0, T]$:

(i) $1 \leq \exp(w(T - \tau)) \leq \exp(wT)$.

(ii) A (random) lower bound for the leftover inventory at time $T$ is

$$Q_T \triangleq (Q_0 - N(\bar{\Lambda}_{H^B}(T)))^+ \leq_{st} (Q_0 - N(\Lambda_{H^B}(T)))^+ = Q_T,$$

where

$$\bar{\Lambda}_{H^B}(\tau) \triangleq \bar{\lambda} \tau \max_{t \in [0, \tau]} \bar{F}(p_h, t),$$

and where “$\leq_{st}$” stands for the stochastic order relational operator.\(^1\)

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\(^1\)Two random variables $X$ and $Y$ are such that $X \leq_{st} Y$ if $P(X > x) \leq P(Y > x), \forall x.
(iii) For a strict ordering rationing rule, \( \Pi_H(\tau) \geq \mathbb{P}(N(\bar{\lambda}T) \leq Q_0 - 1) > 0 \).

(iv) For a strict ordering rationing rule, \( \Pi_H(\tau) \leq \mathbb{P}(Q_\tau > 0|H) \).

(v) \( \mathbb{P}(Q_\tau > 0|H) \geq \mathbb{P}(N(\bar{\Lambda}_H(\tau)) \leq Q_0 - 1) \geq \mathbb{P}(N(\bar{\Lambda}_H(T)) \leq Q_0 - 1) \), where \( \bar{\Lambda}_H(\tau) \) and \( \bar{\Lambda}_H(T) \) are given by (A1).

Proof: Bounds in (i) follow from the monotonicity of the exponential function.

Bound (ii) comes from the fact that an upper bound for the total number of buy-nows is given by \( \bar{\Lambda}_H(T) \).

For bounds (iii) and (iv) observe that

\[
\Pi_H(\tau) = \mathbb{P}(N(\Lambda_H(T) + \Lambda_H(\tau)) \leq Q_0 - 1) \\
\geq \mathbb{P}(N(\bar{\lambda}T) \leq Q_0 - 1) > 0, \quad \text{and} \\
\Pi_H(\tau) \leq \mathbb{P}(N(\Lambda_H(T)) \leq Q_0 - 1) \\
= \mathbb{P}(Q_0 - N(\Lambda_H(T)) > 0) \\
\leq \mathbb{P}(Q_0 - N(\Lambda_H(\tau)) > 0) = \mathbb{P}(Q_\tau > 0|H).
\]

To prove bound (v) recall that \( \mathbb{P}(Q_\tau > 0|H) = \mathbb{P}(N(\Lambda_H(\tau)) \leq Q_0 - 1) \). The first bound now follows from the fact that an upper bound for the mean number of buy-nows up to time \( \tau \) is \( \bar{\Lambda}_H(\tau) \) as defined in (A1). The last bound holds since \( T \geq \tau \), and since the cumulative distribution is decreasing in the mean.

A1.2 Consumer purchasing behavior

Proof of Lemma 1 The proof amounts to computing the derivative of \( \Pi_H(\cdot) \). If the rationing rule is based on a strict ordering, then \( \Pi_H(\tau) = \mathbb{P}(N(\Lambda_H(\tau) + \Lambda_H(T)) \leq Q_0 - 1) \) and by Lemma A1(i),

\[
\left| \frac{d}{d\tau} \Pi_H(\tau) \right| = \left| \mathbb{P}(N(\Lambda_H(T) + \Lambda_H(\tau)) = Q_0 - 1) \frac{d\Lambda_H(\tau)}{d\tau} \right| \\
\leq \left| \frac{d}{d\tau} \left( \int_0^T \mathbb{1}_{\{\xi(t) > \xi(\tau)\}} \lambda(t)F(H(t), t)dt \right) \right|.
\]

Note that

\[
\left| \frac{d}{d\tau} \left( \int_0^T \mathbb{1}_{\{\xi(t) > \xi(\tau)\}} \lambda(t)F(H(t), t)dt \right) \right| = \lim_{d\tau \to 0} \left| \int_0^T \left( \mathbb{1}_{\{\xi(t) > \xi(\tau + d\tau)\}} - \mathbb{1}_{\{\xi(t) > \xi(\tau)\}} \right) \lambda(t)F(H(t), t)dt \right| \\
\leq \lim_{d\tau \to 0} \left| \sum_{i=1}^N \lambda(\tau_i)F(H(\tau_i), \tau_i)d\tau + o(d\tau) \right| \leq \bar{\lambda}N,
\]

where \( \tau_1, \ldots, \tau_N \) are such that \( \xi(\tau) = \xi(\tau_i), \quad i = 1, \ldots, N \). Since \( \xi \in \Xi \), it has a finite number of local extrema and hence \( N \) is finite. Thus, \( \left| \frac{d}{d\tau} \Pi_H(\tau) \right| \leq \bar{\lambda}N. \)
Proof of Lemma 2  First, we need to show that \( g_H(\tau) \) in (7) is a well defined function in \([0, T]\). By (i), (iii), and (iv) of Lemma A2, the following upper and lower bounds can be established:

\[
1 \leq g_H(\tau) \leq \frac{\exp(wT)}{\mathbb{P}(N(\lambda T) \leq Q_0 - 1)}.
\]

Because the LHS in (7) is decreasing in \( v_\tau \) and has a vertical asymptote at \( v_\tau = p_h \), and since \( g_H(\tau) \) is bounded above, we can assert that there exists such \( \bar{v} > p_h \).

Proof of Proposition 2  Lemmas 1 and 2 imply continuity of \( \mathcal{R}(H)(\tau) \) in \( \tau \). Recall from the definition of \( v_H \) that \( \mathcal{R}(H)(\tau) > p_h \) for all \( \tau \in [0, T] \). Also, recall that \( \mathcal{R}(H)(\tau) = 1 \) when \( g_H(\tau) \leq 1/(1-p_h) \). So, we can concentrate on proving the \( K \)-Lipschitz property on the intervals where \( g_H(\tau) \) is differentiable and \( g_H(\tau) > 1/(1-p_h) > 1 \). Take \( \tau_1, \tau_2 \in (\bar{v}, \bar{v}) \) such that \( g_H(\tau) > 1/(1-p_h) \) for all \( \tau \in (\bar{v}, \bar{v}) \). We have that

\[
|\mathcal{R}(H)(\tau_1) - \mathcal{R}(H)(\tau_2)| = p_h \left| \frac{g_H(\tau_1) - g_H(\tau_2)}{g_H(\tau_1) - 1} - \frac{g_H(\tau_2)}{g_H(\tau_2) - 1} \right|
= p_h \left| \int_{\tau_2}^{\tau_1} \frac{d}{dx} \left( \frac{g_H(x)}{g_H(x) - 1} \right) \, dx \right| = p_h \left| \int_{\tau_2}^{\tau_1} \frac{g_H'(x)}{(g_H(x) - 1)^2} \, dx \right| \quad \text{(A2)}
\]

For the numerator in the integrand, by expanding the derivatives we get:

\[
g_H'(\tau) = \frac{(\exp(w(T - \tau))\mathbb{P}(Q_\tau > |H|))' \mathcal{I}_{\Pi H}(\tau) - \exp(w(T - \tau))\mathbb{P}(Q_\tau > |H|)\mathcal{I}_{\Pi H}'(\tau)}{\mathcal{P}_H^2(\tau)},
\]

where from the fact that

\[
\mathbb{P}(Q_\tau > 0) = \mathbb{P}(N(\Lambda_{H_{\beta}}(\tau)) \leq Q_0 - 1),
\]

and property (i) in Lemma A1,

\[
\frac{d}{d\tau} (\exp(w(T - \tau))\mathbb{P}(Q_\tau > |H|)) = -\exp(w(T - \tau)) \left[ w\mathbb{P}(Q_\tau > |H|) + \mathbb{P}(N(\Lambda_{H_{\beta}}(\tau)) = Q_0 - 1) \frac{d}{d\tau} \Lambda_{H_{\beta}}(\tau) \right] = -\exp(w(T - \tau)) \left[ w\mathbb{P}(Q_\tau > |H|) + \mathbb{P}(N(\Lambda_{H_{\beta}}(\tau)) = Q_0 - 1) \lambda(\tau) \mathcal{H}(H(\tau), \tau) \right].
\]

Therefore,

\[
g_H'(\tau) = -\frac{\exp(w(T - \tau))}{\mathcal{P}_H^2(\tau)} \left( \mathcal{H}(\tau) \lambda(\tau) \mathcal{H}(H(\tau), \tau) + \mathbb{P}(N(\Lambda_{H_{\beta}}(\tau)) = Q_0 - 1) + \mathbb{P}(Q_\tau > |H|)(w\mathcal{I}_H(\tau) + \mathcal{P}_H(\tau)) \right)
\]

Upper bounding the probabilities in the numerator by 1, and using bounds (i) and (iii) of Lemma A2, and Lemma 1, we get

\[
|g_H'(\tau)| \leq \frac{\exp(w(T - \tau))}{\mathcal{P}_H^2(\tau)} \left( \mathcal{H}(\tau) \lambda(\tau) + \mathcal{P}_H(\tau) \right) \leq \exp(wT) \left( \frac{\lambda + w}{\mathbb{P}(N(\lambda T) \leq Q_0 - 1)} + \frac{K_{\Pi}}{\mathbb{P}(N(\lambda T) \leq Q_0 - 1)^2} \right). \quad \text{(A3)}
\]
For the denominator in the integrand of (A2), since \( g_H(\tau) > 1/(1-p_h) \), we have
\[
(g_H(\tau) - 1)^2 > \left( \frac{p_h}{1-p_h} \right)^2. \tag{A4}
\]
Plugging bounds (A3) and (A4) back into (A2), we get
\[
|\mathcal{R}(H) (\tau_1) - \mathcal{R}(H) (\tau_2)| = K |\tau_1 - \tau_2|,
\]
where the constant \( K = \exp(wT) \left( \frac{(1-p_h)^2}{p_h} \frac{1 + w}{\mathbb{P}(N(\lambda T) \leq Q_0 - 1)} + \frac{K_H}{\mathbb{P}(N(\lambda T) \leq Q_0 - 1)^2} \right) \) is independent of \( \tau_1 \) and \( \tau_2 \), and is guaranteed to be finite because \( \mathbb{P}(N(\lambda T) \leq Q_0 - 1) > 0 \).

**Proof of Theorem 1** To prove that \( \mathcal{H} \) has the fixed-point property, we apply the Schauder-Tychonoff Fixed-Point Theorem (see Cheney (2001), Chapter 7 for details). For this, we need to show that \( \mathcal{H} \) is a compact convex set. Convexity is immediate from the definition of \( \mathcal{H} \). To check compactness, we apply the Arzelà-Ascoli Theorem II (Cheney (2001), Chapter 7), that is, we need to show that \( \mathcal{H} \) is closed, bounded, and equicontinuous. Take a sequence \( \{H^{(n)}\}_{n \geq 1} \) of strategies in \( \mathcal{H} \) that converges point-wise to \( H \). In order to verify the \( K \)-Lipschitz property of \( H \) note that from the Proposition, for \( n \geq 1 \) and \( \tau_1, \tau_2 \in [0, T] \),
\[
|H^{(n)}(\tau_1) - H^{(n)}(\tau_2)| \leq K |\tau_1 - \tau_2|.
\]
By the continuity of the absolute value and the point-wise convergence of \( H^{(n)} \) to \( H \) we conclude
\[
|H(\tau_1) - H(\tau_2)| \leq K |\tau_1 - \tau_2|,
\]
which proves the closedness of \( \mathcal{H} \). The boundedness of \( \mathcal{H} \) follows from the fact \( H(\tau) \in [0, 1] \) for all \( H \in \mathcal{H} \). Equicontinuity, on the other hand, follows directly from the fact that the elements of \( \mathcal{H} \) are \( K \)-Lipschitz continuous. In fact, to prove equicontinuity of \( \mathcal{H} \) we need to show that for \( \epsilon > 0 \) there is \( \delta > 0 \) such that:

For all \( H \in \mathcal{H} \) and \( \tau_1, \tau_2 \in [0, T] \), such that \( |\tau_1 - \tau_2| < \delta \) then \( |H(\tau_1) - H(\tau_2)| < \epsilon \).

For this, take \( \delta = \frac{\epsilon}{K} \) and use the \( K \)-Lipschitz continuity of \( H \) as follows:

For all \( H \in \mathcal{H} \) and \( \tau_1, \tau_2 \in [0, T] \) such that \( |\tau_1 - \tau_2| < \delta \), \( |H(\tau_1) - H(\tau_2)| \leq K |\tau_1 - \tau_2| < K \delta = \epsilon \).

This proves that \( \mathcal{H} \) has the fixed-point property.

We now prove that the best-response \( \mathcal{R} \) mapping is continuous in \( \mathcal{H} \). We start by observing that we can rewrite the best response mapping in (9) as
\[
\mathcal{R}(H)(\tau) = \frac{p_h \max\{g_H(\tau), 1/(1-p_h)\}}{\max\{g_H(\tau), 1/(1-p_h)\} - 1}.
\]
Therefore,

\[
|\mathcal{R}(H) - \mathcal{R}(\hat{H})| = \left| p_h \max \{ g_H(\tau), 1/(1-p_h) \} \right| - \left| p_h \max \{ g_{\hat{H}}(\tau), 1/(1-p_h) \} \right| \\
= p_h \left| \max \{ g_H(\tau), 1/(1-p_h) \} \right| - \max \{ g_{\hat{H}}(\tau), 1/(1-p_h) \} - 1) | \\
\times \frac{1}{\max \{ g_H(\tau), 1/(1-p_h) \} - 1) \left( \max \{ g_{\hat{H}}(\tau), 1/(1-p_h) \} - 1) \right| \\
\leq (1-p_h)^2 \left| \max \{ g_H(\tau), 1/(1-p_h) \} - \max \{ g_{\hat{H}}(\tau), 1/(1-p_h) \} \right| \\
\leq (1-p_h)^2 |g_H(\tau) - g_{\hat{H}}(\tau)|, 
\]  
(A5)

where the first inequality is verified from the fact that \( \max \{ g(\tau), 1/(1-p_h) \} \geq 1/(1-p_h) > 1 \), and the second inequality holds from Lemma A1, part (iii).

We proceed as follows:

\[
|g_H(\tau) - g_{\hat{H}}(\tau)| = \exp(w(T - \tau)) \left| \frac{\mathbb{P}(Q_\tau > 0|H) - \mathbb{P}(Q_\tau > 0|\hat{H})}{\Pi_H(\tau)} \right| \\
\leq \exp(wT) \left| \frac{\mathbb{P}(Q_\tau > 0|H)\Pi_{\hat{H}}(\tau) - \mathbb{P}(Q_\tau > 0|\hat{H})\Pi_H(\tau)}{\Pi_H(\tau)\Pi_{\hat{H}}(\tau)} \right| \\
\leq \frac{\exp(wT)}{\mathbb{P}(N(\lambda T) \leq Q_0 - 1)^2} \left[ \mathbb{P}(Q_\tau > 0|H)\Pi_{\hat{H}}(\tau) - \mathbb{P}(Q_\tau > 0|\hat{H})\Pi_H(\tau) \right] \\
\leq \frac{\exp(wT)}{\mathbb{P}(N(\lambda T) \leq Q_0 - 1)^2} \left[ |\mathbb{P}(Q_\tau \geq 1|H) - \mathbb{P}(Q_\tau > 0|\hat{H})| + |\Pi_H(\tau) - \Pi_{\hat{H}}(\tau)| \right], 
\]  
(A6)

where the first inequality holds from Lemma A2 part (i), the second one holds from Lemma A2 part (iii), and the last one from the fact that for any pair of reals \( a, b \), and for any reals \( c, d \), such that \(|c|, |d| \leq 1\),

\[
|ad - cb| = |(a - c)d + (d - b)c| \leq |a - c| + |d - b|. 
\]

Taking the first module in equation (A6), we have

\[
\left| \mathbb{P}(Q_\tau > 0|H) - \mathbb{P}(Q_\tau > 0|\hat{H}) \right| = \left| \mathbb{P}(N(\Lambda_H(\tau)) \leq Q_0 - 1) - \mathbb{P}(N(\Lambda_{\hat{H}}(\tau)) \leq Q_0 - 1) \right| \\
\leq \beta(Q_0 - 1) |\Lambda_H(\tau) - \Lambda_{\hat{H}}(\tau)| \\
\leq \beta(Q_0 - 1) \int_0^T \lambda(t)(F(\hat{H}(t), t) - F(H(t), t))dt \\
\leq \beta(Q_0 - 1) \lambda K_F \int_0^T |\hat{H}(t) - H(t)|dt \\
\leq \beta(Q_0 - 1) \lambda K_F T |\hat{H} - H|, 
\]  
(A7)

where the first inequality holds from Lemma A1, part(ii); the second inequality holds from the \( K_F \)-Lipschitz continuity of the c.d.f. \( F \), the third one from the definition of uniform norm for the function space \( \mathcal{H} \), and the last one from the observation: \( \tau \leq T \).
For the second module in equation (A6), note that the expressions for $\Pi_H(\cdot)$ and $\Pi_{\tilde{H}}(\cdot)$ depend on the parameter $\tau$. Using (6) and Lemma A1(ii), and following arguments similar to the ones in (A7), we get:

$$
|\mathbb{P}(N(\Delta_{HR}(\tau) + \Delta_{HB}(T)) \leq Q_0 - 1) - \mathbb{P}(N(\Delta_{HR}(\tau) + \Delta_{HB}(T)) \leq Q_0 - 1)| \\
\leq \beta(Q_0 - 1) \left| \Delta_{HR}(\tau) + \Delta_{HB}(T) - (\Delta_{HR}(\tau) + \Delta_{HB}(T)) \right| \\
\leq \beta(Q_0 - 1) \left( |\Delta_{HR}(\tau) - \Delta_{HR}(\tau)| + |\Delta_{HB}(T) - \Delta_{HB}(T)| \right) \\
\leq 2 \beta(Q_0 - 1) \lambda K_F T ||\tilde{H} - H||. 
$$

Plugging bounds (A7) and (A8) into (A6), and then (A6) into (A5), we get:

$$
|\mathcal{R}(H)_{\tau} - \mathcal{R}(\tilde{H})_{\tau}| \leq \frac{3(1 - p_h)^2 \exp(wT) \beta(Q_0 - 1) \lambda K_F T}{p_h \mathbb{P}(N(\lambda T) \leq Q_0 - 1)^2} ||\tilde{H} - H||,
$$

where $\beta(Q_0 - 1) = \mathbb{P}(N(Q_0 - 1) = Q_0 - 1)$.

From this result, we conclude that $\mathcal{R}$ is continuous which together with the fixed-point property of the set $\mathcal{H}$ guarantee the existence of a SPE, if the rationing rule is based on a strict priority ordering (as defined in Section 4).

A1.2.1 RA rationing rule

**Lemma A3** For all $H \in \mathcal{H}$ and $\tau \in [0, T_S]$: $g_{\mathcal{H}}^{RA}(\tau) > 1$, and there is a valuation $v_H > p_h$, such that $R(H)(\tau) \geq v_H$. The infimum $\tilde{v} = \inf_{H \in \mathcal{H}} \{v_H\}$ satisfies $\tilde{v} = \inf_{H \in \mathcal{H}} \{v_H\} \geq \frac{p_h g^{RA}_H}{\bar{g}^{RA}}$, where $\bar{g}^{RA} = \exp(wT_S) \mathbb{P}(N(\lambda T_S) \leq Q_{T_S} - 1)$ and $Q_{T_S} \triangleq (Q_0 - N(\bar{\Delta}_{HB}(T_S)))^+$, for $\bar{\Delta}_{HB}(T_S)$ given by (A1).

**Proof:** Note that $c(H) < \sum_{k=1}^{Q_0} \mathbb{P}(Q_{T_S} = k|H) = \mathbb{P}(Q_{T_S} > 0|H)$, hence

$$
g_{\mathcal{H}}^{RA}(\tau) > \frac{\mathbb{P}(Q_{T_S} \geq 1|H)}{\mathbb{P}(Q_{T_S} > 0|H)} > 0.
$$

To prove the bound

$$
g_{\mathcal{H}}^{RA}(\tau) \leq \frac{\exp(wT_S)}{\mathbb{P}(N(\lambda T_S) \leq Q_{T_S} - 1)}, 
$$

observe that

$$
c(H) \geq \sum_{k=1}^{Q_0} \mathbb{P}(Q_{T_S} = k|H) \sum_{n=0}^{k} \mathbb{P}(N(\Delta_{HR}(T_S) = n) \\
= \sum_{k=1}^{Q_0} \mathbb{P}(Q_{T_S} = k|H) \mathbb{P}(N(\Delta_{HR}(T_S) \leq n) \\
= \mathbb{P}(N(\Delta_{HR}(T_S)) \leq Q_{T_S}) \\
\geq \mathbb{P}(N(\Delta_{HR}(T_S)) \leq Q_{T_S} - 1) \\
\geq \mathbb{P}(N(\bar{\lambda} T_S) \leq Q_{T_S} - 1). 
$$
The bound (A9) follows from (i) in Lemma A2, and the argument concludes as in the proof of Lemma 2.

Proposition A1 For the exponential utility function (1) and for all $H \in \mathcal{H}$, there is a positive constant $K$ (independent of $H$) such that the best-response strategy $\mathcal{R}(H)(\tau)$ is a $K$-Lipschitz continuous function.

Proof: The proof is analogous to Proposition 2. Representations (A2) and (A3) hold, $\Pi_H(\tau) = c(H)$ and $\Pi'_H(\tau) = 0$. From this and (A4):

$$|\mathcal{R}(H)(\tau_1) - \mathcal{R}(H)(\tau_2)| \leq \frac{(1 - p_h)^2}{p_h} \frac{(w + \bar{\lambda}) \exp(w T_S)}{\mathbb{P}(N(\lambda T_S) \leq Q_{T_S} - 1)} |\tau_1 - \tau_2| \triangleq K|\tau_1 - \tau_2|,$$

where the bound on $c(H)$ follows from Lemma A3. The constant $K$ is independent of $\tau_1$ and $\tau_2$, and is guaranteed to be finite because $\mathbb{P}(N(\lambda T_S) \leq Q_{T_S} - 1) > 0$.

Theorem A1 For the RA rationing rule, the set of strategies $\mathcal{H}$ equipped with the uniform norm $\|X\| = \sup_{0 \leq \tau \leq T_S} \{|X(\tau)|\}$ in $[0, T_S]$ exhibits the fixed-point property. In addition, for all $H, \tilde{H} \in \mathcal{H}$, the mapping $\mathcal{R}$ satisfies:

$$\|\mathcal{R}(H) - \mathcal{R}(\tilde{H})\| \leq K_H \|H - \tilde{H}\|,$$

where $K_H$ is finite. Therefore, $\mathcal{R}$ is a continuous mapping and there always exists a SPE.

Proof: From the proof of Theorem 1, $\mathcal{H}$ has the fixed-point property. To prove the continuity of the best-response mapping $\mathcal{R}(\mathcal{H})$, representation (A5) holds, and therefore

$$|\mathcal{R}(\mathcal{H})(\tau) - \mathcal{R}(\tilde{H})(\tau)| \leq \frac{(1 - p_h)^2}{p_h} |g_H^{RA}(\tau) - g_{\tilde{H}}^{RA}(\tau)|.$$

Now

$$|g_H^{RA}(\tau) - g_{\tilde{H}}^{RA}(\tau)| = \exp(w(T_S - \tau)) \left| \frac{\mathbb{P}(Q_{\tau} > 0|H)}{c(H)} - \frac{\mathbb{P}(Q_{\tau} \geq 1|H)}{c(H)} \right| \leq \exp(w T_S) \left| \frac{\mathbb{P}(Q_{\tau} \geq 1|H)c(H) - \mathbb{P}(Q_{\tau} > 0|H)c(H)}{c(H)c(H)} \right| \leq \frac{\exp(w T_S)}{\mathbb{P}(N(\lambda T_S) \leq Q_{T_S} - 1)^2} \left| \mathbb{P}(Q_{\tau} > 0|H)c(H) - \mathbb{P}(Q_{\tau} > 0|\tilde{H})c(H) \right| \leq \frac{\exp(w T_S)}{\mathbb{P}(N(\lambda T_S) \leq Q_{T_S} - 1)^2} \left[ |\mathbb{P}(Q_{\tau} > 0|H) - \mathbb{P}(Q_{\tau} > 0|\tilde{H})| + |c(H) - c(H)| \right].$$

Let

$$\Lambda_{RA}(T_S) \triangleq \int_0^{T_S} \lambda(t) \tilde{F}(H(t), t) dt.$$
The first module in brackets can be bounded by (A7). Consider the second module and recall $c(H) = \sum_{k=1}^{Q_0} \mathbb{P} (Q_{T_S} = k | H) \sum_{n=0}^{k} \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = n) + \sum_{n=k+1}^{\infty} \frac{k}{n} \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = n)$. Therefore:

$$|c(H) - c(H)| = \left| \sum_{k=1}^{Q_0} \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = Q_0 - k) \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) \leq k) + \sum_{k=1}^{Q_0} \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = Q_0 - k) \sum_{n=k+1}^{\infty} \frac{k}{n} \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = n) \right|$$

$$- \left| \sum_{k=1}^{Q_0} \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = Q_0 - k) \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) \leq k) - \sum_{k=1}^{Q_0} \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = Q_0 - k) \sum_{n=k+1}^{\infty} \frac{k}{n} \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = n) \right|$$

$$\leq |A| + |B|,$$

where

$$|A| = \left| \sum_{k=1}^{Q_0} \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = Q_0 - k) \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) \leq k) - \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = Q_0 - k) \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) \leq k) \right|$$

$$\leq \sum_{k=1}^{Q_0} \left[ \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = Q_0 - k) \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) \leq k) - \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = Q_0 - k) \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) \leq k) \right]$$

$$\leq \sum_{k=1}^{Q_0} \left[ \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) \leq Q_0 - k) - \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) \leq Q_0 - k) \right]$$

$$\leq \sum_{k=1}^{Q_0} \left[ \beta(Q_0 - k) + \beta(Q_0 - k - 1) + |\Lambda_{HR}^{RA}(T_S) - \Lambda_{HR}^{RA}(T_S)| \sum_{k=1}^{Q_0} \beta(k) \right]$$

$$\leq \hat{\lambda} K_F T_S K_A \|\bar{H} - H\|,$$

with $K_A = \sum_{k=1}^{Q_0} [\beta(Q_0 - k) + \beta(Q_0 - k - 1) + \beta(k)]$ being finite. We are assuming in the previous derivation $\beta(-1) = 0$.

Consider term $|B|:

$$|B| = \left| \sum_{k=1}^{Q_0} \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = Q_0 - k) \sum_{n=k+1}^{\infty} \frac{k}{n} \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = n) \right|$$

$$- \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = Q_0 - k) \sum_{n=k+1}^{\infty} \frac{k}{n} \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = n) \right|$$

$$\leq \sum_{k=1}^{Q_0} \left[ \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = Q_0 - k) \sum_{n=k+1}^{\infty} \frac{k}{n} \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = n) - \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = Q_0 - k) \sum_{n=k+1}^{\infty} \frac{k}{n} \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = n) \right]$$

$$\leq \sum_{k=1}^{Q_0} \left[ \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = Q_0 - k) - \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = Q_0 - k) \right]$$

By the same argument as for term $|A|$, $\sum_{k=1}^{Q_0} [\mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = Q_0 - k) - \mathbb{P} (N(\Lambda_{HR}^{RA}(T_S)) = Q_0 - k)] \leq \hat{\lambda} K_F T_S \|\bar{H} - H\| \sum_{k=1}^{Q_0} [\beta(Q_0 - k) + \beta(Q_0 - k - 1)].$
Consider \( \sum_{k=1}^{Q_0} \sum_{n=k+1}^{\infty} \frac{k}{n} \left( \mathbb{P}(N(\Lambda_{HR}^{RA}(T_S)) = n) - \mathbb{P}(N(\Lambda_{HR}^{RA}(T_S)) = n) \right) \). We have:

\[
\left| \sum_{n=k+1}^{\infty} \frac{k}{n} \left( \mathbb{P}(N(\Lambda_{HR}^{RA}(T_S)) = n) - \mathbb{P}(N(\Lambda_{HR}^{RA}(T_S)) = n) \right) \right| \\
\leq \sum_{n=k+1}^{\infty} \frac{k}{n} \left[ \mathbb{P}(N(\Lambda_{HR}^{RA}(T_S)) \leq n) - \mathbb{P}(N(\Lambda_{HR}^{RA}(T_S)) \leq n) \right] + \left| \mathbb{P}(N(\Lambda_{HR}^{RA}(T_S)) \leq n-1) - \mathbb{P}(N(\Lambda_{HR}^{RA}(T_S)) \leq n-1) \right] \\
\leq k|\Lambda_{HR}^{RA}(T_S) - \Lambda_{HR}^{RA}(T_S)| \sum_{n=k+1}^{\infty} \frac{1}{n} (\beta(n) + \beta(n-1)) \leq k\lambda K_F T_S \|\hat{H} - H\| \sum_{n=k+1}^{\infty} \frac{1}{n} (\beta(n) + \beta(n-1)).
\]

From Stirling’s approximation, we know that \( n! > \sqrt{2\pi n^{n+\frac{1}{2}}} e^{-n} n^{\frac{1}{2}} > \sqrt{2\pi n^{n+\frac{1}{2}}} e^{-n} \) (see Feller (1968), Section 2.9). Hence,

\[
\sum_{n=k+1}^{\infty} \frac{1}{n} (\beta(n) + \beta(n-1)) = \sum_{n=k+1}^{\infty} \frac{1}{n} \left( \frac{e^{-n} n^n}{n!} + \frac{e^{-(n-1)}(n-1)^{(n-1)}}{(n-1)!} \right) \\
\leq \frac{1}{\sqrt{2\pi}} \sum_{n=k+1}^{\infty} \left( \frac{1}{n^{3/2}} + \frac{1}{n^{1/2}} \right) \\
\leq \frac{1}{\sqrt{2\pi}} \sum_{n=k+1}^{\infty} \left( \frac{1}{n^{3/2}} + \frac{1}{(n-1)^{3/2}} \right) \triangleq K_k < \infty. \quad (A10)
\]

Therefore,

\[
|B| \leq \lambda K_F T_S \|\hat{H} - H\| \sum_{k=1}^{Q_0} [\beta(Q_0 - k) + \beta(Q_0 - k - 1) + kK_k].
\]

Finally,

\[
|\mathcal{R}(\mathcal{H}) - \mathcal{R}(\hat{\mathcal{H}})| \leq \frac{(1 - p_h)^2}{p_h} \frac{\exp(wT_S)}{\mathbb{P}(N(\Lambda T_S) \leq Q_{TS} - 1)^2} \lambda K_F T_S \\
\times \left( \beta(Q_0 - 1) + \sum_{k=1}^{Q_0} [2\beta(Q_0 - k) + 2\beta(Q_0 - k - 1) + \beta(k) + kK_k] \right) \|\hat{H} - H\|,
\]

where \( \beta(n) = \mathbb{P}(N(n) = n) \) and \( K_k \) is given by (A10).

From this result, we conclude that \( \mathcal{R} \) is continuous, which together with the fixed-point property of the set \( \mathcal{H} \) guarantees existence of an PE.

**A1.3** Asymptotic analysis of the game

**A1.3.1** FIFO rationing rule
Proof of Theorem 2  Recalling that $\lambda^{(n)}(t) = n\lambda(t)$, we start by defining
\[
\Lambda^{(n)}_{\mu B}(\tau) \equiv \int_0^\tau \lambda^{(n)}(s) F(H(s), s) ds \quad \text{and} \quad \Lambda^{(n)}_{\mu P}(\tau) \equiv \int_0^\tau \lambda^{(n)}(s) F(H(s), s) ds.
\]
For part (i), we follow the argument in Maglaras and Meissner (2006), Section 4.2.2. The functional strong-law-of-large-numbers (FSLLN) for Poisson processes states that if $N_1 \overset{d}{=} N$ is a unit rate Poisson process, then as $n \to \infty$, \[
\frac{N(n\tau)}{n} \to \tau \quad \text{a.s., uniformly in } \tau \in [0, T].
\]
Noting that \[
\Lambda^{(n)}_{\mu B}(\tau) = \int_0^\tau n\lambda(s) F(H(s), s) ds = n\Lambda^{(n)}_{\mu B}(\tau),
\]
and that the counting process of a nonhomogeneous Poisson process with rate $\Lambda^{(n)}$, $N^{(n)}_{\mu B}(\tau)$, is equal in distribution to $N(n\Lambda^{(n)}_{\mu B}(\tau))$, we get
\[
\frac{N^{(n)}_{\mu B}(\tau)}{n} \overset{d}{=} \frac{N(\Lambda^{(n)}_{\mu B}(\tau))}{n} \to \Lambda^{(n)}_{\mu B}(\tau) \quad \text{a.s., uniformly in } \tau \in [0, T].
\]
The result $N(\Lambda^{(n)}_{\mu R}(\tau))/n \to \Lambda^{(h)}(\tau)$ is verified similarly.

From the continuity of the max function, it follows that as $n \to \infty$, and for all $\tau \in [0, T]$, \[
\frac{Q^{(n)}_{\tau}}{n} = \frac{(Q^{(n)}_0 - \Lambda^{(n)}_{\mu B}(\tau))^+}{n} \to Q_{\tau} \overset{d}{=} (Q_0 - \Lambda^{(n)}_{\mu B}(\tau))^+ \quad \text{a.s., uniformly in } \tau \in [0, T].
\]

For part (ii), note that $\mathbb{I}\{N(\Lambda^{(n)}_{\mu B}(\tau)) \leq Q^{(n)}_T - 1\} = \mathbb{I}\{N(\Lambda^{(n)}_{\mu B}(\tau))/n \leq (Q^{(n)}_T - 1)/n\}$. Taking limit as $n \to \infty$, we get that $\mathbb{I}\{N(\Lambda^{(n)}_{\mu R}(\tau)) \leq Q^{(n)}_T - 1\}$ converges a.s. to the constant $\Pi^{\infty}_{HR}(\tau)$, with \[
\Pi^{\infty}_{HR}(\tau) = \begin{cases} 1 & \text{if } \Lambda^{(n)}_{\mu R}(\tau) \leq Q_T \\ 0 & \text{if } \Lambda^{(n)}_{\mu R}(\tau) > Q_T, \end{cases}
\]
where $Q_T \overset{d}{=} (Q_0 - \Lambda^{(n)}_{\mu B}(T))^+$. Thus, from the bounded convergence theorem, as $n \to \infty$, the sequence of probabilities $\mathbb{P}(N(\Lambda^{(n)}_{\mu R}(\tau)) \leq Q^{(n)}_T - 1)$ converges to $\Pi^{\infty}_{HR}(\tau)$, i.e., \[
\Pi^{(n)}_{HR}(\tau) \overset{d}{=} \mathbb{P}(N(\Lambda^{(n)}_{\mu R}(\tau)) \leq Q^{(n)}_T - 1) \to \Pi^{\infty}_{HR}(\tau),
\]
completing the proof. \]

A1.3.2 RA rationing rule

Proof of Theorem 3  The probability of getting an item through a reservation is:
\[
c^{(n)}(H) \overset{d}{=} \min \left\{ \frac{(Q^{(n)}_0 - N(\Lambda^{(n)}_{HR}(T_S)))^+}{N(\Lambda^{(n)}_{HR}(T_S))}, 1 \right\}.
\]

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Dividing by \( n \) the numerator and denominator of the first argument, and using the convergence results in Theorem 2, we get
\[
c^{(n)}(H) \longrightarrow \min \left\{ \frac{(Q_0 - \Lambda_{HR}(T_S))^+}{\Lambda_{HR}^R(T_S)}, 1 \right\}, \quad \text{as} \; n \longrightarrow \infty,
\]
which completes the proof. \( \blacksquare \)

**Proof of Proposition 3** For convenience denote \( k \triangleq c^{\infty}(H) \) and define
\[
G(k) \triangleq k - 1 + \frac{(1 - \rho) \int_0^T \lambda(t)dt}{\Lambda_{HR}^R(T_S, k)}, \quad \text{for} \; 0 \leq k \leq 1,
\]
where
\[
\Lambda_{HR}^R(T_S, k) \triangleq \int_0^{T_S} \lambda(t)F \left( \min \left\{ \frac{p_h \exp(w(T_S - t))}{\exp(w(T_S - t)) - k}, 1 \right\}, t \right) dt. \tag{A11}
\]
Next, we compute the extreme values:
\[
G(0) = -1 + \frac{(1 - \rho) \int_0^{T_S} \lambda(t)dt}{\Lambda_{HR}^R(T_S, 0)} = -1 + \int_0^{T_S} \lambda(t)dt - Q_0 \int_0^{T_S} \lambda(t)F(p_h, t)dt < -1 + \int_0^{T_S} \lambda(t)dt - \int_0^{T_S} \lambda(t)\bar{F}(p_h, t)dt = 0;
\]
\[
G(1) = \frac{(1 - \rho) \int_0^{T_S} \lambda(t)dt}{\Lambda_{HR}^R(T_S, 1)} > 0.
\]
Since \( G(k) \) is continuous in \([0, 1]\), and \( G(0) < 0 < G(1) \), the result follows. \( \blacksquare \)

**Proof of Proposition 4** Taking the definition of \( \Lambda_{HR}^R(T_S, k) \) in (A11), the minimum in the integrand equals to one when \( k \geq \exp(w(T_S - \tau))/p_h \). In particular, \( H^*(\tau) = 1 \) when \( k \geq \exp(wT)(1 - p_h) \). Under this situation, \( \Lambda_{HR}^R(T_S, k) = \int_0^{T_S} \lambda(t)dt \), and equation (17) becomes \( c^{\infty}(H^*) = k = \rho \). \( \blacksquare \)

### A2 Priority rules

We consider time-based priority rules defined over the set \( \Xi \triangleq \{ \xi \in D : [t, T] \rightarrow [t, \infty], \; \text{s.t. finite number of local optima} \} \). A function \( \xi \in \Xi \) gives more priority to an arrival at \( \tau_1 \) than an arrival at \( \tau_2 \) (given that both place a reservation) if \( \xi(\tau_1) > \xi(\tau_2) \). The priority ordering defined here is assumed to be strict, in the sense that the number of customer arrivals that share the same priority is finite. Figure A1 shows three cases of priority rules that satisfy the strict ordering (the plots at the top), and two that do not (the two plots at the bottom).
The three strict priorities in the top are: FIFO (first-in-first-out, or first-come-first-served), LIFO (last-in-first-out), and FLIFO (first and last in, first out; which gives same priorities to earliest and latest arrivals, and less to intermediate time arrivals). Of course, there are priorities that would be easier to sustain in the retail practice because (e.g., FIFO), but our model allows for these more general cases.

Note that these are not the unique functional forms that lead to priorities exhibited there. For example, FIFO could also be represented by a function \( \xi(t) = \frac{(t-T)^2}{T^2}, t \in [0, T] \).

![Image of priority rules](image)

Figure A1: Illustrations of strict (top) and weak (bottom) priority rules.

### A3 Analysis of the market composition effect

In this section, we analyze the case where the market consists of two different types of consumers: strategic and myopic. The former arrival rate \( \lambda(t) \) is now split between a fraction \( \gamma, 0 < \gamma < 1 \), of myopic consumers, and a fraction \( 1 - \gamma \) of strategic consumers. We assume that \( \gamma \) is time invariant and common knowledge.

#### A3.1 Consumers’ behavior

The myopic consumers behave according to the simple strategy “buy now if own valuation is higher than \( p_h \), and reserve otherwise”. The strategic ones choose the buying channel to maximize their expected utility. Both types of consumers participate in the clearing of excess inventory at the end of the selling season. In what follows we analyze the FIFO and RA rationing rules for allocating leftover inventory to reservations.
A3.1.1 FIFO rationing rule

Under FIFO rationing rule, reservations placed by strategic or myopic consumers get priority based on the time stamp of their arrivals. Strategic, forward-looking consumers are aware of the myopic strategy, and internalize it when assessing their own expected utilities for buying now and reserving. The analysis then follows the outline described in Section 4.1, but with the following modification of equation (6) there, now specialized for FIFO and the mixed market composition. The probability that a reservation consumer gets a reserved unit is:

\[ \Pi_H(\tau) = \mathbb{P} \left( B(\Lambda_{HB}^S(T) + \Lambda_{HB}^M(T) + \Lambda_{HR}^S(\tau) + \Lambda_{HR}^M(\tau)) \leq Q_0 - 1 \right), \]  

where

- Mean strategic buy-nows: \( \Lambda_{HB}^S(\tau) = (1 - \gamma) \int_0^\tau \lambda(t) \bar{F}(H(t), t) dt \).
- Mean myopic buy-nows: \( \Lambda_{HB}^M(\tau) = \gamma \int_0^\tau \lambda(t) \bar{F}(p_h, t) dt \).
- Mean strategic reservations: \( \Lambda_{HR}^S(\tau) = (1 - \gamma) \int_0^\tau \lambda(t) F(H(t), t) dt \).
- Mean myopic reservations: \( \Lambda_{HR}^M(\tau) = \gamma \int_0^\tau \lambda(t) F(p_h, t) dt \).

Our previous equilibrium results for strict priorities rules (though with a single stream of consumers) can be adapted to this mixed market case. Using Lemma A1(i), and following the argument in the proof of Lemma 1, it can be verified that for the \( \Pi_H(\tau) \) defined in (A12), \( \left| \frac{d}{d\tau} \Pi_H(\tau) \right| \leq K_H \). Therefore, a result analogous to Theorem 1 holds. Figure A2 shows how the strategy \( H(\tau) \) played by forward-looking consumers changes with the fraction \( \gamma \) of myopic consumers. Clearly, the more myopic consumers, the more strategic consumers will tend to buy-now, since units will be depleting faster through the buy-now channel.

The asymptotic regime that we consider is the same as the one introduced in Section 5. The following result characterizes the asymptotic strategy of forward-looking consumers under the mixed market framework. It can be proved similarly to Theorem 2.

**Theorem A2** Suppose that the purchasing strategy \( H(\tau) \) is given. Then, in the limit as \( n \to \infty \):

(i) The following convergence results hold almost surely (a.s.), and uniformly in \( \tau \):

\[
N(\Lambda_{HB}^{S,n}(\tau))/n \to \Lambda_{HB}^S(\tau), \quad N(\Lambda_{HR}^{M,n}(\tau))/n \to \Lambda_{HR}^M(\tau), \quad Q_T^{(n)}/n \to Q_T \triangleq (Q_0 - \Lambda_{HB}^S(\tau) - \Lambda_{HR}^M(\tau))^+, \\
N(\Lambda_{HB}^{S,n}(\tau))/n \to \Lambda_{HB}^S(\tau), \quad N(\Lambda_{HR}^{M,n}(\tau))/n \to \Lambda_{HR}^M(\tau).
\]

(ii) The probability \( \Pi_H^{(n)}(\tau) \) defined in (A12) converges to the two-point distribution:

\[
\Pi_H^{(n)}(\tau) = \begin{cases} 
1 & \text{if } \Lambda_{HR}^S(\tau) + \Lambda_{HR}^M(\tau) \leq Q_T \\
0 & \text{if } \Lambda_{HR}^S(\tau) + \Lambda_{HR}^M(\tau) > Q_T,
\end{cases}
\]

for \( Q_T \triangleq (Q_0 - \Lambda_{HB}^S(T) - \Lambda_{HR}^M(T))^+ \).
Figure A2: Effect of different proportions $\gamma$ of myopic consumers in the strategy played by forward-looking consumers. Value of parameters: $Q_0 = 9, T = 1, \lambda = 10, w = 1, p_l = 0, p_h = 0.4$, and valuations $\text{Unif}[0,1]$.

Now we are interested in the strategies played by forward-looking consumers in this asymptotic regime. Similarly to Section 5, we need to distinguish three possible cases.

i) **Limited supply.** Suppose that the initial inventory is limited i.e. $Q_0 \leq \int_0^T \lambda(t) \bar{F}(p_h, t) dt$.

Consider a strategic consumer arriving with valuation $v$, and suppose that all other forward-looking consumers choose the strategy $H^*(t) = p_h, \forall t \in [0, T]$. Given that the supply scarcity ensures no leftover inventory at time $T$, the arriving player can follow the strategy $H^*(\tau) = p_h$.

ii) **Intermediate supply.** Suppose that initial supply is intermediate in the sense that $\int_0^T \lambda(t) \bar{F}(p_h, t) dt < Q_0 < \int_0^T \lambda(t) dt$. In this case, $Q_T > 0$ and some strategic consumers with valuation smaller than $p_h$ get units by placing reservations. Under FIFO, the leftover inventory will be allocated to early strategic or myopic arrivals who placed reservations. Therefore, all consumers (myopic and strategic) who arrived before the threshold time $\tau^*$ (defined in (14)) will get an item through one of the channels. After $\tau^*$ only consumers with valuations higher than $p_h$ can get an item, i.e., $H(t) = p_h$ if $t > \tau^*$.

The early arriving strategic consumers, with $\tau \in [0, \tau^*)$, must decide which channel to purchase from, and therefore need to solve the limiting version of equation (7):

$$\frac{v_\tau}{v_\tau - p_h} \geq \frac{\exp[w(T - \tau)]\mathbb{P}(Q_\tau > 0|H)}{\Pi_H^\infty(\tau)}.$$

Under intermediate supply and asymptotic regime, $\mathbb{P}(Q_\tau > 0|H) = 1$, for all $\tau$. Thus, it can be shown that $\Pi_H^\infty(\tau) = 1$ for all $\tau < \tau^*$, or equivalently, $\Lambda_H^S(\tau) + \Lambda_H^S(T) + \Lambda_H^M(\tau) + \Lambda_H^M(T) < Q_0$, where

$$\Lambda_H^S(\tau) = \int_{\tau}^T \lambda(s) \bar{F}(p_h, s) ds,$$

$$\Lambda_H^S(T) = \int_{T}^\infty \lambda(s) \bar{F}(p_h, s) ds,$$

$$\Lambda_H^M(\tau) = \int_{\tau}^T \lambda(s) \bar{F}(p_m, s) ds,$$

$$\Lambda_H^M(T) = \int_{T}^\infty \lambda(s) \bar{F}(p_m, s) ds.$$
for all $\tau < \tau^*$. Indeed,

$$
\Lambda^S_{HR}(\tau) + \Lambda^S_{HB}(T) + \Lambda^M_{HR}(\tau) + \Lambda^M_{HB}(T) < \\
\Lambda^S_{HR}(\tau^*) + \Lambda^S_{HB}(T) + \Lambda^M_{HR}(\tau^*) + \Lambda^M_{HB}(T)
$$

$$
= (1 - \gamma) \int_0^{\tau^*} \lambda(t) F(H(t), t) dt + (1 - \gamma) \int_{\tau^*}^T \lambda(t) F(H(t), t) dt + \gamma \int_0^{\tau^*} \lambda(t) F(p_h, t) dt + \gamma \int_{\tau^*}^T \lambda(t) F(p_h, t) dt
$$

$$
= (1 - \gamma) \int_0^{\tau^*} \lambda(t) dt + (1 - \gamma) \int_{\tau^*}^T \lambda(t) F(H(t), t) dt + \gamma \int_0^{\tau^*} \lambda(t) dt + \gamma \int_{\tau^*}^T \lambda(t) F(p_h, t) dt
$$

$$
= \int_0^{\tau^*} \lambda(t) dt + \int_{\tau^*}^T \lambda(t) F(p_h, t) dt = Q_0.
$$

We conclude that in this intermediate case the unique PE $H^*(\tau)$ is given by:

$$
H^*(\tau) = \begin{cases} 
\min \left\{ \frac{p_h \exp(w(T - \tau))}{\exp(w(T - \tau)) - 1}, 1 \right\} & \text{if } \tau \in [0, \tau^*) \\
p_h & \text{if } \tau \in [\tau^*, T].
\end{cases}
$$

iii) Abundant supply. Suppose that the initial supply is abundant, i.e., $Q_0 \geq \int_0^T \lambda(t) dt$. In this case all reservations will be satisfied with probability one, hence, every consumer will get an item from the channel he chooses. The unique optimal strategy for strategic consumers is given by

$$
H^*(\tau) = \min \left\{ \frac{p_h \exp(w(T - \tau))}{\exp(w(T - \tau)) - 1}, 1 \right\}.
$$

(A13)

Overall, even though the exact stochastic strategy depends on the fraction $\gamma$ of myopic consumers, the strategy played by forward-looking consumers in the mixed market environment is the same as that played under FIFO in the homogeneous market consisting only of strategic consumers. This somewhat surprising result is anchored in the following observations:

1. Consumers can asses the time of the last marginal arrival who will get a unit (i.e., the value of $\tau^*$ defined in (14)). This value does not depend on $\gamma$; just on the total arrival rate $\lambda$.

2. For $\tau \in [\tau^*, T]$, strategic consumers play the strategy $H(\tau) = p_h$, which is the same as the one played by the myopic. Hence, after $\tau^*$, strategies of all consumers are the same in both homogeneous and mixed market cases.

3. Before $\tau^*$, strategic consumers know that they can get a unit w.p.1 in both mixed and homogeneous markets and so they play the same strategy in both cases, whereas under the stochastic case the probability of getting an item through reservations is affected by the proportion $\gamma$ of myopic consumers.

In other words, in the context of an asymptotic mixed market, strategic consumers can ignore the fraction $\gamma$ of myopic consumers and are still able to compute an (optimal) equilibrium strategy. Figure A3 (left) illustrates the quality of the asymptotic approximation for FIFO over an intermediate supply case.
A3.1.2 RA rationing rule.

Under RA rationing rule all reservations placed by strategic or myopic consumers have the same priority and will be allocated at random. Therefore the probability of getting an item through a reservation is

\[ c(H) \triangleq \min \left\{ \frac{Q_{Ts}}{\# \text{ of consumers that reserved an item}}, 1 \right\}, \]

where \( Q_{Ts} = (Q_0 - N(S^{S,RA}(T_s) + M^{M,RA}(T_s)))^+ \) and the number of placed reservations is \( N(S^{S,RA}(T_s) + M^{M,RA}(T_s)) \). The mean values involved are

- Mean strategic buy-nows: \( \Lambda^{S,RA}_{H_H}(\tau) = (1 - \gamma) \int_0^\tau \lambda(t) \bar{F}(H(t), t) \, dt \).
- Mean myopic buy-nows: \( \Lambda^{M,RA}_{H_H}(\tau) = \gamma \int_0^\tau \lambda(t) \bar{F}(p_h, t) \, dt \).
- Mean strategic reservations: \( \Lambda^{S,RA}_{H_H}(\tau) = (1 - \gamma) \int_0^\tau \lambda(t) F(H(t), t) \, dt \).
- Mean myopic reservations: \( \Lambda^{M,RA}_{H_H}(\tau) = \gamma \int_0^\tau \lambda(t) F(p_h, t) \, dt \).

The probability \( c(H) \) is constant in \( \tau \), and hence differentiable. The existence of equilibrium can be verified following the guidelines of Section 4.2.

Regarding the asymptotic regime, a result analogous to Theorem 3 can be proved, and a limiting probability \( c^\infty(H) \) of getting a unit through the reservation channel (analogous to (16)) can be established. Consider the following three supply cases:

i) **Abundant supply** (i.e., \( Q_0 \geq \int_0^{T_S} \lambda(t) \, dt \)): Here \( c^\infty(H) = 1, \) and the optimal strategy is given by (15).

ii) **Intermediate supply** (i.e., \( \int_0^{T_S} \lambda(t) \bar{F}(p_h, t) \, dt < Q_0 < \int_0^{T_S} \lambda(t) \, dt \)): We can rewrite (16) as

\[ c^\infty(H) = 1 - \frac{(1 - \rho) \int_0^{T_S} \lambda(t) \, dt}{\Lambda^{S,RA}_{H_H}(T_s) + \Lambda^{M,RA}_{H_H}(T_s)}, \]  \hspace{1cm} (A14)

where \( 0 < c^\infty(H) < 1 \). The condition for reservation for strategic consumers is given by (11), where now \( P(Q_\tau > 0 | H) = 1 \) for all \( \tau \in [0, T_S] \). Therefore, the optimal purchasing strategy is defined by (18). By substituting (18) into (A14), we get the following fixed-point equation for \( c^\infty(H) \):

\[ c^\infty(H^*) = 1 - \frac{(1 - \rho) \int_0^{T_S} \lambda(t) \, dt}{(1 - \gamma) \int_0^{T_S} \lambda(t) F\left( \min \left\{ \frac{p_h \exp(w(T_s - t))}{\exp(w(T_s - t) - c^\infty(H^*)), 1} \right\}, t \right) \, dt + \gamma \int_0^{T_S} \lambda(t) F(p_h, t) \, dt}. \]  \hspace{1cm} (A15)

Under the intermediate supply case there is always at least one solution to (A15). In case of multiple solutions, it can be shown that the one with the highest \( c^\infty(H) \) is Pareto dominant.
iii) **Limited supply** (i.e., \( Q_0 \leq \int_0^{T_S} \lambda(t) \bar{F}(p_h, t)dt \)): Multiple equilibria are also possible in this case. The purchasing strategy \( H^*(\tau) = p_h \) for all \( \tau \in [0, T_S] \), and more generally, any strategy given by (13) is an equilibrium. The probability of getting an item through a reservation is \( c^\infty(H) = 0 \) in this case.

In addition, there could be another type of equilibria in which consumers can get an item through a reservation with \( c^\infty(H) > 0 \). In this scenario, an equilibrium is given by (18), where \( c^\infty(H) \) is defined by (A15).

Unlike the FIFO case, the introduction of myopic customers does affect the limiting strategy played by strategic consumers under the RA rule. This is due to the fact that the limiting probability of getting an item is not zero or one, in general. Figure A3 (right) illustrates the quality of the asymptotic approximation for RA over an intermediate supply case.

Figure A3: Performance of the asymptotic approximation for the case \( \gamma = 0.5 \) under FIFO (left) and RA (right). Value of parameters: \( Q_0 = 8, T = 1, \lambda = 10, w = 1, p_l = 0, p_h = 0.5 \), and time homogeneous valuations Unif[0,1].

### A3.2 Seller’s revenue optimization problem

For the clarity and fair comparison of performance, we assume time homogeneous valuations in this revenue optimization study. To simplify notation for the mix of strategic and myopic consumers, define \( h(\tau) \triangleq (1 - \gamma) \int_0^\tau F(H(t))dt + \gamma \tau F(p_h) \) and \( \bar{h}(\tau) \triangleq \tau - h(\tau) \). Then, after re-scaling the parameters to their original values, the revenue optimization problem is formulated as follows:

\[
V^\gamma(\bar{Q}) = \max_{T, Q_0, p_l, p_h} \left\{ p_h \lambda \int_0^T e^{-\alpha t} 1\{Q_t > 0\} ((1 - \gamma) \bar{F}(H(t)) + \gamma \bar{F}(p_h))dt \\
+ p_l e^{-\alpha T} \min\{(Q_0 - \lambda \bar{h}(T))^+, \lambda h(T)\}, \quad \text{subject to } p_l \leq p_h, Q_0 \leq \bar{Q} \right\}.
\]
The benchmark case problem under the mixed market framework becomes:

\[
V_{RA}^\gamma(Q) = \max_{T, Q_0, p_l, p_h} \left\{ p_h \lambda \int_0^{T_S} e^{-\alpha t} \mathbb{1}\{Q_t > 0\} \left( (1 - \gamma) F(H(t)) + \gamma F(p_h) \right) dt \\
+ p_l e^{-\alpha T_S} \min\{(Q_0 - \lambda h(T_S))^+, \lambda h(T_S)\} + V_C, \quad \text{subject to } p_l \leq p_h, Q_0 \leq \bar{Q} \right\},
\]

where \( V_C \) is given by (22).

References


