Bid-Price Controls for Network Revenue Management: Martingale Characterization of Optimal Bid Prices

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Abstract

We consider a continuous-time, rate-based model of network revenue management. Under mild assumptions, we construct a simple ε-optimal bid-price control, which can be viewed as a perturbation of a bid-price control in the classical sense, cf. Williamson (1992). We show that the associated bid-price process forms a martingale and the corresponding booking controls converge in an appropriate sense to an optimal control as $\varepsilon \searrow 0$. Moreover, we show that there exists an optimal generalized bid-price control, where the bid-price process forms a martingale and is used in conjunction with a capacity usage limit process. We also discuss its connection to the bid-price controls in the classical sense and sufficient conditions for the (near) optimality of the latter.

Key Words: Bid-Price Controls, Network Revenue Management, Martingales.

1 Introduction

Network revenue management problems arise naturally in airline, railway, cruise-line and hotel revenue management, more generally, whenever, customers buy bundles of resources under various terms and conditions. In such settings, bid-price controls represent an intuitively appealing and powerful approach to quantity-based revenue management. Given a network of resources, a bid-price control assigns a threshold price, that is, a bid price, for each resource dynamically over time. Then, the decision to fulfill a booking request is made based on the availability of various resources and whether or not the revenue associated with the request exceeds the sum of the bid-prices of the resources it uses. Bid-price controls simplify decision making in network revenue management by reducing the number of parameters required for implementation (one bid price is specified for each resource) since evaluating a booking request requires only a simple comparison of the fare to the sum of the bid prices for the requested resources.

Bid-price controls were introduced by Simpson (1989) and further analyzed by Williamson (1992). Williamson computes the bid prices of various resources by means of mathematical programming formulations and interprets the bid-price of a resource as the opportunity cost of using one additional unit of the resource. This reflects the intuitive notion that a booking request should be accepted only if its fare exceeds the opportunity cost of the reduction in resource capacities required to satisfy that request. Thus, bid prices retrieved from a revenue management system may facilitate decision making in other areas of management such as capacity planning or pricing.

Talluri & van Ryzin (1998) considers bid price controls in a discrete time model of network revenue management, where the discretization is fine enough such that in each period at most one request arrives. In this context, a bid-price control is implemented by specifying one bid price for each resource (leg) for each time period and capacity vector, and the request is accepted if the fare of the request is higher than the sum of the bid prices it uses. The authors show that the optimal policy need not correspond to a bid-price control, and provide a two-period counter example which shows that the bid-price policy as defined immediately above may not result in optimal accept/deny decisions. The insight they provide for why bid-price controls may not be optimal is that the bid price for a resource may not correspond to the opportunity cost of using one additional unit of that resource due to two reasons: First, selling one unit of capacity might be a large change in the capacity of several resources at the same time if the remaining capacity is low and hence the interpretation of the bid prices as the marginal value of one unit of additional capacity may not be correct. Second, the revenues may depend on the remaining capacity in a nonlinear way.

Despite this counter example, bid-price controls are widely used in practice, cf. Phillips (2005), as they provide a simple, yet powerful approach to quantity-based network revenue management. In this vein, researchers have worked on various practical heuristic methods to derive bid-price controls. Bertsimas & Popescu (2003) proposed a new method based on approximate dynamic programming. Their method computes adaptive and nonadditive bid prices based on a linear programming approximation to the value function of a dynamic programming formulation. The authors provide a comparison of their method and the bid-price control based on deterministic linear programming approach and show that their algorithm results in higher revenues and more robust performance.

Topaloglu (2007*b*) revisits the network revenue management problem studied in Talluri & van Ryzin (1998) and proposes a new method to compute bid prices. Topaloglu explicitly considers the temporal dynamics of the customer arrivals and generates bid prices that depend on the remaining leg capacities. His method is based on relaxing certain capacity constraints that link decisions for different flight legs by associating Lagrange multipliers with them. Then the problem is decomposed by flight legs and one can concentrate on one flight leg at a time, which simplifies the problem tremendously. Topaloglu also shows through a numerical study that his method outperforms the standard heuristics significantly.

Topaloglu & Kunnumkal (2007) follows a similar approach but uses a different relaxation of the capacity constraints which yields time-dependent prices. Their approach provides an upper bound on the optimal objective value of the problem, which is tighter than the one obtained from the so-called deterministic linear program. The authors also show that the bid prices they propose are asymptotically optimal as leg capacities and demand grow proportionally to infinity. Moreover, they discuss how to adapt their method to incorporate cancellations. Finally, the authors demonstrate through numerical examples that their method can improve on the existing methods. Another related paper is Adelman (2007). Adelman considers the dynamic programming formulation of the network revenue management problem. Then assuming an affine functional form for the value function and using the linear programming representation of the dynamic programming formulation, the author computes time-dependent (deterministic) bid prices. He also shows that his approach yields an upper bound tighter than the one obtained from the deterministic linear program. Both Adelman (2007) and Topaloglu & Kunnumkal (2007) observe that their (approximate) method yields (deterministic) bid prices which are decreasing over time.

Another paper related to ours is Kleywegt (2001), where the author considers a stylized (deterministic) fluid model of a general dynamic pricing problem for selling a network of resources. In Kleywegt's model prices are chosen dynamically to sell products (or itineraries) to multiple customer classes over time. Kleywegt's model is very general in terms of problem primitives and allows order cancellations. Moreover, Kleywegt observes that his model readily extends to incorporate probabilistic customer choice behavior. The author also develops a solution method and tests it with some numerical examples. Among other things, Kleywegt shows through an exact analysis that in his setting the opportunity cost of capacity under an optimal policy remains constant, which is in line with the martingale property of optimal bid prices in our setting. Indeed, looking more carefully at the numerical examples of Adelman (2007) and Topaloglu & Kunnumkal (2007) reveals that the bid prices seem to be constant except toward the end of planning horizon, which may be due to their approximate mode of analysis.

Topaloglu (2007*a*) presents a stochastic approximation method to compute bid prices in network revenue management problems by viewing the total expected revenue as a function of bid prices and using sample path derivatives to identify a good set of bid prices. The author demonstrates through numerical examples that the bid prices obtained by his method outperform the ones by the standard methods especially when bid prices are not computed frequently. Farias (2007) applies approximate dynamic programming ideas to revenue management problems. Reiman & Wang (2007) develops a novel diffusion approximation to the network revenue management and advances a policy which is asymptotically optimal under the so-called diffusion scaling.

An important antecedent of this paper Akan & Ata (2008) studies a discrete-time network revenue management model, where a system manager observes the evolution of information continuously, and exerts control at the end of each period. The authors identify a class of adapted bid-price controls, where a bid-price control involves a bid-price process and a capacity usage limit process. Akan & Ata (2008) shows that there *does* exist an optimal policy within that class. Moreover, optimal (adapted) bid prices form a martingale. These results are proved without making *any* assumptions on the stochastic structure of demand, allowing non-stationary demand with an arbitrary dependence structure, including both inter-temporal and cross-product dependencies. Akan & Ata (2008) also proposes a predictable bid-price control where bid prices used in the current period are last updated in the previous period. The authors develop an upper bound on the optimality gap for this predictable bid-price control in terms of the (quadratic) variations of the demand and bid-price processes, and the updating frequency of bid prices, which quantifies the relationship between the updating frequency of the bid prices and their performance, hence, provides insights on how to optimally choose the updating frequency of bid prices. Moreover, the authors consider an asymptotic regime where periods get small, while the planning horizon and the underlying probabilistic primitives remain unchanged. The novel feature of this asymptotic regime is that it preserves the stochastic nature of demand unlike most asymptotic regimes considered previously in the revenue management literature. Then using the upper bound they advanced, the authors show that the optimality gap vanishes in two important special cases where either the incremental information or the incremental demand is small over short time intervals.

In this paper, we analyze a continuous-time, rate-based model of network revenue management. Our main contribution is to prove ε-optimality of a simple bid-price control. The proposed bid-price control only uses bid-prices associated with various resources, hence, it is easy to implement. In what follows, we also construct an optimal generalized bid-price control which consists of a bid-price process and a capacity usage limit process, where the bid-price process forms a martingale. Although the generalized bid-price control we introduce here resembles the bid-price control of Akan & Ata (2008), it yields new insights in this setting.

The martingale property of (optimal) bid prices was first established in Akan & Ata (2008) for a discrete-time model. We provide further insights and implications of the martingale property of the (near) optimal bid prices, which become more transparent in our continuous-time model. Although the martingale property is primarily a theoretical contribution, it has surprising implications, which are explored in Akan (2008). For instance, exploiting the martingale property one can connect the optimal bid prices to Forward-Backward Stochastic Differential Equations (FBSDE). Our paper paves the way for this novel connection by studying the network revenue management problem through a continuous-time stochastic fluid model. Given the readily available numerical methods for computing solutions FBSDEs, one may borrow that machinery to compute bid prices. Thus, this connection sets the stage up for a novel and analytically sound computational approach and is explored in Akan (2008).

From a methodological perspective, our analysis builds on the convex analysis framework of Rockafellar & Wets (1997) and the duality results of Bismut (1973) and illustrates the utility of stochastic duality techniques and their applicability in the revenue management context. Bismut (1973) develops a new approach to problems of stochastic optimal control using convex duality. In particular, Bismut (1973) defines the dual problems in stochastic optimal control and the coextremality conditions associated with the dual optima by applying general methods of convex analysis introduced by Rockafellar (1968), Rockafellar (1969), Rockafellar (1970*a*) and Rockafellar (1970*b*). Bismut also provides results on the existence of optimal solutions for a general class of convex stochastic control problems, which include the stochastic control problems studied in this paper.

The rest of the paper is structured as follows: Section 2 presents the model. Precise definition of several bid-price controls are introduced in Section 3. A dual formulation to the network revenue management problem and the associated coextremality conditions are provided in Section 4. An optimal generalized bid-price mechanism is defined in Section 5. In Section 6, we discuss a perturbed network revenue management and its dual, based on which we also define an ε-optimal bid-price control. In Section 7, some concluding remarks are provided along with future research directions. The proofs, derivations and auxiliary results are relegated to Appendices A through D throughout the paper.

2 The Model

We analyze a continuous-time, rate-based model of network revenue management. There are K resources and J products. In an airline setting a resource is a flight leg and a product is a specific itinerary. A primitive of our model is a $K \times J$ non-negative capacity consumption matrix A, where A_{kj} denotes the amount of resource k capacity consumed by one unit of product j. The j^{th} column of A is denoted by A^j. The definition of a product contains all terms and conditions associated with the purchase. Thus, there may be more than one product that use the same amount of each resource but differ in price, purchase restrictions etc. Therefore, in practice, the number of products will be large compared to the number of resources. In our model, at each point in time the system manager observes the demand rate and chooses the corresponding booking rate for each product. The booking rates for the products translate into consumption rates for the resources through the capacity consumption matrix A. The objective is to maximize expected revenues over the time horizon $[0, T]$ subject to capacity and demand constraints.

Uncertainty is modeled by a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the evolution of information is modeled through the increasing collection $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$ of complete sub-σ-fields of $\mathcal{F}$. In particular, $\mathcal{F}_t$ represents the information available to the system manager at time t. All stochastic processes to appear will be adapted to the filtration $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$. We assume that $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$ is right-continuous and has no time discontinuity as in Bismut (1973). An information structure that has no time discontinuity is also referred to as a quasi-continuous information structure in Huang (1985), which also proves that the natural filtrations of most of the commonly encountered processes are quasi-continuous, including the natural filtrations generated by the Poisson process and Brownian motion. As a matter of fact, Bismut (1978) extends the framework and results in Bismut (1973) to the more general setting of the control of semi-martingales where the quasi-continuity assumption is also dropped.

The demand for the various products is generated by the J-dimensional demand rate process $\{d(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$. In particular, $d_j(\omega, t)$ is the rate at which demand for product j arrives at the system at time t along the sample path ω. Then, the cumulative demand observed by the system manager for product j over the interval $[t_1, t_2]$ is

$$\int_{t_1}^{t_2} d_j(\omega, s)ds,$$

if sample path $\omega \in \Omega$ is realized. The following are the only two assumptions we make on the demand rate process: We assume that the demand rate process $\{d(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ is bounded and adapted to the filtration $\{\mathcal{F}_t : 0 \leq t \leq T\}$. In particular, we allow for non-stationary demand with an arbitrary dependence structure, including both inter-temporal and cross-product dependencies.

As the system evolves, the system manager exerts control on the system by selecting a nonnegative vector of booking rates at each point in time. That is, for each $\omega \in \Omega$ and $0 \leq t \leq T$, the system manager chooses the J-dimensional vector of booking rates, denoted by $u(\omega, t)$. In particular, $u_j(\omega, t)$ denotes the booking rate for product j at time t along the sample path ω for $j = 1, \ldots, J$. Then, under the control $\{u(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$, the cumulative bookings for product j up to time t along sample path ω is given by

$$U_j(\omega, t) = \int_0^t u_j(\omega, s) ds. \tag{1}$$

The system state at time $t \in [0, T]$ for the realization ω is the K-dimensional vector of remaining capacities denoted by $x(\omega, t)$. The component $x_k(\omega, t)$ denotes the remaining capacity for resource $k = 1, \ldots, K$ at time t. Given a control $\{u(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$, the system state evolves according to the following system dynamics equation

$$x(\omega, t) = C - AU(\omega, t) \text{ for } (\omega, t) \in \Omega \times [0, T], \tag{2}$$

where C is the initial capacity vector and $U(\omega, t)$ is the vector of cumulative bookings up to time t whose j^{th} component is given by (1). We use the shorthand notation x to denote the stochastic process $\{x(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$. Similarly, u denotes the booking rate process $\{u(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$.

A booking rate process u is feasible only if it satisfies demand and capacity restrictions. The demand restriction on bookings is that for each $(\omega, t) \in \Omega \times [0, T]$, the booking rate for each product should be less than or equal to the demand rate for that product. The capacity restriction on bookings is that the remaining capacity for each resource at the terminal time T should be nonnegative almost surely.

A booking rate vector $u(\omega, t)$ at time t given a sample path ω results in an instantaneous revenue rate of $f(\omega, t) \cdot u(\omega, t)$, where $f(\omega, t)$ is the exogenously set vector of fares. The process of fares $\{f(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ is bounded, non-negative and adapted to the information structure $\{\mathcal{F}_t,\ t \in \mathbb{R}_+\}$. The fare process can be non-stationary and arbitrarily correlated with the demand process, which, in turn, allows us to model dependencies between the fare and demand process and potentially capture demand substitution across products and over time.

The objective is to choose a booking rate process u so as to maximize expected revenue subject to the demand and capacity restrictions. That is, choose booking rate vector $u(\omega, t)$ for each $(\omega, t) \in \Omega \times [0, T]$ so as to

$$
\begin{aligned}
&\text{maximize} \;\; \mathbb{E}\left[\int_0^T f(\omega,t)\cdot u(\omega,t)\,dt\right] \\
&\text{subject to} \\
&x(\omega,t) = C - AU(\omega,t), \quad (\omega,t)\in\Omega\times[0,T], \\
&U(\omega,t) = \int_0^t u(\omega,s)ds, \quad (\omega,t)\in\Omega\times[0,T], \\
&0 \le u(\omega,t) \le d(\omega,t), \quad (\omega,t)\in\Omega\times[0,T], \\
&x(\omega,T) \ge 0, \quad \omega\in\Omega,
\end{aligned} \tag{P}
$$

where the first and second constraints describe how capacity evolves over time, and the third and fourth constraints are the demand and capacity restrictions, respectively. Throughout the rest of the paper we will refer to the formulation (P) as the network revenue management problem.

As an aside, since the fare process can be an arbitrary adapted stochastic process, the formulation (P) of the network revenue management problem subsumes possible discounted formulations.

3 Bid-Price Control Definitions and Summary of Results

In our setting, a bid-price process is a K-dimensional, non-negative stochastic process $\pi = \{\pi(\omega,t) : (\omega,t)\in\Omega\times[0,T]\}$, where $\pi_k(\omega,t)$ denotes the bid-price, or the shadow price, associated with resource k at time t along the sample path ω. Next, we introduce three closely related definitions of bid-price controls, which will be used in subsequent sections.

Definition 1 *(Bid-Price Control): Given a bid-price process $\pi = \{\pi(\omega,t) : (\omega,t)\in\Omega\times[0,T]\}$ and a booking function $\phi : \mathbb{R}^3_+ \to \mathbb{R}_+$, the pair (π,ϕ) is called a bid-price control, where the corresponding booking rates for each product $j = 1,\ldots,J$ are determined as follows:*

$$u_j(\omega,t) = \phi(\pi(\omega,t)A^j,\; f_j(\omega,t),\; d_j(\omega,t)) \;\; \textit{for } (\omega,t)\in\Omega\times[0,T]. \tag{3}$$

Given a bid-price control (π,ϕ), for each product j and each (ω,t), the booking function ϕ compares the fare $f_j(\omega,t)$ with the sum of the bid prices for the resources used by product j, $\pi(\omega,t)A^j$, and dictates how much of the demand rate $d_j(\omega,t)$ to book. Our definition allows for non-linear booking functions. In particular, it may not result in a "bang-bang" booking process; u is called a bang-bang booking process if $u_j(\omega,t)\in\{0,d_j(\omega,t)\}$ for almost all $(\omega,t)\in\Omega\times[0,T]$ and $j=1,\ldots,J$. In this sense, our definition of a bid-price control is more generous than the classical definition; and the latter can be viewed as a special case of ours. In particular, the booking decisions under a bid-price control in the classical sense necessarily result in a bang-bang booking process. The next definition introduces the bid-price control in the classical sense in our setting.

Definition 2 *(Classical Bid-Price Control) For each bid-price process π, define the corresponding classical bid-price control, denoted also by π, such that it dictates the following booking rates for each product $j = 1, \dots, J$ and $(\omega, t) \in \Omega \times [0, T]$:*

$$u_j(\omega, t) = \begin{cases} d_j(\omega, t) & \text{if } f_j(\omega, t) \geq \pi(\omega, t)A^j, \\ 0 & \text{otherwise.} \end{cases}$$

One can view a classical bid-price control π as a specific bid-price control (π, ϕ) where $\phi(z) = z_3 \mathbf{1}_{\{z_2 \geq z_1\}}$ for $z \in \mathbb{R}^3_+$, from which it follows that

$$\phi(\pi(\omega, t)A^j,\ f_j(\omega, t),\ d_j(\omega, t)) = \begin{cases} d_j(\omega, t) & \text{if } f_j(\omega, t) \geq \pi(\omega, t)A^j, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

A bid-price control (π, ϕ) is called optimal if the booking rates resulting from (π, ϕ), cf. (3), constitute an optimal solution to the network revenue management problem (P). Given $\varepsilon > 0$, a bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$ is called ε-optimal if the revenue associated with the resulting booking process u^ε is within ε of the optimal objective value of the network revenue management problem (P).

An important virtue of bid-price controls is that they offer a tractable solution for a complex problem of allocating a network of resources to a large number of products. Bid-price controls simplify the control in network revenue management by reducing the number of parameters required for implementation (one bid price is specified for each resource.) In addition, a bid-price control decomposes the problem across time, sample paths and products. That is, given a bid-price control (π, ϕ), at each point in time and for every sample path, the booking rates are determined only as a function of the current bid-prices, fares and demand rates without having to account for the future impact of current decisions. Moreover, the booking decisions for each product can be made in isolation, independently of the booking decisions for other products.

Next, we consider a simple example to illustrate the classical bid-price controls. This example is indeed the continous-time, rate-based version of the counterexample provided by Talluri & van Ryzin (1998). Surprisingly, a classical bid-price control is optimal in our continuous time setup, leading to new insights.

Example 1. There are two resources and three products with the associated capacity consumption matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The planning horizon is $[0, 2]$. Each resource has initial capacity of one, that is, $C = (1, 1)'$. There is no uncertainty or non-stationarity in product fares. In particular, the vector of product fares is given by

$$f = (250, 250, 500)'.$$

The only uncertainty is in the demand rate process. The evolution of uncertainty is suitably represented by an information tree in Figure 1. The terminal nodes of the tree correspond to specific sample paths. The intermediate set of nodes represent the resolved uncertainty by time $t = 1$. On each arc of the information tree displayed is the corresponding demand rate vector. There are six sample paths and the probability of each sample path is also displayed in Figure 1, from which one can deduce the probabilities of various events. In particular, during $[0, 1]$ we will see demand for only one type of product (at rate 1). The probability of having demand for product 3 is 0.4; for product 1 it is 0.3 and for product 2 it is 0.3. On the other hand, during $[1, 2]$, we see either demand for product 3 (at rate 1) with probability 0.8 or no demand with probability 0.2. To be more specific, the demand rate process displayed in Figure 1 is given as follows:

$$
d(\omega_1, t) = (1,1)' \text{ for } t \le 2 \qquad \text{and} \qquad d(\omega_2, t) = \begin{cases} (1,1)', & t \le 1, \\ (0,0)', & t > 1, \end{cases}
$$

$$
d(\omega_3, t) = \begin{cases} (1,0)', & t \le 1, \\ (1,1)', & t > 1, \end{cases} \qquad \text{and} \qquad d(\omega_4, t) = \begin{cases} (1,0)', & t \le 1, \\ (0,0)', & t > 1, \end{cases}
$$

$$
d(\omega_5, t) = \begin{cases} (0,1)', & t \le 1, \\ (1,1)', & t > 1, \end{cases} \qquad \text{and} \qquad d(\omega_6, t) = \begin{cases} (0,1)', & t \le 1, \\ (0,0)', & t > 1. \end{cases}
$$

It is easy to see that the optimal solution is to book only product 3, while denying all other requests. This results in the expected revenue of 440. Formally, the solution is given as follows:

$$
u(\omega_1, t) = u(\omega_2, t) = \begin{cases} (0,0,1)', & t \le 1, \\ (0,0,0)', & t > 1, \end{cases} \tag{5}
$$

$$
u(\omega_3, t) = u(\omega_5, t) = \begin{cases} (0,0,0)', & t \le 1, \\ (0,0,1)', & t > 1, \end{cases} \tag{6}
$$

$$
u(\omega_4, t) = u(\omega_6, t) = \begin{cases} (0,0,0)', & t \le 1, \\ (0,0,0)', & t > 1. \end{cases} \tag{7}
$$

As an aside, this solution corresponds to the optimal solution of the Talluri-van Ryzin example, which also books only product 3 and results in expected revenue of 440.

Consider the following classical bid-price control: $\pi(\omega_1, t) = \pi(\omega_2, t) = (250, 250)'$ for all t and

$$
\pi(\omega_3, t) = \begin{cases} (251, 150)', & t \le 1, \\ (312.5, 187.5)', & t > 1, \end{cases} \qquad \text{and} \qquad \pi(\omega_4, t) = \begin{cases} (251, 150)', & t \le 1, \\ (5, 0)', & t > 1, \end{cases}
$$

$$
\pi(\omega_5, t) = \begin{cases} (150, 251)', & t \le 1, \\ (187.5, 312.5)', & t > 1, \end{cases} \qquad \text{and} \qquad \pi(\omega_6, t) = \begin{cases} (150, 251)', & t \le 1, \\ (0, 5)', & t > 1. \end{cases}
$$

Observe that the classical bid-price control π results in the optimal bookings given in (5)-(7), and hence yields expected revenue of 440. It is also easy to check that the bid-price process π forms a martingale. Also note that the choice of the optimal bid-price control is not unique. One can easily come up with other optimal classical bid prices.

The bid-price control given immediately above shows that a classical bid-price control is indeed optimal for this particular example. Thus, it sheds light onto reasons for non-optimality in the Talluri-van Ryzin

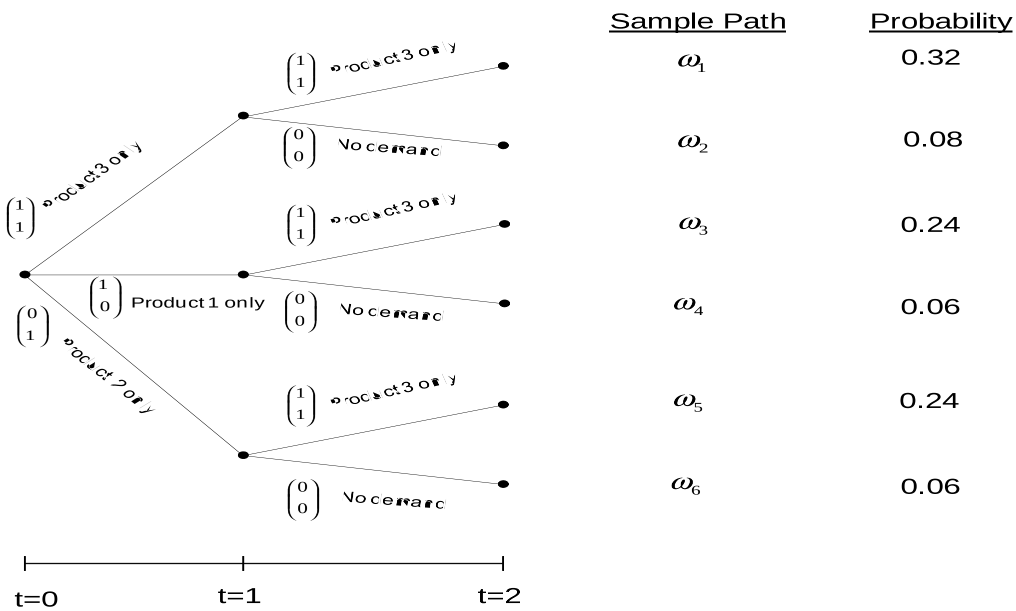

Figure 1: Evolution of uncertainty in Example 1.

example. As pointed out earlier in the literature, the non-optimality of classical bid-price controls stems from discreteness, and, in particular, from the fact that each booking consumes a large fraction of remaining capacity. In contrast, in our setting the bookings at each point in time consumes only an infinitesimal amount of capacity. We also allow frequent (indeed continuous) updating of bid prices. These allow the bid-price controls to perform optimally. Therefore, in addition to discreteness of the problem, infrequent updating of bid prices may be another reason for non-optimality of classical bid-price controls.

Although the classical bid-price controls are optimal for the example immediately above, we next present an example where no classical bid price control can be optimal. Indeed, we show a stronger result that no bang-bang control can be optimal.

Example 2. We have a single resource and two products. As in the earlier example the fares are constant. In particular, we have

$$C = 1, \quad A = [1, 1] \quad \text{and} \quad f = (100, 200)'.$$

The planning horizon is $[0, 1]$. The underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is defined as follows: $\Omega = [0, 1]$ and $\mathcal{F}$ is the collection of Borel subsets of $[0, 1]$ (suitably completed). Let τ be a random variable defined on $(\Omega, \mathcal{F})$ with strictly positive density g on $[0, 1]$. Then $\mathbb{P}$ is the probability measure induced by g; and each sample path can be thought of as a particular realization of τ. Demand for the two products is given as follows:

$$d_1(t) = \begin{cases} 2 & \text{if } t \leq \tau, \\ 0 & \text{if } \ t > \tau, \end{cases} \quad \text{and} \quad d_2(t) = \begin{cases} 0 & \text{if } t \leq \tau, \\ 1 & \text{if } \ t > \tau. \end{cases}$$

The information $\mathcal{F}_t$ available at time t is the σ-algebra generated by $\{d(s) : 0 \le s \le t\}$, suitably completed with the null sets of $\mathcal{F}$.

In this setting, we prove that no classical bid-price control π adapted to $\{\mathcal{F}_t,\ t \ge 0\}$ can be optimal. First, observe that the revenue management problem (P) has a pathwise solution in this example, which is given as follows:

$$u_1(t) = \begin{cases} 1 & \text{if } t \le \tau, \\ 0 & \text{if } \ t > \tau, \end{cases} \quad \text{and} \quad u_2(t) = \begin{cases} 0 & \text{if } t \le \tau, \\ 1 & \text{if } \ t > \tau. \end{cases}$$

In particular, the optimal revenue is $200 - 100\tau$ along each sample path which results in the expected revenue of

$$200 - 100\,\mathbb{E}[\tau] = 200 - 100 \int_0^1 g(s)ds.$$

We argue by contradiction to conclude that no classical bid-price control can be optimal. Suppose that there exists an optimal bid-price control π. First, note that under π we must have

$$2 \int_0^\tau \mathbf{1}_{\{\pi(t) \le 100\}} dt = \tau \ \text{ almost surely.} \tag{8}$$

That is, we must book exactly half of the requests for product 1. Suppose (8) does not hold. Then, with positive probability we have at least one of the following:

$$2 \int_0^\tau \mathbf{1}_{\{\pi(t) \le 100\}} dt \quad < \quad \tau, \tag{9}$$

$$2 \int_0^\tau \mathbf{1}_{\{\pi(t) \le 100\} dt} \quad > \quad \tau. \tag{10}$$

If we have (9) with positive probability, then for those sample paths the total revenue is given by

$$200 \int_0^\tau \mathbf{1}_{\{\pi(t) \le 100\}} dt + 200(1 - \tau) < 200 - 100\,\tau.$$

Thus, if (9) arises with positive probability, then the expected revenues under π will be strictly less than $200 - 100\,\mathbb{E}[\tau]$, the optimal objective. Similarly, if (10) happens with positive probability, then the revenue along such a sample path is

$$200 \int_0^\tau \mathbf{1}_{\{\pi(t) \le 100\}} dt + 200 \left[1 - 2 \int_0^\tau \mathbf{1}_{\{\pi(t) \le 100\}} dt \right] < 200 - 100\,\tau.$$

Thus, the expected revenue is strictly less than $200 - 100\,\mathbb{E}[\tau]$ in this case too, contradicting optimality. Therefore, we must have (8). That is, almost surely

$$2 \int_0^\tau \mathbf{1}_{\{\pi(t) \le 100\}} dt = \tau. \tag{11}$$

Then since τ has a strictly positive density on $[0,1]$ and both sides are absolutely continuous functions of τ, we conclude by differentiating both sides of (11) with respect to τ that for almost every t and sample path

$$\mathbf{1}_{\{\pi(t) \leq 100\}} = \frac{1}{2},$$

which clearly is a contradiction. Thus, no classical bid-price control can be optimal.

As this example shows no classical bid-price control can achieve optimal bookings in general. In this specific example, if we impose the upper bound of 1 on the capacity consumption rate and use the bid price of $\pi = 100$ at all times, then the resulting bookings will be optimal. In Section 5, we show that this idea works in greater generality. That is, using bid prices in conjunction with limits on capacity consumption rates results in optimal bookings. To this end, the next definition introduces a generalized bid-price control along the lines of Akan & Ata (2008). In our setting, a generalized bid-price control involves a pair of stochastic processes (π, λ), where π is a bid-price process and $\lambda = \{\lambda(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ is a K-dimensional, nonnegative stochastic process; we will refer to λ as the capacity usage limit process.

Definition 3 *(Generalized Bid-Price Control) Given a bid-price process π and a capacity usage limit process λ, the pair (π, λ) is called a generalized bid-price control. For each $(\omega, t) \in \Omega \times [0, T]$, the booking rate vector $u(\omega, t)$ corresponding to (π, λ) is given by the solution to the following linear program: Choose booking rate vector u so as to*

$$\begin{aligned} &\textit{maximize} \quad (f(\omega,t) - A^{'}\pi(\omega,t)) \cdot u + \eta(Au - \lambda(\omega,t)) \cdot \mathbf{e} \\ &\textit{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (P(\omega,t)) \\ &Au \leq \lambda(\omega,t), \\ &0 \leq u \leq d(\omega,t), \end{aligned}$$

where $\eta > 0$ is arbitrarily small and $\mathbf{e}$ is the K-dimensional vector of ones.

As before, $\pi_k(\omega, t)$ is the bid-price or the shadow price for resource k at time t along the sample path ω. Similarly, $\lambda_k(\omega, t)$ is associated with resource k, and will be used as an upper bound on the consumption rate of resource k at time t along the sample path ω. The linear program (P(ω, t)) is lexicographic in the following sense. The system manager first solves (P(ω, t)) by setting $\eta = 0$. In the case of multiple optimal solutions, she selects the one that maximizes $(Au - \lambda(\omega, t)) \cdot \mathbf{e}$. For concreteness, tie breaking is done as follows in case of multiple such solutions. The potential choices of basis matrices are numbered up-front. For each (ω, t), the optimal solutions are characterized by the extreme points, each of which corresponds to a basis matrix. In case of multiple optimal solutions, the system manager picks the (extreme) optimal solution which corresponds to the basis matrix with the lowest index. Ideally, the system manager tries to choose a "maximal" booking rate u that has $Au = \lambda(\omega, t)$. A generalized bid-price control (π, λ) is said to be optimal if the booking rate process u resulting from its execution is optimal for the network revenue management problem (P). As will be seen below (cf. Proof of Theorem 1), if π and λ are chosen optimally, then the bookings under (π, λ) satisfy $Au = \lambda(\omega, t)$ for a.e. (ω, t). In particular, the capacity usage limit process λ ensures that the system state x follows an optimal trajectory, which is crucial for optimality because of potential issues due to degeneracy or multiplicity of solutions, cf. Section 5.

As pointed out earlier, it is easy to see that setting

$$\pi(\omega, t) = 100 \quad \text{and} \quad \lambda(\omega, t) = 1 \quad \text{for } (\omega, t) \in \Omega \times [0, 2]$$

results in optimal bookings in Example 2.

In what follows, we first show that there exists an optimal generalized bid-price control (π, λ) for the network revenue management problem. We also show that the optimal bid-price process π forms a martingale. Next, by the help of a perturbed version of the network revenue management problem, we construct an ε-optimal bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$ for each $\varepsilon > 0$, where the associated bid-price process π^ε forms a martingale. The bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$ can be viewed as a perturbation of the classical bid-price control corresponding to the bid-price process π^ε. In particular, it does not involve any capacity usage limits, and hence, is easier to implement. Finally, we show that for small values of ε, the booking process u^ε corresponding to the bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$ is close to an optimal booking process for (P). These results are proved without making *any* assumptions on the stochastic structure of demand and fare processes, allowing non-stationary demand and fare processes with an arbitrary dependence structure, including both inter-temporal and cross-product dependencies.

To facilitate our analysis of bid-price controls, in the next section we present a stochastic control problem that is dual to the network revenue management problem (P) in the sense of Bismut (1973).

4 Dual Network Revenue Management Problem

In this section we present the dual problem formulation (D) of the network revenue management problem (P) laid out in Section 2, and the coextremality results between the two formulations. The dual problem associated with the network revenue management problem (P) is obtained using the stochastic duality theory of Bismut (1973). Bismut (1973) develops a new approach to problems of stochastic optimal control using convex duality, which enables us to express the network revenue management problem in an equivalent way but in a completely different context. In particular, Bismut (1973) defines the dual problems in stochastic optimal control and the coextremality conditions associated with the dual optima by applying general methods of convex analysis introduced by Rockafellar (1968), Rockafellar (1969), Rockafellar (1970*a*) and Rockafellar (1970*b*).

Following Bismut (1973) the dual problem of control associated with the network revenue management problem (P) can be stated as follows (see Appendix B for its derivation): Choose a K-dimensional, square integrable, random vector $y_0 \in \mathcal{F}_0$ and a K-dimensional, square-integrable martingale M, which is null at zero, stopped at time T and adapted to the filtration $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$, so as to

$$\begin{aligned}
&\text{minimize } \ \mathbb{E}\left[\int_0^T d(\omega,t)\cdot[f(\omega,t)-y(\omega,t)A]^+ dt + C\cdot y_0(\omega)\right]\\
&\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(D)}\\
&y(\omega,t) = y_0(\omega) + M(\omega,t), \quad (\omega,t)\in\Omega\times[0,T],\\
&y(\omega,T)\geq 0, \ \ \omega\in\Omega,
\end{aligned}$$

where $[z]^+ = (\max\{0,z_1\},\ldots,\max\{0,z_J\})'$ for $z \in \mathbb{R}^J$.

In the dual problem formulation, y is the dual state variable. The value of the state variable y at time zero is given by $y_0 \in \mathcal{F}_0$, which is constant if there is no randomness at time zero. The dual state vector $y(\omega,t)$ can be interpreted as the shadow price for or the value assigned to the resources at time t along the sample path ω. Then the objective of the dual problem formulation can be interpreted as the value attributed to the network of resources by a given choice of the shadow price process. Thus, the dual problem formulation has the following interpretation: The system manager chooses the shadow price process y, which is a non-negative martingale, so as to minimize the expected value she attributes to her network of resources.

In the dual problem formulation (D), M is a predictive term, null at zero, or the best estimate at time t of the unresolved uncertainty. M is a term that integrates the information on environmental factors. That is, it contains the relevant information from the future and serves the purpose of integrating into the dual variable y this necessary information.

A remarkable feature of the dual problem is that the dual state variable y can have jumps, corresponding to the jumps of M. To elaborate on this, consider the setting where the information is revealed continuously over time. In particular, consider a stopping time τ which is defined as the first time an event happens. Under a continuous information structure this event is foretellable by a sequence of events. Intuitively, no event takes us by surprise under a continuous information structure. As put by Dellacherie & Meyer (1983), "We are forewarned by a succession of precursory signs, of the exact time the phenomenon will occur"; see Huang (1985) for a precise definition of a continuous information structure.

An equivalent characterization of continuous information structures is that all martingales have continuous sample paths, cf. Huang (1985). Moreover, the martingale term M of the dual problem formulation (D) will have jumps only if the information arrives discontinuously, say, because of unpredictable changes in the business environment, political situation etc. in which case the value of the resources reflected by the shadow prices has to be adjusted abruptly, in a discontinuous manner. That is, the new information can significantly increase or decrease the bid prices. Hence, continuity of M under a continuous information structure is the reflection of the continuous flow of information into the system.

To elaborate further, suppose that any martingale, and hence the optimal bid-prices can be represented as a stochastic integral with respect to a given Brownian motion w plus a martingale M that is orthogonal to w. Then, loosely speaking w contains the short term uncertainties and M is a prediction of the long-term uncertainties. The interpretation of the martingale term M is intuitively appealing since it enables us to express formally the distinction in the decision making process between continuous information processing which is done on a routine basis and discontinuous information processing done by reassessing the predictions.

The dual problem (D) and the primal problem (P) are closely linked to each other. Above all, the objective function values of (P) and (D) are equal. Moreover, any optimal primal solution and any optimal dual solution satisfy a set of coextremality conditions, which are necessary and sufficient conditions for optimality. The following proposition summarizes the duality results between the two formulations that are relevant for our purposes; its proof is given in Appendix B.

Proposition 1 *The network revenue management problem (P), that is, the primal problem, and the dual problem (D) have the same optimal objective value. Moreover, letting u be a feasible control for (P) with the corresponding state trajectory x, and (y_0, M) be a feasible control for (D) with the corresponding state trajectory y, the controls u and (y_0, M) are optimal for (P) and (D), respectively, if and only if they satisfy the coextremality conditions (12) and (13) given below:*

$$y(\omega, T) \cdot x(\omega, T) = 0, \;\; a.e. \;\; \omega \in \Omega, \tag{12}$$

and for $j = 1, \ldots, J$ and almost all $(\omega, t) \in \Omega \times [0, T]$ with $d_j(\omega, t) > 0$,

$$\begin{aligned}
&\text{if} \;\; u_j(\omega, t) = 0, && \text{then} \;\; y(\omega, t)A^j - f_j(\omega, t) \geq 0, \\
&\text{if} \;\; 0 < u_j(\omega, t) < d_j(\omega, t), && \text{then} \;\; y(\omega, t)A^j - f_j(\omega, t) = 0, \\
&\text{if} \; u_j(\omega, t) = d_j(\omega, t), && \text{then} \;\; y(\omega, t)A^j - f_j(\omega, t) \leq 0.
\end{aligned} \tag{13}$$

The following corollary is immediate from Proposition 1 and it provides an upper bound on the objective function value of the network revenue management problem (P).

Corollary 1 *For any non-negative, square integrable martingale y adapted to the filtration $\{\mathcal{F}_t, \, t \in \mathbb{R}_+\}$,*

$$\mathbb{E}\left[\int_0^T d(\omega, t) \cdot [f(\omega, t) - y(\omega, t)A]^+ dt + C \cdot y_0(\omega)\right]$$

provides an upper bound on the objective function value of the network revenue management problem (P).

5 An Optimal Generalized Bid-Price Control

In this section we show the existence of an optimal generalized bid-price control (π, λ) defined as in Section 3 such that the optimal bid-price process $\{\pi(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ forms a martingale. Recall that a generalized bid-price policy (π, λ) is said to be optimal if the booking rate process $\{u(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ resulting from the execution of (π, λ) is optimal for the network revenue management problem (P), cf. Section 3. The following theorem is the main result of this section and is proved in Appendix C.

Theorem 1 *There exists an optimal generalized bid-price control* (π, λ) *such that the optimal bid-price process* $\{\pi(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ *is a martingale adapted to* $\{\mathcal{F}_t : t \in [0, T]\}$.

As can be seen from the proof of Theorem 1, we construct an optimal generalized bid-price control (π, λ) by using an optimal state trajectory y for the dual problem (D) as the bid-price process π. To elaborate on its connection to the classical bid-price controls, fix $(\omega, t) \in \Omega \times [0, T]$ and suppose that

$$f_j(\omega, t) \neq \pi(\omega, t)A^j \text{ for } j = 1, \ldots, J.$$

Then it can be seen from the coextremality conditions (12) and (13) that the booking rate vector $u(\omega, t)$ is uniquely determined by the bid-price vector $\pi(\omega, t)$ as follows: For $j = 1, \ldots, J$,

$$u_j(\omega, t) = \begin{cases} d_j(\omega, t) & \text{if} \quad f_j(\omega, t) > \pi(\omega, t)A^j, \\ 0 & \text{if} \quad f_j(\omega, t) < \pi(\omega, t)A^j. \end{cases} \tag{14}$$

Our construction of the generalized bid-price control (π, λ) will choose the capacity usage limit process $\lambda(\omega, t) = Au(\omega, t)$ in this case, and the optimal generalized bid-price control results in the same booking decisions as a classical bid-price control would, cf. (14). Moreover, arguing heuristically, one can simply set $\eta = 0$ and $\lambda(\omega, t) = \infty$, in which case the execution of the generalized bid-price control reduces to a classical bid-price control, because the problem $(\mathrm{P}(\omega, t))$ decomposes across products.

Differences may arise, though, when $f_j(\omega, t) = \pi(\omega, t)A^j$ for some products and the capacity usage limit vector $\lambda(\omega, t)$ can be crucial in separating an optimal booking rate from a non-optimal one. In this sense, the need for a capacity usage limit process λ for optimality is linked to the multiplicity of optimal solutions u to (P) associated with a given optimal dual solution y to (D). Recall that one reason cited for the non-optimality of the classical bid-price controls is the discreteness of demand and bookings, cf. Talluri & van Ryzin (1998). Our analysis suggests that discreteness may not be the only reason for non-optimality as we avoid it by adapting a rate-based model, or a stochastic fluid model. Another reason for potential non-optimality of the classical bid-price controls in our setting seems to be the multiplicity or degeneracy which arises when $f_j(\omega, t) = \pi(\omega, t)A^j$ for some product $j = 1, \ldots, J$ over a non-negligible set of (ω, t). Further discussion of potential reasons for the nonoptimality of classical bid-price controls is deferred to Section 7.

In the next section, we pursue an alternative path and construct an ε-optimal bid-price control for any given $\varepsilon > 0$. To this end, we introduce a perturbed version of the network revenue management problem. The perturbed problem for $\varepsilon > 0$, in turn, gives rise to an ε-optimal bid price control, which can be viewed as a perturbation of a classical bid-price control. The perturbation results in a strictly concave problem which has a unique solution. This is a form of regularization that gives the strict complementarity needed to ensure that the optimal primal solution can be derived directly from the dual problem.

6 An ε-Optimal Bid-Price Control

In this section, we introduce a perturbed version of the network revenue management problem (P) and its dual. Then we derive the coextremality conditions between the two formulations, which eventually gives rise to an ε-optimal bid-price control. For each $\varepsilon > 0$, the perturbed problem (P^ε) can be stated as follows: Choose a booking rate vector $u(\omega, t)$ for each $(\omega, t) \in \Omega \times [0, T]$ so as to

$$\begin{aligned}
&\text{maximize } \ \mathbb{E}\left[\int_0^T [f(\omega,t)\cdot u(\omega,t) - \frac{1}{2}\sum_{j=1}^{J} \varepsilon_j(\omega,t)\, u_j^2(\omega,t)]\, dt\right] \\
&\text{subject to} \\
&x(\omega,t) = C - AU(\omega,t), \quad (\omega,t) \in \Omega\times[0,T], \\
&U(\omega,t) = \int_0^t u(\omega,s)ds, \quad (\omega,t)\in\Omega\times[0,T], \\
&0 \le u(\omega,t) \le d(\omega,t), \quad (\omega,t)\in\Omega\times[0,T], \\
&x(\omega,T) \ge 0, \quad \omega\in\Omega,
\end{aligned} \tag{P^ε}$$

where $\varepsilon_j(\omega, t)$ is defined as follows: For $j = 1, \ldots, J$ and $(\omega, t) \in \Omega \times [0, T]$, let

$$\varepsilon_j(\omega,t) = \begin{cases} \frac{\varepsilon}{d_j(\omega,t)} & \text{if } d_j(\omega,t) > 0, \\ \varepsilon & \text{otherwise.} \end{cases} \tag{15}$$

The perturbed problem (P^ε) is the same as the network revenue management problem (P), except for the quadratic term in its objective, which makes it a strictly concave problem. Thus, the perturbed problem (P^ε) has a unique solution. Consequently, by this simple perturbation we avoid the issues of multiplicity of the solution or degeneracy issues encountered earlier, which in turn leads to an easy characterization of the solution to the perturbed problem (P^ε) in terms of the optimal shadow price process derived from its dual formulation.

The dual problem (D^ε) of the perturbed network revenue management problem (P^ε) can be stated as follows (see Appendix D for its derivation): Choose a K-dimensional, square integrable, random vector $y_0 \in \mathcal{F}_0$ and a K-dimensional, square-integrable martingale M, which is null at zero, stopped at time T, and adapted to the filtration $\{\mathcal{F}_t,\, t \in \mathbb{R}_+\}$, so as to

$$\begin{aligned}
&\text{minimize } \ \mathbb{E}\left[\int_0^T g_\varepsilon(f(\omega,t) - y(\omega,t)A, d(\omega,t))dt + C\cdot y_0(\omega)\right] \\
&\text{subject to} \\
&y(\omega,t) = y_0(\omega) + M(\omega,t), \quad (\omega,t)\in\Omega\times[0,T], \\
&y(\omega,T) \ge 0,\ \omega\in\Omega,
\end{aligned} \tag{D^ε}$$

where $g_\varepsilon(z,d) = \sum_{j=1}^J h_\varepsilon(z_j,d_j)$ and for $j = 1,\dots,J$, h_ε is given by

$$h_\varepsilon(z_j,d_j) = \begin{cases} 0 & \text{if} \quad z_j \le 0, \\ z_j d_j - \frac{\varepsilon}{2} d_j & \text{if} \quad z_j \ge \varepsilon, \\ \frac{z_j^2 d_j^2}{2\varepsilon} & \text{if} \quad 0 < z_j < \varepsilon. \end{cases} \tag{16}$$

As pointed out earlier, the difference between (P^ε) and (P) is that (P^ε) has the strictly concave term $-\frac{1}{2}\sum_{j=1}^J \varepsilon_j(\omega,t)u_j^2(\omega,t)$ in its objective function in addition to the revenue term $f(\omega,t)\cdot u(\omega,t)$, which makes ($\mathrm{P}^\varepsilon$) a strictly concave problem. As a result, there is a unique optimal solution u^ε for (P^ε), cf. Proposition 3, which can be determined through the coextremality conditions between (P^ε) and (D^ε), cf. Proposition 2.

The following proposition summarizes the duality results between the two formulations that are relevant for our purposes; its proof is given in Appendix D.

Proposition 2 *The perturbed primal problem (P^ε) and its dual (D^ε) have the same optimal objective value. Moreover, letting u^ε be a feasible control for (P^ε) with the corresponding state trajectory x^ε, and $(y_0^\varepsilon, M^\varepsilon)$ be a feasible control for (D^ε) with the corresponding state trajectory y^ε, the controls u^ε and $(y_0^\varepsilon, M^\varepsilon)$ are optimal for (P^ε) and (D^ε), respectively, if and only if they satisfy the following coextremality conditions (17) and (18) given below:*

$$y^\varepsilon(\omega,T)\cdot x^\varepsilon(\omega,T) = 0, \ \textit{a.e.}\ \omega \in \Omega, \tag{17}$$

and for $j = 1,\dots,J$ and almost all $(\omega,t) \in \Omega \times [0,T]$ with $d_j(\omega,t) > 0$,

$$\begin{array}{lll} \textit{if} \ \ u_j^\varepsilon(\omega,t) = 0, & \textit{then} & f_j(\omega,t) - y^\varepsilon(\omega,t)A^j \le 0, \\ \textit{if} \ u_j^\varepsilon(\omega,t) = d_j(\omega,t), & \textit{then} & f_j(\omega,t) - y^\varepsilon(\omega,t)A^j \ge \varepsilon, \\ \textit{if} \ \ 0 < u_j^\varepsilon(\omega,t) < d_j(\omega,t), & \textit{then} & \frac{f_j(\omega,t) - y^\varepsilon(\omega,t)A^j}{\varepsilon} d_j(\omega,t) = u_j^\varepsilon(\omega,t). \end{array} \tag{18}$$

One would expect that for small values of ε the objective value of the perturbed problem (P^ε) is close to that of the network revenue management problem (P). Building on this intuition, we next construct a bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$ which is ε-optimal for the network revenue management problem (P) for $\varepsilon > 0$. To facilitate our construction, fix an optimal state trajectory y^ε for the perturbed dual problem (D^ε) for each $\varepsilon > 0$. Then, for $\varepsilon > 0$, let $\pi^\varepsilon = y^\varepsilon$ and

$$\phi^\varepsilon(z_j, f_j, d_j) = \begin{cases} 0 & \text{if} \quad f_j < z_j, \\ d_j & \text{if} \quad f_j > z_j + \varepsilon, \\ \frac{f_j - z_j}{\varepsilon} d_j & \text{if} \quad z_j \le f_j \le z_j + \varepsilon. \end{cases} \tag{19}$$

It follows from Definition 1, cf. (3), that for each $\varepsilon > 0$, the bookings under the bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$ are given as follows.

$$u_j^\varepsilon(\omega,t) = \begin{cases} 0 & \text{if} \quad f_j(\omega,t) - \pi^\varepsilon(\omega,t)A^j < 0, \\ d_j(\omega,t) & \text{if} \quad f_j(\omega,t) - \pi^\varepsilon(\omega,t)A^j > \varepsilon, \\ \frac{f_j(\omega,t) - \pi(\omega,t)A^j}{\varepsilon} d_j(\omega,t) & \text{if} \quad 0 \le f_j(\omega,t) - \pi^\varepsilon(\omega,t)A^j \le \varepsilon \end{cases} \tag{20}$$

for $(\omega, t) \in \Omega \times [0, T]$ and $j = 1, \ldots, J$. The following proposition states the optimality of the control u^ε given in (20) for the perturbed problem (P^ε) and is proved in Appendix D.

Proposition 3 *For each $\varepsilon > 0$, the booking control $\{u^\varepsilon(\omega, t) : (\omega, t) \in \Omega \times [0, T]\}$ given in (20) is the unique optimal control for the perturbed problem (P^ε).*

One would hope that for small values of $\varepsilon > 0$, the performance of the bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$ is close to the optimal objective value of (P). Indeed, the following theorem establishes the ε-optimality of the bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$. It also shows that the booking controls u^ε resulting from the bid-price controls $\{(\pi^\varepsilon, \phi^\varepsilon) : \varepsilon > 0\}$ are close to an optimal solution to the network revenue management problem (P) for small values of ε. Viewing the booking controls u^ε for $\varepsilon > 0$ as an element of L^2, the space of square integrable functions on $\Omega \times [0, T]$, it is easy to see that the controls u^ε for $\varepsilon > 0$ are uniformly bounded in L^2. Thus, it follows from Alaoglu's Theorem, cf. Dunford & Schwartz (1957), that the collection of booking controls $\{u^\varepsilon : \varepsilon > 0\}$ is weak* compact. Defining $\mathcal{U}$ as the collection of weak limit points of the sequences of booking controls $\{u^{\varepsilon_n} : n \geq 1\}$ where $\varepsilon_n \searrow 0$ as $n \to \infty$, the following theorem establishes the optimality of every weak limit $u \in \mathcal{U}$, and its proof is given in Appendix D.

Theorem 2 *The collection of bid-price controls $\{(\pi^\varepsilon, \phi^\varepsilon) : \varepsilon > 0\}$ and the associated booking controls $\{u^\varepsilon : \varepsilon > 0\}$ satisfy the following:*

a) For each $\varepsilon > 0$, the bid-price process π^ε is a non-negative martingale.

b) For each $\varepsilon > 0$, the bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$ is $\kappa\varepsilon$-optimal, where

$$\kappa = \sum_{j=1}^{J} \int_0^T \mathbb{E}[d_j(\omega, t)]dt. \tag{21}$$

c) Every weak limit $u \in \mathcal{U}$ of the booking controls $\{u^\varepsilon : \varepsilon > 0\}$ as $\varepsilon \searrow 0$ is an optimal booking control for the network revenue management problem (P).

For $\varepsilon > 0$ and product $j = 1, \ldots, J$, the bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$ behaves in the same way as a classical bid-price control as long as $f_j(\omega, t) - \pi^\varepsilon(\omega, t)A^j$ does not fall in the interval $(0, \varepsilon)$, in which case a classical bid-price control would dictate booking all of the demand. In contrast, the bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$ results in a booking rate of

$$\frac{f_j(\omega, t) - \pi^\varepsilon(\omega, t)A^j}{\varepsilon} d_j(\omega, t) < d_j(\omega, t).$$

Figure 2 displays $u_j^\varepsilon(\omega, t)$ as a function of the difference $f_j(\omega, t) - \pi^\varepsilon(\omega, t)A^j$. The slope of the line segment in the middle of the figure is $d_j(\omega, t)/\varepsilon$, and as $\varepsilon \searrow 0$, the graph looks more and more like a step function, corresponding to a bang-bang control, which would result from a classical bid-price control.

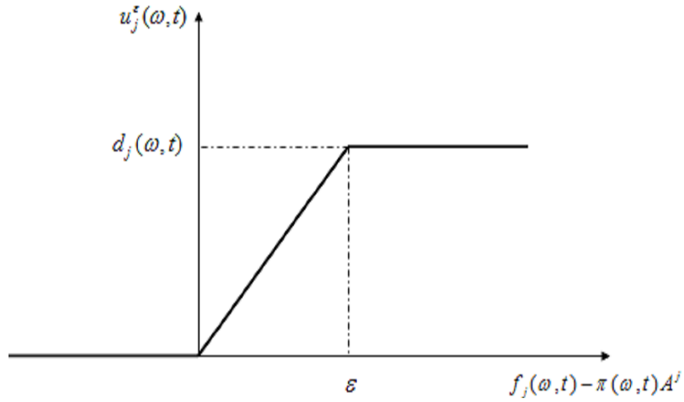

Figure 2: The bookings corresponding to the ε-optimal bid-price control $(\pi^\varepsilon, \phi^\varepsilon)$.

7 Discussion and Concluding Remarks

We consider a continuous-time stochastic fluid model of network revenue management. First, we prove that there exists an optimal generalized bid-price control, where the bid-price process forms a martingale. A generalized bid-price control consists of a bid-price process and a capacity usage limit process, which creates limits on the instantaneous capacity usage rate of the resources. Next, we analyze a perturbed version of the network revenue management problem and its dual, using which we construct an ε-optimal bid-price control. The bid-price process associated with the ε-optimal bid-price control forms a martingale, too. Finally, we show that every weak* limit of the sequence of booking processes resulting from the ε-optimal bid-price controls as $\varepsilon \searrow 0$ is an optimal solution to the network revenue management problem.

Practical insights and implementation issues. Although we study a stylized rate-based model, or a stochastic fluid model, the insights we provide carry over to more practical settings. For instance, one can model the demand for various products as a multidimensional, doubly-stochastic Poisson process, where the intensity of the Poisson process is given by a constant multiple of the demand rate process $\{d(\omega,t) : (\omega,t) \in \Omega \times [0,T]\}$ of Section 2. For such systems with high initial capacity and high demand, the model introduced in Section 2 is a good approximation. Indeed, it is the so-called associated fluid model. It is important, however, to point out that our fluid model is a stochastic fluid model, and it captures all key trade-offs faced by the system manager unlike most fluid models considered in the literature which are deterministic models. Moreover, viewing our rate-based network revenue management model as a fluid model of a more practical system and using its solution, one can propose near optimal policies. To elaborate further, one can implement the ε-optimal bid-price controls proposed in the preceding sections in a practical setting as follows. At every point in time, given the ε-optimal bid prices π^ε for our rate based model, a request for product j is accepted if $f_j \geq \pi^\varepsilon A^j + \varepsilon$, and it is rejected if $f_j \leq \pi^\varepsilon A^j$, while the system manager flips a coin with success probability $(f_j - \pi^\varepsilon A^j)/\varepsilon$ to decide when $0 < f_j - \pi^\varepsilon A^j < \varepsilon$. That is, she accepts the request with probability $(f_j - \pi^\varepsilon A^j)/\varepsilon$ and rejects it with probability $1 - (f_j - \pi^\varepsilon A^j)/\varepsilon$ when $0 < f_j - \pi^\varepsilon A^j < \varepsilon$.

Moreover, as updating the bid-prices continuously may not be practical, one could use a discrete review policy with sufficiently small review periods. We conjecture that such policies can be shown to be near optimal for systems with large capacity and high demand. Although we provide no proof of these assertions, the literature on large call centers and their analysis via fluid models make these claims plausible; see for example Bassamboo, Harrison & Zeevi (2006*a*) and Bassamboo, Harrison & Zeevi (2006*b*).

Connection to Dynamic Programming. Recall that the stochastic primitives of our model are very general, allowing an arbitrary dependence structure both across time and across products. In particular, the underlying demand process need not be a Markovian process. Davis (1979) notes that "... Bellman equation approach is essentially limited to Markovian systems." Therefore, analyzing the network revenue management problem (P) under general probabilistic assumptions by dynamic programming does not seem to be a viable approach. Thus, we adopt the convex analysis approach of Bismut (1973). Nonetheless, Bismut (1973) formally derives a connection between his approach and the dynamic programming approach for controlled Ito processes, and shows formally (under various technical assumptions) that the optimal dual state variable equals the gradient of the value function obtained by dynamic programming along the optimal trajectory. Even if we look at the restrictive version of the network revenue management problem, which is Markovian, the state space constraints prevent us from making such a connection. Nonetheless, we intuitively expect in a Markovian setting that the optimal bid prices (or shadow prices) derived from the dual problem correspond to (generalized) gradients of the value function obtained from the dynamic programming formulation.

Connection to Forward-Backward Stochastic Differential Equations (FBSDEs). Akan (2008) shows that through another perturbation, the optimal primal and dual state trajectories can be expressed as a solution to a FBSDE; see El Karoui, Peng & Quenez (1997) and the references therein for an overview of FBSDEs. This intriguing connection is useful in two regards: First, the numerical methods for solving FBSDEs can be adapted to our setting to compute (near) optimal bid prices; see, for example, Ma, Protter & Yong (1994) and Duffie, Ma & Yong (1995) for PDE methods and Bouchard & Touzi (2004) and Bender & Zhang (2008) for Monte-Carlo methods for solving FBSDEs. Indeed, the latter studies report encouraging results for solving FBSDEs in high dimensions. Second, the question of whether there exists an ε-optimal bid-price control in the classical sense can equivalently be stated as a question of the existence of a solution to a FBSDE; see Akan (2008) for further discussion.

Sufficient conditions for existence of optimal classical bid-price controls. Examples 1 and 2 potentially provide some insights into the reasons for possible non-optimality of classical bid-price controls. Recall that Example 1 studies a continuous-time, rate-based version of the counterexample of Talluri & van Ryzin (1998). Our setup has two important features which lead to optimality of classical bid-price controls in this example. First, the bookings at each point in time consume only an infinitesimal amount of capacity. Second, we allow frequent (indeed continuous) updating of bid prices. The setup of Talluri and van Ryzin does not have these features. Therefore, in addition to discreteness of the problem, the example potentially suggests that infrequent updating of bid prices may be another reason for non optimality of classical bid-price controls. Moreover, Example 2 indicates that the classical bid-price controls are not optimal in general, where the non-optimality stems from the fact that no bang-bang solution is optimal in that example. Thus, the optimality of classical bid-price controls also requires existence of optimal bang-bang solutions to the network revenue management problem (P). Finally, the notion of strict complementary slackness may help identify conditions under which the classical bid-price controls are optimal. It may also suggest additional insights on the potential reasons for nonoptimality of classical bid-price controls; see Akan (2008) for further discussion. Nonetheless, identifying conditions on problem primitives that render classical bid-price controls optimal remains an open problem for future research.

A An Auxiliary Weak* Convergence Lemma

As a preliminary, first let $L^2(\Omega \times [0,T],\, \mathcal{F}_T \otimes \mathcal{B}[0,T],\, \mathbb{P})$ denote the set of functions $X : \Omega \times [0,T] \to \mathbb{R}^J$ that are measurable with respect to the product σ-algebra $\mathcal{F}_T \otimes \mathcal{B}[0,T]$ such that

$$\mathbb{E}\int_0^T |X(\omega,t)|^2\, dt < \infty,$$

where $\mathcal{B}[0,T]$ is the Borel σ-algebra on $[0,T]$. The space $L^2(\Omega \times [0,T],\, \mathcal{F}_T \otimes \mathcal{B}[0,T],\, \mathbb{P})$ is endowed with the usual inner product $\langle \cdot,\cdot \rangle$ given by

$$\langle X, Y\rangle = \mathbb{E}\left[\int_0^T X(\omega,t)\cdot Y(\omega,t)dt\right]$$

for $X, Y \in L^2(\Omega \times [0,T],\, \mathcal{F}_T \otimes \mathcal{B}[0,T],\, \mathbb{P})$ so that it is a Hilbert space.

We also view the stochastic processes as mappings from $\Omega \times [0,T]$ into $\mathbb{R}^J$. To be more specific, we view the adapted stochastic processes as the elements of $L^2(\Omega \times [0,T],\, \mathcal{J}_T^*,\, \mathbb{P})$, where $\mathcal{J}_T^*$ is the completion for the measure $d\mathbb{P} \otimes dt$ of the σ-field $\mathcal{J}_T$ generated by the adapted processes on $\Omega \times [0,T]$ which are right-continuous with left limits. Then, $L^2(\Omega \times [0,T],\, \mathcal{J}_T^*,\, \mathbb{P})$ is also a Hilbert space endowed with the inner product $\langle \cdot,\cdot \rangle$. In particular, $L^2(\Omega \times [0,T],\, \mathcal{J}_T^*,\, \mathbb{P})$ can be viewed as a closed subset of $L^2(\Omega \times [0,T],\, \mathcal{F}_T \otimes \mathcal{B}[0,T],\, \mathbb{P})$, For completeness, we next state the definition of weak* convergence, which is followed by the main result of this section.

Definition 4 *The sequence $\{X_n\}$ of elements of $L^2(\Omega \times [0,T],\, \mathcal{F}_T \otimes \mathcal{B}[0,T],\, \mathbb{P})$ is said to converge in the weak* topology to an element X of $L^2(\Omega \times [0,T],\, \mathcal{F}_T \otimes \mathcal{B}[0,T],\, \mathbb{P})$ if for all $Y \in L^2(\Omega \times [0,T],\, \mathcal{F}_T \otimes \mathcal{B}[0,T],\, \mathbb{P})$,*

$$\langle X^n, Y\rangle \to \langle X, Y\rangle \ \text{ as } n \to \infty.$$

Lemma 1 *Let $\{u^n : n \geq 1\}$ be a sequence of feasible controls for the network revenue management problem (P) which converges to u in the weak* topology. Then, u is a feasible booking control for (P). Moreover, the* *expected revenue under u^n converges to that under u as $n \to \infty$. That is,*

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T f(\omega,t)\cdot u^n(\omega,t)dt\right] = \mathbb{E}\left[\int_0^T f(\omega,t)\cdot u(\omega,t)dt\right].$$

Proof. To prove that u is a feasible control for the network revenue management problem (P), we need to check that u is adapted and satisfies demand and capacity restrictions. First, we prove that $u \in L^2(\Omega\times[0,T],\, \mathcal{J}_T^*,\, \mathbb{P})$, and hence u is adapted to the filtration $\{\mathcal{F}_t,\, t\in\mathbb{R}_+\}$. To this end, note that for all $n\geq 1$ and $v\in (L^2(\Omega\times[0,T],\, \mathcal{J}_T^*,\, \mathbb{P}))^{\perp}$, we have $\langle u^n, v\rangle = 0$. Then, we also have

$$\lim_{n\to\infty}\langle u^n, v\rangle = \langle u, v\rangle = 0,$$

which implies that $u\in L^2(\Omega\times[0,T],\, \mathcal{J}_T^*,\, \mathbb{P})$.

Second, we show that u satisfies the demand restrictions, i.e.

$$0\leq u(\omega,t)\leq d(\omega,t) \text{ for a.e. } (\omega,t)\in\Omega\times[0,T].$$

Note that $\langle u^n, v\rangle \geq 0$ for all $n\geq 1$ and $v\in L^2(\Omega\times[0,T],\, \mathcal{F}_T\otimes\mathcal{B}[0,T],\, \mathbb{P})$ such that $v\geq 0$ for a.e. $(\omega,t)\in\Omega\times[0,T]$. Then, we have that $\langle u,v\rangle\geq 0$ as well for all $v\in L^2(\Omega\times[0,T],\, \mathcal{F}_T\otimes\mathcal{B}[0,T],\, \mathbb{P})$ such that $v\geq 0$, and hence $u(\omega,t)$ is non-negative for a.e. $(\omega,t)\in\Omega\times[0,T]$. To show that $u(\omega,t)\leq d(\omega,t)$ for a.e. $(\omega,t)\in\Omega\times[0,T]$, observe that for all $v\in L^2(\Omega\times[0,T],\, \mathcal{F}_T\otimes\mathcal{B}[0,T],\, \mathbb{P})$ such that $v\geq 0$ for a.e. $(\omega,t)\in\Omega\times[0,T]$, we have

$$\begin{aligned} 0 &\leq \langle d-u^n, v\rangle, \\ &= \langle d,v\rangle - \langle u^n, v\rangle. \end{aligned}$$

Then, we have $\langle d,v\rangle - \langle u,v\rangle = \langle d-u, v\rangle \geq 0$ and hence $d(\omega,t)-u(\omega,t)\geq 0$ for a.e. $(\omega,t)\in\Omega\times[0,T]$.

Third, we show that $A\int_0^T u(\omega,t)dt\leq C$ for a.e. $\omega\in\Omega$. It suffices to show that

$$\mathbb{E}\left[(C - A\int_0^T u(\omega,t))\cdot\alpha(\omega)\right]\geq 0$$

for all square integrable α such that $\alpha\in\mathcal{F}_T$ and $\alpha(\omega)\geq 0$ for a.e. $\omega\in\Omega$. From feasibility of u^n, we have for all n,

$$\mathbb{E}\left[(C - A\int_0^T u^n(\omega,t))\cdot\alpha(\omega)\right]\geq 0.$$

Assume without loss of generality that the capacity consumption matrix A has rank K. If not, we can simply consider a new capacity consumption matrix $\widetilde{A} = [I\ A]$ of dimension $K\times(K+J)$, where the demand for the first K products is equal to zero for all $(\omega,t)\in\Omega\times[0,T]$ and the demand for the rest of the products is given as before. Then, the definitions of $L^2(\Omega\times[0,T],\, \mathcal{F}_T\otimes\mathcal{B}[0,T],\, \mathbb{P})$ and the inner product are modified accordingly and the analysis to follow carry over to this problem.

Let $\beta \in \mathbb{R}^J$ such that $A\beta = C$. Such β exists since the column space of A is $\mathbb{R}^K$. Then, we have that $\left\langle \frac{\beta}{T} - u^n, A'\alpha \right\rangle = \mathbb{E}\left[(C - A\int_0^T u^n(\omega,t)) \cdot \alpha(\omega) \right]$. To see this, note that

$$\begin{aligned} \left\langle \frac{\beta}{T} - u^n, A'\alpha \right\rangle &= \mathbb{E}\left[\int_0^T (\frac{\beta}{T} - u^n(\omega,t)) \cdot A'\alpha(\omega) dt \right], \\ &= \mathbb{E}\left[\int_0^T (\frac{A\beta}{T} - Au^n(\omega,t)) \cdot \alpha(\omega) dt \right], \\ &= \mathbb{E}\left[(C - A\int_0^T u^n(\omega,t)) \cdot \alpha(\omega) \right]. \end{aligned}$$

Then, $\left\langle \frac{\beta}{T} - u^n, A'\alpha \right\rangle \geq 0$ for all n and square integrable $\alpha(\omega) \geq 0$ such that $\alpha \in \mathcal{F}_T$. Since

$$\lim_{n\to\infty} \left\langle \frac{\beta}{T} - u^n, A'\alpha \right\rangle = \left\langle \frac{\beta}{T} - u, A'\alpha \right\rangle,$$

this implies that for all square integrable $\alpha(\omega) \geq 0$ such that $\alpha \in \mathcal{F}_T$, we have

$$\mathbb{E}\left[(C - A\int_0^T u(\omega,t)) \cdot \alpha(\omega) \right] \geq 0.$$

Finally, we show that for all adapted fare processes $\{f(\omega,t) : (\omega,t) \in \Omega \times [0,T]\}$,

$$\mathbb{E}\left[\int_0^T f(\omega,t) \cdot u^n(\omega,t) dt \right] \to \mathbb{E}\left[\int_0^T f(\omega,t) \cdot u(\omega,t) dt \right] \text{ as } n \to \infty.$$

This follows simply from the definition of the weak limit and the fact that $f \in L^2(\Omega \times [0,T], \mathcal{F}_T \otimes \mathcal{B}[0,T], \mathbb{P})$.

■

B Derivation of the dual network revenue management problem and the coextremality results

Derivation of the dual network revenue management problem (D). We will follow the road map provided by Bismut (1973) to derive the dual problem of control associated with the network revenue management problem (P). In particular, we first append the penalty expressions corresponding to the demand and capacity restrictions on bookings in the objective function by defining the convex, extended real valued integrand L and the convex functional l. We also formulate the problem towards minimization. Next, we compute the conjugate convex functions associated with L and l so as to define the dual integrand M and the dual functional m. The dual problem of control is defined using M and m.

The system dynamics equation for the network revenue management problem is given by

$$x(\omega,t) = C - \int_0^t Au(\omega,s)ds, \quad (\omega,t) \in \Omega \times [0,T]. \tag{22}$$

Comparing (22) with the set of admissible controls for the primal problem in the framework of Bismut (1973), cf. Proposition I-1 of Bismut (1973), first thing to note is that there is no stochastic integration term and martingale term in (22).

To facilitate the analysis to follow, define the indicator function $\chi_F(\cdot)$ for a given set F by

$$\chi_F(x) = \begin{cases} 0 & \text{if } x \in F \text{ a.s.,} \\ \infty & \text{otherwise.} \end{cases}$$

We express the network revenue management problem (P) in terms of the convex integrand L and the convex lower semi-continuous functional l which are defined as follows. Define L on $\Omega \times [0, T] \times \mathbb{R}^K \times \mathbb{R}^K$ as

$$L(\omega, t, x, \dot{x}) = \begin{cases} -f(\omega, t) \cdot u + \chi_{\mathbb{R}^J_+}(u) + \chi_{\mathbb{R}^J_-}(u - d(\omega, t)) & \text{if } \dot{x} = -Au, \\ \infty & \text{otherwise.} \end{cases} \tag{23}$$

In (23), $\dot{x}$ denotes the rate of change of x, where x is the state variable denoting the vector of remaining capacities. The integrand L serves the purpose of eliminating the hard constraints of the network revenue management problem (P) by appending them to the objective function as penalty expressions. In this sense, the penalty expression $\chi_{\mathbb{R}^J_+}(u) + \chi_{\mathbb{R}^J_-}(u - d(\omega, t))$ is the demand restriction on bookings and replaces the constraint

$$0 \leq u \leq d(\omega, t).$$

Notice also that we have reformulated the problem towards minimization and $-f(\omega, t) \cdot u$ is the negative of the rate at which revenue is generated. The system dynamics equation (22) is incorporated in L by the fact that we require $\dot{x}$ to be equal to $-Au$.

Next step is to define the functional l on $L^0_2 \times L^T_2$ with values on $\mathbb{R} \cup \{\infty\}$ so as to initiate the problem with capacity vector C and dictate non-negativity of remaining capacity at the terminal time T. The functional l is defined as

$$l(x_0, x_T) = l_0(x_0) + l_T(x_T), \tag{24}$$

where the convex, lower semi-continuous functionals l_0 and l_T are given by

$$l_0(x_0) = \chi_{\{C\}}(x_0), \quad l_T(x_T) = \chi_{\mathbb{R}^K_+}(x_T). \tag{25}$$

The functional l_0 replaces the constraint that $x(\omega, 0) = C$ for a.e. $\omega \in \Omega$ and l_T replaces the capacity constraint $x(\omega, T) \geq 0$ for a.e. $\omega \in \Omega$. Then, the network revenue management problem (P) can equivalently be stated as a problem of minimizing

$$\mathbb{E}\left[\int_0^T L(\omega, t, x(\omega, t), \dot{x}(\omega, t))dt + l(x_0, x_T)\right].$$

As our second step in deriving the dual problem of control, we compute the conjugates to the functions L and l. Let L^* denote the conjugate to L. To be specific,

$$L^*(\omega, t, s, p) = \sup_{z \in \mathbb{R}^K,\ y \in \mathbb{R}^K} \{z \cdot s + y \cdot p - L(\omega, t, z, y)\} \quad \text{for} \quad s, p \in \mathbb{R}^K. \tag{26}$$

We can express L^* more explicitly as follows. Note that $L(\omega, t, z, y) < \infty$ only if there exists some $u \in \mathbb{R}^J$ such that $y = -Au$ and $0 \leq u \leq d(\omega, t)$. Then, for $s, p \in \mathbb{R}^K$, we can write L^* as

$$\begin{aligned} L^*(\omega, t, s, p) &= \sup_{z \in \mathbb{R}^K,\ 0 \leq u \leq d(\omega,t)} \{z \cdot s - pAu - (-f(\omega, t) \cdot u)\}, \\ &= \sup_{z \in \mathbb{R}^K} \{z \cdot s\} + \sup_{0 \leq u \leq d(\omega,t)} \{(f(\omega, t) - pA) \cdot u\}, \\ &= \chi_{\{0\}}(s) + [f(\omega, t) - pA]^+ \cdot d(\omega, t). \end{aligned}$$

The first line is obtained by replacing y with $-Au$ for $0 \leq u \leq d(\omega, t)$ and noting that $L(\omega, t, z, y) = -f(\omega, t) \cdot u$. The second line follows from the observation that we can take the supremum in the first line separately for z and u. To get the third line, note that

$$\sup_{0 \leq u \leq d(\omega,t)} \{(f(\omega, t) - pA) \cdot u\} = [f(\omega, t) - pA]^+ \cdot d(\omega, t),$$

by simple constrained maximization. Finally, we have $\sup_{z \in \mathbb{R}^K} \{z \cdot s\} = \chi_{\{0\}}(s)$, since $\sup_{z \in \mathbb{R}^K} \{z \cdot s\}$ takes the value ∞ if $s_k \neq 0$ for $k = 1, \ldots, K$.

Using the conjugate L^* of the primal integrand L, we calculate the dual integrand M. For $(\omega, t) \in \Omega \times [0, T]$ and $s, p \in \mathbb{R}^K$, the dual integrand M is given by

$$M(\omega, t, p, s) = L^*(\omega, t, s, p).$$

That is, for $(\omega, t) \in \Omega \times [0, T]$ we have

$$\begin{aligned} M(\omega, t, y(\omega, t), \dot{y}(\omega, t)) &= L^*(\omega, t, \dot{y}(\omega, t), y(\omega, t)), \\ &= \chi_{\{0\}}(\dot{y}(\omega, t)) + [0 \vee (f(\omega, t) - y(\omega, t)A)] \cdot d(\omega, t), \end{aligned}$$

where the expression $\chi_{\{0\}}(\dot{y}(\omega, t))$ in the second line forces $\dot{y}(\omega, t) = 0$ for a.e. $(\omega, t) \in \Omega \times [0, T]$. Then, the dynamics of the dual variable y is given by

$$\begin{aligned} y(\omega, t) &= y_0 + \int_0^t \dot{y}(\omega, s) ds + M(\omega, t), \\ &= y_0 + M(\omega, t), \end{aligned}$$

where M is a square integrable martingale null at zero.

What remains is to derive the terminal conditions associated with the dual problem. To that end, define the functional m on $L_2^0 \times L_2^T$ as follows:

$$m(y_0, y_T) = l_0^*(y_0) + l_T^*(-y_T),$$

where l_0^* and l_T^* are the conjugates of l_0 and l_T. We calculate l_0^* as follows:

$$\begin{aligned} l_0^*(y) &= \sup_x \{y \cdot x - l_0(x)\}, \\ &= \sup_{x \in \{C\}} \{y \cdot x\}, \\ &= C \cdot y. \end{aligned}$$

A similar calculation yields $l_T^*(y) = \chi_{\mathbb{R}^K_-}(y)$. From Bismut (1973), the functional m for the dual problem is given by

$$\begin{aligned} m(y_0, y_T) &= l_0^*(y_0) + l_T^*(-y_T), \\ &= C \cdot y_0 + \chi_{\mathbb{R}^K_-}(-y_T), \\ &= C \cdot y_0 + \chi_{\mathbb{R}^K_+}(y_T), \end{aligned} \tag{27}$$

where the expression $\chi_{\mathbb{R}^K_+}(y_T)$ imposes that $y(\omega, T) \geq 0$ for a.e. $\omega \in \Omega$.

The dual problem of control is then to minimize

$$\mathbb{E} \int_0^T M(\omega, t, y(\omega, t), \dot{y}(\omega, t))dt + m(y_0, y_T),$$

which is equivalent to minimizing

$$\begin{aligned} &\mathbb{E}\left[\int_0^T d(\omega, t) \cdot [f(\omega, t) - y(\omega, t)A]^+ dt + C \cdot y_0(\omega)\right] \\ &\text{subject to} \\ &y(\omega, t) = y_0(\omega) + M(\omega, t), \quad (\omega, t) \in \Omega \times [0, T] \\ &y(\omega, T) \geq 0, \ \omega \in \Omega, \end{aligned} \tag{D}$$

where M is a square integrable martingale stopped at T, null at zero and adapted to the filtration $\{\mathcal{F}_t, t \in \mathbb{R}_+\}$. Since the network revenue management problem (P) is trivially feasible (simply let $u(\omega, t) = 0$ for all $(\omega, t) \in \Omega \times [0, T]$), the objective function values of (P) and (D) are equal to each other, cf. Theorem IV-1 of Bismut (1973). ■

Proof of Proposition 1. The network revenue management problem (P) and the dual problem (D) have the same optimal objective value by Theorem IV-1 of Bismut (1973). Moreover, by Theorem IV-2 of Bismut (1973), letting u be a feasible control for (P) with the corresponding state trajectory x, and (y_0, M) be a feasible control for (D) with the corresponding state trajectory y, the controls u and (y_0, M) are optimal for (P) and (D), respectively, if and only if they satisfy the coextremality conditions stated in Definition IV-1 of Bismut (1973). To be more specific about the coextremality conditions for the network revenue management problem and its dual problem, we derive the subgradients of L, l_0 and l_T, where L is a convex integrand and l_0 and l_T are convex functionals as in the derivation of the dual network revenue management problem (D).

First, we calculate the subgradient of L from its epigraphical normals. To that end, we use Theorem 8.9 of Rockafellar & Wets (1997) which proves that for $h : \mathbb{R}^n \to [-\infty, +\infty]$ and any point $\overline{x}$ at which h is finite, one has

$$\partial h(\overline{x}) = \{v : (v, -1) \in N_{\text{epi } h}(\overline{x}, h(\overline{x}))\},$$

where, epi h denotes the epigraph of h defined as

$$\text{epi } h := \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} : \alpha \geq h(x)\},$$

and $N_{\text{epi } h}(\overline{x}, h(\overline{x}))$ is the set of vectors normal to the set epi h at $(\overline{x}, h(\overline{x}))$ in the general sense as in Definition 6.3 of Rockafellar & Wets (1997).

For $(\omega, t) \in \Omega \times [0, T]$, the epigraph of the integrand L at (ω, t) is given by

$$\text{epi } L(\omega, t) = \{(x, \dot{x}, \alpha) \in \mathbb{R}^{2K} \times \mathbb{R} : \ \dot{x} = -Au, \ 0 \leq u \leq d(\omega, t), \ \alpha \geq -f(\omega, t) \cdot u\},$$

since the points $(x, \dot{x}) \in \mathbb{R}^{2K}$ where $L(\omega, t, x, \dot{x}) = \infty$ are such that the vertical line $(x, \dot{x}) \times \mathbb{R}$ misses epi $L(\omega, t)$. Then, we can write

$$\partial L(\omega, t, \overline{x}, \overline{\dot{x}}) = \{(v^1, v^2) \in \mathbb{R}^{2K} : (v^1, v^2, -1) \in N_{\text{epi } L(\omega,t)}(\overline{x}, \overline{\dot{x}}, L(\omega, t, \overline{x}, \overline{\dot{x}}))\}. \tag{28}$$

First, note that for $(\omega, t) \in \Omega \times [0, T]$, epi $L(\omega, t)$ is a convex set and the point $(\overline{x}, \overline{\dot{x}}, L(\omega, t, \overline{x}, \overline{\dot{x}})$ is an element of epi $L(\omega, t)$ for $(\overline{x}, \overline{\dot{x}}) \in \mathbb{R}^{2K}$. Let $\mathbf{v}$ denote an arbitrary element of $\mathbb{R}^{2K+1}$, where the first K components of $\mathbf{v}$ is denoted as v^1, the subsequent K components by v^2 and the last component by v^α. That is, $\mathbf{v} = [v^1, v^2, v^\alpha]'$, where $v^1, v^2 \in \mathbb{R}^K$ and $v^\alpha \in \mathbb{R}$. Then, Theorem 6.9 of Rockafellar & Wets (1997), gives

$$N_{\text{epi } L(\omega,t)}(\overline{x}, \overline{\dot{x}}, L(\omega, t, \overline{x}, \overline{\dot{x}})) = \{\mathbf{v} \in \mathbb{R}^{2K+1} : \ [(x, \dot{x}, \alpha) - (\overline{x}, \overline{\dot{x}}, L(\omega, t, \overline{x}, \overline{\dot{x}}))] \cdot \mathbf{v} \ \leq 0, \ \forall (x, \dot{x}, \alpha) \in \text{epi } L(\omega, t)\}. \tag{29}$$

We next establish the following properties of $N_{\text{epi } L(\omega,t)}(\overline{x}, \overline{\dot{x}}, L(\omega, t, \overline{x}, \overline{\dot{x}}))$ for $(\omega, t) \in \Omega \times [0, T]$, which will assist us in finding the subgradients of L.

Property 1 *For $(\omega, t) \in \Omega \times [0, T]$, if $\mathbf{v} = (v^1, v^2, v_\alpha)' \in N_{epi\ L(\omega,t)}(\overline{x}, \overline{\dot{x}}, L(\omega, t, \overline{x}, \overline{\dot{x}}))$, then $v^1 = 0$.*

To verify Property 1, first note that any $\mathbf{v} = (v^1, v^2, v_\alpha)'$ such that $v^1_k < 0$ cannot be in $N_{\text{epi } L(\omega,t)}(\overline{x}, \overline{\dot{x}}, L(\omega, t, \overline{x}, \overline{\dot{x}}))$. Suppose not. Then, we could find an element $(\widetilde{x}, \widetilde{\dot{x}}, \widetilde{\alpha})$ of epi $L(\omega, t)$ such that it is equal to $(\overline{x}, \overline{\dot{x}}, L(\omega, t, \overline{x}, \overline{\dot{x}}))$ except the k^{th} component of $\widetilde{x}$, where we have $\widetilde{x}_k < \overline{x}_k$. However, we have

$$[(\widetilde{x}, \widetilde{\dot{x}}, \widetilde{\alpha}) - (\overline{x}, \overline{\dot{x}}, L(\omega, t, \overline{x}, \overline{\dot{x}})] \cdot \mathbf{v} = (\widetilde{x}_k - \overline{x}_k) v^1_k > 0,$$

contradicting the fact that $(v^1, v^2, v_\alpha) \in N_{\text{epi } L(\omega,t)}(\overline{x}, \overline{\dot{x}}, L(\omega, t, \overline{x}, \overline{\dot{x}}))$, cf. (29). Similarly, any (v^1, v^2, v_α) such that $v^1_k > 0$ cannot be an element of $N_{\text{epi } L(\omega,t)}(\overline{x}, \overline{\dot{x}}, L(\omega, t, \overline{x}, \overline{\dot{x}}))$, which proves Property 1. Coupled with (28) and Theorem IV-1 of Bismut (1973), Property 1 proves that $\dot{y}(\omega, t) = 0$ for a.e. $(\omega, t) \in \Omega \times [0, T]$.

Property 2 *For $(\omega, t) \in \Omega \times [0, T]$ and $j = 1, \ldots, J$ with $d_j(\omega, t) > 0$, if $\overline{\dot{x}} = -A\overline{u}$ for $\overline{u}$ such that $0 \leq \overline{u} \leq d(\omega, t)$ and $\mathbf{v} = (v^1, v^2, v_\alpha) \in N_{epi\ L}(\overline{x}, \overline{\dot{x}}, L(\omega, t, \overline{x}, \overline{\dot{x}}))$, then, the following conditions hold.*

$$\begin{aligned}
(v^2 A + v^\alpha f(\omega, t))_j &\geq \ 0, \ \ \textit{if} \ \ \overline{u}_j = 0, \\
(v^2 A + v^\alpha f(\omega, t))_j &= \ 0, \ \ \textit{if} \ \ 0 < \overline{u}_j < d(\omega, t), \\
(v^2 A + v^\alpha f(\omega, t))_j &\leq \ 0, \ \ \textit{if} \ \ \overline{u}_j = d_j(\omega, t).
\end{aligned}$$

To establish Property 2, first recall that for any $(x, \dot{x}, \alpha) \in$ epi $L(\omega, t)$, there exists some $u \in \mathbb{R}^J$ such that $\dot{x} = -Au$, $0 \leq u \leq d(\omega, t)$ and $\alpha \geq -f(\omega, t) \cdot u$. Consider now an element $(\overline{x}, \dot{x}, -f(\omega, t) \cdot u)$ of epi

$L(\omega,t)$, where $\dot{x} = -Au$. Then, we the following holds for $\mathbf{v} = (v^1, v^2, v_\alpha) \in N_{\text{epi }L}(\overline{x}, \dot{\overline{x}}, L(\omega,t,\overline{x},\dot{\overline{x}}))$.

$$\begin{aligned}
[(\overline{x}, \dot{x}, -f(\omega,t)\cdot u) - (\overline{x}, \dot{\overline{x}}, L(\omega,t,\overline{x},\dot{\overline{x}})] \cdot \mathbf{v} &= v^1 \cdot (\overline{x} - \overline{x}) + v^2 \cdot (\dot{x} - \dot{\overline{x}}) + v^\alpha(-f(\omega,t)\cdot u - L(\omega,t,\overline{x},\dot{\overline{x}})), \\
&= v^2 \cdot (A\overline{u} - Au) + v^\alpha(-f(\omega,t)\cdot u + f(\omega,t)\cdot \overline{u}), \\
&= (v^2 A + v^\alpha f(\omega,t)) \cdot (\overline{u} - u).
\end{aligned}$$

First, consider the case when $\overline{u}_j = 0$ for some $j = 1,\ldots,J$. If $u = \overline{u}$ except for the j^{th} component, we have $(v^2 A + v^\alpha f(\omega,t)) \cdot (\overline{u} - u) \le 0$, only if $(v^2 A + v^\alpha f(\omega,t))_j \ge 0$. From (29), since $(\overline{x}, \dot{x}, -f(\omega,t)\cdot u)$ is an element of epi $L(\omega,t)$, this proves the first part of Property 2, namely, if $\overline{u}_j = 0$, then $(v^2 A + v^\alpha f(\omega,t))_j \ge 0$. The argument is similar for the cases when $0 < \overline{u}_j < d(\omega,t)$ and $\overline{u}_j = d(\omega,t)$ and this completes the proof of Property 2.

To summarize, for $(\overline{x}, \dot{\overline{x}})$ such that $L(\omega,t,\overline{x},\dot{\overline{x}}) < \infty$ and $\dot{\overline{x}} = -A\overline{u}$, if $(v^1, v^2, v_\alpha) \in N_{\text{epi }L(\omega,t)}(\overline{x}, \dot{\overline{x}}, L(\omega,t,\overline{x},\dot{\overline{x}}))$, then from Property 2, the following conditions hold for $j = 1,\ldots,J$.

$$\begin{aligned}
(v^2 A + v^\alpha f(\omega,t))_j &\ge 0, \text{ if } \overline{u}_j = 0, \\
(v^2 A + v^\alpha f(\omega,t))_j &= 0, \text{ if } 0 < \overline{u}_j < d_j(\omega,t), \\
(v^2 A + v^\alpha f(\omega,t))_j &\le 0, \text{ if } \overline{u}_j = d_j(\omega,t).
\end{aligned}$$

Recall that the subgradient of L is related to the normal cone of its epigraph as follows.

$$\partial L(\omega,t,\overline{x},\dot{\overline{x}}) = \{(v^1, v^2) : (v^1, v^2, -1) \in N_{\text{epi }L}(\overline{x}, \dot{\overline{x}}, L(\omega,t,\overline{x},\dot{\overline{x}}))\},$$

The coextremality conditions in Definition IV-1 of Bismut (1973), state that $d\mathbb{P} \otimes dt$ a.s.

$$(\dot{y}(\omega,t), y(\omega,t)) \in \partial L(\omega,t,x(\omega,t),\dot{x}(\omega,t)).$$

That is, for a.e. $(\omega,t) \in \Omega \times [0,T]$,

$$(\dot{y}(\omega,t), y(\omega,t), -1) \in N_{\text{epi }L}(x(\omega,t), \dot{x}(\omega,t), L(\omega,t,x(\omega,t),\dot{x}(\omega,t)))\}.$$

This implies that for $j = 1,\ldots,J$ and a.e. $(\omega,t) \in \Omega \times [0,T]$, we have

$$(y(\omega,t)A - f(\omega,t))_j \ge 0, \text{ if } u_j(\omega,t) = 0, \tag{30}$$

$$(y(\omega,t)A - f(\omega,t))_j = 0, \text{ if } 0 < u_j(\omega,t) < d_j(\omega,t), \tag{31}$$

$$(y(\omega,t)A - f(\omega,t))_j \le 0, \text{ if } u_j(\omega,t) = d_j(\omega,t), \tag{32}$$

which establishes the coextremality conditions stated in (13).

To complete the proof of Proposition 1, we calculate the subgradients $\partial l_0(x_0)$, and $\partial l_T(x_T)$, and derive the coextremality condition (12). First, consider l_0, which is defined as $l_0(x_0) = \chi_{\{C\}}(x_0)$.We will use Theorem 8.9 of Rockafellar & Wets (1997) to calculate $\partial l_0(\overline{x})$. At any point $\overline{x}$ for which l_0 is finite, we have

$$\partial l_0(\overline{x}) = \{v : (v, -1) \in N_{\text{epi }l_0}(\overline{x}, l_0(\overline{x}))\},$$

where epi l_0 is given by

$$\begin{aligned} \text{epi } l_0 &= \{(x,\alpha) \in \mathbb{R}^K \times \mathbb{R} : x = C,\ \alpha \geq 0\}, \\ &= C_1 \times \ldots \times C_K \times \mathbb{R}_+. \end{aligned}$$

Notice that epi l_0 is a box, and hence, we can use Example 6.10 of Rockafellar & Wets (1997) to calculate its normal cone. As a result,

$$N_{\text{epi } l_0}(\overline{x}, l_0(\overline{x})) = N_{C_1}(\overline{x}_1) \times \ldots \times N_{C_K}(\overline{x}_K) \times N_{\mathbb{R}_+}(l_0(\overline{x})),$$

where $N_{C_k}(\overline{x}_1) = (-\infty, \infty)$ for $k = 1, \ldots, K$ since C_k is a one-point interval. Finally, $N_{\mathbb{R}_+}(l_0(\overline{x})) = (-\infty, 0]$ because for a feasible $\overline{x}$, we have $l_0(\overline{x}) = 0$, and in that case we are at the left end point of the interval $\mathbb{R}_+$, which implies through Example 6.10 of Rockafellar & Wets (1997) that $N_{\mathbb{R}_+}(l_0(\overline{x})) = (-\infty, 0]$. In consequence, the coextremality condition $y_0 \in \partial l_0(x_0)$, cf. Definition IV-1 of Bismut (1973), for a feasible y_0 is equivalent to

$$(y_0, -1) \in N_{\text{epi } l_0}(y_0, l_0(y_0))$$

and places no further restrictions on y_0.

Finally, we calculate the subgradient of $l_T(x_T)$ where $l_T(x_T) = \chi_{\mathbb{R}_+^K}(x_T)$.Again, epi l_0 is a box. Indeed, epi $l_0 = \mathbb{R}_+^{J+1}$, and we can resort to Example 6.10 of Rockafellar & Wets (1997). We have

$$N_{\text{epi } l_T}(\overline{x}, l_T(\overline{x})) = N_{\mathbb{R}_+}(\overline{x}_1) \times \ldots \times N_{\mathbb{R}_+}(\overline{x}_K) \times N_{\mathbb{R}_+}(l_T(\overline{x})).$$

Consequently, $N_{C_k}(\overline{x}_1) = (-\infty, 0]$ for $\overline{x}_1 \geq 0$ and $N_{C_k}(\overline{x}_1) = \{0\}$ if $\overline{x}_1 > 0$ for $k = 1, \ldots, K$. Finally, $N_{\mathbb{R}_+}(l_T(\overline{x})) = (-\infty, 0]$ since $l_T(\overline{x}) = 0$ for a feasible $\overline{x}$. Thus, the coextremality condition $-y_T \in \partial l_T(x_T)$ in Definition IV-1 of Bismut (1973), implies that $y(\omega, T) \geq 0$ and $y(\omega, T) \cdot x(\omega, T) = 0$ for a.e. $\omega \in \Omega$. This establishes the coextremality condition 12 and completes the proof of Proposition 1. ■

C Proofs in Section 5

Proof of Theorem 1. We interpret the dual variables in dual network revenue management problem (D) as the opportunity cost of resources and construct optimal bid-price and capacity usage limit processes for the network revenue management problem (P) using them. To this end, fix an optimal solution u to (P) and an optimal solution y to its dual (D). Given the optimal solutions u and y, define bid-price process π and the capacity usage limit process λ as follows:

$$\pi(\omega, t) := y(\omega, t) \ \text{ and } \ \lambda(\omega, t) := Au(\omega, t) \text{ for } (\omega, t) \in \Omega \times [0, T]. \tag{33}$$

Then, the bid-price process π is a martingale adapted to $\{\mathcal{F}_t,\ t \in \mathbb{R}_+\}$ since y is so. Having defined π and λ, let $u^{(\pi,\lambda)}(\omega, t)$ denote the booking rate vector under the generalized bid-price control (π, λ) for $(\omega, t) \in \Omega \times [0, T]$. That is, $u^{(\pi,\lambda)}(\omega, t)$ solves $(\text{P}(\omega, t))$ for $(\omega, t) \in \Omega \times [0, T]$. Observe that the booking rate process $\{u^{(\pi,\lambda)}(\omega,t) : (\omega,t) \in \Omega \times [0,T]\}$ is clearly feasible for the network revenue management problem (P). To see this, note that since $u^{(\pi,\lambda)}(\omega,t)$ solves (P(ω,t)), it clearly satisfies the demand constraints. Moreover, we have

$$Au^{(\pi,\lambda)}(\omega,t) \leq \lambda(\omega,t).$$

Then integrating both sides of this over $[0,T]$, using the definition of $\lambda(\omega,t)$, cf. (33), and the fact that u is feasible for (P), we conclude that

$$AU^{(\pi,\lambda)\cdot}(\omega,t) \leq \int_0^T \lambda(\omega,t)dt = A\int_0^T u(\omega,t) = AU(\omega,T) \leq C,$$

where $U^{(\pi,\lambda)}(\omega,t)$ denotes the vector of cumulative bookings under (π,λ) up to time T. Thus, the booking policy $u^{(\pi,\lambda)}$ is feasible for (P).

To establish the optimality of $u^{(\pi,\lambda)}$, we will show that $u^{(\pi,\lambda)}$ maximizes the expected revenues, that is,

$$\mathbb{E}\int_0^T f(\omega,t)\cdot u^{(\pi,\lambda)}(\omega,t)dt = \mathbb{E}\int_0^T f(\omega,t)\cdot u(\omega,t)dt.$$

To that end, first note that $u(\omega,t)$ is feasible for (P(ω,t)). To see this, note that $0 \leq u(\omega,t) \leq d(\omega,t)$, which follows because u solves (P), and that $Au(\omega,t) = \lambda(\omega,t)$ by definition of λ, cf. (33). Then $u(\omega,t)$ solves (P(ω,t)) for a.e. $(\omega,t) \in \Omega \times [0,T]$ because $u_j(\omega,t) = d_j(\omega,t)$ whenever $y(\omega,t)A^j < f_j(\omega,t)$ by the coextremality conditions, cf. Proposition 1. That is, $u(\omega,t)$ is an optimal solution to (P(ω,t)), which in particular exhausts the capacity usage limit, *i.e.* $Au(\omega,t) = \lambda(\omega,t)$ by construction of λ. In other words, $u(\omega,t)$ not only maximizes the first term in the objective of (P(ω,t)) but also the second term by setting it to zero, which is the maximum it can be since we require $Au \leq \lambda(\omega,t)$ in (P(ω,t)). Then since (P(ω,t)) is lexicographic any bookings under (π,λ) must not only maximize the first term but also it must set the second term to zero. In particular, we must have

$$Au^{(\pi,\lambda)}(\omega,t) = \lambda(\omega,t) \quad \text{for a.e. } (\omega,t) \in \Omega \times [0,T].$$

Moreover, $Au(\omega,t) = \lambda(\omega,t)$ by construction of (π,λ), cf. (33). In other words, the primal controls u and $u^{(\pi,\lambda)}$ result in the same state trajectory.

Since both $u(\omega,t)$ and $u^{(\pi,\lambda)}(\omega,t)$ are optimal solutions for (P(ω,t)), for a.e. $(\omega,t) \in \Omega \times [0,T]$, we have

$$(f(\omega,t) - A'\pi(\omega,t))\cdot u^{(\pi,\lambda)}(\omega,t) = (f(\omega,t) - A'\pi(\omega,t))\cdot u(\omega,t). \tag{34}$$

Moreover, as argued immediately above we also have

$$Au^{(\pi,\lambda)} = \lambda(\omega,t) = Au(\omega,t). \tag{35}$$

Then combining (34)-(35), we conclude that

$$f(\omega,t)\cdot u^{(\pi,\lambda)}(\omega,t) = f(\omega,t)\cdot u(\omega,t) \quad \text{for a.e. } (\omega,t) \in \Omega \times [0,T].$$

Thus, the expected revenue generated by $u^{(\pi,\lambda)}$ is equal to the expected revenue generated by u, proving the optimality of the generalized bid-price control (π, λ) for the continuous network revenue management problem (P). ■

D Proofs in Section 6

Derivation of the dual problem (D^ε). We follow the same steps as in the derivation of (D). That is, we first append the penalty expressions associated with the demand and capacity restrictions on bookings in the objective function by defining the convex, extended real valued integrand L_ε and the convex functional l_ε. The problem is also formulated towards minimization. Next, we compute the conjugate convex functions associated with L_ε and l_ε and define the dual integrand M_ε and the dual functional m_ε, by the help of which we define the dual problem of control.

Define L_ε, the normal convex integrand in the sense of Rockafellar (1968), on $\Omega \times [0, \infty) \times \mathbb{R}^K \times \mathbb{R}^K$, as follows:

$$L_\varepsilon(\omega, t, x, \dot{x}) = \begin{cases} -f(\omega,t)\cdot u + \sum_{j=1}^{J} \frac{\varepsilon_j(\omega,t)u_j^2}{2}\cdot u + \chi_{\mathbb{R}^J_+}(u) + \chi_{\mathbb{R}^J_-}(u - d(\omega,t)) & \text{if } \dot{x} = -Au, \\ \infty & \text{otherwise.} \end{cases} \tag{36}$$

Since the terminal conditions of (P^ε) are the same as the terminal conditions of (P), the functional l_ε, which specifies the terminal conditions and the initial system parameters, is the same as l, cf. (24) and (25). Hence, all the terminal conditions in the dual formulation (D^ε) are the same as the terminal conditions of (D). Then, (P^ε) is equivalent to minimizing

$$\mathbb{E}\int_0^T L_\varepsilon(\omega, t, x(\omega,t), \dot{x}(\omega,t))dt + l_\varepsilon(x_0, x_T).$$

In order to define the dual problem of control, we derive the conjugate to the function L_ε. Let L_ε^* be the dual integrand of L_ε. That is,

$$\begin{aligned} L_\varepsilon^*(\omega,t,s,p) &= \sup_{z\in\mathbb{R}^K,\ y\in\mathbb{R}^K} \{z\cdot s + y\cdot p - L_\varepsilon(\omega,t,z,y)\} \quad \text{for} \quad s,p\in\mathbb{R}^K \\ &= \sup_{0\le u\le d(\omega,t)} \{z\cdot s - pAu - (-f(\omega,t)\cdot u + \sum_{j=1}^{J}\frac{\varepsilon_j(\omega,t)u_j^2}{2})\}, \\ &= \sup_{z\in\mathbb{R}^K}\{z\cdot s\} + \sup_{0\le u\le d(\omega,t)}\{(f(\omega,t)-pA)\cdot u - \sum_{j=1}^{J}\frac{\varepsilon_j(\omega,t)u_j^2}{2})\}, \\ &= \chi_{\{0\}}(s) + g_\varepsilon(f(\omega,t)-pA, d(\omega,t)), \end{aligned}$$

where $g_\varepsilon(z,d) = \sum_{j=1}^{J} h_\varepsilon(z_j, d_j)$ and h_ε is given by (16). The second line is obtained by replacing y with $-Au$ for $0 \le u \le d(\omega,t)$ and noting that $L_\varepsilon(\omega,t,z,y) = -f(\omega,t)\cdot u + \sum_{j=1}^{J}\frac{\varepsilon_j(\omega,t)u_j^2}{2}$. The third line follows from the observation that we can take the supremum in the second line separately for z and u. To get the fourth line, note that $g_\varepsilon(z, d(\omega, t))$ is the value function of the following maximization problem.

$$\text{Maximize}_{0 \le v \le d(\omega,t)} \; z \cdot v - \sum_{j=1}^{J} \frac{\varepsilon_j(\omega, t)}{2} v_j^2,$$

whose solution for $j = 1, \ldots, J$ is given by

$$v_\varepsilon^*(z, \omega, t) = (v_{1,\varepsilon}^*, \ldots, v_{J,\varepsilon}^*),\; v_{j,\varepsilon}^*(z_j, \omega, t) = \begin{cases} 0 & \text{if} \quad z_j \le 0, \\ d_j(\omega, t) & \text{if} \quad z_j \ge \varepsilon, \\ \frac{z_j}{\varepsilon} d_j(\omega, t) & \text{if} \quad 0 < z_j < \varepsilon. \end{cases}$$

The terminal conditions associated with the dual problem to (P^ε) are derived as follows. As mentioned above, we have $l_\varepsilon = l$ and, hence $l_\varepsilon^* = l^*$. This, in turn implies that $m_\varepsilon = m$, where m is defined in (27). The dual problem of control is then to minimize

$$\mathbb{E} \int_0^T M_\varepsilon(\omega, t, y(\omega, t), \dot{y}(\omega, t))dt + m_\varepsilon(y_0, y_T),$$

which is equivalent to minimizing

$$\begin{aligned} & \mathbb{E}\left[\int_0^T g_\varepsilon(f(\omega, t) - y(\omega, t)A, d(\omega, t))dt + C \cdot y_0(\omega)\right] \\ & \text{subject to} \\ & y(\omega, t) = y_0(\omega) + M(\omega, t), \quad (\omega, t) \in \Omega \times [0, T] \\ & y(\omega, T) \ge 0,\; \omega \in \Omega, \end{aligned} \tag{D}$$

where M is a square integrable martingale stopped at T, null at zero and adapted to the filtration $\{\mathcal{F}_t, t \in \mathbb{R}_+\}$. Since the primal problem (P^ε) is trivially feasible (simply let $u(\omega, t) = 0$ for all $(\omega, t) \in \Omega \times [0, T]$), the objective function values of (P^ε) and (D^ε) are equal to each other, cf. Theorem IV-2 of Bismut (1973). ■

Proof of Proposition 2. The perturbed problem (P^ε) and its dual problem (D^ε) have the same optimal objective value by Theorem IV-1 of Bismut (1973). From Theorem IV-2 of Bismut (1973), letting u^ε be a feasible control for (P^ε) with the corresponding state trajectory x^ε, and $(y_0^\varepsilon, M^\varepsilon)$ be a feasible control for (D^ε) with the corresponding state trajectory y^ε, the controls u^ε and $(y_0^\varepsilon, M^\varepsilon)$ are optimal for (P^ε) and (D^ε), respectively, if and only if they satisfy the coextremality conditions stated in Definition IV-1 of Bismut (1973). Recall that the terminal conditions of the perturbed problem (P^ε) and the network revenue management problem (P) are the same. Thus, the coextremality conditions for the problems (P^ε) and (D^ε) regarding the terminal conditions are the same as those for the problems (P) and (D). This, in turn, establishes the coextremality condition (17). To compute the coextremality conditions stated in Definition IV-1 of Bismut (1973), the subgradient of the convex integrand L_ε defined in (36) needs to be calculated.

To calculate the subgradient of L_ε, first note that for $(\omega, t) \in \Omega \times [0, T]$, we can write the following:

$$L_\varepsilon(\omega, t, x, \dot{x}) - L(\omega, t, x, \dot{x}) - \sum_{j=1}^{J} \frac{\varepsilon_j(\omega, t) u_j^2}{2} = 0 \;\text{ if } \dot{x} = -Au, \tag{37}$$

where L is the convex integrand for the network revenue management problem (P) given in (23). Let $\partial_u L_\varepsilon(\omega,t,x,\dot{x})$ and $\partial_{\dot{x}} L_\varepsilon(\omega,t,x,\dot{x})$ denote the subgradients of L_ε with respect to u and $\dot{x}$, respectively. That is, $\partial_{\dot{x}} L_\varepsilon(\omega,t,x,\dot{x})$ is the projection of the set $\partial L_\varepsilon(\omega,t,x,\dot{x})$ on $\dot{x}$ axis. $\partial_u L_\varepsilon(\omega,t,x,\dot{x})$ is similarly defined by viewing $\dot{x}$ as a function of u. Notice that L and L_ε are convex and piecewise linear quadratic, cf. Definition 10.20 of Rockafellar & Wets (1997). Therefore, from Exercise 10.22 of Rockafellar & Wets (1997), we can calculate the subgradients of each term in (37) separately. The subgradient of L with respect to $\dot{x}$ is already calculated in the proof of Proposition 1. To calculate $\partial_u L(\omega,t,x,\dot{x})$ and $\partial_u L_\varepsilon(\omega,t,x,\dot{x})$ at $(x,\dot{x})$ such that $L(\omega,t,x,\dot{x}), L_\varepsilon(\omega,t,x,\dot{x}) < \infty$, we will use the basic chain rule for subgradients as in Theorem 10.6 of Rockafellar & Wets (1997), which implies that

$$\partial_{\dot{x}} L_\varepsilon(\omega,t,x,\dot{x})^*(-A') - \partial_{\dot{x}} L(\omega,t,x,\dot{x})^*(-A') - \Delta^\varepsilon(\omega,t,u) = \{0\},$$

where $\Delta_j^\varepsilon(\omega,t,u) = \varepsilon_j(\omega,t)u_j$. Then, $y(\omega,t) \in \partial_{\dot{x}} L_\varepsilon(\omega,t,x,\dot{x})$ if and only if

$$(-y(\omega,t)A - \Delta^\varepsilon(\omega,t,u)) \in -A^*\partial_{\dot{x}} L(\omega,t,x,\dot{x}). \tag{38}$$

Definition IV-1 of Bismut (1973) and the coextremality conditions in Proposition 1 imply that if $v \in A^*\partial_{\dot{x}} L(\omega,t,x,\dot{x})$, then for $j = 1,\ldots,J$ with $d_j(\omega,t) > 0$,

$$v_j - f_j(\omega,t) \geq 0, \;\text{ if }\; u_j = 0, \tag{39}$$

$$v_j - f_j(\omega,t) = 0, \;\text{ if }\; 0 < u_j < d_j(\omega,t), \tag{40}$$

$$v_j - f_j(\omega,t) \leq 0, \;\text{ if }\; u_j = d_j(\omega,t). \tag{41}$$

Then, together with (38), the conditions (39)-(41) establish the coextremality conditions (18). ■

Proof of Proposition 3. For each $\varepsilon > 0$, the optimality of the control $\{u^\varepsilon(\omega,t) : (\omega,t) \in \Omega \times [0,T]\}$ given in (20) for the perturbed problem (P^ε) follows from Theorem IV-2 of Bismut (1973), and the fact that u^ε and y^ε satisfy the coextremality conditions stated in Proposition 2 where y^ε is an optimal state trajectory for (D^ε). We next argue that for each $\varepsilon > 0$, (P^ε) has a unique solution. Suppose not. Then, there exists booking controls $\widetilde{u}$ and $\overline{u}$ that are optimal for (P^ε) and yet are not equal on a set of strictly positive $d\mathbb{P} \otimes dt$ measure. From Theorem IV-2 of Bismut (1973), both $\widetilde{u}$ and $\overline{u}$ should satisfy the coextremality conditions (17)-(18) with y^ε. However, this implies that $\widetilde{u}(\omega,t)$ and $\overline{u}(\omega,t)$ are equal for a.e. $(\omega,t) \in \Omega \times [0,T]$. Contradiction. Hence, for each $\varepsilon > 0$, u^ε is the unique solution for (P^ε). ■

Proof of Theorem 2. The bid-price process π^ε is defined as $\pi^\varepsilon = y^\varepsilon$, where y^ε is an optimal state trajectory for the perturbed dual problem (D^ε). Since y^ε forms a martingale, so does π^ε. The terminal condition $y^\varepsilon(\omega,T) \geq 0$ for a.e. $\omega \in \Omega$ and the fact that y^ε is a martingale guarantees that $\pi^\varepsilon(\omega,t) \geq 0$ for a.e. $(\omega,t) \in \Omega \times [0,T]$, which proves the first part of Theorem 2.

Next, we prove that the bid-price policy $(\pi^\varepsilon, \phi^\varepsilon)$ is κ-optimal, where κ is given by (21). To that end, let u be an optimal booking control for the network revenue management problem (P). Notice that u is also feasible for the perturbed problem (P^ε). From Proposition 3, the control $\{u^\varepsilon(\omega,t) : (\omega,t) \in \Omega \times [0,T]\}$

resulting from the bid-price policy $(\pi^\varepsilon, \phi^\varepsilon)$ is the unique optimal control for the perturbed problem (P^ε) for each $\varepsilon > 0$. Then, the objective value of u^ε is greater than or equal to the objective value of u for the perturbed problem (P^ε). Thus, we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^T [f(\omega,t)\cdot u(\omega,t)]\,dt\right] - \varepsilon\sum_{j=1}^J \int_0^T \mathbb{E}[d_j(\omega,t)]dt \quad &\le \quad \mathbb{E}\left[\int_0^T [f(\omega,t)\cdot u(\omega,t) - \frac{1}{2}\sum_{j=1}^J \varepsilon_j(\omega,t)\, u_j^2(\omega,t)]\,dt\right] \\
&\le \quad \mathbb{E}\left[\int_0^T [f(\omega,t)\cdot u^\varepsilon(\omega,t) - \frac{1}{2}\sum_{j=1}^J \varepsilon_j(\omega,t)\,(u_j^\varepsilon(\omega,t))^2]\,dt\right], \\
&\le \quad \mathbb{E}\left[\int_0^T [f(\omega,t)\cdot u^\varepsilon(\omega,t)]\,dt\right]. \qquad (42)
\end{aligned}$$

The first inequality follows from the definition of $\varepsilon_j(\omega,t)$, cf. (15), and the fact that $u(\omega,t) \le d(\omega,t)$ for a.e. $(\omega,t) \in \Omega \times [0,T]$. The second inequality is given by feasibility of u for (P^ε) and optimality of u^ε. The last inequality holds since $\sum_{j=1}^J \varepsilon_j(\omega,t)\,(u_j^\varepsilon(\omega,t))^2 \ge 0$, and it proves that u^ε is $\kappa\varepsilon$-optimal since u^ε is also feasible for (P) and completes the proof of the second part of Theorem 2.

Finally, we show that every weak limit $\widetilde{u} \in \mathcal{U}$ of the booking controls $\{u^\varepsilon : \varepsilon > 0\}$ is an optimal booking control for the network revenue management problem (P). Let $\{u^{\varepsilon_n} : n \ge 1\}$ be a sequence of feasible controls for the network revenue management problem (P) which converges to $\widetilde{u}$ in the weak* topology as $\varepsilon_n \searrow 0$. From Lemma 1, $\widetilde{u}$ is a feasible booking control for (P) and, we have

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T f(\omega,t)\cdot u^{\varepsilon_n}(\omega,t)dt\right] = \mathbb{E}\left[\int_0^T f(\omega,t)\cdot \widetilde{u}(\omega,t)dt\right].$$

The optimality of $\widetilde{u}$, then follows from (42). ■

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