Price Dispersion in Electricity Auctions: Strategic Analysis and Economic Implications

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The paper examines two interrelated questions: (a) the effect of capacity on wholesale pricing decisions in competitive settings, when both final demand and supply are price insensitive and (b) comparison of performance of two auctions (uniform and discriminatory) in such an environment. The problem is motivated by wholesale electricity market and the model allows us to explain high price volatility which is its common feature. The presented model focuses on the structural impacts of inelasticity and randomness of demand, variable production costs, and fixed capacity. We show that price dispersion stems from suppliers' randomized bidding and the variance of the price dispersion is mainly influenced by capacity utilization and "technological" asymmetry among suppliers. Introduction of demand uncertainty increases the chance of price dispersion but not necessarily the magnitude of price variance. Empirical data from the New England Power Pool (NEPOOL) illustrates our theoretical predictions related to price dispersion. The comparison between discriminatory and uniform auctions indicates that, at symmetric equilibria, they yield the same average price but discriminatory auction results in lower price volatility and, thus, might be more desirable. This insight continues to hold with uncertain demand and well describes cases with asymmetric bidders.

1. Introduction

Since the year 2000, high price volatility, including occasional extreme price shocks, has been the most prominent characteristic of wholesale electricity markets. Figure 1 displays hourly spot electricity prices in the New England market (NEPOOL) over the period January 2004 through June 2006, with prices ranging from \$0/MWh to large price spikes above \$200/MWh. The price behavior observed in NEPOOL is far from unique. For the same time period, the prices in other major electricity trading hubs (PJM, NYISO, and MISO) range from -\$10/MWh[1] to above \$250/MWh, including NYISO, the New York City hub, where price has a record low of -\$279/MWh and a record high of 1894/MWh.

In this paper we focus on wholesale markets, where retailers are facing price insensitive demand and suppliers have constant costs and capacities. While many examples have some features of such

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[1] Negative electricity prices may result from suppliers' incentive to maintain power generation in order to overcome high start-up costs.

Figure 1: NEPOOL Internal Hub Hourly Electricity Spot Prices (Jan.2004-Jun.2006)

markets, its extreme case is deregulated wholesale electricity market, which both motivated this study and is our focus throughout the paper.

Major electricity buyers are the energy distributors who procure power from wholesale markets and transmit it to their customers (residential, commercial, and a portion of industrial users). The retail prices paid by electricity end-consumers are regulated and, thus, very stable in short-to-medium term, with instantaneous demand almost inelastic to the wholesale prices. On the supply side, each electricity generation unit with specific technology has a fixed capacity associated with it and a fairly stable variable cost.

Our first objective is to characterize the drivers and structural reasons behind price variability. In order to understand their significance, we examine effects of individual factors such as capacity, end-customer demand variability, and suppliers' asymmetry. Our approach is to answer these questions through a theoretical model that captures the critical inter-dependencies of energy market. While previous models assumed a competitive equilibrium outcome, where all suppliers are pricing at their marginal costs, we relax this assumption and examine whether price dispersion is driven by structural factors and whether it might exist even without demand uncertainty (critical for the existing models). Later we use data from wholesale energy markets to evaluate how consistent they are with our theoretical model.

Our second objective, that builds on the structural form of pricing strategies, is to evaluate and compare two dominant forms of auctions, uniform and discriminatory, in order to provide justification which of them is more appropriate for electricity markets, with emphasis both on efficiency (average prices) as well as price dispersion (price volatility). This is motivated by recent policy changes and the debates surrounding energy market. Specifically, in March 2001, seeking a better market performance, the British government implemented a radical reform in the electricity trading arrangements, and replaced uniform auction (UA) with a discriminatory (or pay-as-bid) auction (DA). Also, in November 2000 during the California crisis, the California Power Exchange appointed a panel of significant auction theorists to investigate a similar proposal, which suggested that UA action is preferred. This view is not, however, uniformly shared in literature.

To answer these two questions we model a procurement auction where demand is price-independent and suppliers have fixed capacities and constant marginal production costs. All information about generators' costs and capacities is public. Each supplier submits a bid, which is the price for operating her/his generation unit. If bid is accepted, full capacity or any portion of it can be dispatched.[2] Both types of auctions, uniform and discriminatory, have the same allocation scheme: system operator admits the suppliers one by one, according to increasing bid prices, until either demand is satisfied or all suppliers are dispatched. In case of a price-tie, each supplier has the same probability to be selected first.

The above model simplifies real electricity auctions for analytical tractability. First, we assume symmetric information about suppliers' costs and capacities among auction bidders.[3] Second, we assume constant marginal production costs. Since one firm may own multiple plants with different generating technologies, this assumption implies that the generating plants are the actual auction bidders and they operate as separate profit units. We also omit the startup costs. Arguably, the supplier's unit variable cost may be viewed as a startup-adjusted average cost. Since the demand pattern has strong intraday and weekly patterns and weather forecast is publicly available, the short-term load profile is fairly predictable. Upon bidding, a supplier has a good estimate of how long her generation unit will be used, if the bid is admitted. Finally, we restrict supplier's bid to be only one price and impose that the total capacity must be fully committed.

The main contributions of our paper are as follows. From technical point of view, the paper completely characterizes the equilibrium structure for N-bidder symmetric auctions. The extensions include random demand and two-bidder asymmetric cases. Our discussion, under both auction schemes, focuses on probabilistic properties of the unit prices paid by the electricity buyers and on comparison of DA and UA. With respect to our first objective, interpreting and characterizing price dispersion, we show that price dispersion may stem from suppliers' strategically randomized bidding. The factors directly influencing it are: capacity structure, cost structure, average capacity utilization, and demand uncertainty. Higher capacity utilization yields higher expected price, while price variance is maximized at intermediate levels of capacity. Introduction of demand uncertainty increases the chance of price dispersion (i.e., manifesting itself through mixed-strategy equilibrium), but not necessarily the magnitude of price variance. In order to test the robustness of the above lessons in asymmetric settings, we consider the asymmetric two-bidder case as an extension. The numerical studies indicate that, for given system utilization, increasing capacity asymmetry leads to a higher expected price and an initial increasing and possibly an eventual drop of the price variance.

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[2] In other words, we purposefully rule out the possibility of strategic withholding, which has been identified as a possible measure for suppliers to exercise their market power. It is a relevant issue, but outsides the scope of this paper. See Hogan (2001) for some related discussion. This setup allows us to concentrate on the effects of capacity structure on price dynamics.

[3] Consequently, we do not study information asymmetry, which is central to significant portion of research in auction theory.

The second area of our interest, comparison of the performance of DA and UA, also leads to interesting findings. (Our paper is the first analytical paper to compare the price volatilities resulting from the two auctions.) Most importantly, for N-bidder symmetric auctions with deterministic demand, the unique symmetric equilibria for DA and UA correspond to the same expected price, while DA results in lower price variance. While with no uncertainty other equilibria may exist, reasonably levels of demand uncertainty imply uniqueness of equilibrium for a UA and this unique equilibrium is symmetric. Our conclusions about average prices and their volatility are robust throughout the whole range of possible uncertainty as we focus on symmetric equilibria. Through a numerical study we examine asymmetric two-bidders cases with random demand and show that buyers, on average, pay similar prices under both market designs but UA, in majority of cases we observed, yields higher price variance than DA.

Since the original motivation came from highly volatile electricity market, we attempt to illustrate our theoretical findings using empirical data. As we do not have access to DA for U.K. market, we look at the nature of price dispersion for the US market. Since Quantile-Regression (Q-R) model allows to characterize price dispersion more comprehensively, compared to conditional-moment models, we introduce this tool into empirical auction literature. The empirical observation seem to be consistent with our structural results for price prediction. While we do not have access to any data that would allow comparison of two auction formats, we review a closely-related field – experimental economics, and point that it provides direct support for our conclusions.

The remainder of the paper is organized as follows. Next section describes the relevant literature, Section 3 describes the model. In Section 4 we derive solution for auctions with symmetric bidders and formally establish that the average prices are the same for DA and UA while variability is smaller for DA. Section 5 investigates two extensions – impacts of random demand and asymmetric bidders. Section 6 presents the preliminary empirical tests of price dispersion, and Section **??** concludes the paper and discusses its practical and policy implications.

2. Literature Review

Our focus is on two research questions (a) existence of price dispersion and (b) a comparison of two auction formats. Each of these two questions has its own stream of research associated with it. Two substreams, dealing with price dispersion in wholesale electricity markets, are within financial engineering and economics literature, respectively.

The financial engineering literature directly models the electricity price dynamics as continuous-time diffusion processes, and calibrates the models by fitting actual price data. For modeling the price process, mean-reverting model with jumps has been a popular choice (Kaminski 1997 and Deng 2000). For estimating and forecasting price volatilities, conditional autoregressive heteroskedasticity (ARCH) model and its variations (GARCH, EGARCH, etc.) are widely used (Duffie et al., 1998, and Goto and Karolyi, 2003). For quantifying the probability of extreme events in the electricity markets, extreme-value theory (EVT) is introduced in Bystrom (2005). The above models have a common objective – to capture the probabilistic properties of electricity price dynamics. While they are very popular in firms dealing with risk management, as pointed out by Duffie et al. (1998), changes in volatility are not generated by a mathematical model, but rather by real-world events that have significance which may at first only be apparent to engineers, geologists, economists or geopolitical analyst. Our paper differs with these papers in that we intend to identify the structural reasons for electricity price dispersion rather than to statistically describe the phenomena.

Within economics literature, there are two groups of relevant papers. Those that consider price dispersion based on competitive equilibrium and those where price dispersion is based on and explained within the framework of mixed strategy. Our paper belongs to the second group.

Within the first group, while not concentrating on price dispersion itself, several papers on energy market economics provide insights into the possible reasons for price dispersion. The inelasticity of both electricity demand and supply is identified as a key driver of the volatile prices (Borenstein, 2002 and Wilson 2002). The argument assumes, however, a competitive equilibrium outcome, where all suppliers are pricing at their marginal costs. Switching on new generation units clearly leads to kinks in marginal cost curve, and the changing demand drives price volatility, as different marginal cost of unit called into operation. Our paper does not use competitive outcome (does not assume price is equal marginal cost) and price dispersion exists even with deterministic demand.

Papers in the second group allow suppliers to price strategically and typically consider mixed-strategy equilibria. The primary mechanism behind their argument is that suppliers may randomize their bids (as a result of mixed-strategy equilibrium) above their costs. In other words, suppliers exercise their market power in the competition. [4] XXX

The first relevant papers are Varian (1980) and Burdett and Judd (1983). They are first to directly use mixed-strategy equilibrium to interpret (spatial) price dispersion in retail markets. In Varian (1980), the existence of uninformed customers provides an incentive to randomize prices. In similar spirit, in Burdett and Judd (1983) customers observe a limited number of price quotes,

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[4] Market power is defined as the capability of a firm to raise the market price above the industrial marginal cost level.

which leads ti price dispersion. The critical element, costly search (or customers not observing all prices) does not take place in our paper. Instead, limited capacity plays a pivotal role.

The possibility that capacitated firms may play mixed pricing strategy is first documented in the literature of Bertrand-Edgeworth game – see Vives (2000) for a comprehensive review of the related papers. While the focus of their paper is on deriving equivalence of two pricing games, Kreps and Scheinkman (1987, KS) was the first that presented a complete duopoly solution with asymmetric capacities. FFH (2006) considers both discriminatory and uniform auction, both for two firms. Their results for DA are similar to B-E solution in KS. Similarity is expected as the discriminatory unit-price auction model can be viewed as a B-E game with inelastic demand (Hu et al., 2007). We extend the solution for DA and for UA to symmetric oligopoly. Hu et al (2007) analyze asymmetric oligopoly for DA.

FFH and Hu et al (2007) are clearly the closest papers to ours. Hu et al (2007) considers DA for asymmetric oligopoly with deterministic demand and provides several properties of equilibrium outcomes. These generalize the results for DA listed in this paper. Hu et al (2007), however, does not study the impacts of demand uncertainty, does not consider UA, and does not compare auction formats. We compare our paper with FFH in more detail below, since it is also relevant to our second research objective.

The paper's second objective is to compare the performance of two prevailing market designs for trading energy within wholesale market, discriminatory auctions (DA) and uniform auctions (UA).

The importance of this question is emphasized not only by volume of relevant papers, but also by a public debate. In the process of global energy deregulation, uniform auction has become dominantly adopted electricity procurement mechanism and, as mentioned in the introduction, in March 2001, to improve market efficiency, the British government replaced uniform auction (UA) with a discriminatory auction (DA). However, during the California crisis, the panel of significant auction theorists (Kahn, Cramton, Porter, and Tabors, 2001) (KCPT) rejected discriminatory auction by predicting that the change would introduce new inefficiencies. The panel's prediction was not, however, based on any specific model of interactions. Similar message is expressed in Wilfram 1999. Other theoretical literature (e.g., Febra et al 2006) predicts that DA auctions are more efficient. The question thus remains open.

The two auction formats have been studied in multi-unit and shared auction literature, motivated by the treasury auctions, where both auction formats have been implemented. The theoretical analysis focuses primarily on revenue of auctioneer and allocation efficiency (i.e., whether the goods are awarded to the buyers with highest valuations, or in procurement setting the suppliers with the lowest costs). Binmore and Swierzbinski (2000) and Ausubel and Cramton (2002) both point out that the efficiency and revenue ranking of the two auctions is ambiguous and may be influenced by equilibrium selection, bidders' valuation structure, and asymmetry of the system. Krishna and Perry (1998) establishes revenue equivalence in multiunit auctions. The key distinction of these papers from ours is that they focus on the impacts of asymmetric information. The dominance of auctions is primarily driven by the information rents. Our paper assumes complete information, so the issue of information rents does not exist.

Wilson (1979) and Wang and Zender (2002) consider these two auctions under the setting of perfectly divisible goods. Wang and Zender show that in symmetric-bidders setting, there always exist equilibria of uniform auction with lower expected revenue for the auctioneer, implying superiority of discriminatory auction. As illustrated by Wilson (1979), when allowed to submit continuous demand schedule, bidders can reduce the intensity of their competition and therefore reduce the revenue of the auctioneer. Simply, continuous schedule reduces the benefits (to the auctioneer) of undercutting. In electricity auctions, since a supplier's bidding decision is restricted to limited number of price-quantity pairs, the nature of the competition is significantly different from the Wilson and Wang and Zender's models, which limit the application of their results.

Our paper formally models the two auction formats and confirms Wilfram (1999) and KCPT (2001)'s argument that bidders will *bid* (stochastically) higher in DA than in UA. However, this does not imply that the prices *paid* will be higher. Our analytical results for symmetric settings (and numerical study for asymmetric ones) suggest that the average prices in both auctions are the same (very close to each other).

Since FFH is very close to our paper, we describe it in more detail. Most importantly, FFH argues that uniform auctions result in higher average prices than discriminatory auctions. Our paper does suggest that DA is a better market design, but not due to average price but due to price variability, i.e., both auctions have the same average price but DA yields lower price variances. The difference is driven by different objectives, slightly different setting, and significantly different equilibrium selection criterium. Specifically:

- FFH concentrate on market efficiency and compares average prices, while we compare the stochastic performance of both auctions.

- FFH focuses on duopoly case. Since most deregulated electricity markets are largely decentralized, with no suppliers (or hardly any) dominating the pricing competition, we focus on oligopoly settings, which is more realistic and highlights the role of relative influence of suppliers on pricing policy.

- For UA we select a different equilibrium. FFH's choice requires some form of pre-game communication, while our criterium is based on independent bidding assumption. This leads to important differences.

Extending on the last point: in UA, outside of perfectly competitive solutions, FFH select pure-strategy equilibria, where one player prices at the market cap (chosen by regulator), while the other players set their prices very low. Demand is cleared at price cap and all players are paid at the price equal to market cap. While the predictions of FFH is that market price is always a cost or price cap, in practice a range of market prices is observed. Also, with some randomness, the pure-strategy equilibrium does not exist. Our selection of symmetric equilibrium allows for a consistent choice of equilibrium for cases of no randomness, small randomness, and high randomness of demand. The range of predicted prices is also more consistent with reality (allowing many prices between cost and price cap). Importantly, we disagree with FFH's equilibrium selection, since such a price outcome is a violation of independent bidding assumption in auction setting – the pure-strategy equilibrium needs pregame communication among all suppliers, which is clearly prohibited by anti-trust law.[5]

The empirical literature on the treasury auctions is summarized in Binmore and Swierzbinski (2000) and it seems to be inconclusive with respect to ranking of the two auction formats. For example, Simon (1994) estimates that switching from DA to UA in 1970's resulted in large loss of revenue for the US Treasury; Nyborg and Sundaresan (1996) estimate the effect between a small loss and moderate gains; while Malvey and Archibald (1998) claim small gains. Empirical comparison of the two auction formats in electricity markets could be conducted only in UK and the findings (Evans and Green 2002, Newbery 2003 and Fabra and Toro 2003) are also controversial, due to major structural changes of the markets taking place during the switch from uniform auction to discriminatory auctions.

The comparison of two auctions formats was also studied in the laboratory settings. Motivated by the electricity auctions, Mount et al. (2002) reports that "both uniform auction and discriminatory auction produce average prices fifty percent above the competitive levels. However, the prices for the uniform price auction are more volatile with many price spikes." Similar test was conducted by Rassenti et al. (RSW, 2001), and their experiments indicate that (a) a DA consistently generates lower price volatility; (b) the average prices of the two auctions have no significant difference for high demand, but (c) DA yields higher average price for low demand. [6] Our pa-

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[5] While relative benefits are not modeled neither in our paper nor in FFH, the asymmetric equilibrium analyzed in FFH puts the price setter at a relative disadvantage due to not using whole capacity.

[6] Here both "low" and "high" demand sustain pure-strategy equilibria with competitive price level, so they can be

per is the first analytical paper to compare the price volatilities resulting from the two auctions. Our analytical results provide direct support to the cited above experimental papers dealing with electricity auction design.

3. The Model

Game Description. Consider wholesale electricity procurement auction with N potential suppliers. Supplier i is assumed to have unit variable cost $c_i \geq 0$ and production capacity $k_i > 0$, for $i = 1, 2, \ldots, N$. The random electricity demand ξ is generated by price-insensitive consumers and the auctioneer procures electricity from suppliers to satisfy the demand as much as possible. The suppliers compete to serve demand by submitting unit prices.

The sequence of events is as follows. Distribution of demand ξ is known to all suppliers. During the auction, suppliers *independently* submit (sealed) prices $\{p_i\}_{i=1}^N$ to the auctioneer. It is assumed that supplier i's bid price p_i is bounded by price cap B, imposed by the regulator, i.e., $p_i \in [0, B]$. After demand is realized and aggregated, the auctioneer calls suppliers into operation based on their bids. The lowest-bid supplier is admitted first. If her capacity cannot cover the demand, the auctioneer moves to the next lowest-bid supplier, and so on, until the demand is filled or no capacity is left. We assume ties are broken by first granting orders to the efficient suppliers (those with lower production costs). If suppliers with the same costs form a tie, each supplier gets a demand share proportional to her capacity.[7]

The following notation is used throughout the paper. Let $\mathbf{p}_{-i} \equiv (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$ and supplier i's realized sales as $z_i(p_i, \mathbf{p}_{-i}) = k_i r_i(p_i, \mathbf{p}_{-i})$, where r_i is the fraction of her bid quantity accepted by the auctioneer. The above assumptions lead to

$$r_i(p_i, \mathbf{p}_{-i}) = 1 \wedge \frac{[\xi - \sum_{n \neq i} k_n \delta_{(p_n < p_i)} - \sum_{n \neq i} k_n \delta_{(p_n = p_i, c_n < c_i)}]^+}{k_i + \sum_{n \neq i} k_n \delta_{(p_n = p_i, c_n = c_i)}},$$

where $\delta_{(A)} = 1$ if A is true, 0 otherwise. To investigate a supplier's sales at the proximity of a certain price, we define r_i^- and r_i^+, and simplify[8] them as

$$\text{(a) } r_i^-(p_i, \mathbf{p}_{-i}) \equiv \lim_{p \uparrow p_i} r_i(p, \mathbf{p}_{-i}) = 1 \wedge \frac{[\xi - \sum_{n \neq i} k_n \delta_{(p_n < p_i)}]^+}{k_i}, \tag{1}$$

$$\text{(b) } r_i^+(p_i, \mathbf{p}_{-i}) \equiv \lim_{p \downarrow p_i} r_i(p, \mathbf{p}_{-i}) = 1 \wedge \frac{[\xi - \sum_{n \neq i} k_n \delta_{(p_n \leq p_i)}]^+}{k_i}.$$

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both viewed as low-demand state.

[7] The mixed-strategy equilibrium solutions are independent of such rules, since any forms of rationing will eliminates the chance of price-tie at equilibrium. For Bertrand-like pure-strategy equilibrium to sustain, the more efficient supplier must possess higher priority. See Deneckere and Kovenock (1996) for a discussion of rationing rules in Bertrand-Edgeworth games with *asymmetric* unit costs.

[8] Equations (1) follow $\lim_{p \uparrow p_i} \delta_{(p_k < p)} = \delta_{(p_k < p_i)}$, $\lim_{p \downarrow p_i} \delta_{(p_k < p)} = \delta_{(p_k \leq p_i)}$, and $\lim_{p \to p_i} \delta_{(p_k = p)} = 0$.

It is easy to verify that $r_i^-(p_i, \mathbf{p}_{-i}) \geq r_i(p_i, \mathbf{p}_{-i}) \geq r_i^+(p_i, \mathbf{p}_{-i})$.

Two auction types are considered in our paper. In a discriminatory auction (DA), an admitted supplier is paid at her bid price, while in a uniform auction (UA), all of the selected suppliers are paid at a uniform price equal to the highest bid admitted (i.e., the highest price among all admitted suppliers). We use superscripts (or subscripts when convenient) d and u to denote the two auction formats. Under each of the two auction formats, supplier i maximizes his expected payoff, where

$$\text{(a) } R_i^d(p_i, \mathbf{p}_{-i}) = (p_i - c_i)k_i r_i(p_i, \mathbf{p}_{-i}),$$

$$\text{(b) } R_i^u(p_i, \mathbf{p}_{-i}) = (\max_n \{p_n : r_n(p_n, \mathbf{p}_{-n}) > 0\} - c_i)k_i r_i(p_i, \mathbf{p}_{-i}).$$

Note that $r_i^-(p_i, \mathbf{p}_{-i}) \neq r_i(p_i, \mathbf{p}_{-i})$ (or $r_i(p_i, \mathbf{p}_{-i}) \neq r_i^+(p_i, \mathbf{p}_{-i})$) only if $p_i = \max\{\mathbf{p}_{-i} : r_n(p_n, \mathbf{p}_{-n}) > 0\}$. Hence, we have the following useful observation, for both DA and UA,

$$\begin{aligned} R_i^-(p_i, \mathbf{p}_{-i}) - R_i(p_i, \mathbf{p}_{-i}) &= (p_i - c_i)k_i[r_i^-(p_i, \mathbf{p}_{-i}) - r_i(p_i, \mathbf{p}_{-i})] \\ R_i(p_i, \mathbf{p}_{-i}) - R_i^+(p_i, \mathbf{p}_{-i}) &= (p_i - c_i)k_i[r_i(p_i, \mathbf{p}_{-i}) - r_i^+(p_i, \mathbf{p}_{-i})] \end{aligned} \tag{2}$$

where $R_i^-(p_i, \mathbf{p}_{-i}) \equiv \lim_{p \uparrow p_i} R_i(p, \mathbf{p}_{-i})$ and $R_i^+(p_i, \mathbf{p}_{-i}) \equiv \lim_{p \downarrow p_i} R_i(p, \mathbf{p}_{-i})$.[9]

As we show in Section 4, pure strategy equilibria exist only under restricted conditions. In general, suppliers are forced to play mixed strategies. Mixed strategies also fit the practitioners opinion about the situation they face, which is discussed in Section 7.

Mixed Strategies. The corresponding notation related to mixed-strategy equilibrium analysis is introduced here. Supplier i's mixed-strategy is denoted by σ_i, a random variable with support $[0, B]$. Define $F_i(p; \sigma_i) \equiv \Pr\{\sigma_i \leq p\}$ as the cumulative distribution function for σ_i and $m_i(p; \sigma_i) \equiv \Pr\{\sigma_i = p\}$ the probability mass at price p. Denote $\overline{p}_i(\sigma_i) \equiv \inf\{p : F_i(p; \sigma_i) = 1\}$ and $\underline{p}_i(\sigma_i) \equiv \sup\{p : F_i(p; \sigma_i) = 0\}$ as the upper and lower pricing bounds for σ_i. Given the opponents' mixed-strategy $\sigma_{-i} \equiv (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$, supplier i has random sales and payoff when choosing price p. Let $\hat{z}_i(p, \sigma_{-i}) \equiv \mathsf{E}_{\sigma_{-i}}[z_i(p, \sigma_{-i})]$, $\hat{r}_i(p, \sigma_{-i}) \equiv \mathsf{E}_{\sigma_{-i}}[r_i(p, \sigma_{-i})]$, and $\hat{R}_i(p, \sigma_{-i}) \equiv \mathsf{E}_{\sigma_{-i}}[R_i(p, \sigma_{-i})]$ represent her expected sales, expected sales fraction, and expected payoff, respectively.

Denote $\sigma^* \equiv (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ as a mixed-strategy equilibrium and $ER_i(\sigma^*) \equiv \hat{R}_i(\sigma_i^*, \sigma_{-i}^*)$ supplier i's expected equilibrium payoff. For simplicity, we suppress the equilibrium-associated notation by omitting σ^*. For example, $F_i(p_i) = F_i(p_i; \sigma_i = \sigma_i^*)$ and $\overline{p}_i = \overline{p}_i(\sigma_i^*)$. Similarly, we use shorthand notation $\hat{z}_i(p_i) = \hat{z}_i(p_i, \sigma_{-i}^*)$, $\hat{r}_i(p_i) = \hat{r}_i(p_i, \sigma_{-i}^*)$, $\hat{R}_i(p_i) = \hat{R}_i(p_i, \sigma_{-i}^*)$, and $ER_i = ER_i(\sigma^*)$.

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[9] By (2), the existence of $R_i^-(\mathbf{p})$ and $R_i^+(\mathbf{p})$ for both UA and DA follows (i) boundedness and monotonicity of $r_i(\mathbf{p})$ in p_i, (ii) continuity of $p_i - c_i$, and (iii) continuity of $max\,\{p_n : r_n(\mathbf{p}) > 0\} - c_i$ in p_i.

Corresponding to (1), we also define $\hat{r}_i^-(p_i) \equiv \lim_{p\uparrow p_i} \hat{r}_i(p)$ and $\hat{r}_i^+(p_i) \equiv \lim_{p\downarrow p_i} \hat{r}_i(p)$. $\hat{R}_i^-(p)$ and $\hat{R}_i^+(p)$ are defined similarly. As $r_i(p) \in [0,1]$ for all p, applying *bounded convergence theorem*, we have $\hat{r}_i^-(p) = \mathsf{E}_{\sigma^*_{-i}}[r_i^-(p)]$ and $\hat{r}_i^+(p) = \mathsf{E}_{\sigma^*_{-i}}[r_i^+(p)]$. The following observation is very useful: for any mixed-strategy equilibrium,[10]

$$\begin{aligned}
&\text{(a) } m_i(p) > 0 \text{ implies } \hat{R}_i(p) = ER_i; \\
&\text{(b) } F_i(p) > F_i(p') \text{ for all } p' < p \text{ implies } \hat{R}_i^-(p) \equiv \lim_{p'\uparrow p} \hat{R}_i(p') = ER_i; \\
&\text{(c) } F_i(p) < F_i(p') \text{ for all } p' > p \text{ implies } \hat{R}_i^+(p) \equiv \lim_{p'\downarrow p} \hat{R}_i(p') = ER_i.
\end{aligned} \tag{3}$$

4. Symmetric Auctions with Deterministic Demand

This section derives both key results of the paper in symmetric oligopoly case. We characterize when suppliers have incentive to randomized prices for each auction type. We show that price dispersion may stem endogenously from suppliers' strategic bidding behaviors even in a deterministic economic system. We derive the distribution of prices and show how it depends on capacities and costs. Understanding pricing strategies for both types of auctions allow us to compare them from point of view of prices that buyers pay. We show that they result in the same expected price, but the same variance. The equilibrium structures identified in this section and the following insights are robust and will be extended later to more general settings with asymmetric bidders and random demand.

Our basic model assumes symmetric bidders and deterministic (or perfectly foreseeable) demand. These assumptions make the analysis tractable and yield closed-form equilibrium solutions. Specifically, here we assume $k_i = k$ and $c_i = c$ for all $i = 1, 2, \ldots, N$ and seek Nash equilibria for both types of auctions. The analysis for symmetric DA and UA relies heavily on the order statistics of bids. We introduce the common notation here. Denote $b_m^{(N)}$ the m-th *lowest* bid among N bids and $G_m^{(N)}$ its c.d.f. Reversely, denote $b_{(m)}^{(N)}$ the m-th *highest* bid and $G_{(m)}^{(N)}$ its c.d.f. Clearly, $b_m^{(N)} = b_{(N+1-m)}^{(N)}$ and $G_m^{(N)} = G_{(N+1-m)}^{(N)}$. Superscript (N) is omitted for simplicity when context is clear and especially when other superscripts are needed. In the following subsection, we separately analyze DA and UA, which allows us later compare their performances. We omit those analytical derivations that are well established in theory of order statistics.

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[10] Part (a) is obvious. For part (b), the monotonicity of F_i at p's left neighborhood implies $\Pr\{\sigma_i^* \in [p-\Delta, p)\} > 0$ for any $\Delta > 0$, and consequently, $\tilde{p} \in [p-\Delta, p)$ exists such that $\hat{R}_i(\tilde{p}) = ER_i$. [Otherwise, if $\hat{R}_i(p') < ER_i$ for all $p' \in [p-\Delta, p)$, we must have $\Pr\{\sigma_i^* \in [p-\Delta, p)\} = 0$, which contradicts to the initial assumption.] As Δ converges to zero, we have $\hat{R}_i^-(p) = ER_i$ where existence of $\hat{R}_i^-(p)$ follows Footnote 9. Similarly we can show part (c).

4.1 Equilibrium Analysis

We first analyze discriminatory auction and show general structure of equilibrium (symmetry and uniqueness), then we derive the exact analytical form of equilibrium. Later we follow with the analysis of uniform auction.

Discriminatory Auction. Pure-strategy equilibrium can be achieved only under restricted market conditions. For high demand $\xi \geq Nk$, all suppliers price at the cap B. Thus, they have the maximal possible market power. On the other hand, for low demand $\xi \leq (N-1)k$, the auction becomes very competitive and the demand is cleared at price equal to cost c. If a supplier prices c, the remaining suppliers can still satisfy the whole demand, so the deviation cannot bring any profit. Clearly, $\{p_i^{d*} = c\}_{i=1}^{N}$ is the only symmetric equilibrium.[11]

For the intermediate demand $(N-1)k < \xi < Nk$, pure-strategy equilibrium does not exist. This results from three incompatible tensions in the competition. First, if different prices are chosen, supplier i who prices lower than the highest bid will have an incentive to raise price as close as possible just below the highest bid, because $r_i(p_i) = 1$ for all $p_i < \max\{\mathbf{p}_{-i}\}$. Second, if all suppliers prices are extremely close, the highest-price supplier has an incentive to set her own bid slightly lower than next highest price to achieve $r^-(p) = 1 > r(p)$, which collectively causes the highest price to drop. Third, if a uniform price $p^* = c$ is shared by all suppliers or everybody prices very close to the cost c, everyone obtains a zero payoff, so supplier i will be better off choosing $p_i = B$, because $R_i(B) = (B-c)[\xi - (N-1)k] > 0$. The above three tensions lead to the formation of a mixed-strategy equilibrium. The most important implication is that price dispersion may happen in a DA without exogenous randomness (like demand uncertainty or information asymmetry). The fundamental driver is the imperfect competition among oligopolistic suppliers. While mixed strategies were analyzed in oligopolistic competition (e.g., KS), we use the concept in auction setting and show that it explains price variability when demand and supply are price insensitive. The randomization of the bidding price is consistent with conversation we heard from traders. The following proposition describes full structure of the equilibrium.

Proposition 1 *A symmetric DA has a unique symmetric Nash equilibrium. (a) For* $\xi \geq Nk$, *it is a pure-strategy equilibrium with* $\{p_i^{d*} = B\}_{i=1}^{N}$*; (b) For* $\xi \leq (N-1)k$, *it is a pure-strategy equilibrium with* $\{p_i^{d*} = c\}_{i=1}^{N}$*; (c) For* $(N-1)k < \xi < Nk$, *it is a mixed-strategy equilibrium, with*

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[11] Note that when demand ξ is less than $(N-2)k$, the equilibrium is sustained even if certain player chooses a price higher than c. It only requires $\lceil \xi/k \rceil + 1$ bidders choosing c where $\lceil x \rceil \equiv \min\{i \in \mathcal{Z} : i \geq x\}$. Hence, multiple equilibria may exist, but they are all payoff-equivalent (zero profit for everyone).

equilibrium distribution function

$$F^d(p) = (\frac{k}{Nk-\xi}\frac{p-\underline{p}^d}{p-c})^{\frac{1}{N-1}} \quad \textit{for } p \in [\underline{p}^d, B], \tag{4}$$

where $\underline{p}^d = c + \frac{(B-c)[\xi-(N-1)k]}{k}$.

Below we outline the critical element of the proof (simple derivations and algebraic steps are omitted). According to Theorem 6 in Dasgupta and Maskin (1986)[4], a symmetric mixed-strategy equilibrium exists. To derive the equilibrium distribution F^d, we first establish that, for any symmetric mixed-strategy equilibrium $\{\sigma_i^* = \sigma_d^*\}_{i=1}^N$, the equilibrium distribution function F^d must be continuous and strictly increasing (Lemma A1). It implies that, according to (3-b,c), any $p \in [\underline{p}^d, \bar{p}^d] \cap (c, B]$ yields the expected equilibrium payoff $\hat{R}(p) = ER^d = \hat{R}(\bar{p}^d) = (\bar{p}^d - c)[\xi - (N-1)k]$. The optimality of ER^d requires $\bar{p}^d = B$ and $ER^d = (B-c)[\xi-(N-1)k] > 0$. By (3-c), we have $\hat{R}^+(\underline{p}^d) = ER^d > 0$, implying $\underline{p}^d > c$ and therefore $m(\underline{p}^d) = 0$. It follows that $\hat{r}^+(\underline{p}^d) = \hat{r}(\underline{p}^d) = 1$, and $\underline{p}^d$ can be derived from $ER^d = \hat{R}^+(\underline{p}^d) = (\underline{p}^d - c)k$. For $p \in [\underline{p}^d, B]$, $F^d(p)$ must satisfy

$$\hat{R}(p) = (p-c)\sum_{n=0}^{N-1} \binom{N-1}{n} F^d(p)^n \bar{F}^d(p)^{N-1-n} z_n = ER^d \ \text{ with } z_n = k \wedge [\xi - nk]^+. \tag{5}$$

It is possible to show that the unique solution to (5) is equation (4) above, which completes the proof of the proposition. Note that cdf of price F does not have mass point at the price cap B (by the same logic as for lack of mass points within the price range).

Notice that, the equilibrium outcomes presented in Proposition 1 are determined by five factors $\{k, \xi, N, c, B\}$. Define $\rho \equiv \frac{\xi}{Nk}$ as the *aggregated utilization*, and the determinant factors become $\{\rho, N, c, B\}$. The three cases in Proposition 1 correspond to (a) $\rho \geq 1$, (b) $\rho \leq \frac{N-1}{N}$, and (c) $\frac{N-1}{N} < \rho < 1$, respectively. For case (c), the equilibrium distribution function can be expressed as

$$F^d(p) = [\frac{1}{N(1-\rho)}\frac{p-\underline{p}^d}{p-c}]^{\frac{1}{N-1}} = [\frac{B-c}{p-c} - \frac{B-p}{N(1-\rho)(p-c)}]^{\frac{1}{N-1}} \text{ for } p \in [\underline{p}^d, B]. \tag{6}$$

Since all marginal costs are c, any price above c is a manifestation of pricing power. From equation (6), the equilibrium bids are stochastically increasing in ρ and stochastically decreasing in N. It suggests that suppliers' pricing power is primarily determined by system utilization, represented by ρ, and market decentralization (delegated by N). Higher utilization gives higher pricing power to suppliers, while increasing number of suppliers dilutes it. While Proposition 1 considers only symmetric solution, it is possible that asymmetric equilibrium can exist. Proposition 4 in Appendix A.2, however, excludes such a possibility. We now are ready to move to analysis of uniform auction. As opposite to discriminatory auction, asymmetric equilibrium can exist and our analysis need to be more detailed.

Uniform Auction. The existence of a symmetric mixed-strategy equilibrium for a UA follows again from Dasgupta and Maskin (1986) Theorem 6. For high demand $\xi \geq Nk$ or low demand $\xi \leq (N-1)k$, the symmetric solution reduces to the same pure-strategy equilibria achieved in a DA, with all suppliers pricing at B or c. For intermediate demand $(N-1)d < \xi < Nk$, there is no symmetric pure-strategy equilibrium due to the same incentive to undercut other suppliers as in a DA.

Despite the above similarities, the competitive nature of DA and UA is different. In a UA, those suppliers who price below the highest dispatched bidder have no incentive to raise price, enjoying the benefits of a " free ride" as price takers; while in a DA, a low-price bidder always has an incentive to increase price as long as she can maintain a sales fraction of 1. For $(N-1)k < \xi < Nk$, this difference matters — unlike DA, a UA has multiple asymmetric pure-strategy equilibria. Consider the following bidding outcome. One bidder chooses B, while the rest of the bidders price at cost c. The high bidder's payoff is $R(B) = (B-c)[\xi-(N-1)k] > 0$. If she deviates to any price $p \in (c, B)$, her sales is still the same, but the profit margin $(p-c)$ is strictly decreased. A price cut to c or below increases her market share but reduces the price margin to zero. Hence pricing at B is her optimal strategy. For other bidders, given that the demand is cleared at price B, the profit equals to monopoly-equivalent payoff $(B-c)k$. Thus, a pure-strategy equilibrium is sustained. Note that the price takers may raise price above c, as long as it is *not* high enough to trigger the price maker's defecting from B. A comprehensive characterization of the asymmetric equilibria is presented in the following lemma.

Lemma 1 *For a symmetric UA with $(N-1)k < \xi < Nk$, a pure-strategy equilibrium must satisfy $b^u_{(1)} = B$ and $b^u_{(2)} \leq c + \frac{(B-c)[\xi-(N-1)k]}{k}$.*

Note that the upper bound for $b^u_{(2)}$ is equal to $\underline{p}^d$, the lower pricing bound of the mixed-strategy equilibrium in a DA, because it is the lowest price securing the highest bidder's profit at price B.

Similarly to DA, also for UA we can describe the structure of symmetric Nash equilibrium. The impacts of market power can be characterized by ρ and N.

Proposition 2 *A symmetric UA has a unique symmetric Nash equilibrium. (a) For $\xi \geq Nk$, it is a pure-strategy equilibrium with $\{p^{u*}_i = B\}^N_{i=1}$; (b) For $\xi \leq (N-1)k$, it is a pure-strategy* *equilibrium with* $\{p_i^{u*} = c\}_{i=1}^N$*; (c) For* $(N-1)k < \xi < Nk$*, it is a mixed-strategy equilibrium, with equilibrium distribution function*

$$F^u(p) = (\frac{p-c}{B-c})^{\frac{\xi-(N-1)k}{(N-1)(Nk-\xi)}} = (\frac{p-c}{B-c})^{\frac{N\rho-N+1}{N(N-1)(1-\rho)}} \quad \textit{for } p \in [c, B]. \tag{7}$$

Let us sketch the justification of Proposition 2. Clearly, from Lemma 1, all pure-strategy equilibria are asymmetric for $\xi \in ((N-1)k, Nk)$. Thus, the existing symmetric solution must be a mixed-strategy one. Similarly to DA, we can show that a symmetric mixed-strategy equilibrium must have $F^u(p)$ continuous and strictly increasing in $p \in [\underline{p}^u, \bar{p}^u] \cap (c, B]$. The continuity and monotonicity of F^u implies $\hat{R}(p) = ER^u$ for all $p \in [\underline{p}^u, \bar{p}^u]$. As $m(\bar{p}^u) = 0$, or equivalently, $\Pr\{\sigma_u^* < \bar{p}^u\} = 1$, pricing at $\bar{p}^u$ results in market clearing price $\bar{p}^u$ and sales of $\xi - (N-1)k$ with probability 1. The optimality of $\bar{p}^u$ implies $\bar{p}^u = B$ and $ER^u = \hat{R}(\bar{p}^u) = (B-c)[\xi - (N-1)k]$. For any $p \in [\underline{p}^u, B]$, by monotonicity of F^u, supplier i's expected payoff satisfies,

$$\hat{R}_i(p) = G_{(1)}^{(-i)}(p) \cdot (p-c)[\xi - (N-1)k] + \int_p^B (v-c)k\,dG_{(1)}^{(-i)}(v) = ER^u. \tag{8}$$

Since equation (8) cannot be solved explicitly, we consider its first order condition, $\frac{d\hat{R}_i(p)}{dp} = 0$, which yields an ordinary differential equation (9) of $G_{(1)}^{(-i)}(p)$:

$$\dot{G}_{(1)}^{(-i)}(p) \cdot (p-c)(\xi - Nk) + G_{(1)}^{(-i)}(p) \cdot [\xi - (N-1)k] = 0. \tag{9}$$

The general solution to the ODE is $G_{(1)}^{(-i)}(p) = C_0(p-c)^{\frac{\xi-(N-1)k}{Nk-\xi}}$ where C_0 is a constant. From the boundary conditions $G_{(1)}^{(-i)}(B) = F^u(B)^{N-1} = 1$ and $G_{(1)}^{(-i)}(\underline{p}^u) = F^u(\underline{p}^u)^{N-1} = 0$, we have $C_0 = (B-c)^{-\frac{\xi-(N-1)k}{Nk-\xi}}$ and $\underline{p}^u = c$. Also, from $G_{(1)}^{(-i)}(p) = F^u(p)^{N-1}$, we obtain the equilibrium distribution function described in (7) above, which complete the outline of the proof.

Note that the price range is always $[c, B]$. That is, for all mixed-strategy equilibria (but pure-strategy ones), any price between c and B can be observed in a UA, while in a DA the lower bound $\underline{p}$ increases in utilization ρ and decreases in the number of suppliers N. The equilibrium solution in a UA stochastically increases in ρ and decreases in N, resulting in similar to DA economic interpretation of how utilization and decentralization influence the suppliers' market power, manifesting itself by randomized bidding above the cost.

While for a DA, with $\frac{N-1}{N} < \rho < 1$, the symmetric mixed-strategy equilibrium is the unique Nash solution; for a UA, we have identified both symmetric mixed-strategy equilibrium and asymmetric pure-strategy equilibrium. In any pure-strategy equilibrium, the task of price making is assigned to one supplier; while in symmetric mixed-strategy equilibrium, this responsibility is equally and randomly shared among all bidders. Obviously a continuum of equilibria exist, morphing between the symmetric mixed-strategy equilibrium and any pure-strategy equilibrium. As these equilibria differ in multiple dimensions, it is difficult to provide a general solution form. We exclude mixed-strategy solutions that can be reduced to payoff-equivalent pure-strategy equilibria such as $\{p_i^{u*} = B, \sigma_j^{u*} < B$ almost surely for all $j \neq i\}$. Formally, we define a mixed-strategy equilibrium as *irreducible* if $\Pr\{\sigma_i^{u*} < B\} > 0$ for all i. Proposition 2 (Appendix A.3) presents several structural properties of irreducible equilibria with continuous and monotone distribution functions. Based on these properties in Proposition 3, we construct a family of mixed-strategy equilibria that clearly illustrate increasing asymmetry between the price maker and the remaining suppliers. (Note that, there exist other solutions outside of this family.)

Proposition 3 *For symmetric UA with* $(N-1)k < \xi < Nk$, *any* $(h, m_B^u) \in \{1, 2, \ldots, N\} \times [0, 1)$ *defines an irreducible mixed-strategy equilibrium with distribution functions,*

$$\text{(a) } F_h^u(p) = (1 - m_B^u)\left(\frac{p-c}{B-c}\right)^{\frac{\xi-(N-1)k}{(N-1)(Nk-\xi)}} \quad \text{for } p \in [c, B) \text{ and } m_h^u(B) = m_B^u, \tag{10}$$

$$\text{(b) } F_i^u(p) = \left(\frac{p-c}{B-c}\right)^{\frac{\xi-(N-1)k}{(N-1)(Nk-\xi)}} \quad \text{for } p \in [c, B] \text{ and } i \neq h.$$

The equilibrium payoff is $ER_i^u = (B-c)[\xi - (N-1)k + \delta_{(i \neq h)} m_B^u (Nk - \xi)]$, *for all* i.

The proposition can be easily proved by verifying the expected payoffs for all suppliers and we omit the details. When multiple equilibria exist, equilibrium selection is critical for evaluating different auction formats. Within the whole equilibrium set, the *focal* points are the pure-strategy equilibria(most asymmetric) and the symmetric mixed-strategy equilibrium(most symmetric). The pure-strategy solutions described in Lemma 1 are "attractive" because the bidders' behavior is deterministic and their total payoff is maximized.[12] Price maker has the same profit as in symmetric mixed-strategy equilibrium, while all other players are strictly better off. However, in sealed-bid auction context, these pure-strategy equilibria have major shortcomings. First, from a behaviorial perspective, they rely heavily on pre-game communication — the $N-1$ price takers needs to achieve an agreement to collectively impose a "threat" to the price maker and the price maker must be informed that she is under such threat. Suppliers bidding at c have no profit if nobody ends up bidding high. Since the price maker does not get higher payoff than in a mixed-strategy equilibrium, it is not clear if anyone would agree to become a price maker and play the pure-strategy

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[12] Based on such equilibrium selection, FFH (Proposition 5) omits all mixed-strategy solutions for symmetric oligopoly model and argues that UA yields higher price (deterministic price B) than DA (random price with B as upper bound). Also note that FFH's equilibrium selection and efficiency argument imply that there is no price dispersion in uniform auctions.

Figure 2: Comparison of Discriminatory and Uniform Auctions ($N = 3, c = 2, B = 10$)

equilibrium. Clearly, the communication needed to achieve a pure-strategy equilibrium violates the original assumption of independent bidding and would be considered as violation of antitrust law. Compared with the pure-strategy equilibria, the symmetric mixed-strategy equilibrium is also more informative because its stochastic features are arguably shared by other non-symmetric mixed-strategy equilibria(e.g., the ones described in Proposition 3). Due to the above considerations, we will focus in the rest of the paper primarily on the symmetric equilibrium for UA and then discuss the relationship with asymmetric pure-strategy ones.

4.2 Comparison of Symmetric DA and UA

One of the main objectives of this paper is to compare the stochastic performances of DA and UA. For $\xi \leq (N-1)k$ or $\xi \geq Nk$, both auctions yield the same pure-strategy equilibria involving uniform pricing outcome (c or B). The interesting case is for demand $(N-1)k < \xi < Nk$. Since the payment schemes in DA and UA are different, instead of comparing the distributions of supplier's equilibrium bids, $F_i^d(p)$ and $F_i^u(p)$, we need to compare the average price for the capacities called into operation, which we label as *transaction price* P. For UA, P^u is the market clearing price. For DA, the admitted suppliers are compensated at different rates (equal to their bid prices) but all electricity buyers pay one single price that covers the total revenue of the suppliers. In order to compare prices paid by customers in two auctions, for DA, we consider the randomly sampled *seller price* P_S^d. In seller price, the probability of being certain price is equal to the probability that the corresponding capacity is admitted. Denote by P_B^d average price, or the *buyer price*, which is a sales-weighted average of the realized bids. Formally, for an auction with $(N-1)k < \xi < Nk$, the above prices can be expressed as

$$\begin{aligned}
&\text{(a) } P_S^d \equiv \left\{ b_{(1)}^d, \frac{\xi - (N-1)k}{\xi} \right\} \oplus \left\{ b_{(2)}^d, \frac{k}{\xi} \right\} \oplus \ldots\ldots \oplus \left\{ b_{(N)}^d, \frac{k}{\xi} \right\} \\
&\text{(b) } P_B^d \equiv \frac{\xi - (N-1)k}{\xi} \cdot b_{(1)}^d + \frac{k}{\xi} \cdot b_{(2)}^d + \ldots\ldots + \frac{k}{\xi} \cdot b_{(N)}^d \\
&\text{(c) } P^u \equiv b_{(1)}^u,
\end{aligned} \tag{11}$$

where $\tilde{y} = \oplus_{n=1}^{N} \{\tilde{x}_n, f_n\}$ denotes a random variable $\tilde{y}$, being a mixture of N random variables, such that $\tilde{x}_n$ is chosen with probability f_n.

Denote $H_S^d(\cdot)$, $H_B^d(\cdot)$, and $H^u(\cdot)$ as the c.d.f. of P_S^d, P_B^d and P^u and they are computed in Appendix A.4. Figure 2 illustrates the behavior of equilibrium bids and transaction prices in both

DA and UA. Obviously, suppliers bid stochastically lower in a UA than in a DA, see Figure 2(a). However, since everyone is paid at the highest bid in a UA, the transaction prices illustrated in Figure 2(b) are significantly increased compared to an individual supplier's bids in Figure 2(a); while for a DA, the differences between $H_S^d(\cdot)$ and $F^d(\cdot)$ are less significant. Note that H_S^d is stochastically smaller than F^d, because among the N i.i.d. bids, the highest realized value is selected with a smaller chance $f_{(1)} = \frac{\xi-(N-1)k}{\xi}$ than other ones $f_{(n)} = \frac{k}{\xi}$ for $n > 1$. Note also that when utilization ρ is increased, bids and transaction prices in both auctions increase stochastically, but in a different fashion. For a DA, a higher ρ implies a higher pricing bound $\underline{p}^d$, resulting in condensing of bidding range towards the price cap B; while for F^u and H^u in a UA, increasing utilization leads to the preponderance of probability mass shifting from $\underline{p}^d$ to B, with pricing range remaining $[c, B]$. Finally, Figure 2(b) illustrates comparisons among P_S^d, P_B^d, and P^u for given ρ. First note that, given any realized bid vector $\mathbf{b}$, P_S^d is a random variable with discrete values while P_B^d is a constant. Therefore, P_S^d must be a mean-preserving spread of P_B^d. The more interesting comparison is between the two auction formats. We observe that, for each ρ, there is a single cross between H^u and H_S^d. It suggests that H^u may be a mean-preserving spread of H_S^d. This conjecture is formally established in the following proposition. See Appendix A.5 for its proof.

Theorem 1 *Consider symmetric auctions with $(N-1)k < \xi < Nk$. The transaction prices P_S^d and P_B^d in a discriminatory auction and P^u in a uniform auction satisfy:*
(a) Equal Expected Price: $\mathsf{E}[P^u] = \mathsf{E}[P_S^d] = \mathsf{E}[P_B^d] = (B-c)\frac{N[\xi-(N-1)k]}{\xi} + c$*;*
(b) Variability Ordering: P^u is stochastically more variable than P_S^d and P_S^d is stochastically more variable than P_B^d.

The equality of expected prices can be easily justified. At a symmetric mixed-strategy equilibrium σ_d^* or σ_u^*, a bidder expects to earn the equilibrium payoff when pricing at B. Despite the different payment schemes, a bid equal to B must be the highest one, so a bidder obtains the same expected payoff $ER^d = ER^u = (B-c)[\xi-(N-1)k]$. The intuition behind variability ordering can be established by revisiting the competitive natures of the two auctions. As we explained, one of the key incentives in a DA is missing in a UA — low-price parties' benefit by approaching the highest bidder. In other words, there is stronger force in a DA to contract the bid range, which is consistent with prices in DA being less dispersed.

Proposition 1 has some surprising consequences. First, equality (a) contradicts FFH's claim[13] that uniform auctions yield higher prices. The difference is due to the equilibrium selection for

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[13] FFH(2006) Proposition 5 (indirectly) suggests $\mathsf{E}[P^d] = (B-c)\frac{\xi-(N-1)k}{k} + c < B = \mathsf{E}[P^u]$.

uniform auctions. FFH assume that, in UA, pure-strategy equilibria described in Lemma 1 is chosen; we argue that the asymmetric pure-strategy solution is not consistent with the independent bidding assumption and that the required communication makes the comparison with DA "unfair." However, when symmetric bidding is assumed, DA and UA result in the same expected price. Second, our analysis of the symmetric equilibria predicts that discriminatory auction outperforms uniform auctions by generating lower price volatility.

5. Extensions

This section extends the previous section in two dimensions, demand uncertainty and asymmetry of bidders. We investigate the robustness of the main takeaways from the previous section and also obtain new insights about the influence of the other system factors.

5.1 Random Demand

Sections 4 assumes demand to be known (perfectly foreseeable). In practice, some leadtimes are involved. For instance, in electricity day-ahead (hour-ahead) markets, suppliers submit bids one day (one hour) before the actual dispatch. Therefore, demand uncertainty is an element of decision process, which is the focus of this section. For simplicity, the analysis of this section focuses mainly on symmetric auctions. Selected results for asymmetric auctions are reported as extensions.

Equilibrium Analysis of Symmetric Auctions. Suppose all bidders share a common belief about the demand distribution with cdf $\Phi(\cdot)$. Define the lowest demand level $\underline{\xi} \equiv \inf\{\xi : \Phi(\xi) > 0\}$ and the highest one $\bar{\xi} \equiv \sup\{\xi : \Phi(\xi) < 1\}$. For the ease of exposition, we assume $\Pr\{\xi = \underline{\xi}\} = \Pr\{\xi = \bar{\xi}\} = 0$ and $\Phi(\xi)$ is strictly increasing in $\xi \in [\underline{\xi}, \bar{\xi}]$. We first characterize the equilibrium structure. With the intuition established in Section 4, pure-strategy equilibrium can be easily derived for both DA and UA. For low demand with $\bar{\xi} \leq (N-1)k$, the highest bidder obtains zero sales with probability 1. Both auctions are competitive and sustain pure-strategy equilibrium $\{p_i^* = c\}_{i=1}^N$. Similarly, for high demand with $\underline{\xi} \geq Nk$, all bidders choose $\{p_i^* = B\}_{i=1}^N$. Note that these two cases probably cover a relatively small range of possible demand realizations. The interesting case is when $\frac{\underline{\xi}}{N} < k < \frac{\bar{\xi}}{N-1}$ and no symmetric pure-strategy equilibrium exists. Symmetric equilibria are still our primary focus.

Discriminatory Auctions. It is useful to define the (ex ante) expected sales Z_n for the n-th lowest bidder,

$$Z_n \equiv \int_{(n-1)k}^{nk} [\xi - (n-1)k] d\Phi(\xi) + k\bar{\Phi}(nk) = \int_{(n-1)k}^{nk} \bar{\Phi}(\xi) d\xi.$$

Notice that $\{Z_n\}_{n=1}^{N}$ are constant values for given demand distribution $\Phi(\cdot)$. Monotonicity of $\bar{\Phi}(\xi)$ and $\frac{\xi}{N} < k < \frac{\bar{\xi}}{N-1}$ implies $Z_1 \geq Z_2 \geq \ldots \geq Z_n$ and $Z_{N-1} > Z_n$. We also define the total expected sales $X \equiv \mathsf{E}[\min\{\xi, Nk\}] = \sum_{n=1}^{N} Z_n$. Following the logic of Lemma A.1, we can show the continuity and strict monotonicity of $F^d(\cdot)$ in $[\underline{p}^d, B]$, and

$$ER^d = (B-c)Z_n,\ \bar{p}^d = B,\ \underline{p}^d = \frac{ER^d}{Z_1} + c.$$

By (3b, c) we also have, for any $p \in [\underline{p}^d, B]$,

$$\hat{R}^d(p) = (p-c)\sum_{n=0}^{N-1}\binom{N-1}{n} F^d(p)^n \bar{F}^d(p)^{N-n-1} Z_n = ER^d. \tag{12}$$

Unlike the deterministic case, there is no closed-form solution to equation (12) and we need to solve the equation numerically.

Uniform Auctions. The derivation of the solutions to UA is more complicated. We first compute supplier i's expected payoff $\hat{R}_i^u(p)$, given that other players follow F^u. Define function $\delta_n(\xi) \equiv \begin{cases} \delta_{(nk-k<\xi\leq nk)} & \text{for } n < N \\ \delta_{(\xi>Nk-k)} & \text{for } n = N \end{cases}$, indicating the number of active suppliers associated with realized demand ξ. To simplify the notation, denote $b_n \equiv b_n^{(-i)}$ and $G_n(p) \equiv G_n^{(N-1)}(p)$. We also denote $b_0 < c$ and $b_n > B$, and correspondingly, $G_0(p) = 1$ and $G_n(p) = 0$ for all $p \in [c, B]$. The continuity and strict monotonicity of F^u can be established, implying that possibility of price tie can be ignored. Now we have

$$R_i^u(p|\xi, \mathbf{b}^{-i}) = \sum_{n=1}^{N}\delta_n(\xi)\delta_{(b_{n-1}<p<b_n)}(p-c)z_n(\xi) + \sum_{n=1}^{N-1}\delta_{n+1}(\xi)\delta_{(p<b_n)}(b_n-c)k,$$

where $z_n = \min\{k, [\xi-(n-1)k]^+\}$. Taking expectation over ξ and $\mathbf{b}^{-i}$, we have

$$\hat{R}_i^u(p) = (p-c)\sum_{n=1}^{N}[G_{n-1}(p) - G_n(p)]Y_n + k\sum_{n=1}^{N-1}\Phi_{n+1}\int_p^B (v-c)dG_n(v), \tag{13}$$

where $Y_n \equiv \begin{cases} Z_n - k\bar{\Phi}(nk) & \text{for } n < N \\ Z_n & \text{for } n = N \end{cases}$ and $\Phi_n \equiv \begin{cases} \Phi(nk) - \Phi(nk-k) & \text{for } n < N \\ \bar{\Phi}(Nk-k) & \text{for } n = N \end{cases}$.

The first order condition leads to the following ODE of F^u

$$\begin{aligned} &\sum_{n=0}^{N-1}\tbinom{N-1}{n}(F^u)^n(\bar{F}^u)^{N-(n+1)}Y_{n+1} \\ &= (p-c)\dot{F}^u\sum_{n=1}^{N-1}(N-n)\tbinom{N-1}{n-1}(F^u)^{n-1}(\bar{F}^u)^{N-n-1}(Z_n - Z_{n+1}), \end{aligned} \tag{14}$$

with boundary condition $F^u(B) = 1$. [See Appendix A.6 for the derivation of (13) and (14).] Similarly to DA, (14) can only solved numerically. The equilibrium solutions are summarized in Proposition 2. (See Appendix A.7 for its proof.)

Figure 3: Impacts of Demand Uncertainty on Transaction Prices ($N = 3, B = 10, c = 2$) Note: In the right graph, values associated with DA are meshed on the grey surfaces.

Theorem 2 *Both symmetric DA and symmetric UA have unique symmetric Nash equilibria. (a) For* $\underline{\xi} \geq Nk$, *they are identical pure-strategy equilibria with* $\{p_i^* = B\}_{i=1}^N$; *(b) For* $\bar{\xi} \leq (N-1)k$, *they are pure-strategy equilibria with* $\{p_i^* = c\}_{i=1}^N$; *(c) For* $\frac{\underline{\xi}}{N} < k < \frac{\bar{\xi}}{N-1}$, *they are mixed-strategy equilibria. The equilibrium distribution* $F^d(p)$ *solves equation (12) for a DA; and* $F^u(p)$ *solves ODE (14) for a UA.*

It is easy to see that the demand uncertainty may eliminate asymmetric equilibrium for uniform auctions. Recall that multiple asymmetric pure-strategy equilibria exist for UA with deterministic demand $(N-1)k < \xi < Nk$ (Lemma 1). They all involve a complete separation between price maker and price takers. However, when demand is random with $\underline{\xi} < (N-1)k$, a pure-strategy profile $\{p_1 = B, p_i = c \text{ for } i \neq 1\}$ cannot be sustained, because with probability $\Pr\{\xi < (N-1)k\} > 0$ the price is set other suppliers but the highest one. Now bidding a low price is not a best response for any possible price-setting bidder, who will either raise bid (for better expected margin) or reduce bid slightly to undercut other suppliers (for larger expected sales). It leads to non-existence of pure-strategy solution. In other words, a random demand with $\underline{\xi} < (N-1)k < \bar{\xi}$ guarantees the prevalence of mixed-strategy equilibrium (not pure ones). This serves as additional justification for choosing to analyze the symmetric mixed-strategy equilibria.

Impacts of Demand Uncertainty and Auction Comparison. To compare the performance of the two auctions, we again consider buyer prices P^u and P^d with distributions of $H^d(p)$ and $H^u(p)$. Appendix A.8 explains numerical derivation of both $H^d(p)$ and $H^u(p)$. It is easy to see that Expected Price Equality described in Proposition 1(a) still holds under stochastic demand.

$$\mathsf{E}[P^d] = \mathsf{E}[P^u] = (B-c)\frac{NZ_n}{X} + c \tag{15}$$

We are interested in whether the variability ordering for UA and DA described in Proposition 1 also holds in random demand case. Since there is no closed-form solution for both $H^d(\cdot)$ and $H^u(\cdot)$, we use numerical experiments to answer this question.

The numerical test is set as follows. Consider 3-player symmetric DA and UA, with the total capacity normalized to $Nk = 1$ (i.e., $k = 1/3$), and therefore, $\rho = \frac{\mathsf{E}[\xi]}{Nk} = \mathsf{E}[\xi]$. Let price cap be $B = 10$ and cost $c = 2$. The demand is $\xi = \rho + \epsilon$, where the expected demand $\rho \in [0,1]$ and random shock ϵ has symmetric triangle distribution $\mathsf{Tr}(-r, r)$. This triangle distribution has a standard deviation of $\frac{r}{\sqrt{6}}$ and coefficient of variation $CV = \frac{r}{\sqrt{6}\rho}$, which can be easily controlled by adjusting r and ρ. Triangle distribution is easy to analyze and has a unimodal pdf, which is typical for commodity demand. To enforce $\underline{\xi} \geq 0$, r is controlled to be less than or equal to ρ. [14]

For any given (ρ, CV), the equilibrium solution F^d and F^u to DA and UA can be numerically solved based on (12) and (14), as explained in Appendix A.8. The experiment outcomes are presented in Figure 3. Deterministic demand corresponds to $CV = 0$, and serves as a benchmark for evaluating the impact of demand variability. The following qualitative behaviors are observed.

(a) *Buyer price in DA is stochastically less variable than in UA.* The numerical tests confirm our conjecture. P^d is much more variable than P^u. At intermediate utilizations, the standard deviation of prices in DA is about one third of that in UA.

(b) *Demand variability increases the chance of price dispersion, but not necessarily its magnitude.* In Figure 3(b) the vertical axis shows price variance and zero price variance corresponds to pure-strategy pricing outcomes. Intuitively, we expect price to start to vary for bigger demand variability (CV). As CV increases, for both DA and UA, the range of utilizations with zero price variance is indeed shrinking and eventually disappears.

Our intuition would also suggest, that the bigger demand variability, the bigger price dispersion. This, however, is often not the case. In many cases, price dispersion is nearly constant across various levels of demand variability. For intermediate utilizations, introduction of demand variability may even decrease the price variance in both auctions. This phenomenon appears more significant in UA, see Figure 3(b). This could be referred to as "the mixing role" of demand variability. In a deterministic case, the price variance peaks at an intermediate utilization (point A in Figure 3(b)). With random demand shocks, the price variance becomes an equivalent of weighted average of price variances. (Obviously, the underlying behavior is more complicated as the sellers' bidding strategies change.)

One of the questions we faced in the initial phases of interaction with a major northwest energy trading company was whether the price variance is "primarily" driven by demand variance. Our analytical solutions and numerical illustrations suggest that it is not the case. Even with very small (or none) demand variability price dispersion (traders' gambling) is significant. With bigger demand uncertainty, the range of utilizations where price dispersion (gambling) takes place quickly extends, but the size of price dispersion is not significantly influenced, suggesting that the structure of the interactions (use of auction) are primarily driving price variance in the energy markets.

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[14] We performed limited experiments for other distributions and found that the qualitative performances identified below are fairly robust.

5.2 Asymmetric Bidders with Random Demand

Analyzing asymmetric auctions with multiple bidders is technically challenging. see paper Hu et al. (2007). We summarize known properties and, for the purpose of illustrating the typical behavior, we use numerical examples.

To keep analytical tractability, we only consider the case of two bidders. Without loss of generality, we assume $c_1 \leq c_2$ throughout this section. For given costs (c_1, c_2) and price cap B, the resulting equilibrium depends on the combination of capacities (k_1, k_2) and random demand ξ. The derivation is similar to KS. The complete solution in presented in Appendix A.9. Since FFH contains a subset of these results, this section focuses only on the results not discussed in FFH and highlights the new insights not discussed in the previous sections.

Similarly to the symmetric case, DA always has a unique equilibrium, either pure-strategy or mixed-strategy one. There are, however, some differences. (i) When a mixed-strategy equilibrium prevails, one supplier may, with a positive probability, choose a bid equal to the price cap B. With identical costs, low capacity supplier bids more aggressively (with stochastically lower bids) than high-capacity one.[15] That is, the high-capacity supplier (h) has $m_h^d(B) > 0$ and prices stochastically higher than the low-capacity party (l). The intuition is similar to KS who study a duopoly capacitated pricing game. Supplier with higher price gets residual demand and, thus, only partially utilizes her capacity. This hurts more low-capacity supplier who, therefore, tends to price more aggressively.

For UA, similar to the symmetric case, we may have a continuum of equilibria, which happens outside of two extremes (competitive equilibrium $\{c_2, c_2\}$ and monopoly-like equilibrium $\{B, B\}$). Each of them can be qualitatively viewed as a mixture of a pure-strategy equilibrium and a " pure" mixed-strategy equilibrium (a solution with $m_1^u(B) = m_2^u(B) = 0$). Our "pure" mixed strategy is structurally similar to the symmetric equilibrium for UA.

We acknowledge that in a duopoly setting, FFH's selection of pure-strategy equilibrium based on payoff-dominance has a better standing than in the oligopoly case. This is because it does not require pre-game communication among the price takers (there is only one). However, as each supplier prefers to be a price-taker and there are two asymmetric equilibria, certain " agreement" is still needed for a specific equilibrium to be played, as in the symmetric case. Also, FFH's selection again suggests that there is no price dispersion between the two possible market clearing prices c_2 and B in UA.

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[15] Numerical examples suggest that this remains the case with asymmetric costs, unless low capacity supplier has much higher cost.

Figure 4: Expectations and Variances of Transaction Prices. In both graphs, values associated with DA are represented by the grey surfaces, cost vector $c = [1, 3]$, price cap $B = 10$, demand has (truncated) normal distribution $N(4, 2)$.

With some demand randomness (usually moderate), nonuniqueness of equilibrium for UA disappears. Assuming that price cap is not very small, it is sufficient that there is a chance of demand realizations straddle the smaller of the capacities $\underline{\xi} < \min\{k_1, k_2\} < \bar{\xi}$. The reason for it is as follows. The only possible pure-strategy equilibria are when the price is equal cost or the cap, B. A chance of demand exceeding lower capacity, $\bar{\xi} > \min\{k_1, k_2\}$, removes possibility of selling at $\max\{c_1, c_2\}$. If demand may be smaller than lower capacity, $\bar{\xi} > \min\{k_1, k_2\}$, there is a chance for each supplier to be the price setter (supplier' bid becomes the price) and none of the suppliers is wiling to set the price at its cost. Obviously in these cases both suppliers pricing at price cap B is not sustainable, eliminating possibility of pure strategy equilibria. Thus, demand uncertainty has two impacts on the price variability – on one hand, it reduces the system uncertainty by eliminating multiplicity of equilibria; on the other hand, it enforces the price variability by making a mixed-strategy equilibrium the unique solution to the game.

Average price and price variability. With unique solution for a UA, any ambiguity in comparing the two auctions disappears. Figure 4 shows means and standard deviations for both auction formats, as a function of combinations of capacities (k_1, k_2). Our tests indicate that (a) the expected prices associated with DA and UA are quite similar; the ranking of their values is ambiguous and the differences are small; (b) the standard deviation is significantly higher in a UA than in a DA for most of $\mathbf{k}$. It suggests that, in asymmetric settings, the same lessons hold as in symmetric ones – UA yields nearly the same expected price as DA but higher price volatility.

Capacity Asymmetry We are also interested in how capacity structure influences the price dispersion. Since the qualitative characteristics of two auctions appear to be similar, we only discuss the case of DA. In Figure 4(b), we illustrate the effect of total capacity and the effect of capacity asymmetry. A reduction of total capacity (an increase of the system utilization) is illustrated by moving from S to F. The expected price increases, as expected, while the variance increases first and then decreases. The price variance is maximized at intermediate utilization, implying robustness of the same observation for symmetric case.

Capacity asymmetry is illustrated by considering capacity combinations between L and R. In the nearly symmetric setting, corresponding to point S, both expected buyer prices and their variances have local minima.[16] The set of nearly symmetric capacity combinations SF corresponds

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[16] The trajectories for UA and DA are not necessarily identical, but they are very close to each other as illustrated

to the boundary separating sets Ω_3^d and Ω_4^d in the description of DA's solution structure (see Appendix A.9). For points on this boundary (curve SF), the unique mixed-strategy equilibrium has no mass points, $m_1^d(B) = m_2^d(B) = 0$. Thus, if total capacity is fairly evenly[17] allocated between the two suppliers, the auction is more competitive and we observe lower expected price.

On line LR the total capacity and thus system utilization are constant. If the two suppliers' capacities become unbalanced (i.e., S moves towards either L or R), the supplier with increasing portion of system capacity will bid, with increasing probability, at price cap B, i.e., gains dominant pricing power. Consequently, the average price increases and so does the price standard deviation.When capacity allocation becomes very unbalanced, $\mathsf{Std}[P^d]$ may decrease, since $\mathsf{E}[P^d]$ is approaching B and does not vary so much. Thus, capacity asymmetry reduces the market competitiveness and leads to an increase of expected buyer price and increase and decrease of the price variability.

6. Preliminary Empirical Study

Our paper focuses on two related problems, the rationale for price dispersion and the comparison between DA and UA. Since we do not have access to data for the U.K. electricity market (the only marketplace that adopted DA), we investigate the uniform auction used in the U.S., and therefore, our partial tests refer only to the first problem, the structure of price dispersion. New England Power Pool (NEPOOL) is chosen due to its geographical integrity and availability of data.

Research Design and Data Description. Our objective is to see whether our theoretical predictions are consistent with the observed data. Our theoretical analysis shows that bidding strategies and corresponding price dispersion are linked to the demand level, decentralization of capacity, and asymmetry of production technology (cost and capacity). Since in reality the capacity profile and costs are fairly stable, while demand changes, we will observe the effect of single primary factor, demand level, on price dispersion. Our purpose is to use empirical data to provide *qualitative illustration* of derived dynamics rather than to formally test the model.

In general, there are two approaches to describe pricing policies. One is to directly study the bidding decisions of individual generating units. It is used in Wolfram (1998), which analyzes the daily electricity auction in U.K. The other approach is to estimate the aggregated price mark-up, adopted in Wolfram (1999, on U.K. market) and Borenstein, et al. (2002, on California market).

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in the Figure 4.

[17] Cost asymmetry influences the boundary: according to Appendix A.9, suppliers' capacities are proportional to their maximum profit margins $(B - c_1) : (B - c_2)$. With fairly high price caps, these capacities are nearly symmetric.

In principle, as a preliminary test, this section takes the second approach. Our main distinction from the above papers is that we focus primarily on the stochastic properties of the price (price dispersion), rather than the average price level.

Since we are interested in describing price dispersion as a function of demand, a natural choice would be to estimate the different moments of price, such as mean, variance, and skewness, conditional on the demand. A more comprehensive characterization is to directly estimate the conditional distribution of the price. This can be done through the Quantile Regression (QR) model introduced by Koenker and Bassett (1978). Due to its advantage over the classic conditional-mean models, QR model is gaining popularity in many areas of applied econometrics. See Koenker & Hallock (2001) for a good introduction and Koenker (2005) for detailed coverage. While not used so far in analyzing market power and auction data, we find QR very appropriate in the settings we study, to estimate a family of price quantile curves as functions of the actual demand.

Among possible concerns, the critical ones are potential endogeneity and influence of factors not captured in the data. The possible endogeneity between demand and price is marginal in our setting because electricity demand is price independent and primarily driven by the weather (Engle, Mustafa, and Rice 1992). Among multiple factors that may influence price dispersion, the main ones are fuel prices and generation outage. As the fossil-fuel (for example, natural gas) generating plants are often the infra-marginal units at NEPOOL, the change of fuel price moves the cost curve on a daily basis, and subsequently the realized electricity price. The generation outage influences the price dispersion in two ways – 1) it directly modifies the marginal cost curve; 2) creates information asymmetry, as some of the suppliers may be aware of the outage of other units, but not necessarily all. In the analysis below, we attempt to control the impacts of both factors.

Next we describe the data we selected for the study. We examine the hourly demand and real-time price for NEPOOL. The available data covers the period March 2003 through June 2006.[18] Since the new Standard Market Design (SMD) was initiated at NEPOOL in March 2003, we omit the first year to eliminate (or at lest lessen) the effect of any potential learning. As the daily outage data for NEPOOL and daily natural gas price for a major pricing point in New England, Algonquin Cityhub are available, we control for the impact of outage and of fuel price. Figure 5 displays two time series, daily prices and daily outages. It is standard to classify the data as heating season, cooling season, and shoulder months. (The data presented in Figure 5 includes spring-shoulder months (Mar-May 2004), cooling season (Jun-Aug 2004), fall-shoulder months (Sep-Nov 2004),

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[18] Spot price and load data are obtained from website http://www.iso-ne.com, where additional information about the NEPOOL market can be found.

Figure 5: Natural Gas Prices at Algonquin Cityhub and Electricity Outage at NEPOOL

Figure 6: Numerical Examples Illustrating The Qualitative Hypotheses

heating season (Dec 2004 - Feb 2005), and then another cycle.) Since outages are an additional source of variability and since outages are much higher during the shoulder months than in the heating and cooling seasons, in our empirical analysis we omit all shoulder months and concentrate on the first cooling season (Jun-Aug 2004). We do not control for weather – influence of the changing weather manifests itself in demand changing over time.

Marginal Cost Curve and Qualitative Hypotheses.

In this subsection we first discuss the price dispersion that we expect to observe based on our theoretical analysis and structure of the cost-capacity profile at NEPOOL. A direct implication of our analysis is that price should exhibit heteroscedasticity conditional on different demand levels. A more interesting and relevant question is how price variance behaves as a function of the aggregated marginal cost curve. The discussion in Sections 4 and 5.1 suggests that, given symmetric suppliers, the price tends to be more dispersed for intermediate demands between 0 and capacity of all suppliers, as illustrated in Figure 6 (left). For the symmetric uniform auction, we compute and plot nine quantile curves (for 10%, 20%, ... and 90%), illustrating the trend of price dispersion as demand increases.

Note, however, that in practice suppliers' marginal costs are not equal. In practice, it is convenient to express marginal cost as a function of cumulative capacity, where all units are ordered from lowest to highest marginal cost. This is referred to as marginal cost (MC) curve. MC curve is critical in our empirical illustration. A major northwest energy trading company is building such curves for NEPOOL on daily basis and provides several historical MC snapshots within our sample period. Since we focus on the 2004 cooling season, we use the MC curve for the median date (July 15th 2004), see Figure 7.[19] Note that, for the time period we consider (summer 2004), the influence of natural gas is less severe, because the price of natural gas, a major electricity generating fuel, is quite stable as illustrated in Figure 5. As marked on Figure 7, we have four groups of suppliers.we also plot the the histogram of hourly demands (the lower graph in Figure 7). Thus, we conceptually treat the MC curve as consisting of two cost levels, first being groups I and II, and second being group III, with nearly constant cost in each group, as illustrated in Figure 6(b).

While our model assumes identical cost for many suppliers and we do not provide a general

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[19] The curve-providing company adopts a method similar to Borenstein et al. (2002) for MC estimation.

Figure 7: Marginal Cost Curve (07/15/2004) and Histogram of Hourly Demands (Jun.- Aug. 2004) at NEPOOL

solution for two levels of cost, we numerically verify that the solution for two costs can be represented as two "stacked up" solutions, each for a constant marginal cost, see Figure 6(b). The solution has an intuitive structure. When demand does not activate the high-cost suppliers or only activates a small portion of them, they are under intensive competition and, therefore, expected to price at their costs or very close to it. For the lower-cost suppliers, pricing above the high cost group is a dominated strategy, so the higher cost is effectively a price cap for the low-cost suppliers playing the same role as B in our symmetric models. When demand activate high-cost suppliers, the low cost supplier can secure their position of being a price taker by pricing low (such as at the higher cost or just below). The game is now played effectively only among the high-cost suppliers.

The 9 quantile curves illustrate the solution structure of above analysis and the grey box indicates the area where we expect to observe the trend of price dispersion, based on the histogram of demand distribution for the sample period (Figure 7). This numerical example suggests that, when demand moves towards the next stack of suppliers, the price range becomes narrower, and with demand continuing to increase it eventually expands. Furthermore, for demands in left of the shaded area, price is left skewed (with more heavier weight close to the cost level); while in the right of the shaded area, the opposite is the case. Our objective is to investigate how the actual price distribution behaves and whether the above prediction is supported.

One-Variable Conditional Price Dispersion. In this section we explain our methodology in more detail and then present the main results. Next subsection asks about robustness of the observations.

The individual points in Figure 8(a) are a scatterplot of hourly demand and price data. Clearly, as demand increases, the price tends to increase but exhibits wide dispersion. It also illustrates significant nonlinearity between demand and price. Figure 8(a) also superimposes 9 estimated quantile regression curves (for 10, 20, ..., and 90 percent) on the scatterplot. Each curve is specified as a cubic function of the demand to allow for the intuitive shapes of quantile curves.[20] Each curves is specified as follows: for $\tau = 10\%, 20\%, \ldots, 90\%$,

$$P(\tau|d) = \alpha(\tau) + \beta_1(\tau)d + \beta_2(\tau)d^2 + \beta_3(\tau)d^3. \tag{16}$$

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[20] The trend of demand-price in Figure 8(a) has a concave-convex curvature, which can well be captured by a cubic function. We have tried other nonlinear and nonparametric expressions, and the estimated Q-R curves appear to be robust.

Figure 8: (a) Quantile Regression Curves and (b) Conditional Price Dispersion

Figure 9: Quantile Curves for Residuals of Conditional Mean Models

The curves in Figure 8 show the conditional heteroscedasticity of electricity prices and imply that demand not only determines the average price level, but also significantly influences the price variance and skewness. We can see that the realized prices are less dispersed for middle-range demand and more dispersed at the two ends. This is exactly what our theoretical model suggested. To show this more explicitly, we estimate a family of quantile functions for every 1% and compute the conditional density function of the market price. Figure 8(b) illustrates three probability density curves conditional on demand levels of 11000, 15000, and 20000 MWh.[21] The price distribution for demand $d_2 = 15000$ has narrower domain than the ones $d_1 = 11000$ and $d_3 = 20000$. Furthermore, price for d_1 is left-skewed and price for d_3 is right-skewed, as predicted by our numerical example.

Robustness: Two-Stage Estimation of Conditional Price Dispersion. The discussion in the previous section suggests that, for given MC curve at NEPOOL, the prices tend to be less dispersed for intermediate demand and more dispersed for both low and high demand. Clearly several uncontrolled factors may contribute to the price dispersion, such as fuel price, outage, and load shape (with strong daily periodicity). Also, the observation is based on one cooling season (summer 2004) and it is not clear whether it would hold for other seasons. The purpose of this subsection is to control the impacts of various system factors and examine other time periods.

We design the following two-stage test. In the first stage, we consider a single-equation model, where average price is a function of demand, gas price, outage, and time-of-the-day indicator, as shown below. In the second stage, we run quantile regression on the residuals of first-stage equation. Similar to model (16), we focus on the residuals' distribution conditional on the demand, highlighting the influences of market power. The model is as follows,

$$\begin{aligned} &\text{(a) } P = \alpha + \beta_1 d + \beta_2 d^2 + \beta_3 d^3 + \gamma c_{GAS} + \delta T_{out} + \sum_{i=1}^{23} \theta_i D_i + \varepsilon \\ &\text{(b) } \varepsilon(\tau|d) = \alpha'(\tau) + \beta'_1(\tau) d + \beta'_2(\tau) d^2 + \beta'_3(\tau) d^3 \quad \text{for } \tau = 10\%, 20\%, \ldots, 90\%. \end{aligned} \tag{17}$$

where c_{GAS} is the daily gas price at Algonquin for the trading day, T_{out} is the daily outage at NEPOOL, and D_i denote the hourly dummy.

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[21] We note that Koenker (2005) estimated conditional distributions using Q-R in an analysis of the serial correlation of daily temperatures.

In Figure 9, we illustrate the four families of estimated quantile curves, corresponding to the four electricity peak seasons. They show dispersion of the price around the mean. The first one uses the same sample set as in Figure 8. In Figure 9(a) we observe the same behavior of price dispersion as in Figure 8(a). It suggests that after controlling the impacts of the additional factors, price dispersion has the same characteristics as a function of demand level (as a proxy of market power). The same pattern can be observed in the other three graphs, suggesting consistency of results among the datasets we considered.

7. Discussion and Concluding Remarks

Motivated by the widely observed price dispersion in electricity markets and the ongoing debate about design of wholesale electricity market, this paper investigates the sources of price dispersion and compares the stochastic performance of two prevailing market designs, DA and UA. We model the critical elements of the auction as a game among energy suppliers. The analysis of the game allows us to characterize the equilibrium solutions and evaluate their sensitivity to the underlying assumptions. Specifically, it enables us to investigate the probabilistic properties of prices paid by electricity buyers and to compare their performances under DA and UA. To test the empirical implications of our analysis, we introduce Quantile Regression model. By linking hourly-spot prices for the cooling season in 2004 at NEPOOL with NEPOOL's marginal cost curve, we illustrate the consistency between these observations and behavior predicted by our model.

Our paper has a number of economic implications:

Interpretation of Price Dispersion. Our model indicates that one of important sources of price dispersion is intentional randomization, manifesting itself as a mixed strategy. While settings and mechanism are noticeably different, Varian (1980) uses mixed-strategy equilibrium to explain the empirical failure of "law of one price" in consumer markets. In order to price discriminate between informed and uninformed customers, stores may randomly choose sales, which disables the uninformed consumers from learning about future prices. The two critical elements of that paper, market segmentation and information asymmetry, do not play any role in our auction settings. Instead, limited capacity causes price randomization. (At a high level of abstraction, both stores' randomized sales and electricity suppliers' randomized bidding can be treated as intentional strategies to achieve profitability.)

In economics literature, intentional randomization, as an interpretation of mixed strategy, has been criticized for its lack of behavioral applicability.[22] Obviously, work of Harsanyi (1973) may provide an alternative interpretation. Harsanyi establishes a direct connection between pure-strategy Bayesian equilibrium and mixed-strategy solution. The Harsanyi's argument, adapted for electricity market, would be that an electricity supplier in real world, instead of randomizing the bids, may price deterministically according to certain privately observed signal(s). It could be her actual cost, capacity outage, or even personal view of the system uncertainty (such as electricity demand forecast) that give rise to some price changes. With such information asymmetry, a supplier's mixed strategy at equilibrium is just her competitors' appropriate[23] belief about her possible actions. Thus, price dispersion may well be interpreted by the prevalence of mixed-strategy equilibrium.

While both interpretations (purposeful randomization and reaction to private signals) are possible, our conversations with traders of a major electricity trading company indicated that the traders do use purposeful randomization when deciding their bids. (We expect that use of private information also plays a role in some situations.)

Implication to Energy Risk Management. Central task in any risk management applications is to estimate and forecast price volatilities. The popular models used in practice (ARCH and its variations) estimate the price volatility based on the historical prices. To our best knowledge, there is no risk management literature which incorporates information about other system factors. A practical reason is that availability of such data is more limited compared to price data. Importantly, we point to an additional critical predictor of price volatility. As shown in Section 6, the level of electricity price dispersion is heavily influenced by the system capacity utilization (or demand level). Since, in electricity markets, data for electricity demand is as easily available as the price data, including demand information seems as an appropriate adjustment. Inspired by this paper, Wilson and Hu (2007) propose a multi-variate GARCH model to estimate and predict the price volatility of electricity spot prices, where demand is jointly considered with price.

Implication to Procurement Auction Design. The ongoing debate about electricity auction designs has focused on the efficiency of discriminatory and uniform auctions and, particularly, on the resulting price levels. Two schools formed among several leading auction theorists (represented by KCPT and FFH respectively) point in opposite directions. Our analysis and numerical tests suggest, similarly to FFH, that DA is a "better" market design. However, we do not en-

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[22] For example, Cachon and Netessine (2003), argue that " mixed strategies have not been applied in SCM, in part because it is not clear how a manager would actually implement a mixed strategy. . . . It seems unreasonable to suggest that a manager should 'flip a coin' when choosing capacity."

[23] Harsanyi shows that a mixed-strategy equilibrium is the limit of a sequence of pure-strategy Bayesian equilibria corresponding to diminishing uncertainties of private signals.

dorse FFH's argument that DA yields lower prices compared to UA. Our paper shows the payment equivalence under symmetric setting and illustrates that such relation holds approximately under asymmetric scenarios. Thus, we do differ with both of the two schools in terms of market efficiency. We prefer, however, DA over UA because electricity buyers will experience lower price volatility under DA. Considering the high price volatility in the U.S. market (where UA is adopted) and significant attention paid to it, this message has a potential to influence policy decisions.

Independent qualitative confirmation of our results comes from the area of experimental economics, as recent developments of experimental economics provide an alternative approach to compare the two market designs. The laboratory observations have supported neither schools' view (both claim one market design is more efficient than another). Mount et al. (2002) reports that " both uniform auction and discriminatory auction produce average prices fifty percent above the competitive levels. However, the prices for the uniform price auction are more volatile with many price spikes." It is a direct support of our theoretical finding. Similar test was conducted by Rassenti et al. (RSW, 2001), and their experiments indicate that (a) a DA consistently generates lower price volatility; (b) the average prices of the two auctions have no difference for high demand, but (c) DA yields higher average price for low demand.[24] Our theory is consistent with (a) and (b), but fails to explain (c). Despite lack of perfect consistency, compared to other theoretical papers we are aware off, our paper provides predictions closest to experimental observations.

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A. Selected Analysis and Proofs

Note: When context is clear, we ignore the superscripts of auction formats, d and u.

A.1 A.1 Continuity and Monotonicity for F^d.

A.1 Continuity and Monotonicity for F^d.

Lemma A1 *Consider a symmetric DA with $(N-1)k < \xi < Nk$. For a symmetric mixed-strategy equilibrium, $F^d(p)$ is continuous and strictly increasing in $p \in [\underline{p}^d, \bar{p}^d] \cap (c, B]$.*

Proof. (a) *Continuity of F for $p > c$.* Suppose $m(p) > 0$ for certain $p \in (c, B]$, implying $\hat{R}(p) = ER$. Note that

$$\begin{aligned} \hat{r}^-(p) - \hat{r}(p) &= \mathsf{E}_{\sigma^*_{-i}}[r_i^-(p) - r_i(p)] \\ &\geq \Pr\{(\sigma^*_n = p)_{n \neq i}\} \mathsf{E}_{\sigma^*_{-i}}[r_i^-(p) - r_i(p) \,|(\sigma^*_n = p)_{n \neq i}] \\ &= \Pr\{(\sigma^*_n = p)_{n \neq i}\}[(1 \wedge \frac{\xi}{k}) - (1 \wedge \frac{\xi}{Nk})] \qquad \text{[by (1-a)]} \\ &= m(p)^{N-1}[1 - \frac{\xi}{Nk}] > 0. \quad \text{[by independence of all } \sigma^*_n\text{'s]} \end{aligned}$$

Since $p > c$, by (2), we have $\hat{R}^-(p) - \hat{R}(p) = (p - c)k[\hat{r}^-(p) - \hat{r}(p)] > 0$, implying a contradiction to the optimality of $ER = \hat{R}(p)$.

(b) *Monotonicity of F.* Suppose there exist $\alpha < \beta$ such that $F(p') < F(\alpha) = F(\beta) < F(p'')$ for all $p' < \alpha$ and all $p'' > \beta$. Note that any player gets at least a sales of $\xi - (N-1)k$, so the equilibrium payoff must be positive. From (3), we have $\hat{R}^+(\underline{p}) = ER > 0$, implying $\underline{p} > c$. Part (a) implies $\hat{R}^-(p) = \hat{R}(p) = \hat{R}^+(p)$ for all $p \in [\underline{p}, B]$. Now initial assumption implies two results contradicting to each other, (i) $\hat{R}(\alpha) = ER = \hat{R}(\beta)$ by (3) and (ii) $\hat{r}(\beta) = \sum_{n=0}^{N-1} \binom{N-1}{n} F(\alpha)^n \bar{F}(\alpha)^{N-n-1} \cdot [1 \wedge \frac{(\xi - nk)^+}{k}] = \hat{r}(\alpha)$, implying $\hat{R}(\alpha) < \hat{R}(\beta)$.

A.2 Uniqueness of Mixed-strategy Equilibrium for a Symmetric DA.

Proposition 4 *For a symmetric DA with* $(N-1)k < \xi < Nk$, *if* $F_i(p)$ *is continuous and strictly increasing in* $p \in [\underline{p}_i, \bar{p}_i] \cap (c, B)$ *for all* i *with* $\underline{p}_i < \bar{p}_i$, *the symmetric equilibrium defined in (4) is the unique Nash equilibrium.*

Proof. (a) $ER_i \geq (B-c)[\xi-(N-1)k]$ *and* $\underline{p}_i \geq \frac{(B-c)[\xi-(N-1)k]}{k}+c > 0$. For any supplier i and any price p, we have $\hat{r}_i(p) \in [\frac{\xi-(N-1)k}{k}, 1]$. Therefore, optimality of ER_i implies $ER_i \geq \hat{R}_i(B) \geq (B-c)[\xi-(N-1)k] > 0$. By (3-c), we also have $ER_i = \hat{R}_i^+(\underline{p}_i) \leq (\underline{p}_i - c)k$, implying $\underline{p}_i \geq \frac{ER_i}{k} + c$.

(b) $\underline{p}_i = \underline{p} \equiv \min_n\{\underline{p}_n\}$ *and* $ER_i = (\underline{p}-c)k$ *for all* i. Suppose there is a supplier i such that $\underline{p}_i > \underline{p}$. For supplier $j \neq i$, consider her expected payoff at a price $\tilde{p} < \underline{p}_i$. With probability 1, supplier j achieves a sales fraction of $r_j(\tilde{p}) = 1 \wedge \frac{\xi - \sum_{n\neq i} k\delta_{(p_n<\tilde{p})}}{\sum_{n\neq i} k\delta_{(p_n=\tilde{p})}} = 1$ because $\xi - \sum_{n\neq i} k\delta_{(p_n<\tilde{p})} > \sum_{n\neq i} k\delta_{(p_n=\tilde{p})}$. [by $\xi - \sum_{n\neq i} k\delta_{(p_n<\tilde{p})} - \sum_{n\neq i} k\delta_{(p_n=\tilde{p})} = \xi - \sum_{n\neq i} k\delta_{(p_n\leq\tilde{p})} \geq \xi-(N-1)k > 0$] Therefore, $\hat{r}_j(p) = 1$ and $\hat{R}_j(p) = (p-c)k$ for all $p < \underline{p}_i$. As it is strictly increasing in p, the optimality of σ_j^* requires $\Pr\{\sigma_j^* \in [0, \underline{p}_i)\} = 0$, implying $\underline{p}_j \geq \underline{p}_i$ for all $j \neq i$. It implies $\min_{j\neq i}\{\underline{p}_j\} \geq \underline{p}_i$, a contradiction to $\underline{p}_i > \underline{p} = \min_n\{\underline{p}_n\}$, so we must have $\underline{p}_i = \underline{p}$ for all i. Now continuity of F_i at $\underline{p} > c$ implies $m_i(\underline{p}) = 0$ for all i and therefore $\hat{r}_i^-(\underline{p}) = \hat{r}_i(\underline{p}) = \hat{r}_i^+(\underline{p}) = 1$. Applying (3-c), we obtain $ER_i = \hat{R}_i^+(\underline{p}) = (\underline{p}-c)k\hat{r}_i^+(\underline{p}) = (\underline{p}-c)k$ for all i.

(c) *There is at most one player with* $m(\bar{P}) > 0$ *for* $\bar{P} \equiv \max_n\{\bar{p}_n\}$. If $\bar{P} < B$, the desired result follows directly our initial assumption (continuity of $\{F_n\}$). Suppose $\bar{P} = B$ and there are $M \geq 2$ suppliers having $m(B) > 0$. From (3-a), we have $ER_i = \hat{R}_i(B)$ for all $i \in I_M \equiv \{n : m_n(B) > 0\}$. However, similarly to part (a) of Lemma A1's proof, we can establish

$$\hat{r}_i^-(p) - \hat{r}_i(p) \geq \prod_{n\in I_M\backslash\{i\}} m_n(B) \cdot [1 - \frac{\xi-(N-M)k}{Mk}] > 0,$$

implying $\hat{R}_i^-(B) > \hat{R}_i(B) = ER_i$, a contradiction to the optimality of ER_i.

(d) $ER_i = (B-c)[\xi-(N-1)k]$ *for* $i = 1, 2, \ldots, N$ *and* $\underline{p} = c + \frac{(B-c)[\xi-(N-1)k]}{k}$.
Part (c) implies at price $\bar{P}$, either $m_i(\bar{P}) = 0$ for all i or only one supplier has $m(\bar{P}) > 0$. For both cases, we can find one supplier, say h, such that $m_n(\bar{P}) = 0$ for all $n \neq h$. Clearly, supplier h has $\hat{r}_h^-(\bar{P}) = \hat{r}_h(\bar{P}) = \frac{\xi-(N-1)k}{k}$, and consequently, $ER_h = \hat{R}_h^-(\bar{P}) = (\bar{P}-c)[\xi-(N-1)k]$ by (3-b). Optimality of ER_h requires $\bar{P} = B$ and part (b) yields $\underline{p}$.

(e) *Derivation of equilibrium solution.* Consider supplier i's expected payoff at any $p \in [\underline{p}, \bar{p}] \cap [\underline{p}, B)$ where $\bar{p} \equiv \min_n\{\bar{p}_n\}$. As all F_n's are continuous, we have $m_n(p) = 0$ for all $n \neq i$, and the possibilities of price-tie at p can omitted. Now given any bid vector $\mathbf{p}_{-i}$, as long as $b_{(1)}^{(-i)} = \max\{\mathbf{p}_{-i}\} > p_i$, supplier i sells $z_i = k$, while if $b_{(1)}^{(-i)} < p_i$, $z_i = \xi - (N-1)k$. Note that $G_{(1)}^{(-i)}(p) = \Pr\{\max\{\mathbf{p}_{-i}\} < p\} = \Pi_{n \neq i} F_n(p)$, and we have

$$\hat{R}_i(p) = (p-c)[(\xi - Nk + k)G_{(1)}^{(-i)}(p) + k\bar{G}_{(1)}^{(-i)}(p)] = ER_i = (\underline{p} - c)k.$$

It implies

$$G_{(1)}^{(-i)}(p) = \Pi_{n \neq i} F_n(p) = \frac{k}{Nk - \xi}\frac{p - \underline{p}}{p - c} \text{ for all } i.$$

Therefore, $\Pi_{i=1}^N G_{(1)}^{(-i)}(p) = \Pi_{i=1}^N F_i(p)^{N-1} = [\frac{k}{Nk-\xi}\frac{p-\underline{p}}{p-c}]^N$, implying $\Pi_{i=1}^N F_i(p) = [\frac{k}{Nk-\xi}\frac{p-\underline{p}}{p-c}]^{\frac{N}{N-1}}$. Now we have

$$F_i(p) = \frac{\Pi_{i=1}^N F_i(p)^{N-1}}{G_{(1)}^{(-i)}(p)} = [\frac{k}{Nk - \xi}\frac{p - \underline{p}}{p - c}]^{\frac{1}{N-1}} \text{ for } p \in [\underline{p}, \min\{\bar{p}_n\}]$$

Since $F_i(p)$ is strictly increasing and takes value 1 at B, we must have $\bar{p}_i = B$ for all i.

A.3 Structural Properties of Mixed-Strategy Equilibria for Symmetric UA.

Proposition 5 *Consider a symmetric UA with $(N-1)k < \xi < Nk$. For an irreducible equilibrium σ_u^*, if $F_i(p)$ is continuous and strictly increasing in $p \in [\underline{p}_i, \bar{p}_i] \cap (c, B)$ for all i with $\bar{p}_i > c$, then*
(a) $\bar{P} \equiv \max\{\bar{p}_i\} = B$ and at most one supplier has $m(B) > 0$; moreover, at least one supplier h with $\bar{p}_h = B$ has $ER_h = (B-c)[\xi - (N-1)k]$;
(b) $\underline{p}_i = c$ and $m_i(c) = 0$ for all i with $\bar{p}_i > c$;
(c) for $p \in (c, B)$ and $i \in I_p \equiv \{n : p < \bar{p}_i\}$, $F_i(p) = C_{I_p}^i (p-c)^{\frac{\xi - (N-1)k}{(N_p - 1)(Nk - \xi)}}$, where $N_p \geq 2$ denotes the number of players in I_p and $C_{Ip}^i > 0$ remains constant for given I_p.

Proof. First, due to the same logic as part (c) of Appendix A.2, there is at most one player with $m(\bar{P}) > 0$. Similarly to part (d) of Appendix A.2, we can show there exists supplier h such that $\bar{p}_h = \bar{P}$ and $ER_h = (\bar{P} - c)[\xi - (N-1)k]$. The optimality of ER_h requires $\bar{P} = B$.

Denote $\underline{p}^* = \max\{c, \max_n\{\underline{p}_n\}\}$ and irreducibility implies $\underline{p}^* < B$. For any $p \in (\underline{p}^*, B)$, denote $J_p \equiv \{n : \underline{p}_n < p < \bar{p}_n\}$ and $M_p = \|J_p\|$. We first show $M_p \geq 2$. Otherwise, if $M_p = 1$ ($M_p = 0$ is impossible), then we must have $\underline{p}_h < p$ and $\bar{p}_j < p$ for all $j \neq h$ [by continuity and monotonicity of F_i for $i \in J_p$]. Due to the monotonicity and continuity of F_h in $[p, B]$, we establish a contradiction $ER_h = \hat{R}_h(p) < \hat{R}_h(B) = ER_h$ where the two equalities result from statements (4c) and (4b).

We next derive the functional form of $F_i(p)$ for $i \in J_p$ and $p \in (\underline{p}^*, B)$. By (3) again, we have

$$\hat{R}_i(p) = G_{(1)}^{(-i)}(p)(p-c)[\xi - (N-1)k] + k\int_p^B (v-c)dG_{(1)}^{(-i)}(v) = ER_i.$$

The first order condition is identical to (9), so it has the same solution

$$G_{(1)}^{(-i)}(p) = \prod\nolimits_{n \in J_p \setminus \{i\}} F_n(p) = D_i (p-c)^{\frac{\xi-(N-1)k}{Nk-\xi}} \text{ where } D_i > 0.$$

Now we have

$$F_i(p) = \frac{\sqrt[M_p-1]{\Pi_{i \in J_p} G_{(1)}^{(-i)}(p)}}{G_{(1)}^{(-i)}} = C^i_{J_p}(p-c)^{\frac{\xi-(N-1)k}{(M_p-1)(Nk-\xi)}} \text{ where } C^i_{J_p} > 0.$$

It implies, for all $i \in J_p$, $F_i(p) > 0$ for $p > \underline{p}^*$, i.e., $\underline{p}_n \leq \underline{p}^*$. Note that the functional form has $F_i(p) > 0$ for all $p > c$ and $F_i^+(c) = \lim_{p' \downarrow c} F_i(p) = 0$. As $F_i(p)$ is continuous and monotone in $p \in [\underline{p}_i, \bar{p}_i] \cap (c, B)$, we must have $\underline{p}_i = c$. It further implies $\underline{p}^* = c$ and $J_p = I_p$ everywhere for $p \in [c, B]$. Now part (c) follows.

A.4 Cumulative Distribution Functions of P_S^d, P_B^d, and P^u.

$$\begin{aligned}
H^u &= (F^u)^N, \\
H_S^d &= \sum_{n=2}^{N} \frac{k}{\xi} G_{(n)}^d + \frac{\xi - Nk + k}{\xi} G_{(1)}^d = \frac{\xi - Nk}{\xi} G_n + \frac{k}{\xi} \sum_{n=1}^{N} G_n \quad [\text{by } G_{(n)} = G_{N+1-n}] \\
&= \frac{\xi - Nk}{\xi} F^N + \frac{k}{\xi} \sum_{n=1}^{N} \sum_{m=n}^{N} \tbinom{N}{m} F^m \bar{F}^{N-m} = \frac{\xi - Nk}{\xi} F^N + \frac{k}{\xi} \sum_{m=1}^{N} \sum_{n=1}^{m} \tbinom{N}{m} F^m \bar{F}^{N-m} \\
&= \frac{\xi - Nk}{\xi} F^N + \frac{k}{\xi} \sum_{m=1}^{N} m \tbinom{N}{m} F^m \bar{F}^{N-m} = \frac{\xi - Nk}{\xi} F^N + \frac{kN}{\xi} F \sum_{m=1}^{N} \tbinom{N-1}{m-1} F^{m-1} \bar{F}^{N-m} \\
&= \frac{\xi - Nk}{\xi} (F^d)^N + \frac{kN}{\xi} F^d, \\
H_B^d &= B_1 * B_2 * \ldots * B_n
\end{aligned} \tag{Eq1}$$

where $B_n(p) = \begin{cases} G_{(n)}^d(\frac{\xi}{k} p) & \text{for } n < N \\ G_{(n)}^d(\frac{\xi}{\xi - Nk + k} p) & \text{for } n = N \end{cases}$ and $(*)$ denotes convolution operator.

A.5 Proof of Proposition 1.

Proof. As explained in the paper, for any realized bid vector, P_B^d is a constant while P_S^d is a random variable. Therefore, P_S^d is a mean-preserving spread of P_B^d, so we only need to compare P_S^d and P^u.

(a) *Equal Expected Price.* It results from $\sum_{i=1}^N ER_i^A + c\xi = \mathsf{E}[P^A]\xi$ for $A = d$ and u. Propositions 1 and 2 imply $ER_i^A = \hat{R}_i^A(B) = (B-c)[\xi - (N-1)k]$, so the desired result follows.

(b) *Stochastic Ordering.* We illustrate the proof in the above figure. Define function $S(p) \equiv \int_c^p [H_S^d(v) - H^u(v)]dv$ for $p \in [c, B]$. From part (a), we have $\int_c^B \bar{H}_S^d(v)dv + c = \mathsf{E}[P_S^d] = \mathsf{E}[P^u] =$

Figure 10: Figure for A.5

$\int_c^B \bar{H}^u(v)dv + c$, implying $S(B) = \int_c^B [H_S^d(v) - H^u(v)]dv = \int_c^B [\bar{H}^u(v) - \bar{H}_S^d(v)]dv = 0$. To show P^u is stochastically more variable than P^u, we only need to prove $S(p) \leq 0$ for all $p \in [c, B]$.

For $p \in [c, \underline{p}^d]$, as $H_S^d(p) = 0$ and $H^u(p) > 0$, $S(p) = -\int_c^p H^u(v)dv < 0$. For $p \in [\underline{p}^d, B]$, as $S(\underline{p}^d) < 0$ and $S(B) = 0$, it is sufficient to show that $S(p)$ is quasi-convex in $p \in [\underline{p}^d, B]$, that is, there exists $p^* \in (\underline{p}^d, B)$ such that $H_S^d(p) - H^u(p) < 0$ for $p < p^*$ and $H_S^d(p) - H^u(p) > 0$ for $p > p^*$. Given that $H_S^d(\underline{p}^d) - H^u(\underline{p}^d) < 0$ and $H_S^d(B) - H^u(B) = 0$, we only need to show $H_S^d(p) - H^u(p)$ is quasi-concave in $p \in [\underline{p}^d, B]$. A sufficient condition for quasi-concavity is that there exist $T_1(p)$ and $T_2(p)$ such that $\dot{H}_S^d(p) - \dot{H}^u(p) = T_1(p)T_2(p)$, $T_1(p) > 0$, and $T_2(p)$ is decreasing in p. Next we construct such T_1 and T_2. From (Eq1),

$$\begin{aligned}
\dot{H}_S^d(p) &= \frac{\underline{p}^d - c}{\rho(N - N\rho)^{\frac{1}{N-1}}(N-1)(p-c)^2}[(\frac{p - \underline{p}^d}{p - c})^{\frac{1}{N-1}-1} - (\frac{p - \underline{p}^d}{p - c})^{\frac{1}{N-1}}] \qquad \text{(Eq2)}\\
&= C_d(p - \underline{p}^d)^{\frac{2-N}{N-1}}(p-c)^{-2-\frac{1}{N-1}}\\
\dot{H}^u(p) &= (B-c)^{-\frac{N\rho-N+1}{(N-1)(1-\rho)}}\frac{N\rho - N + 1}{(N-1)(1-\rho)}(p-c)^{\frac{N\rho-N+1}{(N-1)(1-\rho)}-1} = C_u(p-c)^{\frac{\rho}{(N-1)(1-\rho)}-2}
\end{aligned}$$

where $C_d = \frac{(\underline{p}^d - c)^2}{\rho(N-N\rho)^{\frac{1}{N-1}}(N-1)} > 0$ and $C_u = (B-c)^{-\frac{N\rho-N+1}{(N-1)(1-\rho)}}\frac{N\rho-N+1}{(N-1)(1-\rho)} > 0$. Thus,

$$\begin{aligned}
\dot{H}_S^d(p) - \dot{H}^u(p) &= C_d(p - \underline{p}^d)^{\frac{2-N}{N-1}}(p-c)^{-2-\frac{1}{N-1}} - C_u(p-c)^{-2-\frac{\rho}{(N-1)(\rho-1)}}\\
&= (p-c)^{-2-\frac{\rho}{(N-1)(\rho-1)}}[C_d(p - \underline{p}^d)^{-\frac{N-2}{N-1}}(p-c)^{-\frac{1}{(\rho-1)(N-1)}} - C^u].
\end{aligned}$$

Let $T_1(p) = (p-c)^{-2-\frac{\rho}{(N-1)(\rho-1)}}$ and $T_2(p) = C_d(p - \underline{p}^d)^{-\frac{N-2}{N-1}}(p-c)^{-\frac{1}{(\rho-1)(N-1)}} - C^u$. As $\dot{T}_2(p) = -\frac{N-2}{N-1}C_d(p - \underline{p}^d)^{-\frac{N-2}{N-1}-1}(p-c)^{-\frac{1}{(\rho-1)(N-1)}} - \frac{1}{(\rho-1)(N-1)}C_d(p-c)^{-\frac{1}{(\rho-1)(N-1)}-1} < 0$ and $T_1(p) > 0$, we obtain the desired functions T_1 and T_2.

A.6 Derivation of Expressions (13) and (14).

$$
\begin{aligned}
\hat{R}_i^u(p|\mathbf{b}^{-i}) &= \mathsf{E}_\xi[R_i^u(p|\xi,\mathbf{b}^{-i})] = (p-c)\sum_{n=1}^{N}\delta_{(b_{n-1}<p<b_n)}\int \delta_n(\xi)[k\wedge(\xi-nk+k)]d\Phi(\xi) \\
&\quad + \sum_{n=1}^{N-1}\delta_{(p<b_n)}(b_n-c)\int k\delta_{n+1}(\xi)d\Phi(\xi) \\
&= (p-c)\sum_{n=1}^{N}\delta_{(b_{n-1}<p<b_n)}Y_n + k\sum_{n=1}^{N-1}\delta_{(p<b_n)}(b_n-c)\Phi_{n+1} \\
\hat{R}_i(p) &= \mathsf{E}_{\mathbf{b}^{-i}}[\hat{R}_i^u(p|\mathbf{b}^{-i})] = (p-c)\sum_{n=1}^{N}[G_{n-1}(p)-G_n(p)]Y_n \\
&\quad + k\sum_{n=1}^{N-1}\Phi_{n+1}\int_p^B (b_n-c)dG_n(b_n). \text{ [A rearrangement provides (13).]} \\
\frac{d\hat{R}_i(p)}{dp} &= \sum_{n=1}^{N}[G_{n-1}(p)-G_n(p)]Y_n + (p-c)\sum_{n=1}^{N-1}\dot{G}_n(p)(Y_{n+1}-Y_n-k\Phi_{n+1}) \\
&= \sum_{n=1}^{N}[G_{n-1}(p)-G_n(p)]Y_n + (p-c)\sum_{n=1}^{N-1}\dot{G}_n(p)(Z_{n+1}-Z_n) \\
&= \sum_{n=0}^{N-1}\binom{N-1}{n}F^n\bar{F}^{N-n-1}Y_{n+1} + (p-c)\dot{F}\sum_{n=1}^{N-1}(N-n)\binom{N-1}{n-1}F^{n-1}\bar{F}^{N-n-1}(Z_{n+1}-Z_n),
\end{aligned}
$$

where the last equality follows the established results of order statistics, $G_n = \sum_{m=n}^{N-1} F^m\bar{F}^{N-m-1}$ and $\dot{G}_n = (N-n)\binom{N-1}{n-1}F^{n-1}\bar{F}^{N-n-1}\dot{F}$. The F.O.C. $\frac{d\hat{R}_i(p)}{dp} = 0$ yields ODE (14).

A.7 Proof of Proposition 2

Proof. Parts (a) and (b) follow exactly the same argument as in the deterministic case. We only consider the case for $\frac{\underline{\xi}}{N} < k < \frac{\bar{\xi}}{N-1}$. First, it is easy to verify that if F^d satisfying (*12*) exists, it must be an equilibrium solution by noticing (a) $\hat{R}^d(p) < ER^d$ for all $p < \underline{p} = c + \frac{(B-c)Z_n}{Z_1}$ and (b) $\hat{R}^d(p) = ER$ for all $p \geq \underline{p}$. Similarly, it can be verified that a solution to (*14*) defines a symmetric mixed-strategy equilibrium for a UA. Therefore, the only thing we need to show that the solutions to (*12*) and (*14*) exist and are unique.

Unlike the deterministic cases, where unique solutions are identified in closed forms, here we have to study in more abstract forms. First consider symmetric DA. A necessary condition for equation (*12*) to hold is $\frac{d\hat{R}}{dp} = 0$ for $p \in [\underline{p}, B]$. It leads to the following ODE

$$
\begin{aligned}
\dot{F}(p) &= \frac{\sum_{n=0}^{N-1}[\binom{N-1}{n}F^n(1-F)^{N-n-1}Z_n]}{(p-c)\sum_{n=1}^{N-1}[\binom{N-1}{n}nF^{n-1}(1-F)^{N-n-1}(Z_n-Z_{n+1})]} \qquad \text{(Eq3)} \\
F(B) &= 1.
\end{aligned}
$$

Notice that $\frac{\xi}{N} < k < \frac{\bar{\xi}}{N-1}$ implies $Z_1 \geq Z_2 \geq \ldots \geq Z_n$ and $Z_{N-1} > Z_n$. Thus, the difference $Z_n - Z_{n+1} \geq 0$ for all $n \in \{1, \ldots, N-1\}$ and $Z_{N-1} - Z_n > 0$. It implies function $\Pi(p, F)$ defined by RHS of (Eq3) is continuous and $\frac{\partial \Pi}{\partial F}$ is continuous in F for any $\{F, p\} \in (0,1] \times (c, B+\varepsilon)$ with small $\varepsilon > 0$. Therefore ODE (Eq3) has one and only one solution $F(p)$ in some neighborhood of $\{F^*, p^*\} \in (0,1] \times (c, B+\varepsilon)$ satisfying $F(p^*) = F^*$.[25] As boundary condition $\{1, B\} \in (0,1] \times (c, B+\varepsilon)$, the solution to (Eq3) exists uniquely.

For UA, clearly, ODE (*14*) can be rearranged into

$$\begin{aligned} \dot{F}(p) &= \frac{\sum_{n=0}^{N-1} \binom{N-1}{n} (F^u)^n (\bar{F}^u)^{N-(n+1)} Y_{n+1}}{(p-c) \sum_{n=1}^{N-1} (N-n) \binom{N-1}{n-1} (F^u)^{n-1} (\bar{F}^u)^{N-n-1} (Z_n - Z_{n+1})} \\ F(B) &= 1. \end{aligned}$$

Similarly to the above case of DA, we can show the existence and uniqueness of the solution to the above ODE.

A.8 H^d and H^u for Symmetric Auctions with Random Demand

Symmetric DA: (Numerical Scheme) Similarly to Appendix A.4, given equilibrium distribution function F, we have $G^d_{(n)}(p) = \sum_{m=n}^{N} \binom{N}{m} F^m \bar{F}^{N-m}$. Now cdf of P^d, conditional on demand ξ is

$$H^d(\cdot|\xi) = B_1(\cdot|\xi) * B_2(\cdot|\xi) * \ldots * B_{\bar{N}(\xi)}(\cdot|\xi)$$

where $B_n(p) = G^d_{(n)}(\frac{\xi}{z_n} p)$, $z_n = \min\{k, \xi - (n-1)k\}$ and $\bar{N}(\xi) = \left\lceil \frac{\xi}{k} \right\rceil$. Thus, $H^d(p) = \mathsf{E}_\xi[H^d(p|\xi)]$ can be computed.

Symmetric UA: (Analytical Scheme) From $P^u = \sum_{n=1}^{N} b_n^{(N)} \delta_{(nk-k<\xi\leq nk)} + b_{(N)}^{(N)} \delta_{(\xi>Nk)}$,

$$\begin{aligned} H^u(p|\mathbf{b}) = \mathsf{E}_\xi[\Pr\{P^u < p|\mathbf{b}\}] &= \sum_{n=1}^{N} \delta_{(b_n<p)} \Phi_n + \xi_{(b_n<p)} \bar{\Phi}(Nk) \\ H^u(p) = \mathsf{E}_{\mathbf{b}}[H^u(p|\xi, \mathbf{b})] &= \sum_{n=1}^{N} \Phi_n \sum_{m=n}^{N} \binom{N}{m} F^m \bar{F}^{N-m} + \bar{\Phi}(Nk) F(p)^N \\ &= \sum_{m=1}^{N} \binom{N}{m} F^m \bar{F}^{N-m} \sum_{n=1}^{m} [\Phi(nk) - \Phi(nk-k)] + \bar{\Phi}(Nk) F(p)^N. \end{aligned}$$

A.9 Equilibrium Analysis for Two-Bidder Auctions with Random Demand

The following notation is useful for both auctions. For supplier $i = 1, 2$, define $K_i \equiv \mathsf{E}[k_i \wedge \xi] = \int_{\underline{\xi}}^{k_i} \xi d\Phi(\xi) + k_i \bar{\Phi}(k_i) = \int_0^{k_i} \bar{\Phi}(\xi) d\xi$ as her expected sales for $p_i < p_j$. Define $X \equiv \mathsf{E}[(k_1 + k_2) \wedge \xi] =$

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[25] **Existence and Uniqueness Theorem for Regular ODE:** For ODE $\dot{\mathbf{x}} = f(t, \mathbf{x})$, if f is continuous and $\frac{\partial f}{\partial x_i}$ are continuous in $\mathbf{x} \in D$, $t \in I$ for all i, where D is a domain and I is an open interval, then the ODE has a solution $\mathbf{x}(t)$, defined uniquely in some neighborhood of $(\mathbf{x}^* \in D, t^* \in I)$ which satisfies $\mathbf{x}^* = \mathbf{x}(t^*)$.

$\int_0^{k_1+k_2} \bar{\Phi}(\xi)d\xi$ as the expected total sales. It is easy to verify that, in case of $p_i > p_j$, supplier i's expected sales is $X - K_j$.

1. *Discriminatory Auctions.* We partition the space of capacities in the below table and illustrate them the right figure. The unique equilibrium solution is as follows.

(a) Pure strategy equilibrium. (a-i) For $\mathbf{k} \in \Omega_1^d$, it is a pure-strategy equilibrium with $\left\{p_1^{d*} = p_2^{d*} = B\right\}$; (a-ii) For $\mathbf{k} \in \Omega_4^d$, it is a pure-strategy equilibrium with $\left\{p_1^{d*} = p_2^{d*} = c\right\}$.

(b) Mixed-Strategy Solution. For $\mathbf{k} \in \Omega_3^d \cup \Omega_4^d$, it is a mixed-strategy equilibrium (σ_1^*, σ_2^*) with

$$
\begin{aligned}
F_h^d(p) &= \frac{(p - \underline{p}^d)K_l}{(p - c_l)[K_h + K_l - \xi]} \text{ for } p \in [\underline{p}^d, B) \text{ and } m_h^d(B) = 1 - \frac{(B - c_h)K_l}{(B - c_l)(K_h \wedge \xi)}; \\
F_l^d(p) &= \frac{(p - \underline{p}^d)K_h}{(p - c_h)[K_h + K_l - \xi]} \text{ for } p \in [\underline{p}^d, B],
\end{aligned}
$$

where $\underline{p}^d = \frac{(B-c_h)(\xi-K_l)}{K_h \wedge \xi} + c_h$ and $(h,l) = (1,2)$ for $\mathbf{k} \in \Omega_3^d$, while $(h,l) = (2,1)$ for $\mathbf{k} \in \Omega_4^d$.

$$
\begin{aligned}
\Omega_1^d &\equiv \{\mathbf{k} : k_1 + k_2 \leq \underline{\xi}\} \\
\Omega_2^d &\equiv \{\mathbf{k} : k_1 \geq \bar{\xi}, \tfrac{K_2}{B-c_2} \geq \tfrac{X}{B-c_1}\} \\
\Omega_3^d &\equiv \{\mathbf{k} : k_1 + k_2 > \underline{\xi}, \tfrac{K_2}{B-c_2} < \tfrac{X}{B-c_1}, \tfrac{K_1}{B-c_1} \geq \tfrac{K_2}{B-c_2}\} \\
\Omega_4^d &\equiv \{\mathbf{k} : k_1 + k_2 > \underline{\xi}, k_1 < \bar{\xi}, \tfrac{K_1}{B-c_1} \leq \tfrac{K_2}{B-c_2}\}
\end{aligned}
$$

2. *Uniform Auctions.* The partition of capacity combinations is as follows and illustrated by the right figure. Note that set Ω_2^u is denoted by the grey area which may overlap with set Ω_{5d}^u (and possibly Ω_3^u). We formally present the equilibrium solution.

$$
\begin{aligned}
\Omega_1^u &\equiv \{\mathbf{k} : k_1 + k_2 \leq \underline{\xi}\} \\
\Omega_2^u &\equiv \left\{\mathbf{k} : k_1 \geq \bar{\xi}, \tfrac{K_2}{B-c_2} \geq \tfrac{X}{B-c_1}\right\} \\
\Omega_3^u &\equiv \{\mathbf{k} : k_1 \geq \bar{\xi}, k_2 \leq \underline{\xi}\} \\
\Omega_4^u &\equiv \{\mathbf{k} : k_1 \geq \bar{\xi}, k_2 < \bar{\xi}\} \\
\Omega_{5a}^u &\equiv \{\mathbf{k} : k_1 + k_2 > \underline{\xi}, k_1 \leq \underline{\xi}, k_2 \leq \underline{\xi}\} \\
\Omega_{5b}^u &\equiv \{\mathbf{k} : k_1 \leq \underline{\xi}, \underline{\xi} < k_2 < \bar{\xi}\} \\
\Omega_{5c}^u &\equiv \{\mathbf{k} : \underline{\xi} < k_1 < \bar{\xi}, k_2 \leq \underline{\xi}, \} \\
\Omega_{5d}^u &\equiv \{\mathbf{k} : \underline{\xi} < k_1, \underline{\xi} < k_2, (k_1 \wedge k_2) > \bar{\xi}\}
\end{aligned}
$$

(a) Pure-Strategy Solution. (a-1) For $\mathbf{k} \in \Omega_1^u$, it is unique with $\{p_1^{u*} = p_2^{u*} = B\}$; (a-2) For $\mathbf{k} \in \Omega_2^u$, $\{p_1^{u*} = p_2^{u*} = c_2\}$ is an equilibrium; (a-3) For $\mathbf{k} \in \Omega_3^u \cup \Omega_4^u \cup \Omega_{5a}^u \cup \Omega_{5b}^u \cup \Omega_{5c}^u$, $\{p_h^{u*} = B, p_l^{u*} = \frac{(B-c_2)(X-k_l)}{K_h} + c_h\}$ is an asymmetric pure-strategy equilibrium, where $(h,l) = (1,2)$ for $\mathbf{k} \in \Omega_{5a}^u \cup \Omega_{5c}^u \cup \Omega_3^u$; and $(h,l) = (2,1)$ for $\mathbf{k} \in \Omega_{5a}^u \cup \Omega_{5b}^u \cup \Omega_4^u$; (a-4) For $\mathbf{k} \in \Omega_{5d}^u \backslash \Omega_2^u$, there is no pure-strategy equilibrium.

(b) Mixed-Strategy Solution. Irreducible mixed-strategy equilibrium exists for $\mathbf{k} \in \Omega^u_{5(a,b,c,d)}$ and satisfies the following distribution function,

$$F_i^*(p; m_B^i) = \begin{cases} (\frac{\beta_i}{\lambda_i} + 1 - m_B^i)(\frac{p-c_j}{B-c_j})^{\lambda_i} - \frac{\beta_i}{\lambda_i} & \text{if } \lambda_i \neq 0 \\ \beta_i \ln \frac{p-c_j}{B-c_j} + 1 - m_B^i & \text{if } \lambda_i = 0 \end{cases} \quad \text{for } p \in [\underline{P}, B), \tag{Eq4}$$

$$\text{where } \underline{P} = \max\{\underline{p}_1, \underline{p}_2\}, \ \beta_i = \frac{\int_{\underline{\xi}}^{k_j} \xi d\Phi(\xi)}{K_1 + K_2 - X} \text{ and } \lambda_i = \frac{X - K_i - \int_{\underline{\xi}}^{k_j} \xi d\Phi(\xi)}{K_1 + K_2 - X}.$$

Moreover, price bound $\underline{P}$ satisfies $\underline{P} = \max\{L_1(0), L_2(0)\}$, where function $L_i(\cdot)$ and its inverse function $M_B^i(\cdot) = L_i^{(-1)}(\cdot)$ are defined from equation $F_i^*(L_i; m_B^i) = 0$ for $i = 1, 2$,

$$\begin{cases} L_i(m_B^i) \equiv c_j + (B - c_j)[\frac{\beta_i}{\beta_i + \lambda_i(1 - m_B^i)}]^{\frac{1}{\lambda_i}}; & M_B^i(\underline{p}_i) \equiv 1 + \frac{\beta_i}{\lambda_i} - \frac{\beta_i}{\lambda_i}(\frac{B-c_j}{\underline{p}_i - c_j})^{\lambda_i} & \text{if } \lambda_i \neq 0 \\ L_i(m_B^i) \equiv c_j + (B - c_j)\exp(\frac{m_B^i - 1}{\beta_i}); & M_B^i(\underline{p}_i) \equiv 1 + \beta_i \ln(\frac{\underline{p}_i - c_j}{B - c_j}) & \text{if } \lambda_i = 0 \end{cases}.$$

The probability masses m_B^1 and m_B^2 determine the set of irreducible equilibria. (b-1) For $\mathbf{k} \in \Omega^u_{5a}$, $\underline{P} = c_2$. Any $\{m_B^1, m_B^2\} \in [0, 1)^2$ with $m_B^1 \cdot m_B^2 = 0$ defines an equilibrium *satisfying* $F_2(p) \geq F_2^*(p, m_B^2)$ *for* $p < c_2$; (b-2) For $\mathbf{k} \in \Omega^u_{5d}$, the equilibrium is *unique* with $m_B^l = 0$ for $l = \arg\max_i \{L_i(0)\}$ and $m_B^h = M_h(\underline{P})$ for $h \neq l$; (b-3) For $\mathbf{k} \in \Omega^u_{5b}$, $m_B^1 = 0$ and any $m_B^2 \in [0, 1)$ defines an irreducible equilibrium with $m_2(\underline{P}) = F_2(\underline{P}; m_B^2)$; (b-4) For $\mathbf{k} \in \Omega^u_{5c}$ if $c_2 \leq L_2(0)$, then $m_B^2 = 0$ and any $m_B^1 \in [0, 1)$ defines an irreducible equilibrium; if $c_2 > L_2(0)$, the equilibrium is *unique* $m_B^1 = 0$, and $m_B^2 = M_B^2(c_2)$.