

# Technical Supplement to Coalition Stability in Assembly Models

Mahesh Nagarajan • Greys Sošić

Sauder School of Business, University of British Columbia, Vancouver, B.C., Canada V6T 1Z2

Marshall School of Business, University of Southern California, Los Angeles, California 90089

mahesh.nagarajan@sauder.ubc.ca • sosic@marshall.usc.edu

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## 1. Appendix A - Proofs

**Proposition 1** *Let  $\mathcal{Z} = \{Z_1, \dots, Z_l\}$  be a coalition structure with at least two nonempty coalitions.*

(1) *If  $i \in Z_k$ ,  $j \in Z_m$ , and  $|Z_k| > |Z_m|$ , then  $\Pi_i^{\mathcal{Z}} < \Pi_j^{\mathcal{Z}}$ .*

(2) *Suppose  $i \in Z_k$  leaves the coalition, thereby changing the coalition structure to  $\mathcal{Z}' = \{Z_1, \dots, Z_k \setminus \{i\}, \{i\}, \dots, Z_l\}$ . Then  $\Pi_i^{\mathcal{Z}'} \geq \Pi_i^{\mathcal{Z}}$ .*

(3) *In the SS and the AS model,  $\sum_1^n \Pi_i^{\mathcal{Z}} < \sum_1^n \Pi_i^N$ .*

(4) *In the VN model,  $\sum_1^n \Pi_i^{\mathcal{Z}} < \sum_1^n \Pi_i^{\mathcal{Z}^{k, n-k}}$  for any  $0 < k < n$  and  $\mathcal{Z}$  with  $l \neq 2$ .*

### Proof of Proposition 1:

1. Item 1 trivially follows from the fact every coalition makes the same profit and members divide the profit equally.

2. To prove item 2, note that  $\mathcal{Z}$  has  $l$  coalitions and  $\mathcal{Z}'$  has  $l+1$  coalitions, and in the SS model,

$$\Pi_i^{\mathcal{Z}'} = \frac{(a - bC)^2}{2b(l+2)^2} \geq \frac{1}{2} \left[ \frac{(a - bC)^2}{2b(l+1)^2} \right] \geq \Pi_i^{\mathcal{Z}},$$

in the VN model

$$\Pi_i^{\mathcal{Z}'} = \frac{(a - bC)^2}{b(l+3)^2} \geq \frac{1}{2} \left[ \frac{(a - bC)^2}{b(l+2)^2} \right] \geq \Pi_i^{\mathcal{Z}},$$

while in the AS model

$$\Pi_i^{\mathcal{Z}'} = \frac{(a - bC)^2}{4b(l+2)^2} \geq \frac{1}{2} \left[ \frac{(a - bC)^2}{4b(l+1)^2} \right] \geq \Pi_i^{\mathcal{Z}},$$

and that  $2(l+1)^2 > (l+2)^2$  for  $l \geq 2$ , and  $2(l+2)^2 > (l+3)^2$  for  $l \geq 1$ .

3. If there are  $l$  coalitions, at equilibrium every coalition in the SS model makes  $(a - bC)^2/[2b(l+1)^2]$ , and in the AS model it makes  $(a - bC)^2/[4b(l+1)^2]$ . Thus, the total profit in the SS

model is  $(a - bC)^2 l / [2b(l + 1)^2]$ , and in the AS model it is  $(a - bC)^2 l / [4b(l + 1)^2]$ . Since  $J(l) = l / (l + 1)^2$  is decreasing in  $l$ , the grand coalition is the efficient outcome, which proves item 3.

4. If there are  $l$  coalitions, at equilibrium every coalition in the VN model makes  $(a - bC)^2 / [b(l + 2)^2]$ . Thus, the total profit in the VN model is  $(a - bC)^2 l / [b(l + 2)^2]$ . Noting that  $\hat{J}(l) = l / (l + 2)^2$  is decreasing in  $l$  for  $l \geq 2$  implies the result. ■

**Proposition 2** *In the SS model with linear demand,  $LCS = \{N\}$  when  $n < 6$ .*

**Proof of Proposition 2:**

- For  $n = 1, 2$ , the result is trivial.
- For  $n = 3$ , the profit of a coalition member in  $\mathcal{Z}_2^3$  is  $(a - bC)^2 / 36b$ , while the profit of a non-member is  $(a - bC)^2 / 18b$ . The profit in the grand coalition is  $(a - bC)^2 / 24b$ , while in  $\mathcal{Z}_1^3$  each player makes  $(a - bC)^2 / 32b$ . It is obvious that  $\mathcal{Z}_1^3$  cannot be stable, because a defection of all players to form the grand coalition cannot be deterred. Next, suppose that a coalition,  $S$ , defects from the grand coalition, where  $S$  can be any one- or two-member coalition. Denote by  $\bar{S} = N \setminus S$ , which is again a one- or two-member coalition. Let  $i$  be a member of one of the two coalitions,  $S$  or  $\bar{S}$ , which has two members. It follows from Proposition 1 that a defection by  $S$  may always be deterred by the following sequence of deviations:

$$\mathcal{Z}_3^3 \rightarrow_S \mathcal{Z}_2^3 \rightarrow_i \mathcal{Z}_1^3 \rightarrow_{1,2,3} \mathcal{Z}_3^3.$$

Finally, if a player defects from  $\mathcal{Z}_2^3$ ,

$$\mathcal{Z}_2^3 \rightarrow_i \mathcal{Z}_1^3,$$

the only possible further action that can deter this defection (that is, which leads to a final outcome in which his profit does not exceed  $(a - bC)^2 / 36b$ ) is a joint deviation by  $i$  and one of the remaining two players,  $j$ ,

$$\mathcal{Z}_2^3 \rightarrow_i \mathcal{Z}_1^3 \rightarrow_{i,j} \mathcal{Z}_2^3,$$

where his profit would be  $(a - bC)^2 / 36b$  again. However, this defection would reduce the profit of player  $i$  from  $(a - bC)^2 / 32b$  to  $(a - bC)^2 / 36b$ , and is therefore unlikely. As a result,  $\mathcal{Z}_2^3$  cannot be stable, and the grand coalition is the only stable outcome.

- Next, note that in  $\mathcal{Z}_1^n$  each player makes  $(a - bC)^2/[2(n + 1)^2b]$ , while in the grand coalition  $\mathcal{Z}_n^n$  each player makes  $(a - bC)^2/[8nb]$ . In an arbitrary basic coalition structure  $\mathcal{Z}_{n-k+1}^n, 1 < k < n$ , which has  $k - 1$  independent players (which equals  $k$  coalitions), each player makes  $(a - bC)^2/[2(k + 1)^2(n - k + 1)b]$ . It is easy to verify that for  $n = 4, 5$ , each coalition member in  $\mathcal{Z}_{n-k+1}^n, 1 < k < n$ , makes less than in the case where all players act independently,  $\mathcal{Z}_1^n$ . Thus, following an approach similar to the one used for three players, it is easy to show that the grand coalition is stable, while any basic coalition structure  $\mathcal{Z}_{n-k+1}^n, k > 1$ , is not. As a result, we have to check the stability of  $\mathcal{Z}_{2,2}^4$  for  $n = 4$ , and  $\mathcal{Z}_{2,3}^5$  and  $\mathcal{Z}_{1,2,2}^5$  for  $n = 5$ .

Suppose that  $n = 4$  and the status quo is  $\mathcal{Z}_{2,2}^4$ ; then, each player makes  $(a - bC)^2/(36b)$ . Suppose that a one- or a two-member coalition,  $S$ , defects from the status quo,

$$\mathcal{Z}_{2,2}^4 \rightarrow_S \mathcal{Z}_2^4.$$

Now, each coalition member receives  $(a - bC)^2/(64b)$ , while a non-coalition member gets  $(a - bC)^2/(32b)$ . The only possible action that would deter this defection should end up with the final outcome  $\mathcal{Z}_{2,2}^4$ , so that the payoff of defecting players does not exceed their payoff in the status quo. Since the sequence where two independent players form a coalition,

$$\mathcal{Z}_{2,2}^4 \rightarrow_S \mathcal{Z}_2^4 \rightarrow_{i,j} \mathcal{Z}_{2,2}^4,$$

reduces the profit of players  $i$  and  $j$  from  $(a - bC)^2/(32b)$  to  $(a - bC)^2/(36b)$ ,  $\mathcal{Z}_{2,2}^4$  does not indirectly dominate  $\mathcal{Z}_2^4$  and the move will not happen. The other possible sequence is

$$\mathcal{Z}_{2,2}^4 \rightarrow_S \mathcal{Z}_2^4 \rightarrow_k \mathcal{Z}_1^4,$$

where  $k$  is a member of the remaining two-player coalition. Any move from  $\mathcal{Z}_1^4$  to  $\mathcal{Z}_{2,2}^4$  should first include forming a two-player coalition,

$$\mathcal{Z}_{2,2}^4 \rightarrow_S \mathcal{Z}_2^4 \rightarrow_k \mathcal{Z}_1^4 \rightarrow_{p,r} \mathcal{Z}_2^4 \rightarrow_{i,j} \mathcal{Z}_{2,2}^4.$$

As shown above, such move cannot happen because  $\mathcal{Z}_{2,2}^4$  does not indirectly dominate  $\mathcal{Z}_2^4$ . Thus, when  $n = 4$ , the grand coalition is uniquely stable. In a similar way, one can show that the grand coalition is uniquely stable when  $n = 5$ . ■

**Theorem 1** *In the SS model with linear demand, the following statements hold.*

- (1) *The grand coalition is always in the LCS.*
- (2) *The coalition structure where all players act independently is never in the LCS.*

- (3) Let  $\hat{k} = n - \left\lceil \frac{(\sqrt{4n+5}-1)}{2} \right\rceil + 1$ . Then, no basic coalition structure  $Z_k^n, k \leq \hat{k}$ , is in the LCS.
- (4) For large  $n$ ,  $Z_{k,k}^{2k}$  or  $Z_{j,j,1}^{2j+1}$  are in the LCS.

**Proof of Theorem 1:**

1. Consider  $N \rightarrow_S \mathcal{X}$ , a deviation from the grand coalition by a set of players  $S$ . Consider the following sequence:  $N \rightarrow_S \mathcal{X} \rightarrow_{S_1} \mathcal{X}_1 \rightarrow \dots \rightarrow_{S_k} \mathcal{X}_k \rightarrow_{S_{k+1}} \mathcal{Z}_1^n \rightarrow_N N$ , where at each step before the last step a single member splits from the largest coalition (ties broken arbitrarily) to create the new outcome. In any outcome  $\mathcal{X}_i$  we know that the members of the largest coalition must earn lower than in the grand coalition. Thus  $\mathcal{X} \ll N$  and  $N \not\ll_S N$ . This proves item 1.

2. It is easy to show that a player in  $\mathcal{Z}_1^n$  always receives less than a player in the grand coalition. Thus this automatically rules out the grand coalition as a candidate for  $\mathcal{B}$  in the definition of the LCS. Let  $\mathbf{S} = \{\mathcal{Z} : \text{members of the largest coalition in } \mathcal{Z} \text{ make less than in } \mathcal{Z}_1^n\}$ . Clearly, only candidates for  $\mathcal{B}$  must be in  $\mathbf{S}^c$ .

Now, consider  $\mathcal{Z}_i^n \rightarrow_N N$ . To deter this defection, we need to find a  $\mathcal{B}$  such that  $\mathcal{Z}_i^n \not\ll_N \mathcal{B}$  and  $N \ll \mathcal{B}$ . In addition, we need  $\mathcal{B} \in \mathbf{S}^c$ . Clearly, the only  $\mathcal{B} \in \mathbf{S}^c$  that is a feasible candidate is  $\mathcal{B} = \mathcal{Z}_i^n$ . But,  $(a - bC)^2 / (8b) > (a - bC)^2 / (2b(n + 1)^2)$ . Thus,  $N \ll \mathcal{Z}_i^n$ , which implies that  $\mathcal{Z}_i^n$  is not stable.

3. Next, to prove that for large  $n$ , no basic coalition structures are in the LCS, we use an alternate definition of the LCS. The LCS is the fixed point of a suitably defined map on the set  $2^{\mathbf{Z}}$ . More precisely,  $f : 2^{\mathbf{Z}} \rightarrow 2^{\mathbf{Z}}$  is defined such that

$$f(X) = \{\mathcal{Z} \in \mathbf{Z} : \forall \mathcal{V}, S, \text{ such that } \mathcal{V} \in F_S(\mathcal{Z}), \exists \mathcal{B} \in X, \text{ where } \mathcal{V} = \mathcal{B} \text{ or } \mathcal{V} \ll \mathcal{B}, \text{ such that } \mathcal{Z} \not\ll \mathcal{B}\} \quad (\text{A1})$$

It can be shown that  $f$  is isotonic, and the LCS is nothing but  $\mathbf{Y} = \bigcup_{t \in \Sigma} t$ , where  $\Sigma = \{t \subseteq \mathbf{Z} : t \subset f(t)\}$ . This implies that the LCS can be written as  $\bigcap_{i=0}^s \mathbf{Y}_i$ , where  $\mathbf{Z} = \mathbf{Y}_0$ , and inductively  $\mathcal{Z} \in \mathbf{Y}_{i-1}$  belongs to  $\mathbf{Y}_i$  if and only if  $\forall \mathcal{X}$  and  $S$  such that  $\mathcal{X} \in F_S(\mathcal{Z}), \exists \mathcal{W} \in \mathbf{Y}_{i-1}$ , where  $\mathcal{W} = \mathcal{X}$  or  $\mathcal{X} \ll \mathcal{W}$  such that  $\Pi_S^{\mathcal{Z}} < \Pi_S^{\mathcal{W}}$  does not hold.

Let  $\mathbf{J} = \{\text{set of all basic coalitions} \setminus \mathcal{Z}_1^n\}$ . We can show that, if  $j \in \mathbf{J}$ , there are some  $i$  such that  $j \notin \mathbf{Y}_i$  in the inductive process described before. We show this by demonstrating that if  $j \in \mathbf{Y}_i \forall i$ , then  $\mathcal{Z}_1^n \in \mathbf{Y}_i \forall i$  as well. This leads to a contradiction, as we have shown that  $\mathcal{Z}_1^n \notin \text{LCS}$ . To see this, note that in  $\mathcal{Z}_1^n$  each player makes  $\frac{(a-bC)^2}{2b(n+1)^2}$ , and each player in the largest

coalition in  $\mathcal{Z}_{n-l+1}^n$  makes  $\frac{(a-bC)^2}{2b(l+1)^2(n-l+1)}$ . The function  $g(x) = (x+1)^2(n-x+1)$ ,  $x \leq n$  is concave, and  $g(x) \geq (n+1)^2$  when  $x \geq n$  or  $x \leq \frac{\sqrt{5+4n}-1}{2}$ . This implies that, starting with any basic coalition structure  $\mathcal{Z}_{n-l+1}^n$  with  $l > \frac{\sqrt{5+4n}-1}{2}$ , we can build a chain  $\mathcal{Z}_{n-l+1}^n \rightarrow \dots \rightarrow \mathcal{Z}_1^n$  by using single defections, as each defecting player prefers  $\mathcal{Z}_1^n$  to the current status quo. These chains, together with the item 2 in Proposition 1, imply that if  $\mathbf{Y}_i$  contains a basic coalition for every  $i$ , it contains  $\mathcal{Z}_1^n$  as well. This proves item 3.

4. Let  $\mathbf{R} = \{\text{equal-sized coalition structures} \setminus \{\mathcal{Z}_1^n, N\}\}$ . To show item 4, we need the following lemmas.

**Lemma 1**  $\mathbf{R}$  exhibits external stability with respect to  $\ll$ . That is,  $\forall \mathcal{X} \notin \text{LCS}, \exists \mathcal{R} \in \mathbf{R}$  such that  $\mathcal{X} \ll \mathcal{R}$ .

**Proof:** For any  $\mathbf{U} \subseteq \mathbf{Z}$ , define a relation  $\sim_{\mathbf{U}} \subseteq \mathbf{Z} \times \mathbf{Z}$  as follows:

$$(\mathcal{X}, \mathcal{Y}) \in \sim_{\mathbf{U}} \text{ if } \mathcal{Y} \in \mathbf{U} \Rightarrow \mathcal{X} \ll \mathcal{Y}, \text{ and } \mathcal{Z} \in \mathbf{U} \text{ and } \mathcal{Y} \ll \mathcal{Z} \Rightarrow \mathcal{X} \ll \mathcal{Z}.$$

Define  $\triangleleft_{\mathbf{R}} \subseteq \mathbf{Z} \times \mathbf{R}$  as

$$(\mathcal{R}, \mathcal{Y}) \in \triangleleft_{\mathbf{R}} \text{ if } \mathcal{Y} \in \mathbf{R} \Rightarrow \mathcal{R} \ll \mathcal{Y} \text{ and } \mathcal{Z} \in \mathbf{R} \text{ and } \mathcal{Y} \ll \mathcal{Z} \Rightarrow \mathcal{R} \ll \mathcal{Z}.$$

Define  $\diamond^{\mathbf{U}} = \sim_{\mathbf{U}} \cap \triangleleft_{\mathbf{R}} \subseteq \mathbf{Z} \times \mathbf{Z}$ .  $\diamond^{\mathbf{U}}$  is finite and transitive. Given any  $\mathbf{Y} \subseteq \mathbf{Z}$ , let  $M(\mathbf{Y}, \diamond^{\mathbf{Y}}) = \{\mathcal{A} \in \mathbf{Y} : \exists \mathcal{B} \in \mathbf{Y} \text{ such that } (\mathcal{A}, \mathcal{B}) \in \diamond^{\mathbf{Y}}\}$ . We can show that  $M$  is nonempty. Define  $\ell : 2^{\mathbf{Z}} \rightarrow 2^{\mathbf{Z}}$  as  $\ell(\mathbf{A}) = M(\mathbf{A}, \diamond^{\mathbf{A}})$ . To prove Lemma 1, we need to show that  $\ell$  has a fixed point.

Let

$$T_k = \frac{(a-bC)^2}{2b(k+1)^2\Psi},$$

where

$$\Psi = \begin{cases} \frac{n}{k}, & \text{if } n \bmod k = 0, \\ \lfloor \frac{n}{k} \rfloor + 1, & \text{otherwise.} \end{cases}$$

Thus, for any  $k$ ,  $T_k$  is the minimum profit obtained by coalitions when an equal-sized coalition structure with  $k$  members is formed. Note that  $T_k \geq \frac{(a-bC)^2}{2b(n+1)^2} \forall k = 1, 2, \dots, n-2$  when  $n$  is large. This, together with item 2 in Proposition 1, implies that  $\ell^{i+1}(\mathbf{A}) \subset \ell^i(\mathbf{A})$  when  $n$  is sufficiently large. Because  $\mathbf{Z}$  is finite, there is  $\mathbf{W} \neq \emptyset$  such that  $\ell(\mathbf{W}) = \mathbf{W}$ .  $\blacksquare$

**Lemma 2**  $\mathbf{R} \cap \text{LCS} \neq \emptyset$ .

**Proof:** Follows after replacing  $X$  by  $\mathbf{R}$  in the RHS of the definition of  $f$ , (A1). ■

The above analysis yields an interesting corollary. If we remove  $N$  from the LCS, no single element in the set  $\text{LCS} \setminus N$  can indirectly dominate all elements in  $\mathbf{Z}$ . This is because there is no externally stable element  $\mathcal{Y} \in \mathbf{Y}_i \forall i$  such that  $\ell(\mathcal{Y}) = \mathcal{Y}$  other than  $N$ .

Suppose now that  $n = 2k$ . We are now in the position to show that  $\mathcal{Z}_{k,k}^{2k}$  is in the LCS. To do so, we once again employ reductio ad absurdum. We show that if  $\mathcal{Z}_{N/k, \dots, N/k}^N$  is not the LCS, then  $\mathcal{Z}_{N/(k+1), \dots, N/(k+1)}^N$  is also not in the LCS. By previous lemma, this implies that  $\mathcal{Z}_{k,k}^{2k}$  is in the LCS. To see that, note that from an equal-sized coalition structure with  $k$  coalitions, an equal-sized coalition structure with  $k+1$  coalitions can be reached in at most two steps, of which at least one is myopically beneficial. This implies that, if we use  $\triangleleft_{\mathbf{R}}$  with the constraint of two-step moves in the definition of  $\diamond^{\mathbf{U}}$ , we still have that  $\ell$  has a fixed point. Thus,

$$\left( \mathcal{Z}_{\frac{N}{k+1}, \dots, \frac{N}{k+1}}^N \in \bigcap \mathbf{Y}_i \Rightarrow \mathcal{Z}_{\frac{N}{k}, \dots, \frac{N}{k}}^N \in \bigcap \mathbf{Y}_i \right) \Rightarrow \left( \mathcal{Z}_{\frac{N}{k}, \dots, \frac{N}{k}}^N \notin \text{LCS} \Rightarrow \mathcal{Z}_{\frac{N}{k+1}, \dots, \frac{N}{k+1}}^N \notin \text{LCS} \right).$$

Finally, it is not hard to see that, when  $n = 2j+1$ , any defection from  $\mathcal{Z}_{j,j,1}^{2j+1}$  can be effectively deterred by using  $\mathcal{B} = N$  or  $\mathcal{B} = \mathcal{Z}_{j,j,1}^{2j+1}$  in the definition of the LCS, which proves its stability. ■

**Theorem 2** Consider the SS model with linear demand. Let  $\underline{k} = \lceil 4/9n \rceil$ ,  $\bar{k} = \lfloor 5/9n \rfloor$ , and  $\hat{l} = \lfloor 2\sqrt{n} - 1 \rfloor$ .

(1) There exists  $\Delta^1$  such that, for  $1 > \delta > \Delta^1$ ,

$$EP(n) = \{N\} \cup \{\mathcal{Z}_{k, n-k}^n, k \leq \underline{k}\} \cup \{\mathcal{Z}_{j, j, 1}^{2j+1}, \text{ if } n = 2j+1 \text{ odd}\}.$$

(2) For  $1 > \delta > 0.5$ ,

(i)  $\mathcal{Z}_{k, n-k}^n \notin EP(n)$  for  $\underline{k} < k < \bar{k}$ ;

(ii)  $\mathcal{Z} = \{Z_1, \dots, Z_l\} \notin EP(n)$  whenever  $l \geq \hat{l}$ .

**Proof of Theorem 2:**

(1) Let  $|\mathbf{Z}| = t$ , and let  $m$  and  $M$  be minimum and maximum single period payoffs to any player, respectively. Choose  $\Delta^{(1)} \in (0, 1)$  such that for any two states  $\mathcal{X}, \mathcal{Y}$ , and any player  $i$  with  $u_i(\mathcal{X}) > u_i(\mathcal{Y})$ , we have

$$u_i(\mathcal{X}) > [1 - (\Delta^{(1)})^t]M + (\Delta^{(1)})^t u_i(\mathcal{Y}), \text{ and } u_i(\mathcal{X})(\Delta^{(1)})^t + [1 - (\Delta^{(1)})^t]m > u_i(\mathcal{Y}).$$

Such a  $\Delta^{(1)}$  exists by continuity and the fact that we have a finite number of states. Consider any  $\delta \in (\Delta^{(1)}, 1)$  and fix an arbitrary absorbing deterministic EPCF with absorbing states  $\mathbf{X} \subseteq \mathbf{Z}$ . Let  $\mathcal{Z} \in \mathbf{Z}$ , and  $\mathcal{Y} \in F_S(\mathcal{Z})$  for some fixed coalition  $S$ . Let  $\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_r$  be the subsequent PCF path, where  $\mathcal{Y}_1 = \mathcal{Z}$ , and  $\mathcal{Y}_r = \mathcal{Y}$ . Then, there is an  $S_j$  such that  $\mathcal{Y}_{j+1} \in F_{S_j}(\mathcal{Y}_j)$  and  $v_i(\mathcal{Y}_{j+1}) \geq v_i(\mathcal{Y}_j)$ . Note that  $v_i(\mathcal{Y}_j) = u_i(\mathcal{Y}_j) + \delta v_i(\mathcal{Y}_{j+1})$  so that  $(1 - \delta)v_i(\mathcal{Y}_{j+1}) \geq u_i(\mathcal{Y}_j)$  for each  $S_j$  and  $S_{j+1}$ .

Now, since the PCF from  $\mathcal{Y}_{j+1}$  leads to an absorbing state  $\mathcal{Y}_r = \mathcal{Y}$  in at most  $t$  steps (by the choice of  $\Delta^{(1)}$ ), we conclude that  $(1 - \delta^t)M + \delta^t u_i(\mathcal{Y}) \geq u_i(\mathcal{Y}_j)$ . This implies that  $\mathcal{Y} \in \mathbf{X}$  is also an indirect objection to  $\mathcal{Y}_1 = \mathcal{Z}$ .

Now, since  $N$  is a possible move from  $\mathcal{Z}_{k,n-k}^n$  or  $\mathcal{Z}_{j,j,1}^{2j+1}$  and vice versa, and since lone defections are beneficial myopically,  $N$ ,  $\mathcal{Z}_{k,n-k}^n$ , and  $\mathcal{Z}_{j,j,1}^{2j+1}$  belong to the set  $\mathbf{X}$  such that  $\mathbf{X} \subseteq \hat{f}(\mathbf{X})$ , where  $\hat{f}(\mathbf{X})$  is the set of states  $\mathcal{X}$  such that for every coalition  $S$  and any state  $\mathcal{D} \in F_S(\mathcal{X})$ , there is  $\mathcal{Z} \in \mathbf{X}$ , where either  $\mathcal{D} = \mathcal{Z}$  or  $\mathcal{D} \ll \mathcal{Z}$  and  $u_i(\mathcal{X}) \leq u_i(\mathcal{Z})$  for some  $i \in S$ . This shows that  $N, \mathcal{Z}_{j,j,1}^{2j+1}, \mathcal{Z}_{k,n-k}^n, k \leq \underline{k} \in EP(n)$  for  $\delta > \Delta^{(1)}$ .

Let  $\mathcal{X} \in \mathbf{X} \setminus \left[ \{N\} \cup \left\{ \mathcal{Z}_{k,n-k}^n, k \leq \underline{k} \right\} \cup \left\{ \mathcal{Z}_{j,j,1}^{2j+1}, \text{ if } n = 2j + 1 \text{ odd} \right\} \right] \cup \{\text{structures with } l < \hat{l} \text{ coalitions}\}$ . Then, using exactly the same steps as before, we have  $\lim_{\Delta \rightarrow 1^-} u_i(\mathcal{Z}) - \Delta^t u_i(\mathcal{X}) < 0$  for some  $\mathcal{Z} \in \{N\} \cup \left\{ \mathcal{Z}_{k,n-k}^n, k \leq \underline{k} \right\} \cup \left\{ \mathcal{Z}_{j,j,1}^{2j+1}, \text{ if } n = 2j + 1 \text{ odd} \right\}$ . Thus,

$$EP(n) = \{N\} \cup \left\{ \mathcal{Z}_{k,n-k}^n, k \leq \underline{k} \right\} \cup \left\{ \mathcal{Z}_{j,j,1}^{2j+1}, \text{ if } n = 2j + 1 \text{ odd} \right\}.$$

(2) (i) Each coalition in  $\mathcal{Z}_{k,n-k}^n$  makes  $\frac{(a-bC)^2}{18b}$ . When  $\underline{k} < k < \bar{k}$ , each coalition member in either coalition makes strictly less than in the grand coalition. We cannot find a profitable move from the grand coalition that ends in  $\mathcal{Z}_{k,n-k}^n$  when  $\delta > 0.5$ .

(ii) The highest profit in a coalition structure with  $l$  coalitions is realized by a lone player,  $\frac{(a-bC)^2}{2b(l+1)^2}$ . When  $\delta > 0.5$ , it will be dominated for all players by a coalition  $\mathcal{Z}_{k,n-k}^n$  if  $(l+1)^2 > 9 \max\{k, n-k\} = 9n/2$ . Using (i) we now get the result. ■

**Theorem 3** *In the VN model with linear demand, the following statements hold.*

- (1)  $LCS = \{\mathcal{Z}_1^n\}$  when  $n = 2$  or  $n = 3$ .  $\mathcal{Z}_1^n \notin LCS$  for any other  $n$ .
- (2)  $LCS = \{N\}$  when  $n = 5$  or  $n = 7$ .  $N \notin LCS$  for any other  $n$ .
- (3) Equal-sized coalition structures with two or three coalitions are the only members of the LCS when  $n = 4$ ,  $n = 6$ , and  $n \geq 8$ .

**Proof of Theorem 3:** The fact that the grand coalition is not Pareto efficient and  $\mathcal{Z}_{k,n-k}^n, k = \lfloor \frac{n}{2} \rfloor$ , is better than the grand coalition for all players when  $n \notin \{3, 5, 7\}$ , together with the items 1 and 2 in Proposition 1 gives us the following useful lemma.

**Lemma 3** Consider  $\mathcal{X} \in \mathbf{Z}$  and consider an arbitrary move  $\mathcal{X} \rightarrow_S \mathcal{V}$ . To check if this move is deterred, it is sufficient to consider for  $\mathcal{B} \in \mathbf{D} = \{\mathcal{Z}_{k,n-k}^n, k = 1, \dots, \lfloor \frac{n}{2} \rfloor\}$  in the definition of the LCS when  $n \notin \{3, 5, 7\}$ .

**Proof:** Observe that each coalition in any  $\mathcal{Y} \in \mathbf{D}$  makes  $(a - bC)^2/(16b)$ . We need to show that, by the definition of  $f$  given by (A1),  $f(\mathbf{X}) \subseteq f(\mathbf{D}) \forall \mathbf{X} \in 2^{\mathbf{Z}}$ . We show this by demonstrating that  $\mathbf{D}$  indirectly dominates every coalition structure and there is no  $\mathbf{U} \in 2^{\mathbf{Z}}$  such that  $\mathbf{D} \cap \mathbf{U} = \emptyset$  and  $\mathbf{U} \gg \mathbf{D}$ . Here,  $\mathbf{U} \gg \mathbf{D}$  means that, for a  $\mathcal{X} \in \mathbf{D}$  and a given initial move  $\mathcal{X} \rightarrow_S \mathcal{Y}$ , we can find  $\mathcal{U} \in \mathbf{U}$  such that  $\mathcal{X} \rightarrow_S \mathcal{Y} \rightarrow \dots \rightarrow \mathcal{U}$  and  $\mathcal{X} \ll \mathcal{U}$ . This, together with the Pareto efficiency of  $\mathbf{D}$ , demonstrates the lemma.

First, note that, when  $n \notin \{3, 5, 7\}$ , for any  $\mathcal{Z} \in \mathbf{Z}, \mathcal{Z} \notin \mathbf{D}$ , and  $\xi = \lfloor \frac{n}{2} \rfloor$ , the chain  $\mathcal{Z} \rightarrow \mathcal{Y} \rightarrow \mathcal{W} \rightarrow \dots \rightarrow \mathcal{Z}_1^n \rightarrow N \rightarrow \mathcal{Z}_{\xi, n-\xi}^n$ , where starting from  $\mathcal{Z}$  a lone player defects from the largest coalition, until  $\mathcal{Z}_1^n$  is reached, establishes the above claim trivially. Further, to show that there exists no such  $\mathbf{U}$  as described above, for an outcome  $\mathcal{X} \in \mathbf{D}$  we need only consider the initial move  $\mathcal{X} \rightarrow_N N$ . These two facts coupled with Pareto efficiency let us construct a transitive finite relation similar to that of Theorem 1. This leads us to the conclusion that  $f(\mathbf{X}) \subseteq f(\mathbf{D}) \forall \mathbf{X} \in 2^{\mathbf{Z}}$ . ■

This supplies the ingredients to prove the main findings of Theorem 3.

1. When  $n = 2$  or  $n = 3$ , individual players make more than when they all act together, so it is easy to show that the coalition structure without any coalitions is the sole stable outcome.

Note that, for  $n \geq 4$ , any  $\mathcal{X}$  in  $\mathbf{D}$  directly dominates  $\mathcal{Z}_1^n$ , that is,  $\mathcal{Z}_1^n <_i \mathcal{X} \forall i \in N$ . Thus, there is no  $\mathcal{X} \in \mathbf{D}$  which will work as a choice of  $\mathcal{B}$  in the definition of the LCS. This implies that  $\mathcal{Z}_1^n$  is not stable.

2. Suppose  $n \notin \{3, 5, 7\}$ , let  $Z_\xi$  be an arbitrary set with  $\xi$  players, and let  $S \subset N$  be an arbitrary set. Consider a following sequence of defections:

$$N \rightarrow_{Z_\xi} \mathcal{Z}_{\xi, n-\xi}^n \rightarrow_S \mathcal{V},$$

where  $\mathcal{V} \in F_S(\mathcal{Z}_{\xi, n-\xi}^n)$ . Clearly, it is impossible to find  $\mathcal{B} \in \mathbf{D}$  so that  $\mathcal{Z}_{\xi, n-\xi}^n \ll \mathcal{B}$  and  $N \not\ll \mathcal{B}$ . This is clear as any  $\mathcal{B} \in \mathbf{D}, \mathcal{B} \neq \mathcal{Z}_{\xi, n-\xi}^n$  will have one coalition of size larger than  $\xi$ . Thus, any defection from  $\mathcal{Z}_{\xi, n-\xi}^n$  will leave at least some members of any potential defecting coalition  $S$  worse off in  $\mathcal{B}$ . This implies that the grand coalition cannot be stable.

We have shown that the grand coalition cannot be stable when  $n = 3$ . When  $n = 5$  or  $n = 7$ , the members of the larger coalition in  $\mathcal{Z}_{\xi, n-\xi}^n$  receive less than in the grand coalition. Thus,

a move from the grand coalition to  $Z_{\xi, n-\xi}^n$  is deterred if we consider a sequence of individual defections from larger coalition until we reach  $Z_1^n$ , and then the joint move of all players to form the grand coalition. This also shows that  $Z_{\xi, n-\xi}^n$  is not stable.

3. Suppose  $n \geq 8$ , and let  $\mathcal{X}$  be an outcome other than a coalition structure with two of three equal-sized coalitions. Consider the following two defections:

- (a) Defection by all members to the grand coalition,  $\mathcal{X} \rightarrow_N N$ ;
- (b) Let  $Y^M$  be the largest coalition in  $\mathcal{X}$ , and consider a defection by  $|Y^M|/2$  coalition members.

One of the above two defections must be deterred. To see this, let  $T = (a - bC)^2/b$  and denote by  $Y^m$  the smallest coalition in  $\mathcal{X}$ . Note that

$$\frac{T}{(l+2)^2 \frac{n}{l}} - \frac{T}{(l+3)^2 \frac{n}{l+1}} > \min \{ \Pi_j^{\mathcal{X}}, \Pi_i^N - 2\Pi_r^{\mathcal{X}} \},$$

where  $j \in Y^m, r \in Y^M$ . This implies that, if one of the above two defections is not deterred, then the other one must be, because otherwise we get a direct contradiction to Lemma 3. ■

## 2. Appendix B - LCS for Models with Friction and Costs

Recall that we explicitly consider two kinds of costs. First, we assume that there is a cost to being a part of a coalition, which we call the *membership cost* and denote by  $f$ . This cost may include, for instance, a variety of administrative costs entailed in managing an alliance. Next, we assume that every time a coalition is formed, a certain cost is incurred by the members of the defecting coalition. We call it the *friction cost* and denote it by  $g$ . Due to the symmetry in our setting and our assumptions,  $f$  and  $g$  only depend on the size of the coalition on which they are evaluated and are thus simple cardinality functions. Thus, they can be thought of as  $f, g : \mathbb{N} \rightarrow \mathbb{R}$ , where  $\mathbb{N}$  is the set of natural numbers. More precisely, if  $\mathcal{X} \rightarrow_S \mathcal{Y}$ , each member of  $S$  incurs the cost  $\frac{g(S)}{\|S\|}$ . For any  $Z = \{Z_1, \dots, Z_k\} \in P(N)$ , each member of  $Z_i$  incurs  $\frac{f(Z_i)}{\|Z_i\|}$ . If we assume that  $g$  is convex ( $f$  concave), these functions induce set-valued supermodular (submodular) function on the subsets of  $\{1, 2, \dots, n\}$ .

To model membership costs, we define

$$L(x, y) := \frac{f\left(\lceil \frac{x}{y} \rceil\right)}{\lceil \frac{x}{y} \rceil} - \frac{\rho}{2(y+1)^2 \lceil \frac{x}{y} \rceil},$$

where  $\rho := \frac{(a-bC)^2}{b}$ .  $L(n, l) > 0$  implies that in an assembly system such as ours, where there are  $n$  suppliers, a coalition structure with  $l$  coalitions will never occur. In our analysis, we will assume some properties of  $L$ .

Next, to deal with the cost of defections (friction), we use a novel approach. Intuitively, the introduction of friction translates to curbing the length of the chains that one may need consider in calculations. To show that a certain defection is deterred in the context of the LCS, this would imply that one may need to restrict oneself to a bounded number of possible future defections. In the case of the EPCF, intuition suggests that friction costs inflate the value of  $\delta$ . We consider coalition games such as ours, where all coalitions make equal profit, and use the following assumption.

**Assumption 1** *A friction cost  $g : \mathcal{N} \rightarrow \mathbb{R}$  is either strictly concave or strictly convex, and:*

- (1) *in any allowable chain  $N \rightarrow \dots \rightarrow \mathcal{Z}_1^n$ , the players in  $\mathcal{Z}_1^n$  realize non-positive payoffs (obtained as a difference between their profit and accumulated friction cost).*
- (2) *A sequence of defections  $N \rightarrow \mathcal{Z}_{\frac{n}{2}, \frac{n}{2}} \rightarrow \mathcal{Z}_{\frac{n}{4}, \frac{n}{4}, \frac{n}{2}}$  is feasible (i.e., all players in  $\mathcal{Z}_{\frac{n}{4}, \frac{n}{4}, \frac{n}{2}}$  realize positive payoffs after friction costs are deducted).*

Denote the LCS of the game with such a  $g$  by  $LCS_g$ . Next, consider a modified version of the LCS where only  $k$ -steps defections (i.e., chains of length at most  $k$ ) are allowed, and denote this LCS by  $LCS^k$ . Clearly, at one extreme when the length of chain is zero (i.e., no defection can be considered), all outcomes are stable. Allowing a single defection corresponds loosely to the myopic case, while infinite number of defections creates the LCS. In general, restricting the number of allowable moves may lead to surprising results. This is because, in order to show that a certain defection from an outcome is deterred in a farsighted sense, one may be forced to provide a chain of indirect domination that is quite long. Such a long chain may simply not be feasible when friction costs are positive! We can show the following result, which establishes an implicit connection between  $g$  and the length of the chain that one needs to consider in the definition of stability.

**Proposition 3** *Under Assumption 1,  $LCS_g \equiv \bigcup_{k=2}^{\frac{n}{2}} LCS^k$  for sufficiently large  $n$ .*

**Proof of Proposition 3:** Define  $\ll_k$  by considering a chain of length  $k$  (i.e., with exactly  $k$  moves) in the definition of  $\ll$ .  $LCS^k$  is then the fixed point of the map  $T_k : 2^{\mathcal{Z}} \rightarrow 2^{\mathcal{Z}}$ ,

$$T_k(X) = \{\mathcal{Z} \in \mathcal{Z} : \forall \mathcal{V}, S, \text{ such that } \mathcal{V} \in F_S(\mathcal{Z}), \exists \mathcal{B} \in X, \text{ where } \mathcal{V} = \mathcal{B} \text{ or } \mathcal{V} \ll_k \mathcal{B}, \text{ such that } \mathcal{Z} \not\prec \mathcal{B}\}.$$

If a fixed point does not exist, we set it to be  $\emptyset$ . We first note that  $T_k$  has a nonempty fixed point for at least one  $k, 2 < k \leq n/2$ . To see that, observe that  $\bigcup_k^{\frac{n}{2}} LCS^k = \bigcup_k^{\frac{n}{2}} \bigcup_{t \in \Sigma_k} t$ , where

$\Sigma_k = \{t \subseteq \mathbf{Z} : t \subseteq T_k(t)\}$ , and recall that  $g$  is supermodular cardinality-induced function such that  $N \rightarrow Z_{\frac{n}{2}, \frac{n}{2}} \rightarrow Z_{\frac{n}{4}, \frac{n}{4}, \frac{n}{2}}$  is feasible for all players. Thus,  $\bigcup_{t \in \Sigma_k} t$  cannot be empty for all  $k$ .

Now, consider  $\mathcal{X} \in P(N)$  such that  $\mathcal{X} \in \text{LCS}_g$ , and suppose that  $\mathcal{B} \gg \mathcal{X}$  when payoffs are reduced by  $g$ . Since  $g$  is such that  $N \rightarrow Z_{\frac{n}{2}, \frac{n}{2}} \rightarrow Z_{\frac{n}{4}, \frac{n}{4}, \frac{n}{2}}$  is feasible, we can show that, in any of the assembly games, we can find a chain of length less than  $\lfloor \frac{n}{2} \rfloor$  such that  $\mathcal{B} \gg \mathcal{X}$  in the setting with  $g = 0$ . This implies  $\mathcal{X} \in \bigcup_k \text{LCS}^k$ . The other inclusion,  $\bigcup_k \text{LCS}^k \subseteq \text{LCS}_g$ , can be shown using the definition of  $T$ . ■

The actual value of  $n$  used in the proposition depends on function  $g$ .

Note that the symmetry of the game is a required property. Thus, when one discounts the membership cost  $f$  from the payoff, symmetry is destroyed and we may no longer have the above result. Due to this technical difficulty, we analyze two different settings: (I) zero friction costs ( $g = 0, f > 0$ ), and (II) zero membership costs ( $f = 0, g > 0$ ).

CASE I: ZERO FRICTION COSTS ( $g = 0, f > 0$ )

In this case, we analyzed all three models of competition. Clearly, the magnitude of  $f$  dictates what the stable outcomes are. We have the following result.

**Theorem B 1**  $\exists n^*$  such that  $\forall n \geq n^*$ , if  $L(n^\varepsilon, l)$  is increasing in  $l$  for a fixed  $n^\varepsilon$ , then

- (1) For the SS model,  $N \in \text{LCS}$ ,  $\mathcal{Z}_k^n \in \text{LCS}$  for  $k > 1$ ,  $\mathcal{Z}_{k,k}^{2k} \in \text{LCS}$ , and  $\mathcal{Z}_{j,j,1}^{2j+1} \in \text{LCS}$ ;
- (2) For the VN model,  $N \notin \text{LCS}$ , while equal-sized coalition structures with two, three, and four coalitions are in the LCS.
- (3) if  $\lim_{l \rightarrow \infty} \frac{L(n^\varepsilon, l)}{l} = 0$ , then  $\mathcal{Z}_1^n \notin \text{LCS}$  in either model.

**Proof of Theorem B 1:**

1. Consider any  $N \rightarrow_S \mathcal{X}$  and the subsequent chain of defections as in the proof of item 1. in Theorem 1, with the understanding that we stop at  $\mathcal{X}_k$  and consider  $\mathcal{X}_k \rightarrow N$  when  $\mathcal{X}_{k+1}$  is infeasible. Because  $L(\cdot, l)$  is increasing in  $l$ , we have that  $\mathcal{X}_k \ll N$  and  $N \not\prec_S \mathcal{X}_k$ . This shows that the grand coalition is stable.

The first step in showing the stability of the remaining structures is to note that if  $\mathcal{Z}_{\frac{n}{k}, \dots, \frac{n}{k}}$  is not in the  $\text{LCS}^k$ , then neither is  $\mathcal{Z}_{\frac{n}{k+1}, \dots, \frac{n}{k+1}}$ . This holds because we simply compare  $L(n, k)$  instead of comparing  $T_k$  in the proof of Lemma 1. Thus, the function  $\ell : 2^{\mathbf{Z}} \rightarrow 2^{\mathbf{Z}}$ ,  $\ell(\mathbf{A}) = M(\mathbf{A}, \diamond^{\mathbf{A}})$ , has a nontrivial fixed point.

2. The proof idea for the VN model is similar to above.

3. When  $L(n^\varepsilon, l)$  grows slower than  $O(l)$ , the proof of item 2. in Theorem 1 holds. ■

Note that the proof suggests that the derivative of  $\frac{f(\lceil \frac{x}{y} \rceil)}{\lceil \frac{x}{y} \rceil}$  with respect to  $y$  for a fixed, large enough  $x$  bears a profound implication on whether equal-sized coalition structure with coalition sized  $y$  belongs to the LCS. However, we find it theoretically difficult to establish this connection. Further, the assumption that  $L(n^\varepsilon, l)$  is increasing in  $l$  does not imply the stability of the grand coalition in the VN game. However, in this case, it increases the set of stable outcomes. Thus in the VN game, with exactly the same assumptions as in Theorem 1, we see that equal-sized coalition structures with two, three, and four coalitions are stable for the large values of  $n$ . In general, the AS model exhibits the same dynamics as the SS model, and we do not repeat those results.

The above outcomes yield the same results as before when refined by the EPCF and external stability. For the EPCF results, we only have to note that using the same values of  $\delta$  as before yield an indirect objection. The proof of external stability is easy to see because the basic ingredients were as before.

To further get a handle on the effect of membership costs, we analyzed  $f(x) = x^{1/k}$  for large value of  $n$ , such that a non-trivial number of coalition structures were feasible. Our experiments suggest that when the conditions in Theorem 1 are flouted (say, when  $k$  is small so that cost of forming a coalition is significant), the results can be reversed. This is not surprising at all. Naturally, one can find instances when  $\rho$  is small and  $n$  is large and in which cost of alliance formation is prohibitively expensive, so that  $Z_1^n$  or  $Z_2^n$  are the only possible stable outcomes.

CASE II: ZERO MEMBERSHIP COSTS ( $f = 0, g > 0$ )

We now look at the effect of a positive friction cost. We further assume that Assumption 1 holds, and use Proposition 3 ( $LCS_g \equiv \bigcup_{k=2}^{\frac{n}{2}} LCS^k$  for sufficiently large  $n$ ) to analyze our model. Our first result says that we can partially characterize stable outcomes in the Stackelberg models.

**Theorem B 2** *When Assumption 1 holds, in the AS and SS models:*

- (1)  $N \notin LCS_g, Z_1^n \notin LCS_g;$
- (2)  $Z_k^n \in LCS_g$  for  $k > \tilde{k}, 1 < \tilde{k} \leq \hat{k};$
- (3)  $Z_{k,k}^{2k} \in LCS_g$  and  $Z_{j,j,1}^{2j+1} \in LCS_g.$

Thus, the introduction of even reasonable friction cost rules out the grand coalition. The intuition behind this result is that the grand coalition's stability is contingent on players contemplating a fairly long and sophisticated sequence of moves. Friction costs prohibit this and lead to the above

results. It is interesting to note that certain basic coalition structures become stable ( $\tilde{k} \leq \hat{k}$ ). This happens because we again require several future defections in order to produce an indirect dominant move from those basic coalitions. To prove that  $\mathcal{Z}_{k,k}^{2k}$  and  $\mathcal{Z}_{j,j,1}^{2j+1}$  are stable, we needed to consider only two-step moves, which are now not prohibited. The instability of  $\mathcal{Z}_1^n$  is easy to see from the fact that  $\mathcal{Z}_{k,k}^{2k}$  or  $\mathcal{Z}_{j,j,1}^{2j+1}$  can be reached in only few steps from  $\mathcal{Z}_1^n$ .

We also note that analyzing the VN game seems very challenging. We can show that for the VN game, the grand coalition and basic coalitions are not stable. Numerical experiments indicate that coalition structures with two equal or close to equal coalitions are unstable as well.

### 3. Appendix C - EPCF for a Three-Player Model with Friction and Costs

We have shown in Proposition 2 that the grand coalition is the only stable outcome in the SS model with linear demand when  $n = 3$  when operating costs are zero. This statement holds for any value of  $\delta$ . We now assume nonnegative costs, and consider, for instance, the move from the structure in which no coalitions are formed,  $\mathcal{Z}_1^3$ , to the grand coalition,  $N$ . While each supplier prefers the profit before operating costs in  $N$ , the move will result in the cost  $g(3)$  incurred in one period, and  $f(3)$  instead of  $f(1)$  in every period. Thus, for stability of the grand coalition we require

$$0 < \frac{\rho}{96} + f(1) - f(3) - (1 - \delta)g(3). \quad (\text{A2})$$

A similar condition has to be found for other possible moves (from all coalition structures) that eventually lead to the grand coalition as an absorbing state. It can be shown that, in fact, it is sufficient to check condition (A2).

We now consider a few examples for  $f$  and  $g$ . When  $f(x) = \sqrt{x}$ ,  $g(x) = 0$ ,  $N$  is stable  $\forall \delta > 0$  as long as  $\rho \geq 70$  and can never be stable when  $\rho < 70$ . On the other hand, if  $f(x) = 0$ ,  $g(x) = x^2$ ,  $N$  remains stable for all values of  $\delta$  only when  $\rho \geq 862$ . We depict the lower bound for the value of  $\delta$  for which  $N$  is stable with a thick line in Figure A1. As the value of  $\rho$  decreases, the profit before operating costs decreases, and  $N$  remains stable for large values of  $\delta$ . This is because, with a small profit, less farsighted players may prefer not to move from  $\mathcal{Z}_1^3$  and incur the cost of forming a coalition.

When both costs are positive,  $f(x) = \sqrt{x}$ ,  $g(x) = x^2$ ,  $N$  becomes stable for all  $\delta$  only when  $\rho \geq 935$ . In addition,  $N$  is not stable for any  $\delta$  when  $\rho \leq 70.4$ . The lower bound for  $\delta$  is depicted with a dotted line in Figure A1. When the profit before operating costs is small (that is,  $\rho$  is small), the cost of being in the coalition may offset the higher profit that the suppliers

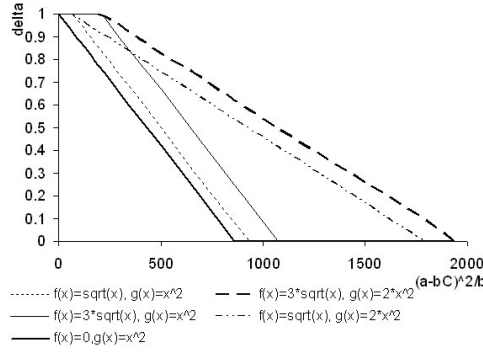


Figure A1: Lower bounds for values of  $\delta$  for which the grand coalition is stable as a function of  $\rho$ .

see as a result of forming  $N$ . If we modify the costs by increasing the cost of coalition formation,  $f(x) = \sqrt{x}, g(x) = 2x^2$ ,  $N$  remains unstable for all  $\delta$  over the same interval (that is,  $\rho \leq 70.4$ ). However, the slope is less steep and  $N$  becomes stable for all  $\delta$  only when  $\rho$  reaches 1800. This trend seems to consistently continue. When we compare this with  $f(x) = 3\sqrt{x}, g(x) = 2x^2$ , the slope remains the same and we observe that  $N$  is unstable for all  $\delta$  when  $\rho \leq 200$ , and  $N$  becomes stable for all  $\delta$  only when  $\rho \geq 1940$ . This is depicted with a thick dashed line in Figure A1.

Thus, the interval of  $\rho$  values over which  $N$  is not stable for any  $\delta$  seems to be highly dependent on the membership cost  $f$  – higher cost implies that  $N$  is not stable for larger interval of  $\rho$ . On the other hand, the “rate” at which  $N$  becomes stable depends more on the friction cost  $g$  – as the cost of forming coalitions increases,  $N$  becomes stable for higher values of  $\delta$  given the same value of  $\rho$ . In other words, as the cost of coalition formation increases, the higher level of farsightedness is required for  $N$  to be stable.

We briefly comment on when some other coalition structures may emerge as stable outcomes. Suppose first that  $f(x) = 0, g(x) = x^2$ . Then,  $\mathcal{Z}_1^3$  is stable only when the discount factor is rather low and  $\rho \geq 170$ . When  $\rho$  is small, the profit before operating costs is small and an individual player who deviates from  $N$  loses a significant portion of his future profits when another player defects from  $\mathcal{Z}_{2,1}^3$ . The same is true when  $\rho$  is large and discount factor  $\delta$  is high. Next, suppose that  $f(x) = \sqrt{x}, g(x) = 0$ . Then, when  $\rho$  is large,  $\mathcal{Z}_1^3$  is stable only for smaller values of  $\delta$ , while for smaller  $\rho$  ( $\rho \leq 145$ ) it becomes stable for any value of the discount factor. When  $\rho$  is small, the suppliers benefit from not incurring the costs of being in the coalition, especially when there is no friction. We also analyzed the case  $f(x) = \sqrt{x}$  and  $g(x) = x^2$ . As the value of  $\rho$  decreases,  $\mathcal{Z}_1^3$  becomes stable for more farsighted suppliers, as the lone supplier leaving  $\mathcal{Z}_{2,1}^3$  benefits more from the additional profit that he can generate in  $\mathcal{Z}_1^3$  and avoids higher cost of being in a larger

coalition. Therefore, when players are quite farsighted,  $\mathcal{Z}_1^3$  is likely to be stable only when it is costly to belong to a coalition and the profit before operating costs is not too high.

Finally, it is also interesting to note that  $\mathcal{Z}_{2,1}^3$  is never stable when the suppliers are farsighted enough; it can only be stable for low values of  $\delta$  when  $\rho$  is large. When the value of  $\rho$  is large, the profit before operating cost generated during the one period in  $N$  is large enough to offset both the cost of forming the coalition and being in the coalition, assuming that the discount factor  $\delta$  is low enough to make the future losses from being in a two-player coalition not significant. When the value of  $\rho$  decreases, the gains from spending one period in  $N$  are not high enough to compensate for both the operating costs and future losses from coalition membership.

## 4. Appendix D - Nonlinear Demand

### 4.1 Nonconstant price elasticity

In this appendix, we first consider two forms of nonlinear demand with nonconstant price elasticity. In the main body of the paper, we have analyzed the case with linear demand,  $D(p) = a - bp$ . Here, we look at  $D_1(p) = (a - bp)^\gamma, \gamma > 0$ , and  $D_2(p) = ae^{-bp}$ .

We focus first on the SS model and then provide a discussion of the VN and AS models. As before, we concentrate on a linear wholesale price contract. Before we can show a result similar to the analysis in Section 3, we need to introduce some notation. Let  $E_i(p) = \frac{-pD'_i(p)}{D_i(p)}$  be the price elasticity of the demand  $D_i(p)$ . We note that for the nonlinear demand models under consideration,  $E_i(p)$  is a ratio of the affine functions of  $p$ . That is,  $E_i(p) = \frac{p}{A+Bp}$ , where  $A$  and  $B$  are constants<sup>1</sup>. We also note that in this case the simple curvature of the demand is given by

$$C_i(p) = \frac{D_i(p)D''_i(p)}{[D'_i(p)]^2} = 1 + B,$$

and is constant. The curvature captures the convexity of the demand – when  $B > 0$  the demand is logconvex, and when  $B < 0$  it is logconcave. We state our next result without a proof; the details can be easily verified.

**Theorem D 1** *Let  $\mathcal{Z} = \{Z_1, \dots, Z_l\}$  be a status quo supplier coalition structure.*

1.  $p^* = \frac{A(1-B)l+A+C}{(1-B)(1-B)l}, \quad Q^* = D_i(p^*), i = 1, 2.$
2.  $w^{Z_k} - c^{Z_k} = w^{Z_j} - c^{Z_j} = \frac{A+BC}{1-B}, \quad k, j \in \{1, \dots, l\}.$
3.  $\Pi_j^Z = \frac{A+BC}{(1-B)l|Z_k|} D_i(p^*), \quad i = 1, 2, \quad j \in Z_k, k \in \{1, \dots, l\}.$

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<sup>1</sup>We abuse notation and write  $A$  and  $B$  irrespective of  $i$ .

$$4. \Pi_{Assembler}^{\mathcal{Z}} = \frac{A+BC}{(1-B)(1-Bl)} D_i(p^*), \quad i = 1, 2.$$

We note that item 3 implies the Pareto dominance of the grand coalition. Further, we can show that individual members immediately benefit from a myopic defection, and it can be easily seen that members of smaller coalitions make more than members of larger coalitions (akin to items 1 and 2 of Proposition 1).

We make two further observations regarding the models with  $D_i(p), i = 1, 2$ . First, the profit expressions, when computed, show similar relative order of magnitudes for the various coalition structures. That is, the “relative incentive to defect” reflected by the ratio of profit of coalition structure with  $l$  coalitions to that of a coalition structure with  $l + 1$  coalitions and captured by the curvature of the demand is similar for those two models. More precisely, if  $\Pi^{\mathcal{Z}(l)}$  denotes profit of a coalition in coalition structure  $\mathcal{Z}(l)$ , which contains  $l$  coalitions, then we let  $RI(l) = -\frac{\Pi^{\mathcal{Z}(l)}}{\Pi^{\mathcal{Z}(l+1)}}$ . The quantity  $\frac{\Pi^{\mathcal{Z}(l)}}{\Pi^{\mathcal{Z}(l+1)}}$  has been used in some network games (captures in some settings an individual agents incentive to form coalitions), but has not been tied to the demand shape. The monotonicity of  $RI(l)$  with respect to  $l$  depends on  $B$ . When  $B \leq 0$ , the demand is logconcave and  $RI(l)$  increases with  $l$ . In our models, we have  $B = -1/\gamma$  for  $i = 1$ , and  $B = 0$  for  $i = 2$ . This implies that players can (and will) contemplate a sequence of individual defections with “significant gains”. Second, we note that across the structures  $i = 1, 2$ , the differences in profit obtained by players in  $N$  and in other coalition structures are similar in structure and magnitude. These two observations drive the fact that the results for  $D_1(p)$  and  $D_2(p)$  are similar. An analogous argument works for the VN and AS models. In the VN model, we obtain a result similar to item 4 in Proposition 1, and the order of magnitude of the relative profits of two coalition structures are similar when  $i = 1, 2$ . This logic applies for the EPCF and external stability.

## 4.2 Isoelastic demand

Suppose that the demand at the assembler is isoelastic and given by  $D_3(p) = ap^{-b}$ , where we assume  $a > 0$  (to ensure nonnegativity of demand) and  $b > 0$  (to reflect price elasticity). Let us first consider  $b > n$  (for the VN model, we assume  $b > n + 1$ ). Following an approach similar to the one used with linear demand, we can find equilibrium prices for the assembler and the suppliers, which are given in Table 1. Our analysis shows that with isoelastic demand,  $w_i^{SS} \leq w_i^{AS} \leq w_i^{VN}$  and  $p^{SS} = p^{AS} \leq p^{VN}$ . Thus, unlike in the model with linear demand, the highest prices are set when no one dominates the market.

If the condition  $b > n$  (for VN model, condition  $b > n + 1$ ) is not satisfied, there is no unique profit-maximizing solution. For every given set of prices  $w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n$ , supplier  $i$  can

Equil.	SS	VN	AS
$w_i - c_i$	$\frac{C}{b-n}$	$\frac{C}{b-n-1}$	$\frac{bC}{(b-1)(b-n)}$
$p$	$\frac{b^2C}{(b-1)(b-n)}$	$\frac{bC}{b-n-1}$	$\frac{b^2C}{(b-1)(b-n)}$

Table 1: Equilibrium prices with isoelastic demand

always increase his profit by raising his price. Thus, in the subsequent analysis we limit our attention to the case  $b > n$  (for VN model,  $b > n + 1$ ). As the elasticity  $b$  in practice does not reach very high values, we will limit our analysis to  $b \leq 6$ ,  $n \leq 6$ .

If we consider an arbitrary coalition structure,  $\mathcal{Z} = \{Z_1, \dots, Z_l\}$ , it is easy to verify that the equilibrium wholesale prices satisfy (7), as in the case with linear demand. Thus, with both linear and isoelastic demand, the net profit obtained by each coalition is the same. As each supplier generates the same marginal profit when no coalitions are formed, we again assume that, in any coalition, members divide the profit equally. At equilibrium, every coalition observes the same margin.

Consider an arbitrary coalition structure,  $\mathcal{Z} = \{Z_1, \dots, Z_l\}$ , and recall that  $c^{Z_k} = \sum_{i \in Z_k} c_i$ . Table 2 gives equilibrium wholesale prices and quantities for an arbitrary coalition,  $Z_k$ , and equilibrium profit for its member,  $i \in Z_k$ . It is interesting to note that the lowest profit for both the

Equil.	SS	VN	AS
$w_k^{\mathcal{Z}} - c^{Z_k}$	$\frac{C}{b-l}$	$\frac{C}{b-l-1}$	$\frac{bC}{(b-1)(b-l)}$
$Q^{\mathcal{Z}}$	$a \left[ \frac{b^2C}{b-1} \right]^{-b} (b-l)^b$	$a(bC)^{-b} (b-l-1)^b$	$a \left[ \frac{b^2C}{b-1} \right]^{-b} (b-l)^b$
$\Pi_i^{\mathcal{Z}}$	$aC \left[ \frac{b^2C}{b-1} \right]^{-b} \frac{(b-l)^{b-1}}{ Z_k }$	$aC(bC)^{-b} \frac{(b-l-1)^{b-1}}{ Z_k }$	$\frac{a}{b} \left[ \frac{b^2C}{b-1} \right]^{-b+1} \frac{(b-l)^{b-1}}{ z_k }$

Table 2: Equilibrium prices and profits in coalition structure  $\mathcal{Z} = \{Z_1, \dots, Z_l\}$  with isoelastic demand

assembler and the suppliers is realized in the VN model (which is again the opposite from the relationship observed with linear demand), and that each party prefers the other one to be the Stackelberg leader. Thus, neither the assembler nor the suppliers directly benefit from the leadership position, but both parties have an indirect incentive to be leaders because the VN model leads to the lowest profits.

As mentioned earlier, because of the requirement  $b \geq n$ , we limit our analysis when demand is isoelastic to the case with at most six suppliers. As a consequence, unlike the case with linear demand, we cannot extend our analysis to asymptotic results. We use results from Proposition 4 and first analyze the LCS.

**Theorem D 2** Consider a model with isoelastic demand and  $n \leq 6$ . The only stable outcomes are given as follows.

- (1) The grand coalition,  $N$ , is in the LCS for any model of competition.
- (2) In SS and AS models, all two-set coalition structures,  $\mathcal{Z}_{j,n-j}^n$ , are in the LCS when  $n \geq 4^2$ .
- (3) When VN model is used, all two-set coalition structures,  $\mathcal{Z}_{j,n-j}^n$ , are in the LCS when  $n \geq 5^3$ .

**Proof of Theorem D 2:** Let us denote by  $\Pi_i^{\mathcal{Z}}(Z_k)$  profit received by  $i \in Z_k$  in the coalition structure  $\mathcal{Z}$ .  $Z_k$  will denote coalition with  $k$  members. Then, we can evaluate relationships between suppliers profits in various coalition structures in the Stackelberg models for  $n \leq 6$  as follows.

- When  $n = 2$ ,  $\Pi_i^{\mathcal{Z}_2^2} > \Pi_i^{\mathcal{Z}_1^2}$ .
- When  $n = 3$ ,  $\max\{\Pi_i^{\mathcal{Z}_3^3}, \Pi_i^{\mathcal{Z}_2^3}(Z_1)\} > \Pi_i^{\mathcal{Z}_2^3}(Z_2) > \Pi_i^{\mathcal{Z}_1^3}$ . In addition,  $\Pi_i^{\mathcal{Z}_3^3} > \Pi_i^{\mathcal{Z}_2^3}(Z_1)$  for  $b < 6.75$  and  $\Pi_i^{\mathcal{Z}_3^3} < \Pi_i^{\mathcal{Z}_2^3}(Z_1)$  for  $b > 6.75$
- When  $n = 4$ ,  $\Pi_i^{\mathcal{Z}_3^4}(Z_1) > \Pi_i^{\mathcal{Z}_4^4} > \Pi_i^{\mathcal{Z}_{2,2}^4} > \Pi_i^{\mathcal{Z}_3^4}(Z_3)$ , and they dominate payoffs of individual suppliers in  $\mathcal{Z}_2^4$  and  $\mathcal{Z}_1^4$ .
- When  $n = 5$ ,

$$\Pi_i^{\mathcal{Z}_4^5}(Z_1) > \Pi_i^{\mathcal{Z}_5^5} > \Pi_i^{\mathcal{Z}_{3,2}^5} > \max\{\Pi_i^{\mathcal{Z}_4^5}(Z_4), \Pi_i^{\mathcal{Z}_3^5}(Z_1) = \Pi_i^{\mathcal{Z}_{2,2,1}^5}(Z_1)\} > \Pi_i^{\mathcal{Z}_{2,2,1}^5}(Z_2) > \Pi_i^{\mathcal{Z}_3^5}(Z_3),$$

and they dominate payoffs in  $\mathcal{Z}_2^5$  and  $\mathcal{Z}_1^5$ . Here,  $\Pi_i^{\mathcal{Z}_4^5}(Z_4) > \Pi_i^{\mathcal{Z}_3^5}(Z_1) = \Pi_i^{\mathcal{Z}_{2,2,1}^5}(Z_1)$  for  $b < 6.46$  and  $\Pi_i^{\mathcal{Z}_4^5}(Z_4) < \Pi_i^{\mathcal{Z}_3^5}(Z_1) = \Pi_i^{\mathcal{Z}_{2,2,1}^5}(Z_1)$  for  $b > 6.46$ .

- When  $n = 6$ ,

$$\Pi_i^{\mathcal{Z}_5^6}(Z_1) > \Pi_i^{\mathcal{Z}_{4,2}^6}(Z_2) > \Pi_i^{\mathcal{Z}_6^6} > \Pi_i^{\mathcal{Z}_{3,3}^6} > \Pi_i^{\mathcal{Z}_4^6}(Z_1) = \Pi_i^{\mathcal{Z}_{3,2,1}^6}(Z_1) > \Pi_i^{\mathcal{Z}_{4,2}^6}(Z_4) > \Pi_i^{\mathcal{Z}_5^6}(Z_5) > \Pi_i^{\mathcal{Z}_{3,2,1}^6}(Z_2) = \Pi_i^{\mathcal{Z}_{2,2,2}^6} > \Pi_i^{\mathcal{Z}_{3,2,1}^6}(Z_3) > \Pi_i^{\mathcal{Z}_{2,2,1,1}^6}(Z_1) = \Pi_i^{\mathcal{Z}_3^6}(Z_1) > \Pi_i^{\mathcal{Z}_4^6}(Z_4) > \Pi_i^{\mathcal{Z}_{2,2,1,1}^6}(Z_2) > \Pi_i^{\mathcal{Z}_3^6}(Z_3)\},$$

and they dominate payoffs in  $\mathcal{Z}_2^6$  and  $\mathcal{Z}_1^6$ .

For the VN model, we have

- When  $n = 2$ ,  $\Pi_i^{\mathcal{Z}_2^2} > \Pi_i^{\mathcal{Z}_1^2}$ .
- When  $n = 3$ ,  $\Pi_i^{\mathcal{Z}_3^3} > \Pi_i^{\mathcal{Z}_2^3}(Z_1) > \Pi_i^{\mathcal{Z}_3^3}(Z_2) > \Pi_i^{\mathcal{Z}_1^3}$ .

<sup>2</sup>When  $n = 3$ ,  $\mathcal{Z}_{2,1}^3$  is stable when  $b > 6.75$ .

<sup>3</sup>When  $n = 4$ , all two-set coalition structures are stable when  $b > 6.46$ .

- When  $n = 4$ ,  $\max\{\Pi_i^{\mathcal{Z}_3^4}(Z_1), \Pi_i^{\mathcal{Z}_4^4}\} > \Pi_i^{\mathcal{Z}_{2,2}^4} > \Pi_i^{\mathcal{Z}_3^4}(Z_3)$ , and they dominate payoffs of individual suppliers in  $\mathcal{Z}_2^4$  and  $\mathcal{Z}_1^4$ . In addition,  $\Pi_i^{\mathcal{Z}_4^4} > \Pi_i^{\mathcal{Z}_3^4}(Z_1)$  for  $b < 6.46$  and  $\Pi_i^{\mathcal{Z}_4^4} < \Pi_i^{\mathcal{Z}_3^4}(Z_1)$  for  $b > 6.46$ .
- When  $n = 5$ ,  $\Pi_i^{\mathcal{Z}_4^5}(Z_1) > \Pi_i^{\mathcal{Z}_5^5} > \Pi_i^{\mathcal{Z}_{3,2}^5} > \Pi_i^{\mathcal{Z}_4^5}(Z_4)$ , and they dominate payoffs in  $\mathcal{Z}_{2,2,1}^5$ ,  $\mathcal{Z}_3^5$ ,  $\mathcal{Z}_2^5$  and  $\mathcal{Z}_1^5$ .
- When  $n = 6$ ,

$$\begin{aligned} \Pi_i^{\mathcal{Z}_5^6}(Z_1) > \Pi_i^{\mathcal{Z}_6^6} > \Pi_i^{\mathcal{Z}_{4,2}^6}(Z_2) > \Pi_i^{\mathcal{Z}_{3,3}^6} > \Pi_i^{\mathcal{Z}_{4,2}^6}(Z_4) > \max\{\Pi_i^{\mathcal{Z}_5^6}(Z_5), \Pi_i^{\mathcal{Z}_4^6}(Z_1) = \Pi_i^{\mathcal{Z}_{3,2,1}^6}(Z_1)\} > \\ & \Pi_i^{\mathcal{Z}_{3,2,1}^6}(Z_2) = \Pi_i^{\mathcal{Z}_{2,2,2}^6} > \Pi_i^{\mathcal{Z}_{3,2,1}^6}(Z_3) > \Pi_i^{\mathcal{Z}_4^6}(Z_4), \end{aligned}$$

and they dominate payoffs in  $\mathcal{Z}_{2,2,1,1}^6$ ,  $\mathcal{Z}_3^6$ ,  $\mathcal{Z}_2^6$  and  $\mathcal{Z}_1^6$ .

The analysis of the LCS membership uses the above relationships and follows the steps similar to those used in the proof of Proposition 2. ■

We can notice the difference in stable outcomes between linear and isoelastic demand. While the grand coalition was the only stable outcome for up to five players with linear demand in Stackelberg models, splits into two coalitions become stable with as little as four players when demand is isoelastic. In addition, the grand coalition belongs to the LCS in the VN model.

It follows from Theorem D 2 that the number of potentially stable outcomes with isoelastic demand can be rather large, even with a small number of players. However, similarly to the case with linear demand, only coalition structures with at most two coalitions are stable. Now, consider the case with four suppliers and coalition structure  $\mathcal{Z}_{2,2}^4$ . Although each player can strictly increase his profit by joint defection to the grand coalition, this defection is deterred since a lone player can further deviate, which would lead to the coalition structure  $\mathcal{Z}_{3,1}^4$ , wherein all players in the three-member coalition would be strictly worse off than in  $\mathcal{Z}_{2,2}^4$ . However, the EPCF can eliminate such inefficient outcomes, and thus it provides a strict refinement of the set of stable outcomes.

**Theorem D 3** *Consider a model with isoelastic demand and  $n \leq 6$ .*

- (1) *The grand coalition,  $N$ , is in  $EP(n)$  for any model of competition.*
- (2) *In SS or AS models, the basic structure with  $k = n - 1$ ,  $\mathcal{Z}_{n-1}^n$ , is in  $EP(n)$  when  $n \geq 4$ .*
- (3) *When VN model is used, the basic structure with  $k = n - 1$ ,  $\mathcal{Z}_{n-1}^n$ , is in  $EP(n)$  when  $n \geq 5$ .*

*There are no other members of the  $EP(n)$ .*

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<sup>4</sup>When  $n = 3$ ,  $\mathcal{Z}_2^3 \in EP(n)$  when  $b > 6.75$ .

<sup>5</sup>When  $n = 4$ ,  $\mathcal{Z}_3^4 \in EP(n)$  when  $b > 6.46$ .

**Proof of Theorem D 3:** We will only show the result for  $n = 4$  and for the Stackelberg models; the remaining cases can be shown in a similar way. Let us denote by  $\{(ij), k, l\}$  coalition structure where  $i$  and  $j$  form an alliance, and  $j$  and  $k$  act independently; other coalition structures can be represented in a similar fashion. Next, define  $p$  as follows:

$$\{(123), 4\} \rightarrow_3 \{(12), 3, 4\}; \{(234), 1\} \rightarrow_4 \{(23), 1, 4\}; \{(134), 2\} \rightarrow_1 \{(34), 1, 2\}; \{(124), 3\} \rightarrow_2 \{(14), 2, 3\},$$

and  $p(\mathcal{Z}, \mathcal{Z}_4^4) = 1$  for all remaining coalition structures  $\mathcal{Z}$ . We want to show that this is an EPCF with absorbing state at the grand coalition. It is easy to verify that players always benefit by deviating from any coalition structure which is not in the LCS. This is also true for  $\mathcal{Z}_{2,2}^4$ : since  $v_i(\mathcal{Z}_4^4, p) = \Pi_i^{\mathcal{Z}_4^4} + \delta v_i(\mathcal{Z}_4^4, p)$ , it follows that  $v_i(\mathcal{Z}_4^4, p) = \Pi_i^{\mathcal{Z}_4^4}/(1 - \delta)$ , while  $v_i(\mathcal{Z}_{2,2}^4, p) = \Pi_i^{\mathcal{Z}_{2,2}^4} + \delta v_i(\mathcal{Z}_4^4, p)$ . Thus,  $v_i(\mathcal{Z}_4^4, p) - v_i(\mathcal{Z}_{2,2}^4, p) = \Pi_i^{\mathcal{Z}_4^4} - \Pi_i^{\mathcal{Z}_{2,2}^4} > 0$ . Lastly, we need to show that player  $i$  has an incentive to deviate from  $\{(ijk), l\}$  to  $\{(jk), i, l\}$ . This is equivalent to  $v_i(\{(ijk), l\}, p) = \Pi_i^{\mathcal{Z}_3^4}(Z_3) + \delta \Pi_i^{\mathcal{Z}_2^4}(Z_1) + \frac{\delta^2}{1-\delta} \Pi_i^{\mathcal{Z}_4^4} \geq \frac{1}{1-\delta} \Pi_i^{\mathcal{Z}_3^4}(Z_3)$ , which is satisfied for  $\delta > \frac{\Pi_i^{\mathcal{Z}_3^4}(Z_3) - \Pi_i^{\mathcal{Z}_2^4}(Z_1)}{\Pi_i^{\mathcal{Z}_4^4} - \Pi_i^{\mathcal{Z}_2^4}(Z_1)}$ .

Next, we want to construct a PCF which is an EPCF with absorbing state at  $\{(123), 4\}$ . For  $i, j, k \in \{1, 2, 3\}, i \neq j \neq k \neq i$ , let us define PCF as follows:

$$\begin{aligned} \{(1234)\} &\rightarrow_4 \{(123), 4\}, & \{(123), 4\} &\rightarrow \{(123), 4\}, & \{(ij4), k\} &\rightarrow_4 \{(ij), k, 4\}, \\ \{(ij), k, 4\} &\rightarrow_{1,2,3} \{(123), 4\}, & \{(i4), j, k\} &\rightarrow_4 \{1, 2, 3, 4\}, & \{(ij), (k4)\} &\rightarrow_4 \{(ij), k, 4\}, \\ \{1, 2, 3, 4\} &\rightarrow_{1,2,3} \{(123), 4\}. \end{aligned}$$

Again, it is easy to verify that players always benefit by deviating from any coalition structure which is not in the LCS, and that player 4 benefits when he deviates from the grand coalition. Next, consider a coalition structure wherein 4 is a member of a three-player coalition, say  $\{(124), 3\}$ . We need to show that 4 has an incentive to deviate from  $\{(124), 3\}$ . This is equivalent to  $v_4(\{(124), 3\}, p) = \Pi_4^{\mathcal{Z}_3^4}(Z_3) + \delta \Pi_4^{\mathcal{Z}_2^4}(Z_1) + \frac{\delta^2}{1-\delta} \Pi_4^{\mathcal{Z}_4^4}(Z_1) \geq \frac{1}{1-\delta} \Pi_4^{\mathcal{Z}_3^4}(Z_3)$ , which is satisfied for  $\delta > \frac{\Pi_4^{\mathcal{Z}_3^4}(Z_3) - \Pi_4^{\mathcal{Z}_2^4}(Z_1)}{\Pi_4^{\mathcal{Z}_4^4}(Z_1) - \Pi_4^{\mathcal{Z}_2^4}(Z_1)}$ . Similarly, 4 has an incentive to deviate from  $\{(12), (34)\}$  if  $\delta > \frac{\Pi_4^{\mathcal{Z}_{2,2}^4} - \Pi_4^{\mathcal{Z}_2^4}(Z_1)}{\Pi_4^{\mathcal{Z}_3^4}(Z_1) - \Pi_4^{\mathcal{Z}_2^4}(Z_1)}$ . In a similar way, it can be shown that the remaining coalition structures of the form  $\mathcal{Z}_3^4$  are absorbing, and that other structures are not. ■

Thus, as mentioned earlier, with isoelastic demand we can consider only a limited number of players, and in these instances the grand coalition is always stable. However, as the number of players increases, lone suppliers may benefit from staying outside the alliance.

Similarly to the case with  $D_1$  and  $D_2$ , we can analyze the curvature and the incentive to defect in the model with isoelastic demand to explain our results. Note that  $B = 1/b > 0$ , hence  $C(p) > 1$ .

As a result, in both Stackelberg models and the VN model  $RI(l)$  decreases in  $l$ , that is,  $D_3(p)$  is logconvex. An immediate implication is that players benefit by aggregation into large coalitions. Note, however, that a myopic defection is still beneficial. Nevertheless, once we get outcomes with  $l > 2$  (that is, three coalitions or more), the magnitude of profit that any player makes relative to profits when  $l = 1$  or  $2$  are smaller than when demand is non-isoelastic. This occurs because benefits from being a Stackelberg leader crucially depend on the ability to manipulate the elasticity of the derived demand, which depends on whether the curvature is logconcave. This may explain the stability of the grand coalition and a basic coalition with one lone member,  $Z_{n-1}^n$ . We also note that, when  $n$  becomes very large,  $C(p) \rightarrow 1$  because we require  $b > n$ , which may then yield similar results to elastic demand models in the Stackelberg cases when  $N$  is Pareto efficient. Our verifications show (we do not demonstrate here) this to be true.