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Working Paper Series

2015/75/TOM

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Keywords: Repricing Algorithms; E-Commerce; Online Competition; Dynamic Pricing

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1. Introduction

On April 18, 2011, a book entitled *The Making of a Fly* was listed on Amazon.com for $23.7 million. In the weeks prior to that day, two book sellers (*profnath* and *bordeebooks*) had been using repricing software that tracked each other's prices. Every day, *profnath*'s repricer would set its price at 0.9983 times the *bordeebooks* price; within a few hours, *bordeebook*'s repricer would automatically change its price to 1.27059 times *profnath*'s. Although both sellers had the same rating score (93% positive), *profnath* had significantly fewer reviews than *bordeebooks*; that difference might explain why *bordeebooks* (resp. *profnath*) was selling at a premium (resp. discount) vis-à-vis its rival. The combination of frequent updating and premium–discount asymmetry resulted in prices quickly spiraling out of control (Eisen 2011).[1]

On December 2014, a different repricing software (Repricer Express) caused items to be unintentionally sold on Amazon for 1¢. The software implemented a simple rule by which it would automatically reprice a seller's items slightly lower than those of its competitors. Absent the safeguard of minimum price rules, on December 12 this software reduced the price of thousands of Amazon-listed items to 1¢. The entire stock of some sellers was wiped out within an hour (Stubbings 2014).

[1] On April 19th, *profnath's* price dropped to $106.23 and *bordeebooks* revised its price soon after to $106.23 $\times$ 1.27059 = $134.97.

It is hard to argue that such pricing rules, in their simplicity, can lead to poor decisions. Despite the potentially expensive consequences of a price glitch, the huge scale of e-commerce (nearly $1.5 trillion (US) in sales for 2014, per EMarketer 2014) combined with the ease of making Internet price comparisons contributed to a large increase in the use of automated repricing tools such as those just described. By *repricing* we simply mean any change in an item's price in response to changes in demand, inventory, or competitors' prices. We define *automated repricing* as price changes implemented by an autonomously operating algorithm, such that no human decision maker sanctions or even monitors the software's price adjustments over the short-to-medium time horizon.

In 2014, Amazon.com—the leading US e-tailer in terms of both sales volume and number of customers[2]—implemented more than 2.5 million automated price changes *daily* (a tenfold increase from the previous year) and sometimes changed an item's price more than ten times in 24 hours. During that same year, BestBuy.com and Walmart.com averaged some 50,000 automated daily price changes (Profitero Price Intelligence 2014).

With many e-tailers managing hundreds of thousands of stock-keeping units (SKUs), it is all but impossible for human decision makers in a highly competitive environment to monitor competitors' prices constantly and to optimize their own pricing decisions manually. That being said, there are readily available technologies in analytics, automation, and optimization that make it feasible to automate many of these pricing decisions.

It is therefore not surprising that the global market for pricing analytics software is estimated to have grown at an average rate of 20% per year (from $180 million in 2008 to $425 million in 2014, according to Fletcher 2009.). This is still a very small part of the worldwide business analytics software market; that market generated $37.7 billion in sales during 2013 and is forecast to reach $59.2 billion by 2018 (Vesset et al. 2014).

The repricing software market, though still not mature, is a competitive one replete with options. Existing algorithms can be roughly divided into two categories: first-generation (rule-based) algorithms and second-generation (rule-free) algorithms. First-generation algorithms (e.g., RepriceIt, RepricerExpress, ChannelMAX) implement simple price rules of the following sort: "Price x dollars (or x%) less (more) than seller Y"; "Price x% and z¢ below (above) competitor Y"; "Always have the lowest price"; "Ignore sellers with a feedback of x% or less"; "Match price to other seller's"; "Set price floor (ceiling) to x dollars"; and so forth.[3] In contrast, second-generation algorithms (e.g., Feedvisor, WisePricer) are more advanced and attempt to "learn"

[2] www.statista.com

[3] At one time, the rule-based repricer ChannelMAX offered almost 60 different repricing rules; for a complete list, see ChannelMAX (2014).

the demand function over time and to maximize the seller's long-term profits. Almost all repricing schemes include such basic functions as the ability to set a minimum and a maximum price, to account for shipping costs, and to incorporate information on competitors' price, product availability, and seller ratings.

In this paper we explore some theoretical aspects of repricing automation as well as their practical implications. More specifically, we propose a class of repricing algorithms based on best-response dynamics and show that their structure is similar to that of the ones most commonly used in practice. We then analyze how these proposed algorithms perform relative to a theoretical benchmark (Nash equilibrium) and fully characterize that benchmark. Lastly, we show that these algorithms are robust to the presence of a strategic seller with perfect knowledge of the competitors' repricing rules.

Our setting is a typical multi-seller platform (e.g., Amazon.com, eBay.com, Buy.com) where competing heterogeneous sellers sell homogeneous products at various prices. The vast majority of repricing algorithms are designed primarily for such marketplaces, where customers can easily compare prices across sellers without incurring a significant search cost. Yet it is well established that, despite the Internet-enabled price transparency (and especially on these platforms) there is significant price dispersion across sellers. In their sample data, for instance, Bounie et al. (2012) find that the average coefficient of variation in the price of a book/CD/DVD ranged between 0.19 and 0.3 while varying across countries and product categories. Traditional models of search cost (e.g., Salop and Stiglitz 1977, Pratt et al. 1979) cannot explain the significant price dispersion on platforms such as Amazon, where search cost is practically negligible. However, evidence has accumulated (starting with Brynjolfsson and Smith 2000) that seller trust can explain price dispersion on the Internet and that seller ratings (Wu 2012) or reputation (Bounie et al. 2012) affect consumers' willingness to pay, thereby explaining the variation in prices—across retailers—for a homogeneous product.

Our model focuses on seller heterogeneity as a driver of price dispersion. We first establish equilibrium results for the single and multi-stage pricing game with vertically differentiated sellers. Then we investigate the structure, performance, and robustness of a class of repricing algorithms based on best-response dynamics. We find that myopic best-response schemes perform well and converge to an equilibrium if one exists; that convergence is at a linear rate for most unimodal distributions of the consumer's willingness to pay. For such distributions, the best response is well approximated by a linear or an affine function depending on whether the seller (or its product) is of relatively low or high quality. This finding helps explain why the most popular repricing rules in practice are linear.

We also investigate whether the automated price response can be "gamed" by a strategic seller with perfect knowledge of competitor repricing schemes. We find that the presence of such a strategic player actually helps the myopic repricer. Each seller's price in such a setting will be higher than its equilibrium level, although a steady state does not always exist. Only for a large enough quality gap between the sellers will prices converge to the stationary policy. If the sellers are not sufficiently differentiated, then prices will cycle and sellers will alternate selling in the market. The more information the repricer takes into account when computing the best response, the more likely a steady state is reached. At the extreme—when the repricer accounts for the entire price history—a steady state always exists irrespective of the quality gap between sellers (provided the discount factor is not too low). The steady-state profits and prices are higher than the equilibrium level for both sellers but are *not* higher than when the repricer uses only the most recent price history of its rival.

Finally, we extend our duopoly analysis to an oligopoly. One insight gleaned from this exercise is that the repricer should pay attention to at most two close competitors—namely, the one slightly above in quality and the one slightly below. This result, too, affirms common practice, whereby only a few competitors are tracked when price changes are considered.

The paper proceeds as follows. In Section 2 we review the literature, and Section 3 presents the model and equilibrium results. In Section 4 we discuss repricing rules based on adaptive learning; Section 5 addresses the possibility of gaming an automated price response. An extension to multiple sellers and all proofs are given in (respectively) Appendix A and Appendix B.

2. Literature Review

The model we propose is one of dynamic price competition for vertically differentiated products. More specifically, we look at the quality of the seller as the differentiating feature—although the model is applicable also to other types of vertical differentiation (e.g., product quality).

There is a burgeoning literature that addresses how seller quality is related to the willingness of consumers to pay for its product (for a comprehensive survey, see Wu 2012). Brynjolfsson and Smith (2000) find that consumers are willing to pay a premium to purchase a book from one of the three most reputable online booksellers (Amazon, Barnes & Noble, and Borders); Resnick and Zeckhauser (2002) show experimentally that products sold by experienced sellers were approximately 8% more expensive than identical products sold by less experienced sellers. Although some studies demonstrate a significant positive relationship between the price a seller can charge and the number of its positive reviews (Standifird 2001, Ba and Pavlou 2002, Livingston 2005, Houser and Wooders 2006, Zhang 2006, Bockstedt and Goh 2011), others find the relationship to be not statistically significant (see e.g. Gilkeson and Reynolds 2003). Regarding alternative metrics, Wu et al. (2013) establish that the *percentage* of positive reviews affect eBay auction prices more strongly than does the *number* of such reviews. Bounie et al. (2012) report a significant reputation premium on transaction prices; the authors measure reputation in terms of average rating score, number of positive reviews, and size of operation (i.e., number of SKUs in the seller's portfolio). We shall rely on the findings from this stream of literature to build a utility model for discriminating consumers who are sensitive to the seller's quality.

For nondifferentiated products and sellers, the classic model of price competition is the Bertrand (1883) model: multiple firms produce an identical product, and customers buy from the firm with the lowest price. In the Bertrand model, the equilibrium is unique and consists of firms pricing at their marginal cost and thus earning zero profit. Edgeworth (1897) extended the Bertrand model by considering capacity constraints. Under that extension, a pure strategy equilibrium does not generally exist, although mixed strategy equilibria do exist (Beckman 1965, Levitan and Shubik 1972).[4] Dudey (1992) analyzes a dynamic version of the Bertrand–Edgeworth model and shows that a unique equilibrium always exists.

Static models of competition have been extended to markets with differentiated products. Caplin and Nalebuff (1991) discuss the general conditions under which pure-strategy equilibria exist for oligopolies selling products differentiated along multiple attributes, and they also stipulate conditions for the uniqueness of an equilibrium. Mizuno (2003) demonstrates not only the existence but also the uniqueness of equilibria for models with demand functions that depend on price differences (e.g., a Hotelling-type model). Closely related to this stream of work is the economics literature on the existence and stability of equilibria for concave games (Rosen 1965), submodular games (Topkis 1979), and games that exhibit strategic complementarity (Milgrom and Roberts 1990). In our paper, we use some of these results to establish the existence and stability of a unique pure-strategy equilibrium in the single-stage pricing game.

Gabszewicz and Thisse (1979) analyze the equilibrium in a market with differentiated products when consumers have identical tastes but different income levels—unlike the original Hotelling (1929) model, in which consumers have identical income levels but preferences that are uniformly distributed. Other static models of imperfect competition examine firms' sequential decisions concerning quality and price. Banker et al. (1998) analyze a two-stage model of competition under which firms decide first on quality and later on price. The equilibrium price and quality are analyzed for firms with asymmetric costs and demand 'potential'. Shaked and Sutton (1982) analyze a three-stage noncooperative game where firms first choose whether or not to enter an industry, then determine (if entering) what quality of product to sell, and finally set the price.

[4] A pure-strategy equilibrium exists for either low or high levels of capacity but not for intermediate levels.

Consumers have identical preferences but different income levels. In equilibrium, exactly two entrants will chose to enter; their products will be differentiated and both firms will earn positive profits. In the operations literature, Gallego et al. (2006) employ an “attraction demand” model with convex costs to establish the existence, uniqueness, and stability of a Nash equilibrium in an oligopoly. In our model, consumers differ only in terms of their sensitivity to seller quality. The quality of the sellers is exogenous to the model and fixed throughout the entire selling horizon.

Dynamic models of competition have been extended to account for differentiated products (Chintagunta and Rao 1996, Gallego and Hu 2008, Levin et al. 2009), strategic consumers (Levin et al. 2009), and stochastic demand (Perakis and Sood 2006, Gallego and Hu 2008, Martinez-de-Albeniz and Talluri 2011). Perakis and Sood study multi-period robust pricing policies for oligopolies that market homogeneous products, maintain limited inventory, and face uncertain demand. The authors use a framework of variational inequalities to prove the existence and uniqueness of a so-called open-loop equilibrium. Martinez-de-Albeniz and Talluri seek to identify subgame-perfect equilibria for oligopolies with fixed capacities and homogeneous products in a multi-period setting with stochastic demand. They show the existence and uniqueness of such equilibria provided that the lowest-capacity seller’s units are sold first. Gallego and Hu study an oligopolistic market with a mix of substitutable and complementary products, limited inventory, fixed selling horizon, and deterministic customer arrival rates. They focus on structural results and find there is a finite set of “shadow” prices reflecting the aggregate capacity externalities exerted by firms on each other. Chintagunta and Rao characterize the open-loop equilibrium prices—that is, where the price path is fixed for the entire horizon and prices need not be subgame perfect—in a duopoly with differentiated products and heterogeneous consumers whose preferences evolve over time. Bernstein and Federgruen (2004) advance single- and multi-period models of competition in which firms compete on both price and level of service. These authors use the theory of supermodular games to show that a Nash equilibrium of infinite-horizon stationary strategies exists under price competition only (i.e., exogenous service levels), under competition on both price and service, and in a two-stage competition where service level is chosen first and prices are decided later. Levin et al. study a dynamic pricing model for oligopolies with differentiated products and multiple segments of strategic consumers who are aware that pricing is dynamic and may time their purchases accordingly; they establish both the existence and the uniqueness of a subgame-perfect equilibrium.

Another stream of related literature is the one addressing revenue management under competition (Netessine and Shumsky 2000, Zhao 2003, Li et al. 2008). In all these models, firms have a limited inventory of perishable items that they sell to two classes of customers. Prices for the two classes are either exogenous (as in Netessine and Shumsky 2000, Li et al. 2008)—in which case each firm must decide how much capacity to reserve for the higher-class customer—or prices are set first with capacity levels being decided later (as in Zhao 2003). Netessine and Shumsky and Li et al. identify sufficient conditions for the existence of a unique pure-strategy Nash equilibrium; Zhao establishes that, when firms compete in terms of both prices and quantities, then prices are lower and also reserved capacity levels are lower.

3. Model

We model a typical online multi-seller platform with vertically differentiated sellers. For simplicity of exposition and clarity of insights, we start by analyzing the duopoly case. The extension to more than two sellers is shown in Appendix A.

We assume there are two sellers with respective quality ratings q_1 and q_2 such that $q_1 > q_2$. All buyers prefer a higher quality to a lower one, but buyers vary in their willingness to pay (w.t.p.) for quality. Suppose that this w.t.p., denoted w, has cumulative distribution function F and probability density function f with support standardized to the interval $[0,1]$.[5] We assume that f is both differentiable on $[0,1]$ and log-concave (i.e., $f'(x)/f(x)$ is decreasing in x).[6] The expected market size is N, and demand is stationary over time.

We also assume that sellers have unlimited inventory of the product in each period. To simplify the exposition and w.l.o.g., we standardize the product's marginal cost to zero. Note that all results can be extended trivially to the case of nonzero marginal cost.

Much as in Bertini et al. (2012), we assume that consumers are heterogeneous in terms of their sensitivity to seller quality. A customer x has sensitivity w_x to seller quality and derives utility $w_x q_i - p_i$ from the product sold by seller i with quality q_i at price p_i. The higher the customer's sensitivity, the greater her willingness to pay for quality. For example: given seller 2's offering (q_2, p_2), a greater sensitivity to quality implies more willingness to pay for the high-quality seller's product: $\text{w.t.p.}_x(q_1) = p_2 + w_x(q_1 - q_2)$. Similarly, if seller 1 offers (q_1, p_1) then a greater sensitivity to quality implies less willingness to pay for the low-quality seller's product: $\text{w.t.p.}_x(q_2) = p_1 - w_x(q_1 - q_2)$.

Each customer solves her own surplus maximization problem and will choose to buy from seller i if and only if the following two conditions are satisfied.

- Individual rationality: $w_x q_i - p_i \geq 0$.

[5] This assumption is without loss of generality (w.l.o.g.) because any closed interval in $\mathbb{R}$ is homeomorphic to the closed unit interval in $\mathbb{R}$.

[6] This assumption is widely made in the economics literature, and many equilibrium results depend on it (see Bagnoli and Bergstrom 2005). We remark that almost all common unimodal distributions satisfy this condition. Examples include the normal, uniform, and exponential distributions; the beta distribution with $a, b \geq 1$; the logistic and Laplace distributions; the gamma distribution with $m \geq 1$; and more (see Bagnoli and Bergstrom 2005, Banciu and Mirchandani 2013).

- Utility maximization: $w_x q_i - p_i \geq w_x q_{-i} - p_{-i}$.

The resulting purchase decisions are plotted in Figure 1. A customer will buy from seller 1 whenever $w_x \geq \max\left\{\frac{p_1-p_2}{q_1-q_2}, \frac{p_1}{q_1}\right\}$; otherwise, she will either buy from seller 2 if $w_x \geq \frac{p_2}{q_2}$ or not buy at all if $w_x < \frac{p_2}{q_2}$.

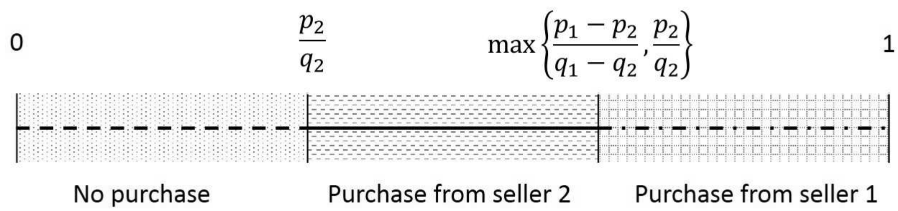

Figure 1 Purchasing decision based on price, quality, and willingness to pay for quality

At this point it is clear that we can restrict the feasible set of p_i to $[0, q_i]$. The deterministic demand functions (i.e., fluid model approximations) for the two sellers can be written as follows:

$$D_1(p_1, p_2) = \begin{cases} NF^c\left(\frac{p_1}{q_1}\right), & p_1 \leq p_2 q_1/q_2, \\ NF^c\left(\frac{p_1-p_2}{q_1-q_2}\right), & p_2 q_1/q_2 \leq p_1 < p_2 + q_1 - q_2, \\ 0, & p_1 \geq p_2 + q_1 - q_2; \end{cases} \tag{1}$$

and

$$D_2(p_1, p_2) = \begin{cases} NF^c\left(\frac{p_2}{q_2}\right), & p_2 \leq p_1 - q_1 + q_2, \\ N\left[F\left(\frac{p_1-p_2}{q_1-q_2}\right) - F\left(\frac{p_2}{q_2}\right)\right], & p_1 - q_1 + q_2 \leq p_2 < p_1 q_2/q_1, \\ 0, & p_2 \geq p_1 q_2/q_1. \end{cases} \tag{2}$$

Because N is a constant and has no effect on the optimization problem, hereafter we omit it from all expressions for product demand.

3.1. Analysis of the Equilibrium

Assuming that perfect information about demand is available, we seek to determine the optimal pricing strategy under competition. We will first describe the best responses of each seller and then demonstrate the equilibrium's existence and uniqueness.

Seller i ($i \in \{1, 2\}$) solves the following revenue maximization problem:

$$\max_{p_i \in [0, q_i]} p_i D_i(p_i, p_{-i}), \tag{3}$$

where p_{-i} represents the competitor's price. We use $B_i(p_{-i}) = \{p_i \in [0, q_i] \mid p_i = \arg\max p_i D_i(p_1, p_2)\}$ to denote the set of solutions to (3).

Let p_i^m be the optimal monopoly price for seller i. Under our assumption that f is log-concave, p_1^m and p_2^m are the unique solutions to

$$p_1^m - q_1 \frac{F^c\left(\frac{p_1^m}{q_1}\right)}{f\left(\frac{p_1^m}{q_1}\right)} = 0 \quad \text{and} \quad p_2^m - q_2 \frac{F^c\left(\frac{p_2^m}{q_2}\right)}{f\left(\frac{p_2^m}{q_2}\right)} = 0.$$

Let $\bar{p}_1$ and $\bar{p}_2$ be the unique solutions to the following equations:

$$\bar{p}_1 - q_1 + q_2 - \frac{1 - F\left(1 - \frac{q_1 - \bar{p}_1}{q_2}\right)}{\frac{f(1)}{q_1 - q_2} + \frac{f\left(1 - \frac{q_1 - \bar{p}_1}{q_2}\right)}{q_2}} = 0 \quad \text{and} \quad \frac{\bar{p}_2}{q_2} - \frac{q_1 - q_2}{q_1} \frac{F^c\left(\frac{\bar{p}_2}{q_2}\right)}{f\left(\frac{\bar{p}_2}{q_2}\right)} = 0. \tag{4}$$

We can interpret $\bar{p}_i$ as the lowest price of seller i in response to which seller $-i$ will price in such a way that seller i will *not* sell. Then the best response of the two sellers can be characterized as follows.

Proposition 1 **(Best-response characterization)**.

(i) $B_1(p_2)$ *is a real-valued function given by*

$$B_1(p_2) = \begin{cases} p_1^m, & p_2 \geq p_1^m q_2/q_1; \\ p_2 q_1/q_2, & \bar{p}_2 \leq p_2 < p_1^m q_2/q_1; \\ p_1^*, & p_2 < \bar{p}_2. \end{cases}$$

Here p_1^* *is the unique solution to*

$$p_1^* - (q_1 - q_2) \frac{F^c\left(\frac{p_1^* - p_2}{q_1 - q_2}\right)}{f\left(\frac{p_1^* - p_2}{q_1 - q_2}\right)} = 0. \tag{5}$$

(ii) $B_2(p_1)$ *is a real-valued function given by*

$$B_2(p_1) = \begin{cases} p_2^m, & p_1 \geq p_2^m + q_1 - q_2; \\ p_1 - q_1 + q_2, & \bar{p}_1 \leq p_1 < p_2^m + q_1 - q_2; \\ p_2^*, & p_1 < \bar{p}_1. \end{cases}$$

Here p_2^* *is the unique solution to*

$$p_2^* - \frac{F\left(\frac{p_1 - p_2^*}{q_1 - q_2}\right) - F\left(\frac{p_2^*}{q_2}\right)}{\frac{1}{q_1 - q_2} f\left(\frac{p_1 - p_2^*}{q_1 - q_2}\right) + \frac{1}{q_2} f\left(\frac{p_2^*}{q_2}\right)} = 0. \tag{6}$$

(iii) $0 \leq B_1'(p_2), B_2'(p_1) < 1$.

There are three aspects of this best-response function's structure. First, if the competitor's price is high enough then each seller can price as a monopolist and be the only one that sells in the market. Second, if the competitor's price is *above* a threshold $\bar{p}_{-i}$ then the seller will price low enough that the competitor will not sell. Third, if the competitor's price is *below* the $\bar{p}_{-i}$ threshold then both sellers will sell a positive quantity. Observe that, since the best response of seller i is increasing in seller $-i$'s price, the pricing game exhibits strategic complementarities. In other words: the lower is seller i's price, the higher is seller $-i$'s marginal revenue. Also note that the slope is less than 1; this outcome reflects that the two sellers' respective products are not perfect substitutes. For a uniform w.t.p. distribution, the closed-form solutions of the best-response functions are given next.

COROLLARY 1 **(Best-response for uniform w.t.p.)**. *If $w \sim U[0,1]$, then the best responses of sellers 1 and 2 will be as follows. For seller 1*:

$$B_1(p_2) = \begin{cases} \frac{q_1}{2}, & p_2 \geq \frac{q_2}{2}; \\ \frac{p_2 q_1}{q_2}, & \frac{q_2(q_1-q_2)}{2q_1-q_2} \leq p_2 < \frac{q_2}{2}; \\ \frac{p_2+q_1-q_2}{2}, & p_2 < \frac{q_2(q_1-q_2)}{2q_1-q_2}. \end{cases}$$

For seller 2:

$$B_2(p_1) = \begin{cases} \frac{q_2}{2}, & p_1 \geq q_1 - \frac{q_2}{2}; \\ p_1 - q_1 + q_2, & \frac{2q_1(q_1-q_2)}{2q_1-q_2} \leq p_1 < q_1 - \frac{q_2}{2}; \\ \frac{p_1 q_2}{2q_1}, & p_1 < \frac{2q_1(q_1-q_2)}{2q_1-q_2}. \end{cases}$$

3.1.1. Nash Equilibrium In equilibrium, the two sellers will simultaneously solve the following revenue maximization problem:

$$\max p_i D_i(p_i, p_{-i}) \quad \text{s.t.} \;\; p_{-i} = \operatorname{argmax}\{p_{-i} D_{-i}(p_i, p_{-i})\}.$$

We will show that the pricing game is log-supermodular. Such games always have a Nash equilibrium (see e.g. Topkis 1979). The following theorem is adapted from Milgrom and Roberts (1990) and based on Topkis (1979).

THEOREM 1. *Suppose there are finitely many players. Each player $n \in N$ has a strategy set S_n with typical element x_n; the competitors' strategies are denoted by x_{-n}. Let $\pi_n(x_n, x_{-n})$ be player n's payoff function, and let $\geq$ signify the usual componentwise ordering. Then, for $i = 1, \ldots, n$, the game $\Gamma = \{n, (S_i, \pi_i), \geq\}$ is supermodular if Conditions C1–C3 are satisfied*:

C1. $S_n = [\underline{y}_n, \overline{y}_n] = \{x \mid \underline{y}_n \leq x \leq \overline{y}_n\}$;

C2. *π_n is twice continuously differentiable on S_n; and*

C3. *$\partial^2 \pi_n / \partial x_n \partial x_m \geq 0$ for all $n \neq m$.*

If these conditions are satisfied for $\log \pi_n$, then the game is log-supermodular and the transformed game has the same equilibria as the original game. If, in addition, the following "dominant diagonal" condition is satisfied for π_n (or $\log \pi_n$), then the game has a unique pure-strategy equilibrium:

$$-\frac{\partial^2 \pi_n}{(\partial x_n)^2} > \sum_{m \neq n} \frac{\partial^2 \pi_n}{\partial x_n \partial x_m}. \tag{7}$$

The pricing game described in this section satisfies Conditions C1–C3 as well as the dominant diagonal condition. Hence a pure-strategy equilibrium always exists and that equilibrium is unique. The following proposition formalizes this result.

PROPOSITION 2. *A Nash equilibrium always exists. In equilibrium, both sellers sell a strictly positive quantity provided that $q_i > 0$. The equilibrium (p_1^N, p_2^N) is given by the solution to the following system of equations*:

$$p_1^N = (q_1 - q_2)\frac{F^c\left(\frac{p_1^N - p_2^N}{q_1 - q_2}\right)}{f\left(\frac{p_1^N - p_2^N}{q_1 - q_2}\right)} \quad \text{and} \quad p_2^N = \frac{F\left(\frac{p_1^N - p_2^N}{q_1 - q_2}\right) - F\left(\frac{p_2^N}{q_2}\right)}{\frac{1}{q_1 - q_2} f\left(\frac{p_1^N - p_2^N}{q_1 - q_2}\right) + \frac{1}{q_2} f\left(\frac{p_2^N}{q_2}\right)}. \tag{8}$$

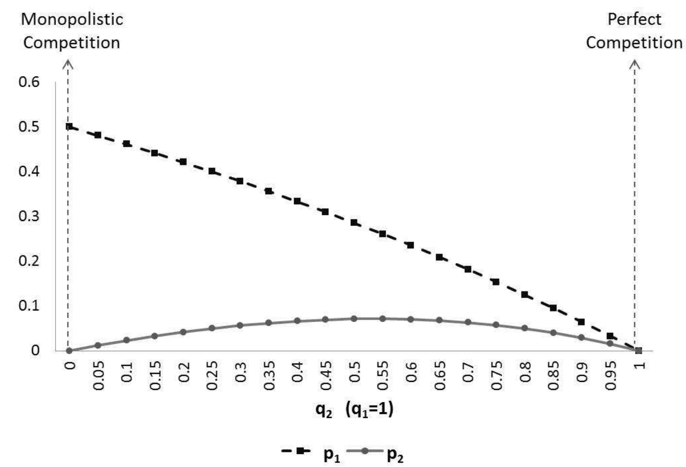

Figure 2 **Equilibrium prices for uniformly distributed willingness to pay and various quality differences**

Corollary 2 **(Nash equilibrium for uniform w.t.p.)**. *If $w \sim U[0,1]$, then the Nash equilibrium of the duopoly competition is given by $p_1^N = \frac{2q_1(q_1-q_2)}{4q_1-q_2}$ and $p_2^N = \frac{q_2(q_1-q_2)}{4q_1-q_2}$.*

In Figure 2 we plot the equilibrium prices for different quality gaps. We note that if the quality difference is large, then sellers are engaged in a monopolistic competition and can therefore charge the maximum price—just as they would under a true monopoly. In other words, the product offerings are so heterogeneous that competition does not affect prices.[7]

If the quality gap is small, however, then we observe the typical Bertrand price competition in which prices converge to the seller's marginal cost.

3.2. Dynamic N-period Game

In the absence of capacity constraints, the dynamic N-period duopoly game where the players maximize the N-period discounted/undiscounted profit reduces to a repeated single-stage competition. Thus, in each period, Seller i will choose price p_i^N defined in (8). The proof is straightforward using backward induction arguments.

4. Automated Repricing

In the previous section we focused on characterizing the equilibrium prices of the strategic pricing game in a duopoly and on proving the existence and uniqueness of that equilibrium. In a strategic game, implementing the equilibrium presumes that each decision maker chooses a plan of action once and for all and that these choices are made simultaneously (Osborne and Rubinstein 1994). Yet neither of these presumptions is valid in real life, where choices are seldom

[7] This phenomenon is often observed in practice. The high-end online fashion retailer Nordstrom.com has a price-matching policy that states: "We will be glad to meet our competitor's pricing if you ever find an item that we offer, in the same color and size, available from a *similar* retailer. We are unable to match prices from auction and outlet stores or websites" (http://shop.nordstrom.com/c/pricing-policy).

final and hardly ever made simultaneously; therefore, attaining the equilibrium is far from guaranteed. In this section we describe the limiting behavior of dynamic repricing schemes based on adaptive learning processes. Such processes are general enough to include not only simple best-response dynamics but also more complex Bayesian learning. For this general class of adaptive learning processes, prices always converge to the unique Nash equilibrium. We find numerically that, for the most common log-concave distributions, repricing rules are well approximated by affine functions of the competitor's price; also, prices converge to the equilibrium point at a linear rate.

4.1. Adaptive Learning

Following Milgrom and Roberts (1990) and Fundenberg and Kreps (1988) we define a learning process for the game $\Gamma = \{n, (S_i, \pi_i)_{i=1,\ldots,n}, \geq\}$ as $\{x_{i,t}\}$, with $x_{i,t} \in S_i$, for all $i = 1, \ldots, n$ and $t \in [0, T]$. Let $P_i(t_1, t_2) = \{x_{i,t} \in S_i | t_1 \leq t \leq t_2\}$ be the set of strategies played by player i in time interval $[t_1, t_2]$. We say that a process is of adaptive dynamics if $\forall t_1$, $\exists t_2 > t_1$ such that $\forall t \geq t_2$, $x_{i,t} \in \bar{U}_i[\inf\{P_{-i}(t_1, t)\}, \sup\{P_{-i}(t_1, t)\}]$, where $U_i(\hat{S}) = \{x_i \in S_i | (\forall x_i' \in S_i)(\exists x_{-i} \in \hat{S})\pi_i(x_i, x_{-i}) \geq \pi_i(x_i', x_{-i})\}$ represents the set of undominated responses of player i to strategies in $\hat{S}$ and $\bar{U}_i(S) = [\inf\{U_i(S)\}, \sup\{U_i(S)\}]$ is the smallest compact interval containing all points in $U_i(\hat{S})$.

Thus, an adaptive learning process is one such that, for each date t_1, there exists a later date t_2 after which each seller selects a strategy that lies in the interval defined by the set of justifiable choices. A strategy choice is *justifiable* if it is not dominated by any other strategy choice — in other words, by any that would have yielded better results against every strategy from the smallest interval containing the competitor's latest play.

Our next theorem is based on Milgrom and Roberts (1990).

Theorem 2 **(Milgrom and Roberts 1990)**. *Let $\{x_{i,t}\}$ be an adaptive learning process for a supermodular game Γ and let $\underline{y} = (\underline{y}_1, \ldots, \underline{y}_n)$ and $\overline{y} = (\overline{y}_1, \ldots, \overline{y}_n)$ be (respectively) the smallest and largest Nash equilibrium strategy profiles of Γ. Then $\liminf x_{i,t} \geq \underline{y}_i$ and $\limsup x_{i,t} \leq \overline{y}_i$ for all $i = 1, \ldots, n$. If the equilibrium is unique, then $\underline{y} = \overline{y}$ and $\lim x_{i,t} \to y_i$ for all $i = 1, \ldots, n$.*

Although the theorem is stated for supermodular games, one can easily see that it holds for log-supermodular games as well. The reason is that no monotone transformation of the payoff function changes the set of undominated player responses.

In our duopoly setting, consider the last two prices of seller 2 to be $p_{2,t-1}$ and $p_{2,t}$ with $p_{2,t-1} > p_{2,t}$. Then seller 1 can justify any choice of $p_{1,t+1} \in [0, q_1]$ that is a best response to some probability distribution over the interval $[p_{2,t}, p_{2,t-1}]$.

Quality Gap	Normal (μ,σ)			Beta (a,b)				Uniform $[0,1]$
$q_1 - q_2$	$(0.5,0.3)$	$(0.5,0.2)$	$(0.5,0.1)$	$(2,5)$	$(5,2)$	$(2,2)$	$(3,3)$	
Low (0.2)								
B_1	0.998	0.999	0.999	0.997	0.999	0.999	0.999	1
B_2	0.963	0.949	0.929	0.992	0.895	0.949	0.949	1
Medium (0.4)								
B_1	0.998	0.999	0.999	0.997	0.999	0.999	0.999	1
B_2	0.996	0.989	0.912	0.949	0.895	0.993	0.993	1
High (0.8)								
B_1	0.999	0.999	0.999	0.999	0.999	0.999	0.999	1
B_2	0.991	0.972	0.979	0.922	0.999	0.994	0.985	1

Table 1 R^2 **for the linear approximation of the best response functions for various distributions and quality gaps**

4.2. Repricing Schemes

We define two types of repricing schemes and show that they are adaptive learning processes. Both these schemes are 'myopic', in the sense that they rely on the assumption that the competitor's price in the current and future periods will not change.

Repricing Scheme 1: *Best Response Dynamics*

In every period, each seller sets a price while assuming that its rival's price from the previous period will remain the same in the current and in future periods. The economics literature refers to this strategy as *best-response dynamics*, and its study dates back to the classic oligopoly works of Cournot (1838) and Edgeworth (1897).

Such dynamics obviously constitute an adaptive learning process because $p_{i,t} = B_i(p_{-i,t-1}) = U_i(p_{-i,t-1})$. Since $p_{-i,t-1} \in P_{-i}(t-k,t)$ for any $k \geq 1$, it follows that for all t_1 there exists a $t_2 = t_1 + 1$ such that $x_{i,t} \in \bar{U}_i[\inf\{P_{-i}(t_1,t)\}, \sup\{P_{-i}(t_1,t)\}]$ for all $t \geq t_2$.

Repricing Scheme 2: *Average-based Response Dynamics*

In every period, seller i computes the average price charged by seller $-i$ over the past k periods. Then seller i computes the best response to this average price. It is easy to see that this is an adaptive process, as $p_{i,t} = B_i(\frac{\sum_{s=t-k}^{t-1} p_{-i,s}}{k})$. We know that the best response function is monotone. Furthermore, $\frac{\sum_{s=t-k}^{t-1} p_{-i,s}}{k} \in [\inf\{P_{-i}(t-k,t-1)\}, \sup\{P_{-i}(t-k,t-1)\}]$, and so $B_i\left(\frac{\sum_{s=t-k}^{t-1} p_{-i,s}}{k}\right) = U_i\left(\frac{\sum_{s=t-k}^{t-1} p_{-i,s}}{k}\right) \in [U_i(\inf\{P_{-i}(t-k,t-1)\}), U_i(\sup\{P_{-i}(t-k,t-1))] = \bar{U}_i[\inf\{P_{-i}(t-k,t-1)\}, \sup\{P_{-i}(t-k,t-1)\}]$. Hence it follows that, for all t_1, there exists a $t_2 = t_1 + k$ such that $x_{i,t} \in \bar{U}_i[\inf\{P_{-i}(t_1,t)\}, \sup\{P_{-i}(t_1,t)\}]$ for all $t \geq t_2$.

Repricing scheme 2 is a more general version of scheme 1, where the former assumes $1 \leq k \leq t$ and the latter assumes $k = 1$.

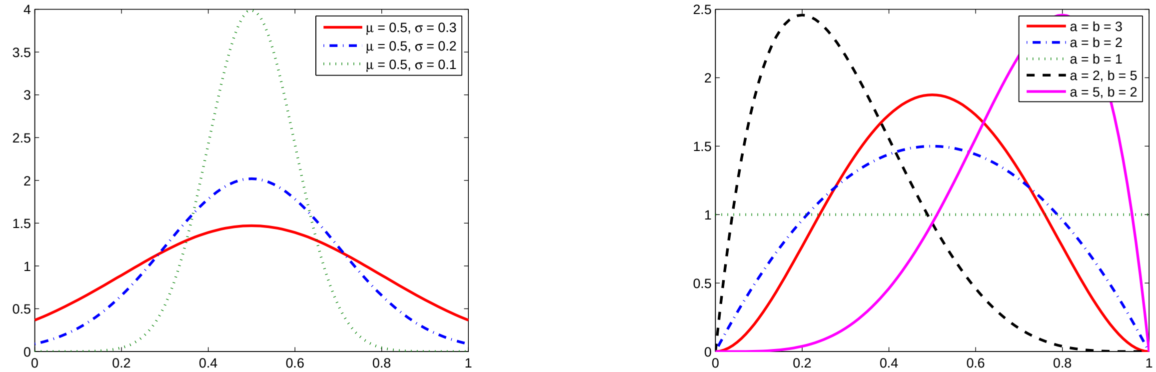

Figure 3 **Different distributions tested: Normal(μ, σ), Beta(a,b).**

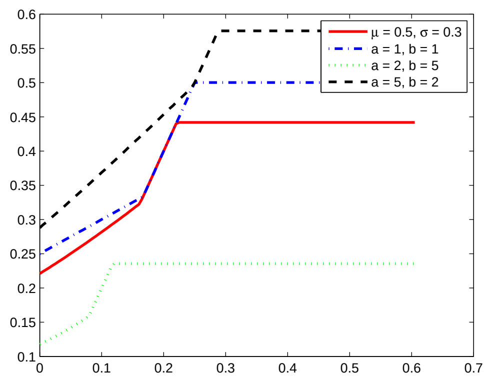

Figure 4 **Best response of seller 1 to seller 2's price: various w.t.p. distributions**

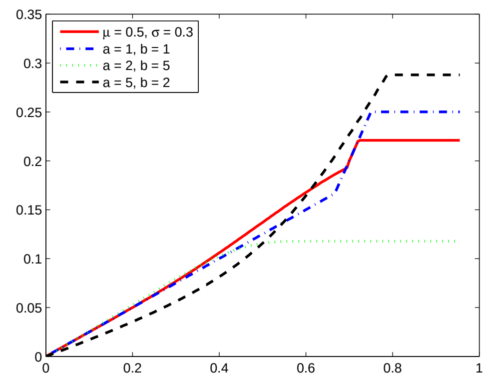

Figure 5 **Best response of seller 2 to seller 1's price: various w.t.p. distributions**

4.3. When Simple Rules Do (and Don't) Work

We computed (numerically) the best-response functions for various well-known log-concave distributions with support on $[0,1]$. In particular, we analyzed the truncated normal distribution ($\mu = 0.5$, $\sigma \in [0.1,\ldots,0.3]$), the beta$(a,b)$ distribution ($a \in [1,5]$, $b \in [1,5]$), and the uniform$[0,1]$ distribution. Figure 3 plots the most representative distributions. In all these cases we found that the best-response functions B_i are well approximated by a piecewise linear function; see Figures 4 and 5. We measured the goodness-of-fit using R^2 and for the majority of cases we note that $R^2 > 0.99$. The values are given in Table 1. Figures 6–9 plot the best-response functions of sellers 1 and 2 for various differences in quality when the distribution of consumers' willingness to pay is normal or beta.

If the repricing algorithm implements a simple best-response function and if that function is piecewise linear, then the rate of convergence to equilibrium is linear.

Proposition 3 **(Convergence to equilibrium)**. *Suppose the best-response functions of two sellers*

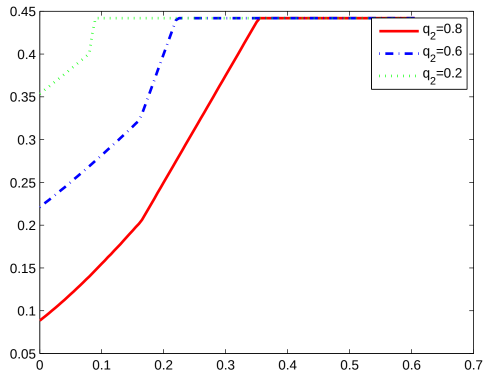

Figure 6 Best response of seller 1 to seller 2's price for different quality differences: normal w.t.p. distribution

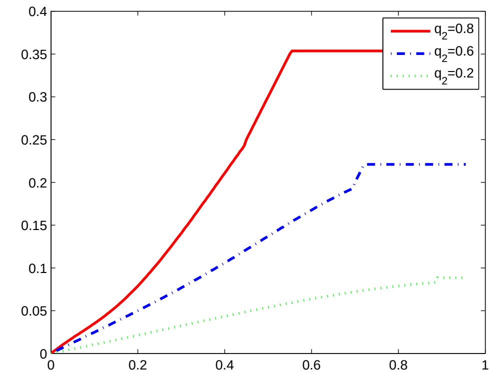

Figure 7 Best response of seller 2 to seller 1's price for different quality differences: normal w.t.p. distribution

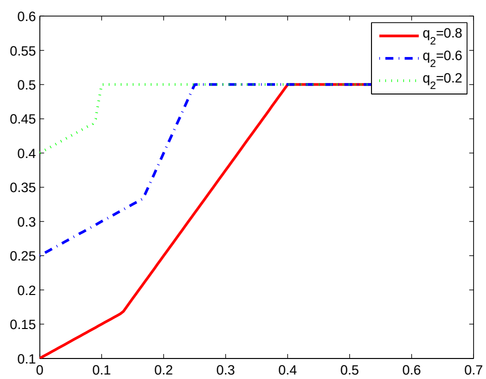

Figure 8 Best response of seller 1 to seller 2's price for different quality differences: beta distribution

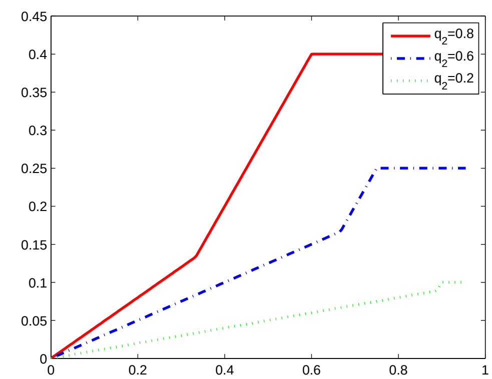

Figure 9 Best response of seller 2 to seller 1's price for different quality differences: beta distribution

are characterized as follows:

- $B_1(p_2) = ap_2 + b$ *if* $p_2 < p_2^s$ *with* $b > 0$ *and* $0 < a < 1$; *otherwise*, $B_1(p_2) = \min(p_1^m, p_2 q_2/q_1)$ *with* $ap_2^s + b = p_2^s q_2/q_1$.
- $B_2(p_2) = cp_1$ *if* $p_1 < p_1^s$ *with* $0 < c < 1$; *otherwise*, $B_2(p_2) = \min(p_2^m, p_1 - q_1 + q_2)$ *with* $cp_1^s = p_1^s - q_1 + q_2$.

Then the sequential best-responses converge to the fixed point of $f(p_1, p_2) = (B_1(p_2), B_2(p_1))$. *The rate of convergence to equilibrium is linear.*

If the starting price is $(p_1, p_2) \leq (p_1^s, p_2^s)$ and so both sellers sell a positive quantity, then the best response of the low-quality seller is a linear function of the high-quality seller's price. Similarly, the best response of the high-quality seller is an affine function of the low-quality seller's price.

This simple linear structure lends some justification to the most popular repricing rules seen in practice (e.g., "Price x dollars (or x%) less than seller Y" or "Price x% and z¢ below (above) competitor Y"). We established in Section 3 that the slope of each best-response function is less than 1. Here we observe that the intercept of the best-response function is zero for the low-quality seller (seller 2), whereas this intercept is positive for the high-quality seller. So a low-quality seller that uses a simple best-response repricer need only estimate the multiplicative adjustment factor $c < 1$; in contrast, a high-quality seller must estimate not only the slope $a < 1$ but also the lowest price that it is willing to charge (i.e., b).

This structure of the best response also indicates certain shortcomings of the rules often employed in practice. If both sellers use multiplicative (i.e., linear) rules without bounds, then prices will quickly converge to zero or infinity—which is what happened in the examples given by Eisen (2011) and Stubbings (2014).

4.4. Gaming the Automated Price Response

In this section we investigate the potential for strategic advantage in a market of automated repricers. Of particular interest is answering this question: when one firm implements an automated ("myopic" and known/learnable) price response while the other firm implements a strategically optimal price response, will the latter's strategy reduce its myopic rival's profit?

For the sake of brevity, we detail only the case in which the low-quality firm (seller 2) reprices automatically while the high-quality firm (seller 1) reprices strategically. All the results in this section will hold likewise when it is the low-quality seller that reprices strategically; the analysis is similar in all respects.

As before, let $B_i(p_{-i})$ denote the single-period profit-maximizing price of seller i given seller $-i$'s price p_{-i}. Then, given seller 1's price in period $t-1$ ($p_{1,t-1}$), seller 2 will always price such that $p^*_{2,t} = B_2(p_{1,t-1})$.

Yet because seller 1 is endowed with complete knowledge of its competitor's repricing scheme, that price response is incorporated into seller 1's infinite-horizon dynamic pricing problem. Its long-term profit maximization problem can be formulated as

$$\Pi_1(p_{1,0}) = \sup_{p_{1,t}\in\Gamma} \sum_{t=0}^{\infty} \delta^t p_{1,t} D_1(p_{1,t}, p^*_{2,t}(p_{1,t-1})) \qquad (9)$$
$$\text{s.t.} \;\; p^*_{2,t}(p_{1,t-1}) = B_2(p_{1,t-1}) \; \forall t \geq 1,$$

where $\delta \in (0,1)$ is the discount factor.[8]

The timeline is as follows. Both sellers start with the exogenous prices $p_{1,0}$ and $p_{2,0}$ respectively. During each period t ($t = 0, 1, \ldots$), seller 2's repricer collects data about seller 1's price in period

[8] For most repricers, the time between updatings is extremely short (i.e., less than a day). For all practical purposes, then, the discount factor should be close to 1.

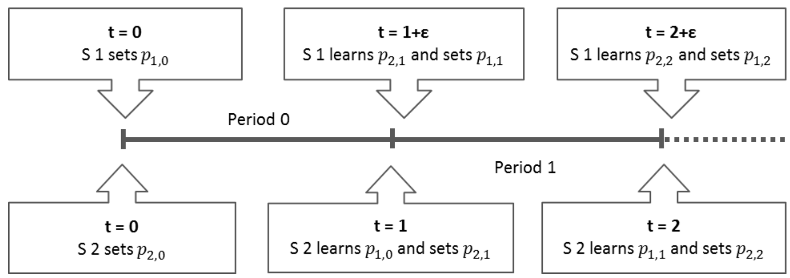

Figure 10 Re-pricing Timeline: An Illustration

t, $p_{1,t}$. At the start of period $t+1$, seller 2 updates its price to $p_{2,t+1}$. Shortly after observing $p_{2,t+1}$, seller 1 updates its price to $p_{1,t+1}$. Figure 10 illustrates this timeline.

There are two important remarks about the setup. First, seller 1 updates his price *immediately after* seller 2 (i.e., at time $t+\epsilon$ with $\epsilon \to 0$). This gives seller 1 the maximum advantage relative to seller 2, which satisfies our assumption that seller 1 is perfectly rational and has full knowledge of the competitor's repricing algorithm. Second, seller 1 only sets one price in each period. This is a simplifying assumption, but one can extend the model to the case where seller 1 can vary the price within a period. However, this extension complicates the analysis, without providing additional insights. Moreover, for best response functions of the type described in Proposition 3, one can show that the single-stage profit function $p_{1,t}D_1(p_{1,t}, p_{2,t})$ is concave in $p_{1,t}$. In this case, it is suboptimal for seller 1 to vary its price within a period.

The profit per stage is bounded, so the value function is the unique solution of the Bellman equation:

$$v(p) = \max_{x \in \Gamma} \{xD_1(x, B_2(p)) + \delta v(x)\}. \tag{10}$$

So if $p_1^* = (p_{1,1}^*, p_{1,2}^*, \ldots)$ is a pricing plan that attains the supremum in (9) for an initial state $p_{1,0}$, then

$$v^*(p_{1,t}^*) = p_{1,t+1}^* D_1(p_{1,t+1}^*, p_{2,t+1}^*(p_{1,t}^*)) + \delta v^*(p_{1,t+1}^*). \tag{11}$$

Thus, the optimal price path $\{p_{1,t}^*\}_{t=1,2,\ldots}$ generated from the initial state $p_{1,0}$ is computed sequentially as $p_{1,t+1}^* = x^*(p_{1,t}^*)$, where $x^*(p)$ is the solution to (10). We have already shown $B_2(p)$ to be a single-valued function that is increasing in p (see Proposition 1). Moreover, $D_1(p_1, p_2)$ is increasing in p_2, thus the profit per stage $xD_1(x, B_2(p))$ is increasing in p. Hence we have the following lemma.

Lemma 1. *The value function $v(p)$ is increasing in p.*

This lemma allows us to exclude the possibility that the strategic player will underprice. We can show that prices will always be higher than those resulting from single-stage myopic play. Also, the presence of one strategic player with perfect knowledge of the other player's pricing algorithm can help maintain prices above the equilibrium level for *both* sellers. Our next proposition formalizes this result.

PROPOSITION 4 **(No "low ball" offers)**. *The single period profit maximizing price is always lower than the strategic infinite-horizon price:* $B_1(B_2(p)) \leq x^*(p)$*, where* $x^*(p)$ *is the solution to (10). Moreover,* $B_{1,t}(p_{1,0}) \leq p^*_{1,t}$ *for all* $t > 0$*, where* $B_{1,t}(p_{1,0})$ *refers to the t-th update of seller 1 in the best response dynamics for a starting price of* $p_{1,0} > p_1^N$ *(i.e.,* $B_{i,t}(p) = B_i[B_{-i}[B_{i,t-1}(p)]]$*). Also,* $p^*_{1,t} \geq p_1^N$ *and* $p^*_{2,t} \geq p_2^N$ *for all* $t \geq 1$*.*

We would also like to investigate the characteristics of the price path in this infinite-horizon problem and to identify conditions for the existence of a steady state. First, we define the stationary policy for (10) to be (p_1^s, p_2^s), where

$$p_1^s = \arg\max_{p_1 \in [0,q_1]} p_1 D_1(p_1, B_2(p_1)), \tag{12}$$

$$p_2^s = \arg\max_{p_2 \in [0,q_2]} p_2 D_2(p_1^s, p_2). \tag{13}$$

Note that the stationary policy need not be the optimal policy.

Recall from Section 3 that p_1^m and p_2^m are the two sellers' respective monopoly prices. Then it is easy to see that: (a) if $q_1 = q_2$ then $p_1^s = p_2^s = 0$; and (b) if $q_1 - q_2 = 1$ (or, equivalently, if $q_1 = 1$ and $q_2 = 0$) then $p_1^s = p_1^m$ and $p_2^s = p_2^m$.

Consider the price sequence from seller 1's perspective. There are two possibilities: either the price sequence is monotone decreasing or it is non-monotone. If the price sequence is monotone decreasing then, since the price is bounded below by zero, the sequence will converge to the steady-state solution described by (12) and (13). A price sequence that is non-monotone does not converge. In Proposition 5 (resp., Proposition 6) we argue that, if the quality difference is small enough (resp., large enough), then the price sequence will not (resp., will) converge.

PROPOSITION 5 **(Cyclic policy for relatively undifferentiated sellers)**. *If the quality gap between sellers is small, then optimal price path follows a cyclic pattern.*

The intuition behind this cyclic pricing policy is as follows. When the quality gap is small, the stationary profits are close to zero for both sellers. Hence it becomes more profitable for seller 1 to set a high price in one period, *not sell* in that period, and thus reset seller 2's price expectations to a higher level. Then seller 1 can set a lower price, sell as a monopolist, and derive maximum monopoly revenues until seller 2 catches up and adjusts its price expectation to a competitive level.

What happens in such cases is that sellers alternate selling in the market: in some periods, only seller 2 will sell; in other periods, only seller 1 will. That seller 1 takes a strategic (long-term) approach, rather than a myopic (short-term) one, helps both sellers.

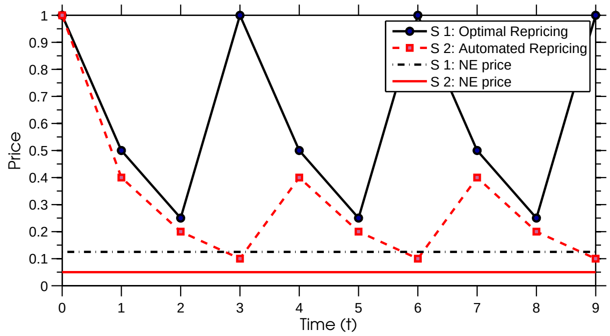

Figure 11 Realized price path for $q_1 = 1, q_2 = 0.8$ with seller 1 as the strategic player;

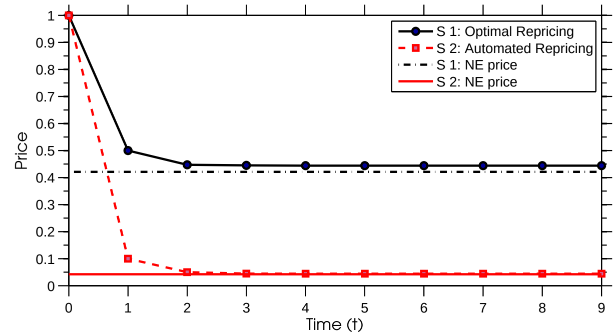

Figure 12 Realized price path for $q_1 = 1, q_2 = 0.2$ with seller 1 as the strategic player;

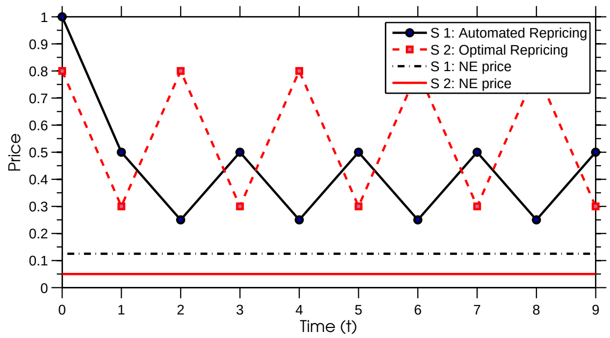

Figure 13 Realized price path for $q_1 = 1, q_2 = 0.8$ with seller 2 as the strategic player;

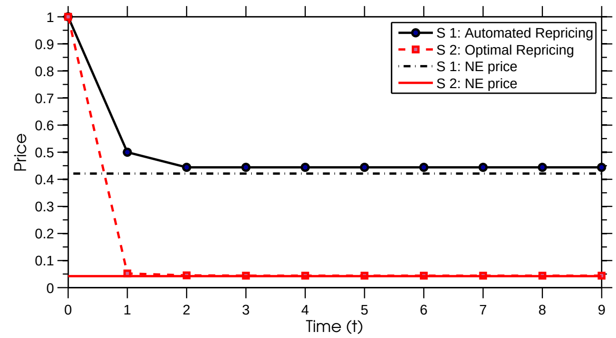

Figure 14 Realized price path for $q_1 = 1, q_2 = 0.2$ with seller 2 as the strategic player;

This pricing strategy makes sense only if there is "tight" competition between the sellers. If the quality gap is large, then both sellers can sustain high profits in each period; in that case it is not necessary to alternate selling in the market. The next proposition formalizes this result.

PROPOSITION 6 **(Monotone policy for relatively differentiated sellers)**. *If the quality difference between sellers is large, then the optimal price path converges to the steady-state policy.*

Figures 11 -14 plot the price paths for different quality gaps depending on which of the two sellers acts strategically. For each graph, the price path of each seller is plotted against time (x-axis). In all cases, each seller's profit is higher than the Nash equilibrium profit. We therefore conclude that both sellers benefit when only one of them reprices strategically.

4.5. Average-based best response repricing

Suppose that seller 2, instead of making a simple best response based on seller 1's price in the previous period, makes a best response based on the average of seller 1's price over the last k_t periods. In this case, seller 1's problem becomes:

$$\Pi(p_{1,0}) = \sup_{p_{1,t} \in \Gamma} \sum_{t=0}^{\infty} \delta^t p_{1,t} D_1(p_{1,t}, p^*_{2,t}(p_{1,t-k_t}, \ldots, p_{1,t-1})) \tag{14}$$

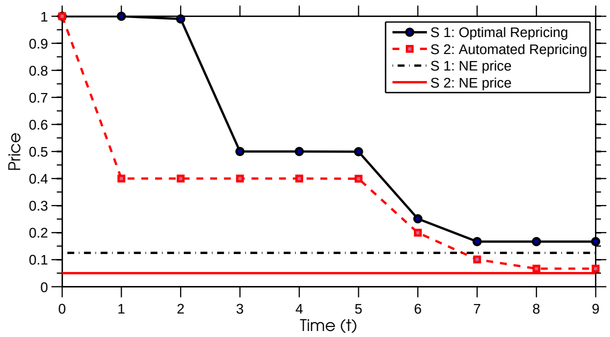

Figure 15 **Realized price path for $q_1 = 1, q_2 = 0.8$ with seller 1 as the strategic player**

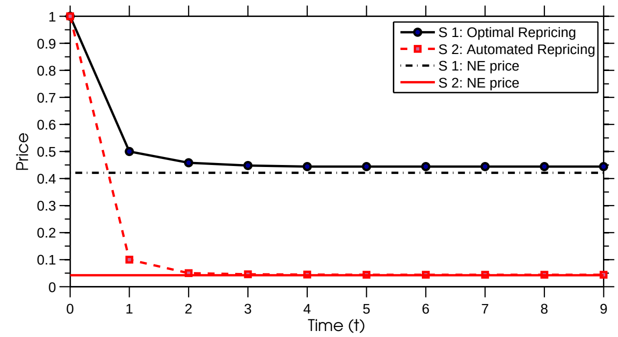

Figure 16 **Realized price path for $q_1 = 1, q_2 = 0.2$ with seller 1 as the strategic player**

$$\text{s.t. } p^*_{2,t}(p_{1,t-k_t}, \ldots, p_{1,t-1}) = B_2\left(\frac{\sum_{s=1}^{k_t} p_{1,t-s}}{k_t}\right) \ \forall t \geq 1.$$

We will analyze the extreme case where $k_t = t$. That is, in every period the repricer uses the entire price history to compute seller 1's average price and then reprices using a best-response rule based on that average. This situation leads to the following result.

Proposition 7. *For large enough δ (i.e., δ close to 1), the path of seller 1's optimal price converges to the stationary policy described in* (12) *and* (13).

If the average is taken over a fixed number of periods then the pricing policy either converges to the stationary policy or cycles—just as in the case of a simple best response, but as the number of periods increases, the pricing path will eventually converge to the stationary policy irrespective of the quality gap between the sellers.

Figures 15, 16 plot the price path for different quality gaps when the myopic repricer uses an average-based best response. For both low and high quality gaps, the price sequence converges to the stationary policy. The result will still hold if seller 2 is the strategic player.

5. Conclusions

In this paper we investigate the performance of myopic best-response pricing algorithms for homogeneous products sold by vertically differentiated sellers in an online marketplace. We characterize the one-stage equilibrium prices and show that the pricing game is log-supermodular—and hence that a unique equilibrium always exists. Moreover, this equilibrium is stable; that is, the dynamic best response will converge to the unique equilibrium.

For a wide class of distributions, we demonstrate that the best response can be approximated numerically by a linear function of slope less than 1 (for the low-quality seller) and by an affine function of slope less than 1 and with positive intercept (for the high-quality seller). These findings lend some credence to the simple repricing rules typically encountered in practice. It also shows that many of these algorithms can be reduced to estimating a multiplicative adjustment factor and setting a price floor.

We then investigate the possible gaming of automated repricing algorithms by a strategic player endowed with complete knowledge of its competitor's repricing scheme. We find that the presence of a strategic player actually helps both sellers. There are no low-ball offers because prices will always be above the Nash equilibrium levels. Prices will also be monotonic and will converge to a steady state if the quality gap is sufficiently large. If the quality gap is small, however, and sellers are not otherwise well differentiated, then prices will cycle: periods of selling by the high-quality seller will alternate with periods of selling by the low-quality seller. The longer the price history that is used by the repricer when updating prices, the more robust the pricing policy will be. At the extreme, that policy is always monotonic and so prices decrease over time and eventually converge to the steady-state solution. But more information is not always helpful, since both the myopic repricer and the strategic player are *worse*-off as more information (i.e., a longer price history) is incorporated into the updating.

When there are more than two sellers, we show (in Appendix A) that each player cares only about its closest rivals (i.e., the competitors immediately above and below). This result, too, is in conformance with the pricing rules frequently encountered in practice.

In summary, we find that correctly implemented repricing tools should result in rapid convergence to equilibrium prices. The presence of a strategic seller helps all sellers, so the possibility of gaming an algorithm is essentially non-existent. However, our analysis relies upon two strong assumptions: stationary demand and unlimited inventory. Relaxing demand stationarity would likely alter our results. Hence future research is needed to analyze the effect of seasonality and/or consumer behavior (strategic behavior, anchoring and other cognitive biases, etc.) on the performance and robustness of repricing algorithms. Assuming unlimited inventory is less problematic in that doing so is less likely to affect our findings. A complication that arises when inventory *is* limited is that a pure-strategy equilibrium no longer exists. In that event, other benchmarks (e.g., open-loop equilibria) would need to be considered.

6. Appendix A - Multiple Sellers Extension

Assume now that there are n sellers with respective qualities $q_1 > q_2 > \ldots > q_n$. For simplicity, we assume the qualities are all different, even if only by a small $\epsilon > 0$. Thus, no two sellers are identical. This is true, as sellers hardly ever have identical feedback score, number of reviews, comments, etc. We will characterize the best response of each seller and show that a unique Nash equilibrium always exists. In equilibrium, all sellers sell a non-negative quantity.

Proposition 8 **(Condition for all sellers to sell)**. *Sellers* $1,2,...,n$ *with quality ratings* $q_1 > q_2 > \ldots > q_n$ *will all sell a positive quantity if and only if* $1 > \frac{p_1-p_2}{q_1-q_2} > \frac{p_2-p_3}{q_2-q_3} > \ldots > \frac{p_{n-1}-p_n}{q_{n-1}-q_n} > 0$ *and* $\frac{p_n}{q_n} < \frac{p_{n-1}}{q_{n-1}} < \ldots < \frac{p_1}{q_1}$.

For every $i = 2, n-1$ define

$$f_j(i) = \begin{cases} \frac{p_n q_i}{q_n}, & j = n; \\ p_j + \frac{(p_j - p_{j+1})(q_i - q_j)}{q_j - q_{j+1}}, & j > i, j+1 \leq n; \\ p_j - \frac{(p_{j-1} - p_j)(q_j - q_i)}{(q_{j-1} - q_j)}, & j < i, j-1 \geq 1; \\ p_1 - q_1 + q_i, & j = 1; \\ \frac{p_{i-1}(q_i - q_{i+1}) + p_{i+1}(q_{i-1} - q_i)}{q_{i-1} - q_{i+1}}, & j = i. \end{cases} \tag{15}$$

Also,

$$f_j(1) = \begin{cases} \frac{p_n q_1}{q_n}, & j = n; \\ p_j + \frac{(p_j - p_{j+1})(q_1 - q_j)}{q_j - q_{j+1}}, & j > i, j+1 \leq n; \\ p_2 - q_1 + q_2, & j = 1; \end{cases} \tag{16}$$

and

$$f_j(n) = \begin{cases} \frac{p_{n-1} q_n}{q_{n-1}}, & j = n; \\ p_j - \frac{(p_{j-1} - p_j)(q_j - q_i)}{(q_{j-1} - q_j)}, & j < i, j-1 \geq 1; \\ p_1 - q_1 + q_n, & j = 1. \end{cases} \tag{17}$$

We sort the elements of $f(i) = \{f_1(i), \ldots, f_n(i)\}$ from smallest to largest. Let $f_{(l)}(i)$ be the l-th largest element of $f(i)$, $l = 1, n$, and let $in\{f_{(l)}(i)\}$ be the index corresponding to this element in the array $\{f_1(i), \ldots, f_n(i)\}$. We have $f_i(i)$ is the largest element of $f(i)$. Also, $f_{j+1}(i) < f_j(i)$ for $j > i$, and $f_{j-1}(i) < f_j(i)$ for $j < i$.

Proposition 9. *Let* p_{-i} *be such that all sellers* $j \in \{1, \ldots, i-1, i+2, \ldots, n\}$ *are 'active' (i.e., satisfy conditions of Proposition 8). Then* $f_j(i)$ *is the highest price of Seller* i *at which Seller* j *sells zero units. That is, if* $p_i \leq f_j(i)$*, then* $D_j(p_i, p_{-i}) = 0$.

Then for $i = 2, n-1$, we can express $D_i(p_i, p_{-i})$ as follows:

$$D_i(p_i, p_{-i}) = \begin{cases} 0, & p_i \geq f_i(i); \\ F\left(\frac{p_{i-1} - p_i}{q_{i-1} - q_i}\right) - F\left(\frac{p_i - p_{i+1}}{q_i - q_{i+1}}\right), & \max\{f_{i-1}(i), f_{i+1}(i)\} < p_i < f_i(i); \\ F\left(\frac{p_{i-k} - p_i}{q_{i-k} - q_i}\right) - F\left(\frac{p_i - p_{i+h}}{q_i - q_{i+h}}\right), & f_{i-k}(i) < p_i < f_{i-k+1}(i), f_{i+h}(i) < p_i < f_{i+h-1}(i); \\ F^c\left(\frac{p_i}{q_i}\right), & p_i < \min\{f_1(i), f_n(i)\}. \end{cases} \tag{18}$$

Also,

$$D_1(p_1, p_{-1}) = \begin{cases} 0, & p_1 \geq f_1(1); \\ F^c\left(\frac{p_1 - p_j}{q_1 - q_j}\right), & f_j(1) < p_1 < f_{j+1}(1); \\ F^c\left(\frac{p_1}{q_1}\right), & p_1 < f_n(1); \end{cases} \tag{19}$$

and

$$D_n(p_n, p_{-n}) = \begin{cases} 0, & p_n \geq f_n(n); \\ F\left(\frac{p_{n-k}-p_n}{q_{n-k}-q_n}\right) - F\left(\frac{p_n}{q_n}\right), & f_{n-k}(n) < p_i < f_{n-k+1}(n); \\ F^c\left(\frac{p_n}{q_n}\right), & p_n < f_1(n). \end{cases} \tag{20}$$

To compute the best response of each seller, we must calculate the optimal price in each interval $[f_{(1)}(i), f_{(2)}(i)], ..., [f_{(n-1)}(i), f_i(i)], [f_i(i), 1]$ and then find the global maximum. Thus, if p_i^k is the optimal price in interval $[f_{(k)}(i), f_{(k+1)}(i)]$, then the global maximum is $\arg\max_{p_i \in \{p_i^1, ..., p_i^n\}} \{p_i D_i(p_i, p_{-i})\}$.

From this, we can characterize the Nash equilibrium of the oligopoly game.

Proposition 10. *The Nash equilibrium of the n-oligopoly competition always exist. In equilibrium, all n sellers earn a non-zero revenue. The equilibrium prices satisfy the following system of equations:*

$$p_1 = (q_1 - q_2)\frac{F^c\left(\frac{p_1-p_2}{q_1-q_2}\right)}{f\left(\frac{p_1-p_2}{q_1-q_2}\right)},\ p_2 = \frac{F\left(\frac{p_1-p_2}{q_1-q_2}\right) - F\left(\frac{p_2-p_3}{q_2-q_3}\right)}{\frac{1}{q_1-q_2} f\left(\frac{p_1-p_2}{q_1-q_2}\right) + \frac{1}{q_2-q_3} f\left(\frac{p_2-p_3}{q_2-q_3}\right)}, \ldots, p_n = \frac{F\left(\frac{p_{n-1}-p_n}{q_{n-1}-q_n}\right) - F\left(\frac{p_n}{q_n}\right)}{\frac{1}{q_{n-1}-q_n} f\left(\frac{p_{n-1}-p_n}{q_{n-1}-q_n}\right) + \frac{1}{q_n} f\left(\frac{p_n}{q_n}\right)}.$$

The proof is based on similar arguments as in the two-seller case. The structure of the equilibrium as well as the demand functions show that, at any point in time, seller i's price depends only on the price of at most two 'active' competitors (the one slightly below and the one slightly above in terms of quality).

7. Appendix B - Proofs

Proof of Proposition 1: First we prove the following two lemmas.

Lemma 2. *If f is log-concave with support on $[0,1]$, then $H(x|\alpha,\beta) = x - \alpha\frac{F^c(\beta x)}{f(\beta x)}$ is a strictly increasing function of x for any $\alpha, \beta > 0$. Moreover, there exists a unique $\bar{x}$ such that $H(\bar{x}|\alpha,\beta) = 0$.*

Proof: If f is log-concave, then f has increasing failure rate (IFR) (see, e.g., Bagnoli and Bergstrom 2005). That implies $\frac{F^c(x)}{f(x)}$ is decreasing in x and thus $-\alpha\frac{F^c(\beta x)}{f(\beta x)}$ is increasing in x. Thus $H(x|\alpha,\beta)$ is *strictly* increasing in x, since it is the sum of two increasing functions of x, one of which is strictly increasing in x. Also, $H(0|\alpha,\beta) < 0$ and $H\left(\frac{x}{\beta}|\alpha,\beta\right) > 0$. Thus, there exists a unique $\bar{x}$ such that $H(\bar{x}|\alpha,\beta) = 0$. Moreover, for $x < \bar{x}$, $H(\bar{x}|\alpha,\beta) < 0$ and for $x > \bar{x}$, $H(\bar{x}|\alpha,\beta) > 0$. □

Lemma 3. *If f is log-concave on $[0,1]$, $\alpha \geq 0$, $\beta \geq 0$ and $\gamma \geq 0$, then the following hold:*

(i) $f'(x)F(x) - f^2(x) \leq 0$ *for all* $x \in [0,1]$.

(ii) $R(x,y|\beta,\gamma) = \frac{F(\gamma(x-y)) - F(\beta y)}{[\gamma f(\gamma(x-y)) + \beta f(\beta y)]}$ *is increasing in x for all $x > (\beta/\gamma+1)y$ and decreasing in y for all $y < \frac{x}{\beta/\gamma+1}$.*

(iii) *For all $x, y \in [0,1]$ such that $x > (\beta/\gamma+1)y$,*

$$f(\gamma(x-y))[\gamma f(\gamma(x-y)) + \beta f(\beta y)] + \gamma f'(\gamma(x-y))F^c(\gamma(x-y)) - \gamma f'(\gamma(x-y))F^c(\beta y) \geq 0. \quad (*)$$

$$-\beta f(\beta y)[\gamma f(\gamma(x-y))+\beta f(\beta y)]+\beta^2 f^{'}(\beta y)F^c(\gamma(x-y))-\beta^2 f^{'}(\beta y)F^c(\beta y)\leq 0. \quad (**)$$

(iv) $J(x|\alpha,\beta,\gamma)=x-\frac{F(\alpha-\gamma x)-F(\beta x)}{[\gamma f(\alpha-\gamma x)+\beta f(\beta x)]}$ *is a strictly increasing function of* x *for any* $\alpha,\beta,\gamma>0$. *Moreover, there exists a unique* $\bar{x}\in\left[0,\frac{\alpha}{\beta+\gamma}\right]$ *such that* $J(\bar{x}|\alpha,\beta,\gamma)=0$.

Proof:

(i) If f is log-concave, then F is log-concave (see Bagnoli and Bergstrom 2005), thus $\frac{f(x)}{F(x)}$ is decreasing on $[0,1]$. That means $f^{'}(x)F(x)-f^2(x)\leq 0$ for all $x\in[0,1]$.

(ii) To show that $R(x,y|\beta,\gamma)$ is increasing in x for all $x>(\beta/\gamma+1)y$, we will show that $R_x(x,y|\beta,\gamma)\geq 0$ for all $x>(\beta/\gamma+1)y$.

$$R_x(x,y|\beta,\gamma)=\gamma\frac{f(\gamma(x-y))[\gamma f(\gamma(x-y))+\beta f(\beta y)]-\gamma f^{'}(\gamma(x-y))[F(\gamma(x-y))-F(\beta y)]}{[\gamma f(\gamma(x-y))+\beta f(\beta y)]^2}. \quad (21)$$

If $f^{'}(\gamma(x-y))\leq 0$, then it is obvious that $R_x(x,y|\beta,\gamma)\geq 0$ for all $(\beta/\gamma+1)y<x$. If $f^{'}(\gamma(x-y))\geq 0$, then the numerator can be re-written as:

$$f(\gamma(x-y))[\gamma f(\gamma(x-y))+\beta f(\beta y)]-\gamma f^{'}(\gamma(x-y))F(\gamma(x-y))+\gamma f^{'}(\gamma(x-y))F(\beta y). \quad (22)$$

But $f^2(\gamma(x-y))-f^{'}(\gamma(x-y))F(\gamma(x-y))\geq 0$ from part (i); $f(\gamma(x-y))f(\beta y)\geq 0, f^{'}(\gamma(x-y))F(\beta y)\geq 0$, so the numerator is always non-negative. To show that $R(x,y|\beta,\gamma)$ is decreasing in y for all $y<\frac{x}{\beta/\gamma+1}$, we will show that $R_y(x,y|\beta,\gamma)\leq 0$ for all $y<\frac{x}{\beta/\gamma+1}$.

$$\begin{aligned} R_y(x,y|\beta,\gamma) &= -\frac{[\gamma f(\gamma(x-y))+\beta f(\beta y)][\gamma f(\gamma(x-y))+\beta f(\beta y)]}{[\gamma f(\gamma(x-y))+\beta f(\beta y)]^2} \\ &+ \frac{[\gamma^2 f^{'}(\gamma(x-y))-\beta^2 f^{'}(\beta y)][F(\gamma(x-y))-F(\beta y)]}{[\gamma f(\gamma(x-y))+\beta f(\beta y)]^2}. \end{aligned} \quad (23)$$

From (22) we know that

$$-\gamma f(\gamma(x-y))(\gamma f(\gamma(x-y))+\beta f(\beta y))+\gamma^2 f^{'}(\gamma(x-y))[F(\gamma(x-y))-F(\beta y)]\leq 0.$$

We need to show that

$$-\beta f(\beta y)[\gamma f(\gamma(x-y))+\beta f(\beta y)]-\beta^2 f^{'}(\beta y)[F(\gamma(x-y))-F(\beta y)]\leq 0. \quad (24)$$

If $f^{'}(\beta y)\geq 0$, then it is obvious that $R_y(x,y|\beta,\gamma)\leq 0$ for all $(\beta/\gamma+1)y<x$. If $f^{'}(\beta y)\leq 0$, then the numerator satisfies the following inequality,

$$\begin{aligned} &-\beta f(\beta y)[\gamma f(\gamma(x-y))+\beta f(\beta y)]-\beta^2 f^{'}(\beta y)[F(\gamma(x-y))-F(\beta y)]\leq \\ &\leq -\beta f(\beta y)[\gamma f(\gamma(x-y))+\beta f(\beta y)]-\beta^2 f^{'}(\beta y)F^c(\beta y)\leq 0, \end{aligned}$$

where the second inequality follows from the log-concavity of f, which implies log-concavity of F^c and hence $f^2(x)+f^{'}(x)F^c(x)\geq 0$ for all $x\in[0,1]$. So the numerator is always negative.

(iii) Note that the numerator in (21) can be written as $f(\gamma(x-y))[\gamma f(\gamma(x-y)) + \beta f(\beta y)] + \gamma f'(\gamma(x-y))F^c(\gamma(x-y)) - \gamma f'(\gamma(x-y))F^c(\beta y)$ and we showed in (ii) that it is always positive. Also note that the expression in (24) can be written as $-\beta f(\beta y)(\gamma f(\gamma(x-y)) + \beta f(\beta y)) + \beta^2 f'(\beta y)F^c(\gamma(x-y)) - \beta^2 f'(\beta y)F^c(\beta y)$ and we showed in (ii) that it is always negative.

(iv) We show first that $K(x|\alpha,\beta,\gamma) = \frac{F(\alpha-\gamma x)-F(\beta x)}{[\gamma f(\alpha-\gamma x)+\beta f(\beta x)]}$ is decreasing in x.

$$K_x(x|\alpha,\beta,\gamma) = \frac{-[\gamma f(\alpha-\gamma x) + \beta f(\beta x)]^2 + [\gamma^2 f'(\alpha-\gamma x) - \beta^2 f'(\beta x)][F(\alpha-\gamma x) - F(\beta x)]}{[\gamma f(\alpha-\gamma x) + \beta f(\beta x)]^2}.$$

We are interested in the sign of the numerator:

$$-\gamma^2 f^2(\alpha-\gamma x) - \gamma^2 f'(\alpha-\gamma x)F^c(\alpha-\gamma x) - \beta^2 f^2(\beta x) - \beta^2 f'(\beta x)F^c(\beta x) - \\ -2\gamma\beta f(\alpha-\gamma x)f(\beta x) + \gamma^2 f'(\alpha-\gamma x)F^c(\beta x) + \beta^2 f'(\beta x)F^c(\alpha-\gamma x).$$

We have

$$-\gamma^2 f^2(\alpha-\gamma x) - \gamma^2 f'(\alpha-\gamma x)F^c(\alpha-\gamma x) - \gamma\beta f(\alpha-\gamma x)f(\beta x) + \gamma^2 f'(\alpha-\gamma x)F^c(\beta x) \le 0, \\ -\beta^2 f^2(\beta x) - \beta^2 f'(\beta x)F^c(\beta x) - \gamma\beta f(\alpha-\gamma x)f(\beta x) + \beta^2 f'(\beta x)F^c(\alpha-\gamma x) \le 0.$$

The first inequality follows from part (iii), inequality (*). The second inequality follows from part (iii), inequality (**). Thus, $K(x|\alpha,\beta,\gamma)$ is decreasing in x. Then $J(x|\alpha,\beta,\gamma)$ is *strictly* increasing in x, since it is the sum of two increasing functions of x, one of which is strictly increasing in x. Also, $J(0|\alpha,\beta,\gamma) < 0$ and $J\left(\frac{\alpha}{\beta+\gamma}|\alpha,\beta,\gamma\right) > 0$. Thus, there exists a unique $\bar{x} \in \left[0, \frac{\alpha}{\beta+\gamma}\right]$ such that $J(\bar{x}|\alpha,\beta,\gamma) = 0$.

□

Now back to proposition 1.

(i) We have

$$\pi_1(p_1,p_2) = p_1 D_1(p_1,p_2) = \begin{cases} Np_1F^c\left(\frac{p_1}{q_1}\right), & p_1 \le p_2q_1/q_2, \\ Np_1F^c\left(\frac{p_1-p_2}{q_1-q_2}\right), & p_2q_1/q_2 \le p_1 < p_2 + q_1 - q_2, \\ 0, & p_1 \ge p_2 + q_1 - q_2. \end{cases} \tag{25}$$

The first order condition (FOC) for $\max_{p_1\in[0,q_1]} p_1F^c\left(\frac{p_1-p_2}{q_1-q_2}\right)$ is:

$$F^c\left(\frac{p_1-p_2}{q_1-q_2}\right) - \frac{p_1}{q_1-q_2} f\left(\frac{p_1-p_2}{q_1-q_2}\right) = 0. \tag{26}$$

Recall that $\bar{p}_2$ was defined in (4) as the unique solution to the equation:

$$\frac{\bar{p}_2}{q_2} - \frac{q_1-q_2}{q_1} \frac{F^c\left(\frac{\bar{p}_2}{q_2}\right)}{f\left(\frac{\bar{p}_2}{q_2}\right)} = 0. \tag{27}$$

From (26) and Lemma 2, we get that if $p_2 \ge \bar{p}_2$ then $p_1F^c\left(\frac{p_1-p_2}{q_1-q_2}\right)$ is decreasing in p_1 on $[p_2q_1/q_2, q_1]$, so it is optimal for Seller 1 to price in such a way that Seller 2 does not sell (i.e.,

$p_1 \le p_2 q_1/q_2$). He would set the price to maximize the monopolist profit with the constraint that $p_1 \le p_2 q_1/q_2$. If $p_1^m < p_2 q_1/q_2$, then the best response for seller 1 is p_1^m. Else, it is $p_2 q_1/q_2$.

If $p_2 < \bar{p}_2$ then $p_1 F^c\left(\frac{p_1-p_2}{q_1-q_2}\right)$ has a unique maximizer $p_1^* \in (p_2 q_1/q_2; q_1)$ which solves (26).

(ii) We have

$$\pi_2(p_1,p_2) = p_2 D_2(p_1,p_2) = \begin{cases} Np_2 F^c\left(\frac{p_2}{q_2}\right), & p_2 \le p_1 - q_1 + q_2, \\ Np_2\left[F\left(\frac{p_1-p_2}{q_1-q_2}\right) - F\left(\frac{p_2}{q_2}\right)\right], & p_1 - q_1 + q_2 \le p_2 < p_1 q_2/q_1, \\ 0, & p_2 \ge p_1 q_2/q_1. \end{cases} \tag{28}$$

The first order condition (FOC) for $\max_{p_2\in[0,q_2]} p_2\left[F\left(\frac{p_1-p_2}{q_1-q_2}\right) - F\left(\frac{p_2}{q_2}\right)\right]$ is:

$$F\left(\frac{p_1-p_2}{q_1-q_2}\right) - F\left(\frac{p_2}{q_2}\right) - \frac{p_2}{q_1-q_2} f\left(\frac{p_1-p_2}{q_1-q_2}\right) - \frac{p_2}{q_2} f\left(\frac{p_2}{q_2}\right) = 0. \tag{29}$$

Recall that $\bar{p}_1$ was defined in (4) as the unique solution to the equation:

$$\bar{p}_1 - q_1 - q_2 + \frac{1 - F\left(1 - \frac{q_1-\bar{p}_1}{q_2}\right)}{\frac{f(1)}{q_1-q_2} + \frac{f\left(1-\frac{q_1-\bar{p}_1}{q_2}\right)}{q_2}} = 0. \tag{30}$$

From (29) and Lemma 3, we get that if $p_1 \ge \bar{p}_1$ then $p_2\left[F\left(\frac{p_1-p_2}{q_1-q_2}\right) - F\left(\frac{p_2}{q_2}\right)\right]$ is decreasing in p_2 on $[p_1 - q_1 + q_2, q_2]$, so it is optimal for Seller 2 to price in such a way that Seller 1 does not sell (i.e., $p_2 \le p_1 - q_1 + q_2$). He would set the price to maximize the monopolist profit under the constraint that $p_2 \le p_1 - q_1 + q_2$. If $p_2^m < p_1 - q_1 + q_2$, then the best response for seller 1 is p_2^m. Else, it is $p_1 - q_1 + q_2$.

If $p_1 < \bar{p}_1$ then $p_2\left[F\left(\frac{p_1-p_2}{q_1-q_2}\right) - F\left(\frac{p_2}{q_2}\right)\right]$ has a unique maximizer $p_2^* \in (p_1 - q_1 + q_2; q_2)$, which solves (29).

(iii) To show that the $B_1(p_2)$ is increasing in p_2 we need to show that if $p_2 < \bar{p}_2$ then the unique solution to (26) is increasing in p_2. Using the implicit function theorem we get that:

$$\frac{\partial p_1^*}{\partial p_2} = g(p_1,p_2) = \frac{\frac{p_1}{q_1-q_2} f'\left(\frac{p_1-p_2}{q_1-q_2}\right) + f\left(\frac{p_1-p_2}{q_1-q_2}\right)}{\frac{p_1}{q_1-q_2} f'\left(\frac{p_1-p_2}{q_1-q_2}\right) + 2f\left(\frac{p_1-p_2}{q_1-q_2}\right)} = 1 - \frac{1}{\frac{p_1}{q_1-q_2} f'\left(\frac{p_1-p_2}{q_1-q_2}\right)/f\left(\frac{p_1-p_2}{q_1-q_2}\right) + 2}. \tag{31}$$

By the IFR property implied by the log-normality of f we have $-f'(x)F^c(x) \le f^2(x)$. Thus, $-\frac{p_1}{q_1-q_2} f'\left(\frac{p_1-p_2}{q_1-q_2}\right) \le \frac{p_1}{q_1-q_2} \frac{f^2\left(\frac{p_1-p_2}{q_1-q_2}\right)}{F^c\left(\frac{p_1-p_2}{q_1-q_2}\right)}$. But from (26) we have $\frac{p_1}{q_1-q_2} \frac{f\left(\frac{p_1-p_2}{q_1-q_2}\right)}{F^c\left(\frac{p_1-p_2}{q_1-q_2}\right)} = 1$, therefore $\frac{p_1}{q_1-q_2} f'\left(\frac{p_1-p_2}{q_1-q_2}\right) + f\left(\frac{p_1-p_2}{q_1-q_2}\right) \ge 0$. Thus, $\frac{p_1}{q_1-q_2} f'\left(\frac{p_1-p_2}{q_1-q_2}\right)/f\left(\frac{p_1-p_2}{q_1-q_2}\right) + 2 \ge 1$, therefore $0 \le \frac{\partial p_1^*}{\partial p_2} < 1$. ■

To show that the $B_2(p_1)$ is increasing in p_1 we need to show that if $p_1 < \bar{p}_1$ then the unique solution to 6 is increasing in p_1. Using the implicit function theorem we get that:

$$\frac{\partial p_2^*}{\partial p_1} = \frac{\frac{1}{q_1-q_2} f\left(\frac{p_1-p_2^*}{q_1-q_2}\right) - \frac{p_2^*}{(q_1-q_2)^2} f'\left(\frac{p_1-p_2^*}{q_1-q_2}\right)}{\frac{2}{q_1-q_2} f\left(\frac{p_1-p_2^*}{q_1-q_2}\right) + \frac{2}{q_2} f\left(\frac{p_2^*}{q_2}\right) - \frac{p_2^*}{(q_1-q_2)^2} f'\left(\frac{p_1-p_2^*}{q_1-q_2}\right) + \frac{p_2^*}{q_2^2} f'\left(\frac{p_2^*}{q_2}\right)} = \frac{R_{p_1}\left(p_1, p_2^* | \frac{1}{q_2}, \frac{1}{q_1-q_2}\right)}{1 - R_{p_2^*}\left(p_1, p_2^* | \frac{1}{q_2}, \frac{1}{q_1-q_2}\right)}. \tag{32}$$

From Lemma 3, $R_x(x,y|\beta,\gamma) \ge 0$ and $R_y(x,y|\beta,\gamma) \le 0$ for all $(\beta/\gamma + 1)y < x$. Thus $\frac{\partial p_2^*}{\partial p_1} \ge 0$. Moreover, we can show that $\frac{\partial p_2^*}{\partial p_1} \le 1$. to see this, we must show that

$$R_{p_1}\left(p_1,p_2|\frac{1}{q_2},\frac{1}{q_1-q_2}\right)+R_{p_2}\left(p_1,p_2|\frac{1}{q_2},\frac{1}{q_1-q_2}\right)\leq 1,$$

or equivalently

$$-\frac{1}{q_2}f\left(\frac{p_2}{q_2}\right)\left[\frac{1}{q_1-q_2}f\left(\frac{p_1-p_2}{q_1-q_2}\right)+\frac{1}{q_2}f\left(\frac{p_2}{q_2}\right)\right]-\frac{1}{q_2^2}f'\left(\frac{p_2}{q_2}\right)\left[F\left(\frac{p_1-p_2}{q_1-q_2}\right)+F\left(\frac{p_2}{q_2}\right)\right]$$
$$\leq\left[\frac{1}{q_1-q_2}f\left(\frac{p_1-p_2}{q_1-q_2}\right)+\frac{1}{q_2}f\left(\frac{p_2}{q_2}\right)\right]^2. \tag{33}$$

But the left hand side in (33) is negative, so the inequality follows. ■

PROOF OF PROPOSITION 2: First we show that in the equilibrium both sellers sell a positive quantity. Assume that $(p_1^*,p_2^*)\in(0,q_1)\times(0,q_2)$ is a NE such that $D_1(p_1^*,p_2^*)=0$. Then Seller 1 , can decrease the price to $p_2^*q_1/q_2+\epsilon$ and make a positive profit. Assume now that (p_1^*,p_2^*) is a NE, such that $D_2(p_1^*,p_2^*)=0$. Then Seller 2 can decrease the price to $p_1^*q_2/q_1-\epsilon$ and make a non-negative profit.

Because in equilibrium both sellers sell a non-negative quantity, a NE must be a solution to the system of equations given in (8). To show that such a solution exists and is unique, we will show that the conditions in Theorem 1 and the dominant diagonal condition hold for our problem. We will focus on conditions C3 and the dominant diagonal condition.

- Seller 1: we check condition C3.

$$\partial^2 log\pi_1/\partial p_1\partial p_2=\frac{\frac{1}{(q_1-q_2)^2}f'\left(\frac{p_1-p_2}{q_1-q_2}\right)F^c\left(\frac{p_1-p_2}{q_1-q_2}\right)+\frac{1}{(q_1-q_2)^2}f^2\left(\frac{p_1-p_2}{q_1-q_2}\right)}{\left(F^c\left(\frac{p_1-p_2}{q_1-q_2}\right)\right)^2}\geq 0. \tag{34}$$

This follows form the log-concavity of f which implies increasing hazard rate. We check the dominant diagonal condition:

$$\partial^2 log\pi_1/(\partial p_1)^2=-\frac{1}{p_1^2}-\frac{\frac{1}{(q_1-q_2)^2}f'\left(\frac{p_1-p_2}{q_1-q_2}\right)F^c\left(\frac{p_1-p_2}{q_1-q_2}\right)+\frac{1}{(q_1-q_2)^2}f^2\left(\frac{p_1-p_2}{q_1-q_2}\right)}{\left(F^c\left(\frac{p_1-p_2}{q_1-q_2}\right)\right)^2}<0. \tag{35}$$

Moreover

$$-\partial^2 log\pi_1/(\partial p_1)^2-\partial^2 log\pi_1/\partial p_1\partial p_2=\frac{1}{p_1^2}>0. \tag{36}$$

- Seller 2: we check condition C3.

$$\partial^2 log\pi_2/\partial p_2\partial p_1=\frac{\partial\left(-\frac{\frac{1}{q_1-q_2}f\left(\frac{p_1-p_2}{q_1-q_2}\right)+\frac{1}{q_2}f\left(\frac{p_2}{q_2}\right)}{F\left(\frac{p_1-p_2}{q_1-q_2}\right)-F\left(\frac{p_2}{q_2}\right)}\right)}{\partial p_1}=\frac{\partial\left(-\frac{1}{R\left(p_1,p_2|\frac{1}{q_1-q_2},\frac{1}{q_2}\right)}\right)}{\partial p_1}\geq 0. \tag{37}$$

This follows form Lemma 3 where we showed that $R(x,y|\beta,\gamma)$ is increasing in x. We check the dominant diagonal condition:

$$\partial^2 log\pi_2/(\partial p_2)^2=-\frac{1}{p_2^2}-\frac{\partial\left(\frac{1}{R\left(p_1,p_2|\frac{1}{q_1-q_2},\frac{1}{q_2}\right)}\right)}{\partial p_2}<0. \tag{38}$$

Moreover

$$-\partial^2 log\pi_2/(\partial p_2)^2 - \partial^2 log\pi_2/\partial p_2 \partial p_1 = \frac{1}{p_2^2} - \frac{R_{p_2}\left(p_1, p_2 \mid \frac{1}{q_1-q_2}, \frac{1}{q_2}\right)}{R^2\left(p_1, p_2 \mid \frac{1}{q_1-q_2}, \frac{1}{q_2}\right)} - \frac{R_{p_1}\left(p_1, p_2 \mid \frac{1}{q_1-q_2}, \frac{1}{q_2}\right)}{R^2\left(p_1, p_2 \mid \frac{1}{q_1-q_2}, \frac{1}{q_2}\right)} \leq 0.$$

This follows from the fact that

$$R_{p_1}\left(p_1, p_2 \mid \frac{1}{q_1-q_2}, \frac{1}{q_2}\right) + R_{p_2}\left(p_1, p_2 \mid \frac{1}{q_1-q_2}, \frac{1}{q_2}\right) \leq 0. \tag{39}$$

■

PROOF OF PROPOSITION 3:

Assume the starting prices are $(p_{1,0}, p_{2,0})$. If both sellers sell a positive quantity and if Seller 2 updates first we have: $p_{2,1} = cp_{1,0}$ and $p_{1,1} = ap_{2,1} + b = acp_{1,0} + b$. Continuing with the update, we have $p_{2,2} = cp_{1,1} = ac^2 p_{1,0} + cb$ and $p_{1,2} = ap_{2,2} + b = (ac)^2 p_{1,0} + acb + b$. After n steps:

$$p_{2,n} = cp_{1,n-1}, \quad p_{1,n} = ap_{2,n} + b = (ac)^n p_{1,0} + (ac)^{n-1} b + \dots acb + b = (ac)^n p_{1,0} + \frac{(ac)^n - 1}{ac - 1} b.$$

We know that the NE satisfies the following:

$$ap_2^N + b = p_1^N, \quad cp_1^N = p_2^N,$$

which implies $b = p_1^N(1 - ac)$. We also know that $a < 1, c < 1$. Then $\lim_{n\to\infty} = p_{1,n} = p_1^N$.

If Seller 2 does not sell (i.e., $D_2(p_{1,0}, p_{2,0}) = 0$) and if Seller 2 updates first we have: $p_{2,1} = p_{1,0} - q_1 + q_2$. Then, if $p_{2,1} > p_2^s$, we have $p_{1,1} = p_{2,1}q_1/q_2$, else $p_{1,1} = ap_{2,1} + b$. If the latter, then we are back to case 1, and the two prices will eventually converge to the NE. If the former, then $p_{2,2} = p_{1,1} - q_1 + q_2 = p_{2,1}q_1/q_2 - q_1 + q_2$. Then

$$p_{2,2} - p_{2,1} = p_{2,1}(q_1 - q_2)/q_2 - (q_1 - q_2) = (q_1 - q_2)(p_{2,1} - q_2)/q_2.$$

Since at each step the price of Seller 2 goes down by at least $\frac{(q_1-q_2)(q_1-p_1^N)}{q_2}$, eventually we will find ourselves in the case where $p_{2,t} < p_2^s$. Then the argument from Case 1 applies.

The cases where Seller 1 updates first are similar.

To show that the convergence rate is linear, we must show that there exists $\lambda \geq 0$ such that:

$$\lim_{n\to\infty} \frac{|B_2(p_{1,n+1}) - p_2^N|}{|B_2(p_{1,n}) - p_2^N|} = \lambda.$$

Note that $B_2(p_{1,n+1}) = cB_1(p_{2,n+1}) = c(ap_{2,n+1} + b)$ and $B_2(p_{1,n}) = p_{2,n+1}$. Then

$$\lim_{n\to\infty} \frac{|B_2(p_{1,n+1}) - p_2^N|}{|B_2(p_{1,n}) - p_2^N|} = ca + \frac{cb - p_2^N + cap_2^N}{p_{2,n+1} - p_2^N} = ca,$$

because $cp_1^N = p_2^N$ and $ap_2^N + b = p_1^N$. For seller 1:

$$\lim_{n\to\infty} \frac{|B_1(p_{2,n+1}) - p_1^N|}{|B_1(p_{2,n}) - p_1^N|} = ca.$$

Note that $B_1(p_{2,n+1}) = aB_2(p_{1,n}) + b$ and $B_1(p_{2,n}) = p_{1,n}$. Then

$$\lim_{n\to\infty} \frac{|B_1(p_{2,n+1}) - p_1^N|}{|B_1(p_{2,n}) - p_1^N|} = ca + \frac{b - p_1^N + cap_1^N}{p_{1,n} - p_1^N} = ca,$$

because $cp_1^N = p_2^N$ and $ap_2^N + b = p_1^N$. ■

PROOF OF LEMMA 1: For a proof of this results see for instance Theorem 4.7 (Stokey et al., 1989). ■

PROOF OF PROPOSITION 4: By Lemma 3, we have that $v(p)$ is increasing in p. Then $x^*(p) = \arg\max_{x\in\Gamma}\{xD_1(x, B_2(p)) + \delta v(x)\} \geq \arg\max_{x\in\Gamma} xD_1(x, B_2(p)) = B_1(B_2(p))$.

By induction, given an initial state $p_{1,0}$ we have $p_{1,1}^* = x^*(p_{1,0}) \geq B_1(B_2(p_{1,0}))$. Suppose $p_{1,t-1}^* \geq B_{1,t-1}(p)$. Then $B_{1,t}(p) = B_1(B_2(B_{1,t-1}(p))) \leq B_1(B_2(p_{1,t-1}^*)) \leq x^*(p_{1,t-1}^*) = p_{1,t}^*$. Thus the 'myopic' repricer will always underprice. ■

PROOF OF PROPOSITION 5: We show that if the quality difference is low enough, then the optimal price path cycles. We prove by contradiction. Assume that it is not. Then the price path is monotone, in which case it converges to the stationary policy.

However, under the stationary policy, if the quality gap between the sellers is close to zero (i.e., $q_1 - q_2 \to 0$), then their respective profits will also be close to zero. Then it is obviously more profitable for the strategic seller not to sell in one period, set the price to $p_{1,t} = q_1$, and then price as a monopolist in the subsequent period (i.e., set $p_{1,t+1} = p_1^m$) and earn the monopolist profit in that period. Seller 1 will be able to do that, as $p_{2,t+1} = B_2(q_1) = p_2^m$ and $D_1(p_1^m, p_2^m) = F^c\left(\frac{p_1^m}{q_1}\right)$. Thus, the price path cannot be monotone, therefore it must be a cyclic policy. ■

PROOF OF PROPOSITION 6: We define the following restricted strategy space $\Gamma_1 = [0, \tilde{p}]$, where $\tilde{p} \leq p_2^N + q_1 - q_2$. This strategy space has the property that for every $p \in \Gamma_1$, and $p_2 \geq p_2^N$, then $D_1(p, p_2) = F^c\left(\frac{p-p_2}{q_1-q_2}\right) \geq 0$. That means both sellers sell a positive quantity. For $q_2 \to 0$ we will show that the policy is monotone on Γ_1. But as $q_2 \to 0$, $\tilde{p} \to q_1$, hence $\Gamma_1 \to \Gamma$.

Consider the value function as defined in (9). Derivating with respect to $p_{1,t}$ we get:

$$\frac{\partial \Pi_1(p_{1,0})}{\partial p_{1,t}} = \frac{\partial \sum_{t=0}^{\infty} \delta^t p_{1,t} F^c\left(\frac{p_{1,t}-B_2(p_{1,t-1})}{q_1-q_2})\right)}{\partial p_{1,t}}. \tag{40}$$

Dividing by δ^t, the FOCs for all $t > 0$ yield:

$$F^c\left(\frac{p_{1,t} - B_2(p_{1,t-1})}{q_1 - q_2}\right) - \frac{p_{1,t}}{q_1 - q_2} f\left(\frac{p_{1,t} - B_2(p_{1,t-1})}{q_1 - q_2}\right) + \frac{\delta p_{1,t+1}}{q_1 - q_2} f\left(\frac{p_{1,t+1} - B_2(p_{1,t})}{q_1 - q_2}\right) \frac{\partial B_2(p_{1,t})}{\partial p_{1,t}} = 0. \tag{41}$$

Let $\{p_{1,t}^*\}_{t=1,2,...}$ be the optimal price path that satisfies the FOCs (41) for all $t > 0$. Then by the implicit function theorem we get:

$$\frac{\partial p^*_{1,t}}{\partial p^*_{1,t-1}} = -\frac{\frac{\partial B_2(p^*_{1,t-1})}{\partial p^*_{1,t-1}}\left[f\left(\frac{p^*_{1,t}-B_2(p^*_{1,t-1})}{q_1-q_2}\right)+\frac{p^*_{1,t}}{q_1-q_2}f'\left(\frac{p^*_{1,t}-B_2(p^*_{1,t-1})}{q_1-q_2}\right)\right]}{-2f\left(\frac{p^*_{1,t}-B_2(p^*_{1,t-1})}{q_1-q_2}\right)-\frac{p^*_{1,t}}{q_1-q_2}f'\left(\frac{p^*_{1,t}-B_2(p^*_{1,t-1})}{q_1-q_2}\right)-G}, \tag{42}$$

where

$$G = -\frac{\delta p^*_{1,t+1}}{(q_1-q_2)^2}f'\left(\frac{p^*_{1,t+1}-B_2(p^*_{1,t})}{q_1-q_2}\right)\left(\frac{\partial B_2(p^*_{1,t})}{\partial p^*_{1,t}}\right)^2. \tag{43}$$

At the limit, if the sellers are highly differentiated (i.e., $q_2 \to 0$), then we can see from (32) that $\frac{\partial B_2(p)}{\partial p} \to 0$. Then $G \to 0$.

To see that $f\left(\frac{p^*_{1,t}-B_2(p^*_{1,t-1})}{q_1-q_2}\right)+\frac{p^*_{1,t}}{q_1-q_2}f'\left(\frac{p^*_{1,t}-B_2(p^*_{1,t-1})}{q_1-q_2}\right) \geq 0$ note that from the FOC equation (41) we get:

$$\frac{p^*_{1,t}}{q_1-q_2} = \frac{F^c\left(\frac{p^*_{1,t}-B_2(p^*_{1,t-1})}{q_1-q_2}\right)+\frac{\delta p^*_{1,t+1}}{q_1-q_2}f\left(\frac{p^*_{1,t+1}-B_2(p^*_{1,t})}{q_1-q_2}\right)\frac{\partial B_2(p^*_{1,t})}{\partial p^*_{1,t}}}{f\left(\frac{p^*_{1,t}-B_2(p^*_{1,t-1})}{q_1-q_2}\right)}. \tag{44}$$

Thus $sign\left\{f\left(\frac{p^*_{1,t}-B_2(p^*_{1,t-1})}{q_1-q_2}\right)+\frac{p^*_{1,t}}{q_1-q_2}f'\left(\frac{p^*_{1,t}-B_2(p^*_{1,t-1})}{q_1-q_2}\right)\right\}$ is given by the sign of the following expression:

$$f^2\left(\frac{p^*_{1,t}-B_2(p^*_{1,t-1})}{q_1-q_2}\right)+\left[F^c\left(\frac{p^*_{1,t}-B_2(p^*_{1,t-1})}{q_1-q_2}\right)+\frac{\delta p^*_{1,t+1}}{q_1-q_2}f\left(\frac{p^*_{1,t+1}-B_2(p^*_{1,t})}{q_1-q_2}\right)\frac{\partial B_2(p^*_{1,t})}{\partial p^*_{1,t}}\right]f'\left(\frac{p^*_{1,t}-B_2(p^*_{1,t-1})}{q_1-q_2}\right).$$

But $\frac{\partial B_2(p)}{\partial p} \to 0$ and from the log-concavity of F^c we have $f^2(x)+F^c(x)f'(x) \geq 0$ for all x. Therefore the expression above is positive and hence $\frac{\partial p^*_{1,t}}{\partial p^*_{1,t-1}} \geq 0$. This implies monotonicity of the price path $\{p^*_{1,t}\}_{t=1,2,\ldots}$ (we have $p^*_{1,t} = x^*(p^*_{1,t-1})$ and $x^*(p)$ is increasing in p, then we cannot have $p^*_{1,t-1} < p^*_{1,t} > p^*_{1,t+1}$ or $p^*_{1,t-1} > p^*_{1,t} < p^*_{1,t+1}$). But the price path is bounded on Γ_1, therefore it must converge to the stationary policy. ■

PROOF OF PROPOSITION 7:

To show this we will prove first that for every $p^*_{1,t}$, we have $p^*_{1,t} \leq \frac{\sum_{i=1}^{t-1} p^*_{1,i}}{t-1}$.

Assume $p^*_{1,t} > \frac{\sum_{i=1}^{t-1} p^*_{1,i}}{t-1}$ for some t. Let t be the fist such index that satisfies the inequality. Then we can increase the profit by interchanging $p^*_{1,t}$ with $p^*_{1,t-1}$. To see this, note that the profit before the interchange is given by:

$$\sum_{s=0}^{t}\delta^s p^*_{1,s}D_1(p^*_{1,s}, p^*_{2,s}(p_{1,0},\ldots,p^*_{1,s-1})) + \delta^{t+1}\Pi(p_{1,0},\ldots,p^*_{1,t}). \tag{45}$$

After the interchange, the profit will be:

$$\sum_{s=0}^{t-2}\delta^s p^*_{1,s}D_1(p^*_{1,s}, p^*_{2,s}(p_{1,0},\ldots,p^*_{1,s-1})) + \delta^{t-1}p^*_{1,t}D_1(p^*_{1,t}, p^*_{2,t-1}(p_{1,0},\ldots,p^*_{1,t-2})) +$$
$$+\ \delta^t p^*_{1,t-1}D_1(p^*_{1,t-1}, p^*_{2,t}(p_{1,0},\ldots,p^*_{1,t-2},p^*_{1,t})) + \delta^{t+1}\Pi(p_{1,0},\ldots,p^*_{1,t}). \tag{46}$$

The difference is:

$$\delta^{t-1}\left[p^*_{1,t}D_1(p^*_{1,t},p^*_{2,t-1}(p_{1,0},\ldots,p^*_{1,t-2}))+\delta p^*_{1,t-1}D_1(p^*_{1,t-1},p^*_{2,t}(p_{1,0},\ldots,p^*_{1,t-2},p^*_{1,t}))-\right.$$
$$\left.-\;p^*_{1,t-1}D_1(p^*_{1,t-1},p^*_{2,t-1}(p_{1,0},\ldots,p^*_{1,t-2}))-\delta p^*_{1,t}D_1(p^*_{1,t},p^*_{2,t}(p_{1,0},\ldots,p^*_{1,t-1}))\right]. \tag{47}$$

We have $\frac{\sum_{i=1}^{t-1}p^*_{1,i}}{t-1}\leq\frac{\sum_{i=1}^{t-2}p^*_{1,i}}{t-2}<\frac{\sum_{i=1}^{t-2}p^*_{1,i}+p^*_{1,t}}{t-1}$. So,

$$p^*_{1,t}D_1(p^*_{1,t},p^*_{2,t-1}(p_{1,0},\ldots,p^*_{1,t-2}))\geq p^*_{1,t}D_1(p^*_{1,t},p^*_{2,t-1}(p_{1,0},\ldots,p^*_{1,t-1})),$$

$$p^*_{1,t-1}D_1(p^*_{1,t-1},p^*_{2,t}(p_{1,0},\ldots,p^*_{1,t-2},p^*_{1,t}))>p^*_{1,t-1}D_1(p^*_{1,t-1},p^*_{2,t-1}(p_{1,0},\ldots,p^*_{1,t-2})).$$

If $\delta\to 1$, then (47) is positive, so by interchanging $p^*_{1,t}$ with $p^*_{1,t-1}$ we can increase the profit, which contradicts the optimality of the price path $\{p^*_{1,t}\}$.

So, for every $p^*_{1,t}$, we have $p^*_{1,t}\leq\frac{\sum_{i=1}^{t-1}p^*_{1,i}}{t-1}$. This, in turn implies $\frac{\sum_{i=1}^{t}p^*_{1,i}}{t}\leq\frac{\sum_{i=1}^{t-1}p^*_{1,i}}{t-1}$, which means that the average price is decreasing over time. Because it is bounded, it means that the sequence $\left\{\frac{\sum_{i=1}^{t}p^*_{1,i}}{t}\right\}_{t=1,\infty}$ converges to a stationary point. From (12) this optimal stationary point is p^s_1.

We still have to show that the sequence $\{p^*_{i,t}\}_{t=1,2,\ldots}$ converge. Assume it does not. This means that $\exists\delta>0$ such that $\forall T>0$, $\exists t_1,t_2\geq T$ such that $p^*_{1,t_1}-p^*_{1,t_2}\geq\delta$. Now, fix $0<\epsilon<\delta/2$ and take T sufficiently large so that $\left|\frac{\sum_{i=1}^{t}p^*_{1,i}}{t}-p^s_1\right|<\epsilon$ for all $t\geq T$. Let $t_1,t_2>T$ be such that $p^*_{1,t_1}-p^*_{1,t_2}\geq p^*_{1,t_3}-p^*_{1,t_3}$ for all $t_3,t_4>T$. If both $p^*_{1,t_1},p^*_{1,t_2}<p^s_1$, then $\lim_{t\to\infty}\frac{\sum_{i=1}^{t}p^*_{1,i}}{t}<p^s_1$. So, $p^*_{1,t_1}>p^s_1+\epsilon>\frac{\sum_{i=1}^{t_1}p^*_{1,i}}{t_1}$ and $p_{1,t_2}<p^s_1+\epsilon$. But this contradicts the fact that for every $p^*_{1,t}$, we have $p^*_{1,t}\leq\frac{\sum_{i=1}^{t-1}p^*_{1,i}}{t-1}$. So, the sequence $\{p^*_{i,t}\}$ must also converge to p^s_1. ■

PROOF OF PROPOSITION 8: We first prove the "⇒" implication. If seller i sells, it means that there exist some customers which strictly prefer seller i to all sellers. Let $j<i$ and $k>i$. A customer with willingness to pay for quality w strictly prefers seller i to seller j if $wq_i-p_i>wq_j-p_j$, which is equivalent to $w<\frac{p_j-p_i}{q_j-q_i}$. Similarly, a customer with willingness to pay for quality w strictly prefers seller i to seller k if $wq_i-p_i>wq_k-p_k$, which is equivalent to $w>\frac{p_i-p_k}{q_i-q_k}$. Then, a customer which strictly prefer seller i to all sellers will have a willingness to pay which satisfies $\max_{k>i}\left\{\frac{p_i-p_k}{q_i-q_k}\right\}<w<\min_{j<i}\left\{\frac{p_j-p_i}{q_j-q_i}\right\}$. If there exists $i,i+1,i+2$ such that $\frac{p_i-p_{i+1}}{q_i-q_{i+1}}<\frac{p_{i+1}-p_{i+2}}{q_{i+1}-q_{i+2}}$ then it is easy to see that seller $i+1$ will not sell. Thus, if all sellers sell, then we must have $\frac{p_1-p_2}{q_1-q_2}>\frac{p_2-p_3}{q_2-q_3}>\ldots>\frac{p_{n-1}-p_n}{q_{n-1}-q_n}$. In addition, a customer will purchase from seller i if his individual rationality constraint is satisfied, that is if $wq_i-p_i>0$ or equivalently $w>\frac{p_i}{q_i}$. Thus, $\frac{p_i}{q_i}<w<\min_{j<i}\left\{\frac{p_j-p_i}{q_j-q_i}\right\}$, which is equivalent to $\frac{p_i}{q_i}\leq\frac{p_j}{q_j}$ for all $j<i$. This holds for all i, so if all sellers sell it must be that $\frac{p_n}{q_n}<\frac{p_{n-1}}{q_{n-1}}<\ldots<\frac{p_1}{q_1}$.

We now prove the "⇐" implication: if $\frac{p_1-p_2}{q_1-q_2}>\frac{p_2-p_3}{q_2-q_3}>\ldots>\frac{p_{n-1}-p_n}{q_{n-1}-q_n}$ and $\frac{p_n}{q_n}<\frac{p_{n-1}}{q_{n-1}}<\ldots<\frac{p_1}{q_1}$, then all sellers sell a positive quantity. All we have to show is that for each $i=1,\ldots,n$ there exists a segment of customers which prefer seller i to all sellers and for which the individual rationality constraint is satisfied. It is easy to see that customers with willingness to pay $w\in\left[\frac{p_i-p_{i+1}}{q_i-q_{i+1}},\frac{p_{i-1}-p_i}{q_{i-1}-q_i}\right]$ have this property. ■

PROOF OF PROPOSITION 9: For simplicity of exposition (and the multitude of cases that need to be analyzed, we will prove this Proposition for $n = 4$. But the same arguments can be extended for any number of sellers.

$$f_1(1) = p_2 - q_1 + q_2;\ \ f_2(1) = p_2 + \frac{(p_2 - p_3)(q_1 - q_2)}{q_2 - q_3};\ \ f_3(1) = p_3 + \frac{(p_3 - p_4)(q_1 - q_3)}{q_3 - q_4};\ \ f_4(1) = \frac{p_4 q_1}{q_4};$$
$$f_1(2) = p_1 - q_1 + q_2;\ \ f_2(2) = \frac{p_1(q_2 - q_3) + p_3(q_1 - q_2)}{q_1 - q_3};\ \ f_3(2) = p_3 + \frac{(p_3 - p_4)(q_2 - q_3)}{q_3 - q_4};\ \ f_4(2) = \frac{p_4 q_2}{q_4};$$
$$f_1(3) = p_1 - q_1 + q_3;\ \ f_2(3) = p_2 - \frac{(p_1 - p_2)(q_2 - q_3)}{(q_1 - q_2)};\ \ f_3(3) = \frac{p_2(q_3 - q_4) + p_3(q_2 - q_3)}{q_2 - q_4};\ \ f_4(3) = \frac{p_4 q_3}{q_4};$$
$$f_1(4) = p_1 - q_1 + q_4;\ \ f_2(4) = p_2 - \frac{(p_1 - p_2)(q_2 - q_4)}{(q_1 - q_2)};\ \ f_3(4) = p_3 - \frac{(p_2 - p_3)(q_3 - q_4)}{(q_2 - q_3)};\ \ f_4(4) = \frac{p_3 q_4}{q_3}.$$

Seller 1:

- If $p_1 > f_1(1)$, then seller 1 does not sell, because every customer with w.t.p. $w \in [0,1]$ prefers seller 2 to seller 1 (i.e., $wq_2 - p_2 > wq_1 - p_1$).
- If $f_2(1) < p_1 < f_1(1)$, then seller 1 sells $F^c\left(\frac{p_1-p_2}{q_1-q_2}\right)$, because $\frac{p_2-p_3}{q_2-q_3} \leq \frac{p_1-p_2}{q_1-q_2} \leq 1$ and from the proof of Proposition 8 we know that customers with w.t.p. $w \in \left[\frac{p_2-p_3}{q_2-q_3}, \frac{p_1-p_2}{q_1-q_2}\right]$ and only them strictly prefer seller 1 to every other seller (and also to the no-purchase option).
- If $f_3(1) < p_1 < f_2(1)$, then seller 2 does not sell, and seller 1 sells $F^c\left(\frac{p_1-p_3}{q_1-q_3}\right)$. The reason is as follows: every customer with w.t.p. $w \in [0,1]$ prefers either seller 1 or seller 3 to seller 2 (i.e., $\frac{p_2-p_3}{q_2-q_3} > \frac{p_1-p_2}{q_1-q_2}$ and we know that for $w \in \left[\frac{p_1-p_2}{q_1-q_2}, 1\right]$ seller 1 is preferred to seller 2; while for $w \in \left[0, \frac{p_2-p_3}{q_2-q_3}\right]$ seller 3 is preferred to seller 2). To see that seller 1 sells, we must show that $p_1 - q_1 + q_3 < p_3$. Note that $p_1 < f_2(1)$ is equivalent to $\frac{p_1-p_3}{q_1-q_3} < \frac{p_2-p_3}{q_2-q_3}$. From the assumption that p_{-1} is such that all sellers sell, we have that $\frac{p_2-p_3}{q_2-q_3} < 1$. So $\frac{p_1-p_3}{q_1-q_3} < 1$.
- $f_4(1) < p_1 < f_3(1)$, then sellers 2 and 3 do not sell and seller 1 sells $F^c\left(\frac{p_1-p_4}{q_1-q_4}\right)$, because every customer prefers either seller 1 or seller 3 to sellers 2 and 4. To see that seller 1 sells, we must show that $p_1 - q_1 + q_4 < p_4$. Note that $p_1 < f_3(1)$ is equivalent to $\frac{p_1-p_4}{q_1-q_4} < \frac{p_3-p_4}{q_3-q_4}$. From the assumption that p_{-1} is such that all sellers sell, we have that $\frac{p_3-p_4}{q_3-q_4} < 1$. So $\frac{p_1-p_4}{q_1-q_4} < 1$.
- If $p_1 < f_4(1)$, then sellers 2, 3 and 4 do not sell and seller 1 sells $F^c\left(\frac{p_1}{q_1}\right)$.

Seller 2:

- If $p_2 > f_2(2)$, then seller 2 does not sell, because for every $w \in [0,1]$, sellers 1 or 3 are preferred to seller 2 (i.e., $\frac{p_2-p_3}{q_2-q_3} > \frac{p_1-p_2}{q_1-q_2}$).
- If $f_1(2), f_3(2) < p_2 < f_2(2)$, then all sellers sell and seller 2 sells $F\left(\frac{p_1-p_2}{q_1-q_2}\right) - F\left(\frac{p_2-p_3}{q_2-q_3}\right)$. This follows from the proof of Proposition 8.
- If $f_1(2) < f_3(2)$ and $f_1(2) < p_2 < f_3(2), f_2(2)$, then seller 3 does not sell, and seller 2 sells $F\left(\frac{p_1-p_2}{q_1-q_2}\right) - F\left(\frac{p_2-p_4}{q_2-q_4}\right)$, because every customer prefers either seller 2 or seller 4 to seller 3 (i.e., $\frac{p_2-p_3}{q_2-q_3} < \frac{p_3-p_4}{q_3-q_4}$).

- If $f_1(2) > f_3(2)$ and $f_3(2) < p_2 < f_2(2), f_1(2)$, then seller 1 does not sell, and seller 2 sells $F^c\left(\frac{p_2-p_3}{q_2-q_3}\right)$, because every customer prefers seller 2 to seller 1. To see that seller 2 sells, we must show that $p_2 - q_2 + q_3 < p_3$. Note that $p_2 < f_2(2)$ is equivalent to $\frac{p_2-p_3}{q_2-q_3} < \frac{p_1-p_3}{q_1-q_3}$. From the assumption that p_{-1} is such that all sellers sell, we have that $\frac{p_1-p_3}{q_1-q_3} < 1$. So $\frac{p_2-p_3}{q_2-q_3} < 1$.
- If $f_4(2) < p_2 < f_3(2), f_1(2)$, then sellers 1 and 3 do not sell and seller 2 sells $F^c\left(\frac{p_2-p_4}{q_2-q_4}\right)$, because every customer prefers either seller 2 or seller 4 to sellers 1 and 3. To see that seller 2 sells, we must show that $p_2 - q_2 + q_4 < p_4$. Note that $p_2 < f_3(2)$ is equivalent to $\frac{p_2-p_4}{q_2-q_4} < \frac{p_3-p_4}{q_3-q_4}$. From the assumption that p_{-1} is such that all sellers sell, we have that $\frac{p_3-p_4}{q_3-q_4} < 1$. So $\frac{p_2-p_4}{q_2-q_4} < 1$.
- If $p_2 < f_4(2)$, then sellers 1, 3 and 4 don't sell and seller 2 sells $F^c\left(\frac{p_2}{q_2}\right)$.

Seller 3: similar to Seller 2 case.

Seller 4:

- If $p_4 > f_4(4)$, then seller 4 does not sell, because for every $w \in [\frac{p_3}{q_3}, 1]$, seller 3 is preferred to seller 4 (i.e., $\frac{p_3}{q_3} > \frac{p_3-p_4}{q_3-q_4}$).
- If $f_3(4) < p_4 < f_4(4)$, then all sellers sell and seller 4 sells $F\left(\frac{p_3-p_4}{q_3-q_4}\right) - F\left(\frac{p_4}{q_4}\right)$ (from Proposition 8).
- If $f_2(4) < p_4 < f_3(4)$, then seller 3 does not sell and seller 4 sells $F\left(\frac{p_2-p_4}{q_2-q_4}\right) - F\left(\frac{p_4}{q_4}\right)$.
- If $f_1(4) < p_4 < f_2(4)$, then sellers 2 and 3 do not sell and seller 4 sells $F\left(\frac{p_1-p_4}{q_1-q_4}\right) - F\left(\frac{p_4}{q_4}\right)$.
- If $p_4 < f_1(4)$, then only seller 4 sells $F^c\left(\frac{p_4}{q_4}\right)$. ■

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