CENTRALIZED CLEARINGHOUSE DESIGN: A QUANTITY-QUALITY TRADEOFF

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Abstract. Stable matching mechanisms are used to clear many two-sided markets. In practice, these mechanisms leave many agents on both sides unmatched. What factors determine the number of unmatched agents, and the quality of matches that do form? This paper answers these questions, with a particular focus on how match outcomes depend on correlations in agent preferences. I consider three canonical preference structures: fully idiosyncratic preferences, common preferences (agents agree on the attractiveness of those on the opposite side), and aligned preferences (potential partners agree on the attractiveness of their match).

I find that idiosyncratic preferences result in more matches than common preferences do. Perhaps more surprisingly, the case of aligned preferences results in the fewest matches. Regarding match quality, the story reverses itself: aligned preferences produce the most high quality matches, followed by common preferences. These facts have implications for the design of priority rules and tie-breaking procedures in school choice settings, and establish a fundamental tradeoff between matching many students, and maximizing the number of students who get one of their top choices.

1. Introduction

One of the most central questions in economics is that of how best to organize markets. An increasingly popular solution (especially in settings where monetary transfers are absent or inflexible) is to use a *centralized clearinghouse* that asks participants for their preferences and then uses this information to recommend a matching. For example, the National Residency Matching Program has been in operation since the 1950's, and in 2015 processed applications from 35,000 medical students for 30,000 residency positions. Recently, similar clearinghouses have been adopted to assign students to public schools in metropolitan areas across the United States and Europe. Centralization of this form thickens the market, while resolving some of the mis-coordination ("congestion") that can occur in decentralized markets.[1]

While there are many possible ways to use lists to produce a matching,[2] the deferred acceptance algorithm of Gale and Shapley (1962) has proven to be quite popular. It produces a match that is *stable*, meaning that no pair of agents can undermine the match by finding a mutually profitable deviation. In the context of school choice, this makes the mechanism transparent and (relatively) simple to explain: any school that a student prefers to their assignment is filled by students who have higher priority at that school. Furthermore, it provides students with incentives to state their

[1]See Abdulkadiroglu et al. (2005) for a discussion of congestion in high school assignment in New York City.
[2]Abdulkadiroglu and Sönmez (2003) propose a mechanism based on Gale's top trading cycles algorithm which is both strategy-proof and Pareto efficient for students. New Orleans used such a mechanism for one year before abandoning it in favor of deferred acceptance. Other alternatives include the so-called "Boston" mechanism and optimization-based approaches such as those used for assigning medical internships in Israel (Alon et al., 2015), although neither of these mechanisms is strategy-proof.

true preferences. [3] Abdulkadiroglu et al. (2006) argue that this "levels the playing field" among parents with varying degrees of sophistication. An additional benefit is that this provides district administrators with more accurate data regarding the popularity of different schools. Perhaps for these reasons, matching procedures based on the deferred acceptance algorithm have recently been adopted in cities across the United States and Europe, including New York, Boston, Denver, New Orleans, Chicago, Paris, and Amsterdam.

Although these clearinghouses have been heavily studied by theorists and practitioners alike, many facts about their operation remain poorly understood. Chief among these is that in all large-scale markets that utilize stable matching mechanisms, many participants on both sides of the market fail to match. For example, in 2015,

- 1,306 residency positions offered through the NRMP (4.3%) did not fill.[4]
- 5,795 students in New York City (7.7%) were initially unassigned.[5]
- 2,325 students in New Orleans (20.4%) failed to match to a listed school. [6]

In each case, these participants were on the "short" side of the market. Why are so many participants failing to match? And what features of the market determine this and other aggregate match outcomes? This chapter addresses these questions.

One salient feature of these marketplaces is that even though the number of *potential* partners is quite large (for example, the New York City match features more than 700 distinct high school programs), the number of options listed by any individual is generally small (often in single digits). This fact clearly contributes to the number of unmatched participants: after all, a student who (hypothetically) listed every school would be assured of receiving a seat at one of them. Of course, short lists don't necessarily imply that many students go unmatched: if student preferences are sufficiently diverse, it may be possible to match almost all students to one of their top choices. Thus, a second important consideration is the extent to which students desire the same schools. Finally, there is the question of how schools rank students. The first major contribution of this work (Theorem 1) is to provide a detailed analysis of aggregate match outcomes in a model that incorporates each of these factors.

My analysis focuses in particular on the effect of school rankings ("priorities") on match outcomes. The importance of school priorities has not been lost on parents. In Boston, for example, there is a vigorous and ongoing debate about whether students who live within the "walk zone" for a school should receive higher priority than those who live farther away. In Amsterdam, school officials recently decided to change the system for determining student priorities in response to a lawsuit filed by parents.[7]

[3]It is worth noting that in most cities, students may only list a limited number of schools. As a result, students may be unable to truthfully report their preferences, and may have an incentive to drop very popular schools from their list. Nonetheless, it seems substantially simpler for students to participate in procedures that use deferred (rather than immediate) acceptance. For one way to formalize this idea, see Pathak and Sönmez (2013).

[4]Source: `http://www.nrmp.org/match-data/main-residency-match-data/`

[5]Source: `http://insideschools.org/blog/item/1000947`

[6]Source: `http://enrollnola.org/enrollnola-annual-report/`

[7]For more details on the walk-on debate in Boston, see Dur et al. (2013). The situation in Amsterdam is addressed in de Haan et al. (2015). Articles in Dutch about the court case (`http://www.trouw.nl/tr/nl/4492/Nederland/article/detail/4084791/2015/06/20/Het-beest-in-de-Amsterdamse-ouder-is-los.dhtml`) and the decision to modify the lottery system (`http://www.verenigingosvo.nl/`) are also available.

Despite this attention, the aggregate effects of proposed policy changes are generally poorly understood. One example is the question of how to break ties among students in the same priority class. Should ties be resolved using a single, district-wide lottery, or should separate lotteries be conducted at each school? Simulations with preferences from New York, Boston, and Amsterdam demonstrate that in all three cities, the former procedure assigns more students to their first choice, but the latter assigns more students overall (Abdulkadiroglu et al., 2009; de Haan et al., 2015). Furthermore, the differences can be significant: in New York, for example, a common lottery resulted in approximately 2,300 more students receiving their top choice.

The second main contribution of this work is a comprehensive comparison of aggregate outcomes under different priority rules. In particular, Theorem 4 offers an explanation for the observations from Abdulkadiroglu et al. (2009) and de Haan et al. (2015). It states that regardless of the ratio of students to seats, the length of student lists, and the relative popularities of different schools, a common lottery will assign more students to their top choice, but will also leave more students unassigned.

I consider three methods for generating priorities:

- **Idiosyncratic**: School priorities are drawn independently, and uniformly at random.
- **Common**: Schools have a common ranking over students.
- **Aligned**: Each student-school pair has an (observable) "match quality", and both sides rank accordingly. Pairwise match qualities are drawn independently and identically.

These are intended to be canonical (if extreme) representations of the types of correlations present in most matching markets: participants on the same side may agree on which partners are desirable, and participants on opposite sides may agree about what makes a good match.[8] In the context of school choice, idiosyncratic priorities can be thought of as assignment via school-specific lottery, and common priorities might arise through consideration of grades, test scores, or a common lottery. The case of aligned priorities is a stylized way of capturing the effect of walk zone priority, sibling priority, and priority for information session attendance: all three give priority to students who are likely to rank that school highly.[9]

To enable analysis, I study outcomes in the limiting case where the numbers of students and schools grow, holding constant the ratio of students to seats, the lengths of student lists, and the distribution of school capacities and popularity. In this regime, a constant fraction of participants on both sides fail to find match partners. Theorem 1 provides exact expressions for this fraction under each priority rule (as a function of the above parameters). In addition, it gives exact expressions for other match outcomes (for example, the number of students receiving a top-k choice, for any value of k).

[8]In the context of the National Residency Matching Program, doctors may have *idiosyncratic* geographical preferences driven by the locations of their friends or family members. They may differ in their level of expertise, causing hospitals to share a *common* assessment of doctors. Additionally, doctor and hospital preferences may be *aligned*, as a doctor who specializes in a particular field may wish to go to a hospital where her expertise is in demand.

[9]Information about priorities in New York is provided in the NYC 2016 High School Directory, available at `http://schools.nyc.gov/ChoicesEnrollment/High/Resources/default.htm`. Information about priorities in Boston is available at `http://www.bostonpublicschools.org/domain/219`.

These expressions can be used to generate a rich set of comparative statics regarding the effect of increasing list length, increasing school capacity, or changing schools' popularities. While certain factors (such as market imbalance and list length) have an effect whose direction is obvious,[10] the results from Theorem 1 make it possible to study the *magnitude* of these effects. Meanwhile, even the *directional* effect of changing from one priority rule to another is unclear. Which priority rule assigns the most students to their top choice? To one of their listed schools?

Theorem 2 answers these questions unambiguously. It states that for any choices of market parameters, idiosyncratic priorities result in the most matches, while aligned priorities produce the fewest. Regarding top choices, the ordering reverses: the most students get their top choice when priorities are aligned, followed by common priorities. In fact, this tradeoff extends beyond top choices: Theorem 3 states that when comparing idiosyncratic priorities to either other case, a "single crossing" property holds: idiosyncratic priorities result in fewer high quality matches, but more matches overall.

The remainder of the chapter is organized as follows. Section 2 situates this work in the context of related literature. Section 3 describes the model. Section 4 presents my main results, and provides intuition using a simple example. Sections 5 and 6 sketch the proofs of Theorems 1 and 2, respectively. Section 7 relaxes the assumption that students all list the same number of schools. It turns out that this assumption is consequential: for some list length distributions, the tradeoff identified in Theorem 2 disappears, so that common priorities result in both more matches and more students receiving their top choice. Theorem 5 establishes sufficient conditions under which idiosyncratic priorities continue to match more students than common priorities, as well as conditions under which the reverse holds. Section 8 concludes and discusses applications of these results to other markets.

2. Related Work

There is a large literature on stable outcomes in two-sided matching markets. In their seminal paper, Gale and Shapley (1962) defined the concept of a stable matching, and proved that the deferred acceptance algorithm will always find such a matching. Roth (1984) identified that the algorithm used by the NRMP to match medical residents to hospitals was in fact the hospital-proposing deferred acceptance algorithm.

In general, there may be many stable matchings: the deferred acceptance algorithm selects a matching which is uniformly optimal (among all stable matches) for the proposing side. One natural question is how "far apart" two stable matchings can be. This question is economically interesting for two reasons:

I. If stable matchings differ substantively, then it may be possible to achieve policy objectives merely by selecting among the set of stable matchings.

[10] It is intuitive that adding students worsens outcomes for existing students, while expanding school capacities improves them. Furthermore, the number of matches should increase as students list more schools. Previous work has established that these statements hold in a very strong sense: adding a student to the marketplace weakly worsens the assignment of *all* other students, whereas adding a seat to a school weakly improves the outcome for all students. Meanwhile, adding a school to the bottom of a student's list weakly increases the number of matches formed.

II. There is a close link between the set of stable matchings and agent incentives. In particular, when agents seek a single match partner, an agent on the side receiving proposals can benefit by misreporting her preferences if and only if she has multiple stable partners. (Roth (1982) showed that reporting truthfully is a dominant strategy for any agent on the proposing side.)

A growing body of evidence suggests that differences between any two stable matchings are typically minor. The so-called "rural hospital theorem" (McVitie and Wilson, 1970; Roth, 1986) states that the set of assigned students and seats is constant across stable matchings. Roth and Peranson (1999) identify that in the data from the NRMP, the vast majority of students have a unique stable match partner. Immorlica and Mahdian (2005) use techniques developed by Pittel (1989) and Knuth et al. (1990) to show that in large random one-to-one matching markets where students submit lists of bounded length, with high probability almost all students have a single match partner. Their result was extended to the case of many-to-one matching by Kojima and Pathak (2009), whose model I adopt in this chapter. More recent work has shown that this result also holds in the presence of a small number of couples (Kojima et al., 2013), and in the case when agents have long lists, so long as the market is not perfectly balanced (Ashlagi et al., 2013).

The above work establishes that in many cases, there is an essentially unique stable outcome. Left largely unanswered, however, is the question of what this outcome actually "looks like." Of course, many clearinghouses publish descriptive statistics of match outcomes, but this data does not explain *why* outcomes differ across markets. Existing theoretical examinations of match outcomes are quite limited: most work studies one-to-one markets with complete preference lists drawn uniformly at random. Pittel (1989) shows that in a large market with an equal number n of men and women and uniform random (complete) preferences, the proposing side is on average assigned a partner of rank approximately $\log(n)$. Ashlagi et al. (2013) identify that adding a single agent to the proposing side dramatically changes this number, to $n/\log(n)$. Neither paper seems to adequately explain outcomes in real markets, where many participants go unmatched and no single agent has a large effect on aggregate match outcomes.

Abdulkadiroglu and Sönmez (2003) were among the first to propose the deferred acceptance algorithm as a method for assigning public schools. They raised the question of how to break ties in priority, and observed that using multiple lotteries may produce an outcome that is not Pareto efficient. Abdulkadiroglu et al. (2009), Pathak (2011), and de Haan et al. (2015) present empirical comparisons between single and multiple tie-breaking procedures. In each case, the authors find that using a common lottery assigns more students to their first choice, while using multiple lotteries assigns more students overall. These papers do not offer an explanation for this fact, nor do they provide insight into how other changes in priorities might affect match outcomes. I address both of these questions.

Concurrently with this work, Ashlagi et al. (2015) and Ashlagi and Nikzad (2015) theoretically examine the question of common versus school-specific tie-breaking. Ashlagi et al. (2015) prove that if students submit complete preference lists drawn uniformly at random, school-specific lotteries result in many fewer students getting their top choices, when compared to outcomes under a common lottery. They conjecture that when lists are short, a common lottery assigns fewer students: this fact is a special case of Theorem 2. Ashlagi and Nikzad (2015) find that when participants submit complete lists and the market has an excess of seats, neither procedure dominates the other, but when there are more students than seats, a common lottery "almost dominates" multiple lotteries. By contrast, I show that when lists are short, there is always a tradeoff between the two procedures.

The recent work of Fack et al. (2015) uses data from high school admissions in Paris to study the effect of school admissions criteria on student outcomes. They ask how outcomes would change if schools adjusted the relative weights of student grades and school-specific lottery draws. Using structural estimates of student utilities, they conclude that ranking based only on grades yields the highest aggregate welfare.

Many of the results in this chapter are formally statements about (specific) matchings in large random graphs. The use of differential equations to describe the behavior of large random graphs was popularized by Wormald (1995, 1999), and Theorem 1 makes use of these techniques. Mastin and Jaillet (2013) also apply these techniques to study the size of matches generated by greedy procedures. In the language of this chapter, they study the size of the stable matching when schools have unit capacity, share a common ranking of students, there are an equal number of students and schools, and the number of schools listed by each student follows a Poisson distribution. Thus, their result is a special case of the extension to Theorem 1 presented in Appendix E (which accommodates variable list lengths).

The case of one-to-one matching in which each student lists exactly ℓ schools has been studied (using very different language) by Dietzfelbinger et al. (2010), Frieze and Melsted (2012) and Fountoulakis and Panagiotou (2012). Rather than studying stable matchings, these papers study the size of the *largest possible* matching. In particular, they focus on the question of how many schools are needed in order for it to be possible to assign all students to *some* school on their list.

3. Model

Informally, a school choice environment is characterized by a set of students and schools, capacities for each school, student preferences and school priorities. My model is effectively identical to that considered in Kojima and Pathak (2009): in particular, both preferences and priorities are drawn randomly, and my results are statements about outcomes in the limit as the number of students and schools increases.

Whenever possible, notation is chosen such that the letter used is a pneumonic for the entity that it represents (a summary of notation is provided below). In an unfortunate coincidence, the words "student" and "school" start with the same letter, so this chapter will study the assignment of students to **H**igh Schools. Despite this fact, most of the ideas presented below apply equally well to settings where students are to be assigned to middle schools, elementary schools, or even kindergarten programs. For a more complete discussion of this extension, please contact the author.

$\mathcal{S}$	The set of **S**tudents.
$\mathcal{H}$	The set of **H**igh schools.
C_h	**C**apacity of high school h.
p_h	**P**opularity of high school h.
$\mathcal{D}$	Joint **D**istribution of popularity and capacity.
ρ	The **R**atio of students to schools.
N	Total **N**umber of applications sent.
$N(t)$	Total **N**umber of applications of priority above t.
λ	Application **L**evel ($= N/n$).
$\Lambda(t)$	**L**evel of applications of priority above t ($= N(t)/n$).
$\mathcal{V}(\cdot)$	Fraction of applications sent to schools with a **V**acancy.
$\mathcal{E}(\cdot)$	The average **E**nrollment, across schools.
$\alpha(\cdot)$	Aggregate **A**cceptance rate ($= \mathcal{E}(\lambda)/\lambda$)
$F(k)$	**F**raction of students who receive a top-k choice.
I	**I**diosyncratic priorities
C	**C**ommon priorities
A	**A**ligned priorities

3.1. **The Basics.** There is a finite set of students $\mathcal{S}$, and a set of high schools $\mathcal{H}$, with $|\mathcal{H}| = n$. Each student $s \in \mathcal{S}$ provides an ordered list of schools that she would be willing to attend (with her most preferred school listed first, followed by her second choice school, and so on). Let ℓ_s be the length of the preference list submitted by student s. Initially, assume that each student lists the same number of schools, $\ell_s = \ell < n$ (Section 7 relaxes this assumption). In addition to an ordinal ranking of schools, each student s is endowed with a cardinal utility $U_s(h) > 0$ for each listed school h. Each high school $h \in \mathcal{H}$ is characterized by a capacity $C_h \in \mathbb{N}$ and a priority rule $U_h(\cdot) : \mathcal{S} \to [0, 1]$. There are no ties, meaning that (for students) if h is ranked above h' by s, then $U_s(h) > U_s(h')$, and (for schools) $s \neq s'$ implies that $U_h(s) \neq U_h(s')$ for all h.

Given student lists and school capacities and priorities, students are assigned to schools using the *student-proposing deferred acceptance algorithm*, which proceeds in a series of rounds. Initially, all students are unassigned. In each round, select a student s who is currently unassigned and has not applied to every school on her list.[11] Student s applies to her most preferred school among those that have not yet rejected her (call this school h). If h has received fewer than C_h applications, s is tentatively accepted. Otherwise, h rejects all but the C_h applicants of highest priority (according to U_h). The algorithm terminates when every student who is unassigned has been rejected from every school on his or her list.

3.2. **Random Preferences and Priorities.** In addition to a capacity C_h, each school h is endowed with "popularity" $p_h > 0$, normalized so that $\frac{1}{n} \sum p_h = 1$. Student lists are drawn iid by sampling schools repeatedly (without replacement) in proportion to their popularity until ℓ schools have been drawn. If student s lists h_1 as her most preferred school, followed by h_2, h_3, and so on,

[11]It is well-known that the choice of student in each round does not affect the matching produced by the algorithm.

the values $U_s(h_1), U_s(h_2), \ldots, U_s(h_\ell)$ are distributed as the order statistics of ℓ independent draws from the uniform distribution on $[0,1]$.[12]

I consider three methods of generating school priorities:

- **I**diosyncratic: priorities are iid uniform over students. $(\forall s, h : U_h(s)\, iid \sim U[0,1])$
- **C**ommon: schools share a common ranking of students. $(U_h(\cdot) = U_{h'}(\cdot))$
- **A**ligned: school priorities correspond with student preferences $(U_h(s) = U_s(h), \forall s, h)$

Note that $U_h(s)$ is observable, meaning that students cannot affect their priority at each school.[13]

3.3. **Large Market Limit.** The results that follow consider outcomes in "large markets." Formally, I consider a sequence of markets, indexed by the number of high schools n. In this sequence, the list length ℓ and the priority scheme (I, C, or A) are held constant. The n^{th} market is characterized by a set of students $\mathcal{S}^n$, a set of schools $\mathcal{H}^n$, and an empirical distribution $\mathcal{D}^n$ of school popularity and capacity. Formally, define $\mathcal{D}^n(p, C)$ to be the fraction of schools with popularity at most p and capacity at most C:

$$\mathcal{D}^n(p, C) = \frac{1}{n} \left| \{h \in \mathcal{H}_n : p_h \leq p, C_h \leq C\} \right|.$$

I will assume that as the number of schools grows, the empirical distribution $\mathcal{D}^n$ converges to some joint distribution $\mathcal{D}$ on $\mathbb{R}_+ \times \mathbb{N}$. Furthermore, I assume that the ratio of students to schools converges to a constant ρ. I define my (standard) notions of convergence below.

- A function $\mathcal{D} : \mathbb{R}_+ \times \mathbb{N} \to [0,1]$ is continuous at (p, C) if, $p_i \to p$, implies $\mathcal{D}(p_i, C) \to \mathcal{D}(p, C)$.
- Given a sequence of functions $\mathcal{D}^n : \mathbb{R}_+ \times \mathbb{N} \to [0,1]$, say that $\mathcal{D}^n \to \mathcal{D}$ if $\mathcal{D}^n(p, C) \to \mathcal{D}(p, C)$ for all (p, C) at which $\mathcal{D}$ is continuous. The convergence is uniform if
$$\sup_{(p,D):\mathcal{D} \text{ is continuous at } (p,C)} |\mathcal{D}^n(p, C) - \mathcal{D}(p, C)| \to 0.$$
- A sequence of random variables $X^n \in \mathbb{R}$ converges in probability to X (written $X^n \xrightarrow{p} X$) if, for any $\epsilon > 0$, $\mathbb{P}(|X^n - X| > \epsilon) \to 0$.

Definition 1. *For fixed ℓ, a sequence of markets $(\mathcal{S}^n, \mathcal{H}^n, \mathcal{D}^n)$ is* **regular** *if $|\mathcal{H}^n| = n$ for all n, and there exists $\rho > 0$ and $\mathcal{D} : \mathbb{R}_+ \times \mathbb{N} \to [0,1]$ such that $|\mathcal{S}^n|/n \to \rho$, and $\mathcal{D}^n \to \mathcal{D}$ uniformly.*

Throughout the remainder of the chapter, I assume that the sequence of markets under consideration is regular.

4. Main Contributions

4.1. **Formal Results.** This section studies match outcomes in a large market under each priority rule. It considers two natural welfare measures: the fraction of students who receive a top-k

[12]The choice of the uniform distribution is without loss of generality, as I give expressions for the probability of getting a match above utility u, for all u. Thus, given any atomless distribution on the positive reals, all results continue to hold with the value u interpreted as the *quantile* of the match according to the distribution of values.

[13]For the case of Aligned priorities, it may seem strange to assume that $U_h(s)$ is observable, given that it is equal to $U_s(h)$. This choice is merely a technical device for correlating school priorities with students' (ordinal) lists. The simplest interpretation of the Aligned priority model is that there are some observable match characteristics which are used to determine priorities, and that students tend to rank highly those schools where they have higher priority. Whether these observables directly correspond to student utilities is largely immaterial.

choice (for varying values of k), and the fraction of students who receive match utility below u, for $u \in (0,1)$.[14] Below, I define notation corresponding to these quantities.

Given $k \in \{1, \dots, \ell\}$, let $F_I^n(k)$ denote the (random) fraction of students are assigned to their k^{th} choice or better in the n^{th} market, when priorities are idiosyncratic. Thus, $F_I^n(1)$ gives the fraction of students who are assigned to their first choice school, and $F_I^n(\ell)$ gives the fraction of students who are assigned to *some* school on their list. Define $F_C^n(k)$ and $F_A^n(k)$ to be the corresponding quantities when priorities are common and aligned, respectively.

Given $u \in (0,1)$, let $G_I^n(u), G_C^n(u), G_A^n(u)$ be the (random) fraction of students who receive utility no greater than u in the n^{th} market, when priorities are (respectively) idiosyncratic, common, and aligned. Thus, $G_I^n(0)$ gives the fraction of students who are left unassigned when priorities are idiosyncratic, and $G_I^n(1) = G_C^n(1) = G_A^n(1) = 1$ (that is, no student receives utility above one).

It turns out that as the market grows, the values $F^n(k)$ and $G^n(u)$ become essentially deterministic (that is, their distribution concentrates). Theorem 1 gives exact limiting expressions for these values. To make these expressions less unwieldy, I define some additional notation.

Let $\mathbf{Po}(\lambda)$ be a poisson random variable with mean λ, and define the functions $\mathcal{E} : \mathbb{R}_+ \to \mathbb{R}_+$, $\mathcal{V}, \alpha : \mathbb{R}_+ \to [0,1]$, and $\mu : [0,1] \to \mathbb{R}_+$ as follows:

$$\mathcal{E}(\lambda) = \mathbb{E}_{(p,C)\sim\mathcal{D}}\left[\mathbb{E}[\min(\mathbf{Po}(p\lambda), C)]\right] \tag{1}$$

$$\mathcal{V}(\lambda) = \mathbb{E}_{(p,C)\sim\mathcal{D}}\left[p\mathbb{P}(\mathbf{Po}(p\lambda) < C)\right] \tag{2}$$

$$\alpha(\lambda) = \mathcal{E}(\lambda)/\lambda. \tag{3}$$

$$\mu(\alpha) = \sum_{k=0}^{\ell-1}(1-\alpha)^k = \frac{1-(1-\alpha)^\ell}{\alpha}. \tag{4}$$

Informally, the quantity λ represents the number of applications sent over the course of the deferred-acceptance algorithm, divided by the number of schools. This is a coarse summary of the "level of competition" in the marketplace. The quantity $\mathcal{E}(\lambda)$ represents the average enrollment across schools (given competition level λ), and $\mathcal{V}(\lambda)$ represents the probability that a student's next application is sent to a school with a vacancy, given λ. The quantity $\alpha(\lambda)$ is simply the number of seats filled divided by number of applications sent, or the aggregate "acceptance rate" in the market. Meanwhile (in light of Fact 1 below), $\mu(\alpha)$ represents the expected number of applications sent by a student for whom each application is accepted with probability α.

I discuss these interpretations in more detail in Section 5. For now, they are used to provide expressions for large-market outcomes under each priority rule.[15]

[14] Both of these welfare measures consider only student outcomes. This is consistent with tradition in the school choice literature, which tends to define properties such as Pareto efficiency with respect to student preferences only. However, in some cases, priorities may reflect actual preferences on the part of the district or schools, in which case it is important to measure school outcomes. In fact, the computation of student outcomes below proceeds by first deriving expressions for school outcomes. Thus, my work fully describes outcomes for both sides of the market, although I do not emphasize school welfare.

[15] The proofs of Theorem 1 in the Appendix are currently written assuming that $p_h = 1$ for all h (meaning that students list schools uniformly at random). The same ideas apply to the case of non-uniform popularity, but the

Theorem 1. *Fix $\rho, \ell, \mathcal{D}$. For any $k \in \{1, \ldots, \ell\}$ and $u \in [0,1]$, as $n \to \infty$,*

(I) Idiosyncratic: $F_I^n(k) \xrightarrow{p} F_I(k)$, $\quad G_I^n(u) \xrightarrow{p} G_I(u)$, *where*

$$G_I(u) = (1 - (1-u)\alpha(\lambda_I))^\ell, \qquad F_I(k) = 1 - (1 - \alpha(\lambda_I))^k,$$

and λ_I is the unique solution to

$$\lambda_I = \rho \cdot \mu(\alpha(\lambda_I)). \tag{5}$$

(C) Common: $F_C^n(k) \xrightarrow{p} F_C(k)$, $\quad G_C^n(u) \xrightarrow{p} G_C(u)$, *where*

$$G_C(u) = \int_0^1 (1 - (1-u)\mathcal{V}(\Lambda_C(t)))^\ell \, dt, \quad F_C(k) = 1 - \int_0^1 (1 - \mathcal{V}(\Lambda_C(t)))^k \, dt,$$

and the function $\Lambda_C : [0,1] \to \mathbb{R}_+$ is given by the differential equation

$$\Lambda_C(1) = 0, \quad \Lambda_C'(t) = -\rho \cdot \mu(\mathcal{V}(\Lambda_C(t)))). \tag{6}$$

(A) Aligned: $F_A^n(k) \xrightarrow{p} F_A(k)$, $\quad G_A^n(u) \xrightarrow{p} G_A(u)$, *where*

$$G_A(u) = \left(1 - \int_u^1 \mathcal{V}(\Lambda_A(t))dt\right)^\ell,$$

$$F_A(k) = 1 - \ell\binom{\ell-1}{k-1}\int_0^1 t^{\ell-k}\left(\int_t^1 1 - \mathcal{V}(\Lambda_A(s))ds\right)^{k-1}(1 - \mathcal{V}(\Lambda_A(t)))dt,$$

and the function $\Lambda_A : [0,1] \to \mathbb{R}_+$ is given by the differential equation

$$\Lambda_A(1) = 0, \quad \Lambda_A'(t) = -\rho\ell\left(1 - \int_t^1 \mathcal{V}(\Lambda_A(u))du\right)^{\ell-1}. \tag{7}$$

On their own, the expressions in Theorem 1 provide little insight or intuition. They can be used, however, to perform comparative statics along several dimensions, including the ratio ρ of students to schools, the distribution of school capacities and popularity $\mathcal{D}$, and the length ℓ of student lists. This chapter focuses on a different comparison. Holding fixed the above parameters, I vary the priority rule used to rank students.

Which priority rule assigns the most students? Does the same rule also give more students a "high quality" match? In principle, the answers could depend on the values of other market parameters. It turns out, however, that the three priority rules studied here can be unambiguously ranked according to these welfare criteria.

Theorem 2. *For any $\rho, \ell, \mathcal{D}$, there exists $u' < 1$ such that the following holds:*

$$G_I(0) \le G_C(0) \le G_A(0), \qquad G_I(u) \ge G_C(u) \ge G_A(u), \text{ for all } u > u',$$

with all inequalities strict if $\ell > 1$.

Theorem 2 states that idiosyncratic school priorities result in the most students being assigned, while aligned priorities assign the *fewest* students. This result may seem surprising, given that aligned priorities are intuitively "good" for students. The second component of the Theorem

proofs become substantially more technical. I am currently working on a complete proof for the general case. Taking the expressions in Theorem 1 as given, the remaining Theorems are proven for general $\mathcal{D}$.

establishes that aligned priorities do, in fact, assign the most students to very "high quality" matches (above u'), and that idiosyncratic priorities result in the fewest high quality matches.

Theorem 2 establishes a tradeoff between the goals of maximizing the number of *total* matches, and maximizing the number of *high-quality* matches, but does not directly compare G_I, G_C, G_A in the interval $(0, u')$.Theorem 3 provides such a comparison: It turns out that there is a single point u^{IC} where the functions G_I and G_C cross, and another point u^{IA} where the functions G_I and G_A cross. Put another way, suppose that we define a "quality" match to be one in which the student receives utility at least u'. Then idiosyncratic priorities yield more quality matches than common priorities if $u' < u^{IC}$, but yield fewer quality matches if $u' > u^{IC}$.

Theorem 3. *For any $\rho > 0, \ell > 1, \mathcal{D}$, there exist $u^{IC}, u^{IA} \in (0,1)$ such that*

- $G_I(u) < G_C(u)$ *for* $u < u^{IC}$
- $G_I(u) \geq G_C(u)$ *for* $u \geq u^{IC}$
- $G_I(u) < G_A(u)$ *for* $u < u^{IA}$
- $G_I(u) \geq G_A(u)$ *for* $u \geq u^{IA}$

Theorems 2 and 3 establish that there is a tradeoff between total matches and high-quality matches, but are not directly testable (as student utilities are not observed in practice). It is, however, possible to observe the fraction of students who receive one of their top$-k$ choices (in other words, the quantities $F_I(k), F_C(k), F_A(k)$ from Theorem 1). It turns out that the comparison between idiosyncratic and common priorities given in Theorems 2 and 3 continues to apply to this measure of student welfare.

Figure 1 plots the functions $F_I(\cdot), F_C(\cdot), F_A(\cdot)$ for a specific market. In this figure, we see that for $F_I(k) < F_C(k)$ for $k \leq 7$, and $F_I(k) > F_C(k)$ for $k \geq 8$. Theorem 4 establishes that this observation holds more generally: common priorities always result in more students receiving their first choice, and the functions F_I and F_C always have a single crossing point.

Theorem 4. *For any $\rho > 0, \ell > 1, \mathcal{D}$, there exists $k' \in \{2, \ldots, \ell\}$ such that*

- $F_I(k) < F_C(k)$ *for* $k < k'$
- $F_I(k) \geq F_C(k)$ *for* $k \geq k'$

I present the main intuition using a simple example in Section 4.2, and sketch proofs of Theorems 1 and 2 in Sections 5 and 6.

4.2. **A Simple Example.** The intuition for Theorem 2 is well-illustrated by the following example. Consider the market shown in Figure 2, in which there are three students and three high schools, each school has a single seat, and each student lists $\ell = 2$ schools uniformly at random. Consider the deferred acceptance algorithm from the perspective of someone who can observe its steps (i.e. proposals and rejections), but not the underlying preferences and priorities. Suppose that both s_1 and s_2 initially apply to h_1, which rejects s_2. If s_2 applies to h_3 (which happens to be the first choice of s_3), h_3 must reject either s_2 or s_3.

First, consider the consequences of each possibility. If h_3 rejects s_2, then s_2 will go unassigned, but s_1 and s_3 will both get their first choices. If h_3 rejects s_3, it is possible that all three students will be assigned, but at most one student will get her first choice. Thus, in this example, there is a tradeoff between matching students to their first choice, and matching more students overall.

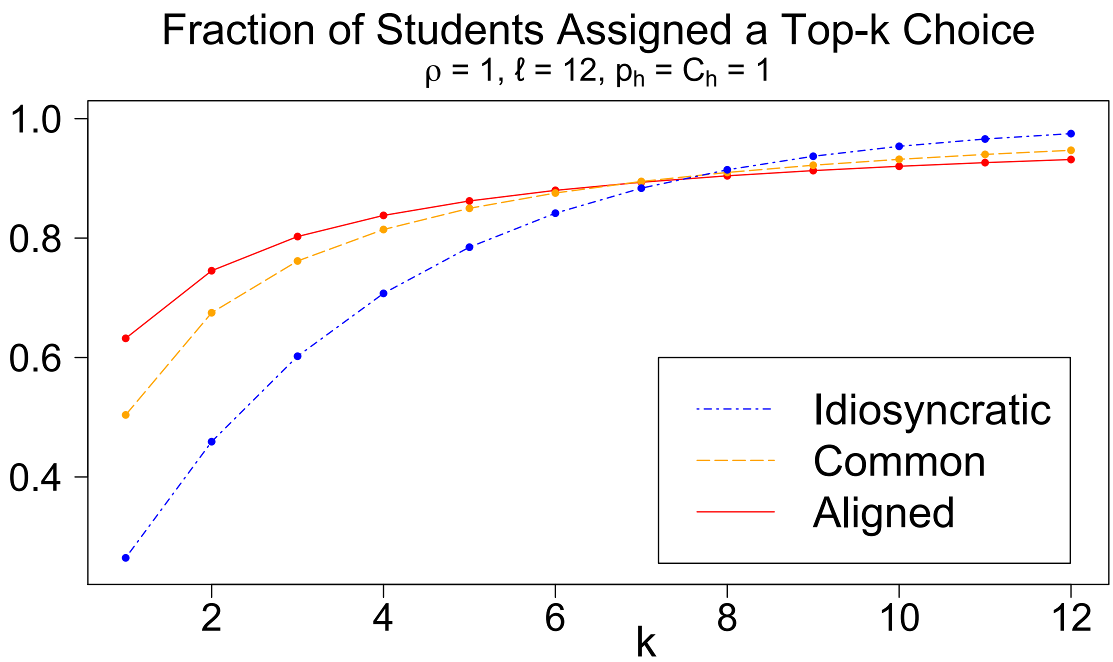

FIGURE 1. The fraction of students assigned a top$-k$ choice under each priority rule, when students list 12 schools uniformly at random, schools have unit capacity, and there are equal number of students and schools. Note that idiosyncratic priorities assign the fewest students to their top choice, but the most students overall. Aligned priorities match the most students to their top choice, but the fewest overall. Furthermore, the lines for idiosyncratic and common priorities cross exactly once. Theorems 2, 3, and 4 generalize these observations.

Next, consider the likelihood that h_3 rejects s_2 under each of the three priority structures. When school priorities are idiosyncratic, h_3 is equally likely to reject s_2 and s_3. When school priorities are common, the observer knows that s_2 ranks below s_1. The conditional probability that s_2 ranks below s_3 is $2/3$. When school priorities are aligned with student preferences, the observer knows (among other things) that h_3 is the worst of two possible matches for s_2, and is the best match for s_3. It can be computed that h_3 rejects s_2 with probability $8/9$. Thus, in this example, idiosyncratic priorities are most likely to match all three students, while aligned priorities are likely to result in more first choices (at the cost of fewer matches).

More generally, in any market, there will be repeated cases where a school must choose between two students s and s', one of which (say, s) ranks this school higher than the other does. The intuition for Theorems 2 and 3 is that

- If s is rejected, then fewer students get a "high quality" match.
- s' has fewer remaining options, and thus is more likely to go unmatched if rejected.
- s' is most likely to be rejected when priorities are aligned with student preferences (she is applying to a low-quality match), followed by common priorities (the fact that s' has been rejected more often suggests that she has lower priority), and is least likely to be rejected when priorities are idiosyncratic (in which case s' and s are on equal footing).

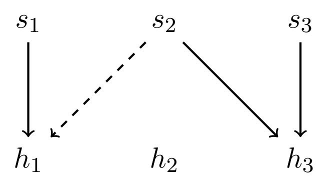

FIGURE 2. An example in which each school has a single seat and each student lists two schools. Students s_1 and s_2 both listed h_1 as their first choice. Student s_2 was rejected, and applied to h_3. If h_3 rejects s_2, then s_2 will go unassigned, but both s_1 and s_3 will get their first choice. If h_3 rejects s_3, it is possible that all three students will be assigned, but at most one student will get her first choice. The first case (h_3 rejects s_2) is most likely when preferences are aligned, and least likely when preferences are idiosyncratic.

5. Proof Outline: Exact Asysmptotics

In this section, I analyze the limiting behavior of the market under each priority rule. Specifically, I derive the asymptotic expressions given in Theorem 1. The key to the analysis is reducing the complex application process to a low-dimensional set of "nearly sufficient" summary statistics. I discuss this reduction in Section 5.1. I follow this discussion with more detailed derivations for each priority rule in Sections 5.2, 5.3, and 5.4. Finally, Section 5.5 briefly discusses the process through which this reasoning is made rigorous.

5.1. **Preliminaries.** Following in the footsteps of Knuth et al. (1990), Immorlica and Mahdian (2005), and Kojima and Pathak (2009), I make use of the "principle of deferred decisions." Roughly speaking, this means that I take the perspective of a Bayesian who observes the steps of the deferred acceptance algorithm (i.e. each application and rejection), without ever seeing the lists themselves. From this perspective, the matching procedure is a randomized algorithm - each time a student is asked to apply to a new school, there is a probability distribution that governs where this application will be sent.

It is quite complicated to compute the exact probability that a student's next application will be accepted: this computation depends on the full history of previous proposals and rejections. However, the assumptions of the model jointly imply that this probability can be well-approximated by a much lower-dimensional state, thereby allowing for tractable analysis. More specifically, if s applies to school h, define the *priority of the application* to be $U_h(s)$. For $t \in [0,1]$, will track the value $\Lambda(t) = N(t)/n$, where $N(t)$ is the total number of applications of priority at least t.

Several features of the model imply that the function $\Lambda(\cdot)$ captures essentially all relevant information about competition in the marketplace. First, the method by which student preferences are drawn implies that the early portion of a student's list conveys (almost) no information about the remaining schools on her list. To see this, fix $k < \ell$, and suppose that s lists schools $h_1, \ldots, h_k$ as her top k choices. Then for $h \notin \{h_1, \ldots, h_k\}$, the probability that h is ranked $(k+1)^{st}$ by s is

$$\frac{p_h}{\sum_{h' \notin \{h_1,\ldots,h_k\}} p'_h} = \frac{p_h}{n - \sum_{i=1}^k p_{h_i}} \approx \frac{p_h}{n}.$$

Thus, asymptotically, it is a reasonable approximation to assume that each new application is sent to school h with probability p_h/n, independently of all previous history.

A second important feature of the model is that schools are ex-ante homogeneous in their assessment of students. Thus, whenever a student s is asked to apply to a new school, how this school views s (i.e. the distribution of $U_h(s)$) is independent of *which* school s applies to.

It follows that if, over the course of the algorithm, there are a total of $N(t)$ applications of priority at least t, then the number of these applications sent to school h is approximately binomial with parameters $N(t)$ and p_h/n. For large n, this is well approximated by a Poisson random variable with mean $N(t)p_h/n = p_h\Lambda(t)$. Furthermore, for any finite set of schools, the number of applications sent to each school should be roughly independent.

An application of priority t to school h is accepted whenever the number of applications from higher-priority students is less than C_h. Given $\Lambda(\cdot)$, the probability of this event is approximately $\mathbb{P}(\mathbf{Po}(p_h\Lambda(t)) < C_h)$. Averaging over schools to which this application might be sent, we see that the "ex-ante" probability that an application of priority t is accepted is approximately

$$\sum_h \frac{p_h}{n}\mathbb{P}(\mathbf{Po}(p_h\Lambda(t)) < C_h) \to \mathbb{E}_{(p,C)\sim\mathcal{D}}\left[p\mathbb{P}(\mathbf{Po}(p\Lambda(t)) < C)\right] = \mathcal{V}(\Lambda(t)).$$

This fact makes it possible to compute aggregate student outcomes. Additionally, these computations can be used to derive consistency conditions that uniquely determine Λ. Each of the following subsections discusses these consistency conditions for a particular priority rule. I use $\Lambda_I(t), \Lambda_C(t), \Lambda_A(t)$ to denote $N(t)/n$ when priorities are idiosyncratic, common, and aligned, respectively. Define $\lambda_I = \Lambda_I(0), \lambda_C = \Lambda_C(0), \lambda_A = \Lambda_A(0)$ to be the (normalized) number of applications sent over the course of each procedure. At points below, I use the following facts.

Fact 1. *Let X be a random variable on $\mathbb{N}$. Then $\mathbb{E}[X] = \sum_{k=0}^{\infty}\mathbb{P}(X > k)$.*

Proof.

$$\mathbb{E}[X] = \sum_{j=0}^{\infty} j\,\mathbb{P}(X = j) = \sum_{j=0}^{\infty}\sum_{k=0}^{j-1}\mathbb{P}(X = j) = \sum_{k=0}^{\infty}\sum_{j=k+1}^{\infty}\mathbb{P}(X = j) = \sum_{k=0}^{\infty}\mathbb{P}(X > k),$$

□

Fact 2. *$\mathcal{E}'(\lambda) = \mathcal{V}(\lambda)$ for all λ.*

Proof. First, note that

$$\begin{aligned}\frac{d}{d\lambda}\mathbb{P}(\mathbf{Po}(\lambda) > C) &= \frac{d}{d\lambda}\left(1 - \sum_{0}^{C}\frac{e^{-\lambda}\lambda^k}{k!}\right)\\ &= \sum_{k=0}^{C}\frac{e^{-\lambda}\lambda^k}{k!} - \sum_{1}^{C}\frac{e^{-\lambda}\lambda^{k-1}}{(k-1)!}\\ &= \frac{e^{-\lambda}\lambda^C}{C!} = \mathbb{P}(\mathbf{Po}(\lambda) = C).\end{aligned}$$

Next, apply Fact 1, from which it follows that

$$\begin{aligned}\frac{d}{d\lambda}\mathbb{E}[\min(\mathbf{Po}(p\lambda), C)] &= \frac{d}{d\lambda}\sum_{k=0}^{C-1}\mathbb{P}(\mathbf{Po}(p\lambda) > k)\\ &= \sum_{k=0}^{C-1} p \cdot \mathbb{P}(\mathbf{Po}(p\lambda) = k)\\ &= p \cdot \mathbb{P}(\mathbf{Po}(p\lambda) < C).\end{aligned}$$

□

Recall that $\mathcal{E}(\lambda)$ represents the number of filled seats, and $\mathcal{V}(\lambda)$ represents the probability that new applications are sent to a school with a vacancy, given that $n\lambda$ applications have been sent so far. Thus, Fact 2 simply states that the marginal change in enrollment due to a small number new applications is equal to the number of such applications, times the probability that each application is sent to a school with a vacancy.

5.2. **Idiosyncratic Priorities.** The key to deriving consistency conditions for the case of idiosyncratic priorities is that regardless of history, each new application has priority distributed uniformly on $[0, 1]$. It follows that for all $t \in [0, 1]$,

$$\Lambda_I(t) = (1-t)\Lambda_I(0) = (1-t)\lambda_I.$$

First, suppose that λ_I is given. From λ_I, it is possible to compute outcomes for an individual student. Recall that an application of priority t is accepted with probability $\mathcal{V}(\Lambda_I(t))$ (averaging across schools to which the application might be sent). Because the priority of each application is uniform on $[0, 1]$, the ex-ante probability that a new application is accepted, given Λ_I, is

$$\int_0^1 \mathcal{V}(\Lambda_I(t))dt = \int_0^1 \mathcal{V}((1-t)\lambda_I)dt = \mathcal{E}(\lambda_I)/\lambda_I = \alpha(\lambda_I),$$

where the second equality follows from Fact 2. Note that this expression is intuitive: it states that the probability that a new application is accepted is equal to the total number of seats filled, divided by the total number of applications sent.

Having computed the anticipated acceptance rate α given the application level λ_I, I now derive λ_I, given a hypothesized acceptance rate α. In particular, if each application is accepted with probability α, then by Fact 1 the average number of applications sent by students is $\mu(\alpha)$. Counting the total number of applications from the perspective of schools and students yields the following consistency equation for λ_I:

$$|\mathcal{H}|\,\lambda_I = |\mathcal{S}|\,\mu(\alpha(\lambda_I)).$$

Noting that $|\mathcal{H}|\,/\,|\mathcal{S}| = \rho$ yields the equation from Theorem 1. To see that there is a unique solution λ_I for any choice of ρ, ℓ, and $\mathcal{D}$, multiply each side by $\alpha(\lambda_I)$ to get

$$\mathcal{E}(\lambda_I) = \rho \cdot \alpha(\lambda_I)\,\mu(\alpha(\lambda_I)).$$

It is easy to verify that the left side above increases (strictly) from 0 (at $\lambda_I = 0$), while the right side above decreases (strictly) from ρ (at $\lambda_I = 0$) to 0 (as $\lambda_I \to \infty$).

Given λ_I, each application sent is accepted with ex-ante probability $\alpha(\lambda_I)$, and therefore

$$F_I(k) = 1 - (1 - \alpha(\lambda_I))^k.$$

Because the priority of student s at school h is independent of $U_s(h)$, it follows that the number of schools h such that $U_s(h) > u$ and h would accept s is distributed as a binomial with ℓ trials and success probability $(1-u)\alpha(\lambda_I)$, and therefore

$$G_I(u) = (1 - (1-u)\alpha(\lambda_I))^\ell.$$

5.3. **Common Priorities.** Note that for a student of priority t, applications from students of priority below t are irrelevant. Thus, it is possible to analyze the market from a "top down" approach. This yields a differential equation for Λ_C, with initial condition $\Lambda_C(1) = 0$.

To see how this differential equation is derived, suppose that $\Lambda_C(t)$ is given, and consider the number of applications sent by applicants of priority t. As explained previously, each application sent by such a student is accepted with ex-ante probability $\mathcal{V}(\Lambda_C(t))$. It follows from Fact 1 that the expected number of applications sent by this student is

$$\sum_{k=0}^{\ell-1} (1 - \mathcal{V}(\Lambda_C(t)))^k = \mu(\mathcal{V}(\Lambda_C(t))).$$

For small Δ, this is also close to the expected number of applications sent by each student with priority between $t - \Delta$ and t. Because the number of such students is $\Delta\,|\mathcal{S}| = \Delta\rho n$, it follows that for small Δ, the number of applications sent by these students is

$$N(t-\Delta) - N(t) \approx n\,\rho\,\Delta\,\mu(\mathcal{V}(\Lambda_C(t))).$$

Divide by $n \cdot \Delta$ and let $\Delta \to 0$ to conclude that $\Lambda_C(t)$ is characterized by the differential equation

$$\Lambda_C'(t) = -\rho\ \mu(\mathcal{V}(\Lambda_C(t))). \tag{8}$$

This allows us to compute the function $\Lambda_C(\cdot)$. Because a student of priority t has each application accepted with ex-ante probability $\mathcal{V}(\Lambda_C(t))$, we can average across students to get the expressions for $F_C(\cdot)$ and $G_C(\cdot)$ given in Theorem 1.

5.4. **Aligned Priorities.** As in the case of common priorities, it is possible to analyze the market from the top down, starting with the match of the highest quality. The key to the analysis is that for any $u \in [0,1]$, the number of matches of quality at least u is equal to the number of matches of priority at least u. Analyzing this number from the perspectives of students and schools yields a differential equation that characterizes Λ_A.

Of course, $\Lambda_A(1) = 0$. Suppose that $\Lambda_A(t)$ is known for $t \geq u$. Student s fails to get a match of quality u if every school h with $U_s(h) > u$ rejects s. Because match qualities are distributed as ℓ iid $U[0,1]$ random variables, and because a match of quality t is accepted with probability $\mathcal{V}(\Lambda_A(t))$, this occurs with probability

$$G_A(u) = \left(1 - \int_u^1 \mathcal{V}(\Lambda_A(t))dt\right)^\ell.$$

It follows that the number of students who receive a match of quality exceeding u is

$$|\mathcal{S}|\left(1-\left(1-\int_u^1 \mathcal{V}(\Lambda_A(t))dt\right)^{\ell}\right).$$

Conversely, viewed from the schools' perspective, the number of matches of priority above u is

$$|\mathcal{H}|\cdot\mathcal{E}(\Lambda_A(u)).$$

Equate these expressions, note that $|\mathcal{S}|=\rho\,|\mathcal{H}|$, and differentiate with respect to u to get

$$\Lambda_A'(u)\mathcal{V}(\Lambda_A(u)) = -\rho\,\ell\left(1-\int_u^1 \mathcal{V}(\Lambda_A(t))dt\right)^{\ell-1}\mathcal{V}(\Lambda_A(u)).$$

Canceling common terms gives a differential equation for Λ_A given in Theorem 1, from which expressions for $F_A(\cdot)$ and $G_A(\cdot)$ follow.

5.5. **Formalizing the Proof.** To make the above logic rigorous, I study stochastic processes X_I, X_C, X_A corresponding to the applications and rejections in the deferred acceptance algorithm. In the case of idiosyncratic and common priorities, one step of the process corresponds to adding a single student to the market. In the case of aligned priorities, one step of the process corresponds to the sending of a single application.

The state space for these processes is very large, as it includes the application history for each student. However, in each case, a lower-dimensional state space "nearly suffices" to describe expected changes. More specifically, I show that there exists a reduced state $Y(t)=y(X(t))$ (which has a dimension that does not depend on n)[16] and a function $f(\cdot)$ such that for any history $H_t=(X(0),\ldots,X(t))$,

$$\mathbb{E}[Y(t+1)-Y(t)|H_t]\approx f(Y(t)/n),$$

with error that vanishes as n grows.

Of course, this does not imply that $Y(t+1)-Y(t)\approx f(Y(t)/n)$, due to random fluctuation. However, over a large number of steps, this randomness should cancel out; we might hope that

$$Y(t+w)-Y(t)\approx\sum_{k=0}^{w-1} f(Y(t+k)/n)\approx wf(Y(t)/n),$$

provided that two conditions hold:

1. $Y(t+1)-Y(t)$ is unlikely to be large. This implies that
   - $Y(t+w)-Y(t)=\sum_{k=0}^{w-1} Y(t+k+1)-Y(t+k)$ is not dominated by any large jumps (so that the "law of large numbers" might apply)
   - $Y(t+k)/n\approx Y(t)/n$, so long as k is small relative to n.
2. f is sufficiently smooth, so that $f(Y(t+k)/n)\approx f(Y(t)/n)$.

If these conditions hold, then standard concentration bounds for submartingales imply that $Y(t)\approx nz(t/n)$, where $z(\cdot)$ is the solution to the differential equation $z'(y)=f(y)$. This technique

[16] For example, in the case of common priorities and uniform popularity, this is simply the number of schools with k vacancies, for each value of k.

has been developed extensively by Wormald (1999), and I make use of a minor modification to his Theorem 5.1 (Theorem 7 in the appendix).

Applying this theorem to each priority rule implies that the variables $Y(t)$ concentrate, but the system $z' = f$ still has very high dimension. The final step of the proof is to demonstrate that the solution to this system can be expressed as a function of a the one-dimensional variable Λ, as given in Theorem 1.

6. Proof Outline: Comparing Priority Rules

6.1. **Preliminaries: Sequential Deferred Acceptance.**

It is well-known that the order of proposals does not affect the matching produced by the deferred acceptance algorithm. I analyze a sequential version of the algorithm which has $|\mathcal{S}|$ "rounds." Each round consists of a (random) number of proposals, and round i terminates with a matching which is stable in the submarket containing only students $\{s_1, \ldots, s_i\}$ (and is student-optimal among all such matchings).

The algorithm for advancing from round i to $i+1$ is as follows. First, invite student $i+1$ to apply to her most-preferred school. This application will result in one of three possible outcomes:

- Sent to a school with a vacancy.
- Accepted, triggering rejection of another student.
- Rejected.

In the first case, s_{i+1} is accepted, and round $i+1$ terminates. In the second and third cases, one student has just been rejected. If this student has no more schools on her list, round $i+1$ terminates. Otherwise, this student applies to her next most-preferred school. This application again leads to one of the three cases listed above. The process continues until either a vacancy is filled or a student is rejected from the last school on her list.

6.2. **Idiosyncratic vs. Common Priorities.**

6.2.1. *Overview of Proof Technique.* The proof that $F_I(\ell) \geq F_C(\ell)$ proceeds as follows. It is intuitively clear that $\rho\, n\, F_I(\ell) = n\,\mathcal{E}(\lambda_I)$, as both sides represent the number of matched students. Formally, this can be shown by noting that

$$\rho\, F_I(\ell) = \rho\, \alpha(\lambda_I)\, \mu(\alpha(\lambda_I)) = \mathcal{E}(\lambda_I),$$

where we have used the definition of F_I from Theorem 1, along with the definitions for μ and $\mathcal{E}$. Analogously,

$$\begin{aligned}
\rho\, F_C(\ell) &= \rho \int_0^1 \mathcal{V}(\Lambda_C(t))\mu(\mathcal{V}(\Lambda_C(t)))dt \\
&= -\int_0^1 \mathcal{V}(\Lambda_C(t))\Lambda_C'(t)dt \\
&= \int_0^{\lambda_C} \mathcal{V}(\lambda)d\lambda = \mathcal{E}(\lambda_C),
\end{aligned}$$

where the first line follows from the definitions of F_C and μ, the second from the characterizing differential equation for Λ_C, and the last from substitution and Fact 2, which states that $\mathcal{E}'(\lambda) =$

$\mathcal{V}(\lambda)$. Because the function $\mathcal{E}(\cdot)$ is monotonic, to show that $F_I(\ell) \geq F_C(\ell)$, it suffices to show that $\lambda_I \geq \lambda_C$. I do this by considering these quantities as functions of ρ (for fixed $\ell, \mathcal{D}$), and showing that (using the obvious notation)

$$\text{If } \rho_I, \rho_C \text{ satisfy } \lambda_I(\rho_I) = \lambda_C(\rho_C), \text{ then } \lambda_I'(\rho_I) > \lambda_C'(\rho_C). \tag{9}$$

Since $\lambda_I(0) = 0 = \lambda_C(0)$, it follows that $\lambda_I(\rho) \geq \lambda_C(\rho)$ for all ρ.

The proof in the Appendix differentiates the implicit expressions for λ_I and λ_C with respect to ρ, and compares the resulting quantities. This process is not particularly illuminating, so here I sketch an alternate proof that sheds more light on the structure of the problem.

Increasing ρ by ϵ corresponds to adding $n\epsilon$ additional students to the market. Thus, the derivative $\lambda'(\rho)$ intuitively corresponds to the expected number of additional applications triggered by adding a single student to a market with $n\rho$ students. We must show that when the number of vacancies in the two markets are identical, the addition of a single student triggers more new applications in the market with idiosyncratic priorities.

When student s is added to the market, she applies to schools until one of the following occurs:

- s applies to a school with a vacancy, and is accepted.
- s applies to a school without a vacancy, is accepted, and triggers the rejection of another student.
- s reaches the end of her list.

If the second case occurs (i.e. a new student s' is rejected), then the process continues with s' playing the role of s. Eventually, the process will terminate with either the filling of a vacant seat or a student reaching the end of her list.

I prove (9) by constructing two Markov chains, (X_t^I, A_t^I) and (X_t^C, A_t^C). These chains approximately describe the sequence of proposals and rejections that result when a single student is added to the market. The values X_t^I, X_t^C belong to the set $\mathbb{N} \cup \{R, V, E\}$. $X_t = k \in \mathbb{N}$ corresponds to the case where the current proposing student has been rejected from her first k schools.

The initial state of the chain is $X_0 = 0$ (as the newly-added student has yet to be rejected from any school). From state $k \in \mathbb{N}$, the possible values for X_{t+1} are $\{k+1, R, V, E\}$. A transition to $k+1$ represents the case where the student applies to a new school and is rejected, while transitions to V, R, E correspond to the three cases above: the student fills a **V**acancy, triggers **R**ejection of another student, or reaches the **E**nd of her list. State E is reachable only from state ℓ (i.e. a student reaches the end of her list if and only if she is rejected ℓ times). If $X_t = R$, then $X_{t+1} \in \mathbb{N}$ represents the number of times that the newly-rejected student has been rejected (in total, not just in this round). The states V and E are absorbing (indicating the end of the round). Define $\tau^I = \min\{t : X_t^I \in \{V, E\}\}$ and $\tau^C = \min\{t : X_t^I \in \{V, E\}\}$ to be the time taken to reach the end of the round (the yet-unspecified transition dynamics imply that both of these quantities are almost surely finite). The value $A_t \in \mathbb{N}$ intuitively represents the number of applications sent during the first t steps of the chain, so that the quantities $A_{\tau^I}^I, A_{\tau^C}^C$ correspond to the number of new applications that are sent during the entire round. A visualization of the Markov chain X^I is given in Figure 3.

For those uneasy with the informal tone above, the proof of (9) involves three steps:

I. $\mathbb{E}[A^I_{\tau^I}] = \lambda'_I(\rho_I)$.
II. $\mathbb{E}[A^C_{\tau^C}] = \lambda'_C(\rho_C)$.
III. $\mathbb{E}[A^I_{\tau^I}] > \mathbb{E}[A^C_{\tau^C}]$.

Although I attempt to make the argument transparent by using terminology that exposes the interpretations of X_t and A_t, the rigor of the proof does not rely on these interpretations being "correct." Instead, it rests only on establishing the three points listed above.

6.2.2. *Transition Dynamics.* First, consider the simpler dynamics of a common priority rule. It is most convenient to assume that students apply in order of decreasing priority. In this case, s never triggers the rejection of another student (i.e. the state R is never reached). Thus, the round simply consists of s applying until she either applies to a school with a vacancy or reaches the end of her list. In expectation, the first $n\rho_C$ students have sent a total of $n\lambda_C(\rho_C)$ applications so far, implying that each application from s is sent to a school with a vacancy with (ex-ante) probability $\mathcal{V}(\lambda_C(\rho_C))$. Based on this reasoning, the transition probabilities for the chain (X^C_t, A^C_t) are defined as follows (to avoid notational clutter, use $\mathcal{V}$ to denote $\mathcal{V}(\lambda_C(\rho_C))$).

- If $X^C_t = \ell$, transition to $(X^C_{t+1}, A^C_{t+1}) = (E, A^C_t)$.
- If $X^C_t = k < \ell$, then $A^C_{t+1} = A^C_t + 1$. X^C_{t+1} is V with probability $\mathcal{V}$, and $k+1$ otherwise.

Recall that A^C_t corresponds to the number of applications sent due to the addition of s to the market. The first bullet above corresponds to the case where s has reached the end of her list (and thus does not apply anywhere). The second corresponds to the case where s applies to a new school (filling a vacancy with probability $\mathcal{V}$ and otherwise is rejected), and thus A^C is incremented.

From these transition probabilities, an application of Fact 1 reveals that the expected number of applications sent by s is

$$\mathbb{E}[A^C_{\tau^C}] = \sum_{k=0}^{\infty} \mathbb{P}(A^C_{\tau^C} > k) = \sum_{k=0}^{\ell-1} (1-\mathcal{V})^k = \mu(\mathcal{V}),$$

and it is straightforward to verify that this expression matches $\lambda'_C(\rho_C)$.

Next, we turn to the (considerably more complicated) case of idiosyncratic priorities. In this case, s may trigger the rejection of another student (and thus the state X_t need not increase monotonically). As before, we expect that each application is sent to a school with a vacancy with (ex-ante) probability $\mathcal{V} = \mathcal{V}(\lambda_I(\rho_I))$. Each application is rejected with probability $1-\alpha = 1-\alpha(\lambda_I(\rho_I))$, implying that it triggers the rejection of another student with probability $\alpha - \mathcal{V}$. Based on this reasoning, define transition probabilities for our chain as follows.

- If $X^I_t = \ell$, transition deterministically to $(X^I_{t+1}, A^I_{t+1}) = (E, A^I_t)$.
- If $X^I_t = k < \ell$, then $A^I_{t+1} = A^I_t + 1$. Transition to $X^I_{t+1} = V$ with probability $\mathcal{V}$, to $X^I_{t+1} = k+1$ with probability $1-\alpha$, and to $X^I_{t+1} = R$ otherwise.
- If $X^I_t = R$, then $A^I_{t+1} = A^I_t$. For $k \in \{1, \ldots, \ell\}$, $\mathbb{P}(X^I_{t+1} = k) = \frac{(1-\alpha)^{k-1}}{\sum_{j=1}^{\ell}(1-\alpha)^{j-1}}$

The transition from R corresponds to selecting the number of times that the newly-rejected student s' has been rejected (in total). Here, I give an explanation for the transition probabilities from R, although understanding this point is not crucial to the remainder of the argument.

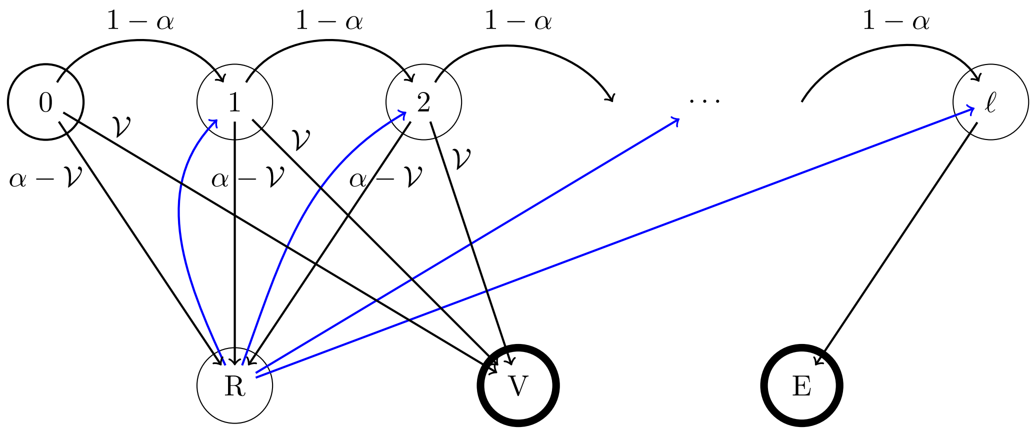

FIGURE 3. A visualization of the Markov chain that summarizes the effect of adding a single student to the market when priorities are idiosyncratic. The states $0, \ldots, \ell$ represent the number of times that the current "applying student" has been rejected. Each application is rejected with probability $1-\alpha$, is sent to a school with a vacancy with probability $\mathcal{V}$, and triggers the rejection of another student with probability $\alpha - \mathcal{V}$. In the latter case, this newly rejected student is asked to apply. The chain has terminal states at V (a student applies to a school with a vacancy) and E (a student reaches the end of her list).

Consider the state of the market before s is added. As given by $F_I(\cdot)$, for $k \in \{1, \ldots, \ell\}$, the fraction of students who are currently assigned to their k^{th} choice is $F_I(k) - F_I(k-1)$. When s triggers a rejection, the rejected student is selected uniformly at random among all matched students, implying that the probability that this student has just been rejected for the k^{th} time is $\frac{F_I(k)-F_I(k-1)}{F_I(\ell)}$, which is equal to the probability given above.

From these transition probabilities, it is possible to compute that

$$\mathbb{E}[A^I_{\tau^I}] = \frac{\mu(\alpha)}{1 + (\alpha - \mathcal{V})\frac{\mu'(\alpha)}{\mu(\alpha)}},$$

and that this expression does in fact correspond to the quantity $\lambda_I'(\rho_I)$, as computed from differentiating the implicit expressions in Theorem 1.

It remains is to show that $\mathbb{E}[A^I_{\tau^I}] > \mathbb{E}[A^C_{\tau^C}]$. The natural "elegant" approach would be to couple the chains so that $A^I_{\tau^I} \geq A^C_{\tau^C}$ with probability one. Unfortunately, this is not feasible, because the distribution of $A^I_{\tau^I}$ does not stochastically dominate the distribution of $A^C_{\tau^C}$. To see this, note that under a common priority rule, the round consists of a single application if and only if the first application is sent to a school with a vacancy. Under an idiosyncratic priority rule, there is also the possibility that the first application is accepted, and triggers the rejection of a student who has now been rejected ℓ times (and therefore cannot reapply). Stated formally,

$$\mathbb{P}(A^C_{\tau^C} = 1) = \mathcal{V} < \mathbb{P}(A^I_{\tau^I}) = \mathcal{V} + (\alpha - \mathcal{V})p_{R\ell},$$

where $p_{R\ell}$ is the probability of transitioning directly from state R to state ℓ (i.e. the probability that a newly-rejected student was just rejected for the ℓ^{th} time).

The challenge is that the number of applications sent by student s has no relationship with the number of times that the student whose rejection is triggered by s has been rejected. To rectify this, I define a new chain $(\hat{X}_t^I, \hat{A}_t^I)$ such that the expected number of applications sent by each student is the same, as is the probability that this student triggers the k^{th} rejection of another student, but in which passing through R never causes $\hat{X}_t$ to increase by more than one. This construction ensures that $\mathbb{E}[\hat{A}^I_{\hat{\tau}^I}] = \mathbb{E}[A^I_{\tau_I}] = \lambda_I'(\rho_I)$, and also implies a natural coupling between $(\hat{X}_t^I, \hat{A}_t^I)$ with (X_t^C, A_t^C) such that

$$\hat{A}^I_{\hat{t}} = A^C_t \Rightarrow \hat{X}^I_{\hat{t}} \leq X^C_t. \tag{10}$$

From this, it follows that $\hat{X}^I$ reaches state E only if X^C does. Because the probability of transitioning to V with a new application is constant across chains and states, it follows that

$$\lambda_C'(\rho_C) = \mathbb{E}[A^C_{\tau^C}] < \mathbb{E}[A^I_{\hat{\tau}^I}] = \mathbb{E}[A^I_{\tau_I}] = \lambda_I'(\rho_I).$$

6.3. **Common vs. Aligned Priorities.** I now sketch the proof that $G_C(0) < G_A(0)$ (or equivalently, that $F_C(\ell) > F_A(\ell)$). Recall from the discussion in Section 5 that under both common and aligned priorities, there is a unique stable matching, which can be constructed through a simple greedy procedure.

P1 Common priorities: the top-ranked student must receive her most preferred option. Form this match, and apply this principle iteratively.

P2 Aligned priorities: the match of highest quality must be in any stable matching. Form this match, and apply this principle iteratively.

To gain intuition, let us first imagine running these procedures on the example from Figure 2, in which three students each list two of three equally popular unit-capacity schools. We will take the perspective of a Bayesian observer who not only sees matches as they are formed, but also, whenever a school fills its last seat, *learns which students listed this school* (but not where on their list this school appeared). From the perspective of this observer, each of P1 and P2 begins by forming a match uniformly at random. Without loss of generality, this match is between s_1 and h_1. Suppose that we now observe that s_2 listed h_1, and that s_3 did not.

This implies that s_3 has listed both remaining schools, but s_2 has listed only one. Thus, if s_2 is the next student to match, all three students will be assigned. By contrast, if the next student to match is s_3, there is a 1/2 chance that s_3 will attend the final school listed by s_2, leaving s_2 unmatched. When priorities are common, s_2 and s_3 are equally likely to be the second student to match. When priorities are aligned, the chance that the highest-priority remaining match involves s_3 is 2/3 (and thus the expected number of matches is lower).

To turn this observation into a proof, I construct two Markov chains, corresponding to procedures P1 and P2. The state space of each chain tracks the number of unfilled seats for each $h \in \mathcal{H}$, as well as the number of students who listed precisely k schools that still have vacancies, for $k \in \{0, \ldots, \ell\}$. The key difference between the matching procedures is that at each step, P1 selects a student uniformly at random (interpreted as the highest-priority remaining student), whereas P2 selects each student *in proportion to the number of high schools on her list that still have vacancies.* Therefore, the greedy procedure corresponding to aligned priorities tends to match students with many remaining options, forcing students with few remaining options to wait (and risk that their final option gets matched). If N_k^C, N_k^A track the number of students who listed k schools that still have vacancies under common and aligned priorities, respectively, I show that it is possible to couple the two chains such that N_k^C stochastically dominates N_k^A. Because the number of assigned students is exactly the number of steps until N_k is zero for all k, this implies that common priorities result in more matches. Figure 4 provides an illustration of each chain, and the coupling between them.

7. Variable List Length

The model above assumes that students all submit lists of the same length, ℓ. In practice, of course, students submit lists of varying lengths. Is the assumption of a common list length an innocuous simplification, or a consequential one?

To answer this question, I consider a model in which student s submits a list of length ℓ_s, with $\mathbb{P}(\ell_s \leq k) = \mathcal{L}(k)$. Let $\overline{\mathcal{L}}(k) = 1 - \mathcal{L}(k)$ be the probability that a student lists more than k schools. Conditioned on the value ℓ_s, student lists are drawn as before.[17] In the appendix, I generalize the results on idiosyncratic and common priorities presented in Theorem 1 to arbitrary list length distributions.[18] In particular, it turns out that the fixed point equation (5) defining λ_I and the differential equation (6) defining $\Lambda_C(\cdot)$ are unchanged, if the function μ is generalized as follows:

$$\mu(\alpha) = \sum_{k=0}^{\infty} (1-\alpha)^k \overline{\mathcal{L}}(k). \tag{11}$$

While generalizing the list length distribution does not notably complicate the asymptotic analysis of Theorem 1, it is enough to change the conclusions of Theorems 2 and 3. In particular, there exist list length distributions $\mathcal{L}$ such that more students are assigned when schools have common priorities than when school priorities are idiosyncratic. Below, I explain why this might occur, and derive a simple sufficient condition on $\mathcal{L}$ which guarantees that Theorem 2 continues to hold, as well as a condition under which common priorities result in more matches than independent priorities.

To see why a common priority rule might result in more matches, return to the example shown in Figure 2. Suppose that student list lengths vary: some students are "picky" and list only a single school, while others list all three options. In this case, when h_3 chooses between s_2 and s_3, the observer knows that s_3 will apply (and be accepted) to h_2 if rejected from h_3. Meanwhile, there is a chance that s_3 listed only h_3, and will go unassigned if rejected. Thus, in this example, the

[17] In practice, it is likely that students with short and long lists differ systematically in *which* schools they list. One could in principle work with a model in which the popularity of each school varied with the length of the student's list. This gives the modeler a great deal of flexibility, and (unsurprisingly) many conclusions are possible with such a model. It is perhaps more surprising that the conclusion $G_I(0) < G_C(0)$ may fail to hold even when school popularity and list length are assumed to be independent.

[18] There are several natural extensions of the model of aligned priorities to the case where list lengths differ. One possibility is that conditioned on the value of ℓ_s, the priority of student s at her k^{th} choice is distributed as the k^{th} largest of ℓ_s independent $U[0,1]$ draws. This model is straightforward to analyze (indeed, the coupling argument used to prove that $G_C(0) < G_A(0)$ continues to apply to this model), but has the (perhaps unnatural) feature that students who submit longer lists are more likely to have high priority at their first choice. If the list length distribution is bounded (say, $\ell_s \leq \overline{\ell}$ for all s), then another possibility is that the priority of student s at her k^{th} choice (for $k \leq \ell_s$) is distributed as the k^{th} largest of $\overline{\ell}$ $U[0,1]$ draws (regardless of the value of ℓ_s). To avoid taking a stand on this question, this section focuses on the analysis of idiosyncratic and common priorities.

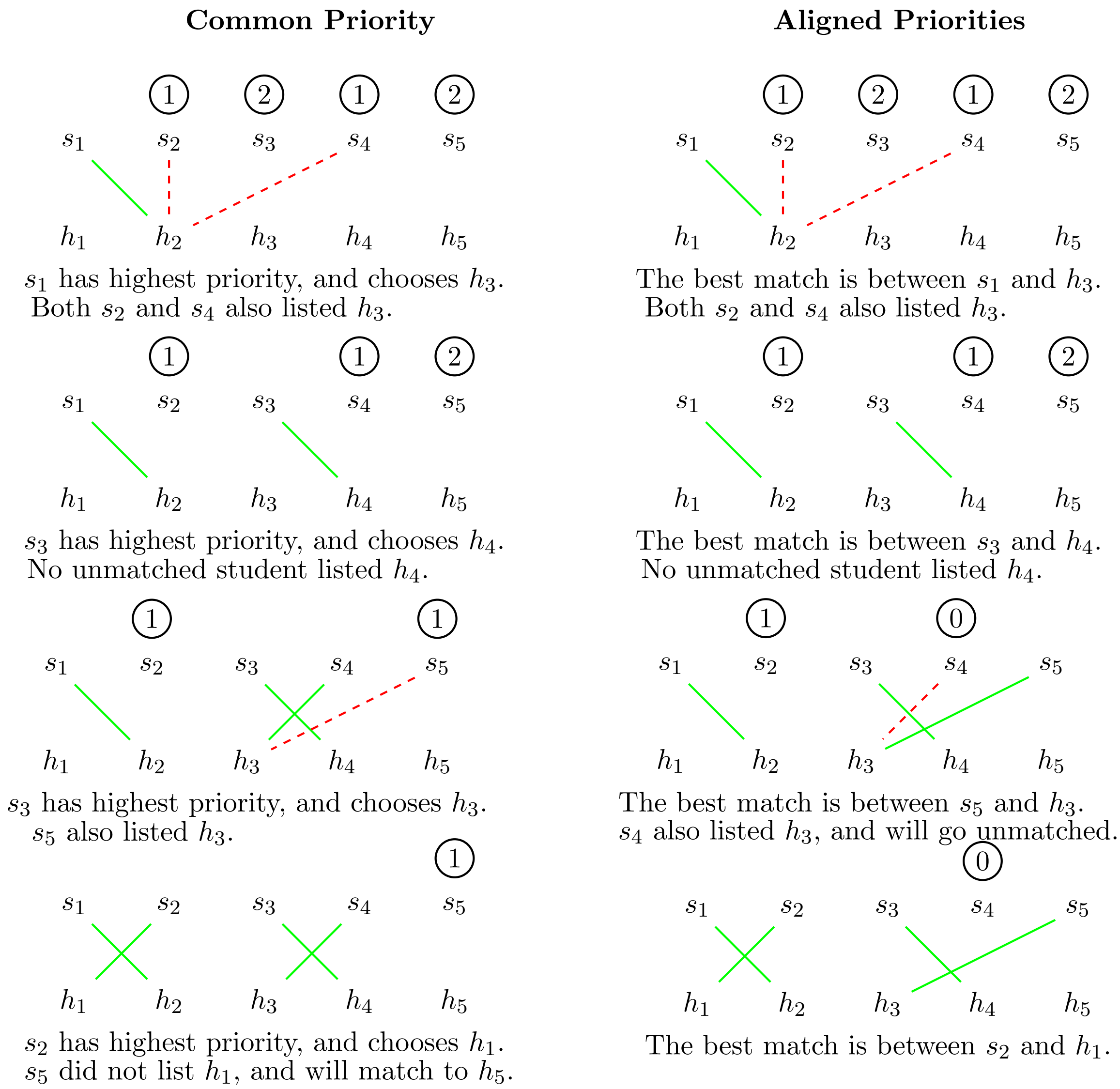

FIGURE 4. A visualization of the coupling used to prove that more students match when priorities are common, rather than aligned. Each step selects one unmatched student and matches this student to her most preferred remaining high school. If this causes the school to reach its capacity, we reveal which yet-unmatched students also listed this school (marked with red dashed lines). The numbers above each student denote the number of schools on that student's list which still have vacancies. Under a common priority rule, students are selected in a uniform random order, whereas when priorities are aligned, students are selected in proportion to the number of schools on their list that still have vacancies. In Appendix B, I prove that the processes can be coupled such that the vector of "listed schools with vacancies" under common priorities dominates the corresponding vector for aligned priorities.

goals of "matching many students" and "matching many students with their first choice" are not in conflict, and a common priority rule is more likely to achieve both goals (i.e. reject s_2).

Why does the tradeoff between quantity and quality disappear in this example? The intuition in Section 4.2 relied on two facts:

1. A common priority rule tends to reject students that have already been rejected.
2. These students have fewer remaining options (and thus are less likely to eventually apply to a school with a vacancy).

When student lists differ in length, the first of these conclusions continues to hold, but the second may not. In the example above, s_2 is actually *more* likely than s_3 to apply to a school with a vacancy (since it has already been revealed that s_2 listed more than one, and therefore all three, schools). Therefore a common priority rule will result in more matches.

This reasoning suggests that so long as students who have been rejected more often tend to have fewer *remaining* options than those that have been rejected less often, Theorem 2 should continue to hold, and idiosyncratic priority rules should result in more matches. Theorem 5 formalizes this intuition, for a suitable interpretation of the phrase "tend to have fewer remaining options." More specifically, define the *hazard rate* of $\mathcal{L}$ at k to be $1-\overline{\mathcal{L}}(k)/\overline{\mathcal{L}}(k-1)$ (this is simply $\mathbb{P}(\ell_s = k | \ell_s \geq k)$). It turns out that if the hazard rate is increasing, then common priorities result in fewer matches than idiosyncratic priorities; if the hazard rate is decreasing, the opposite conclusion holds.

Theorem 5. *If $\mathcal{L}$ has a non-decreasing hazard rate, then for any $\rho, \mathcal{D}$, it holds that $F_I(\infty) \leq F_C(\infty)$. If $\mathcal{L}$ has a non-increasing hazard rate, then for any $\rho, \mathcal{D}$, it holds that $F_I(\infty) \geq F_C(\infty)$.*

The proof is nearly identical to the case where all lists are of length ℓ. Modify the Markov chains $(\hat{X}^I_t, \hat{A}^I_t)$ and (X^C_t, A^C_t) described in Section 6.2 (and shown in Figure 3) so that from state k, the chain transitions to E with probability $1 - \overline{\mathcal{L}}(k)/\overline{\mathcal{L}}(k-1)$, and the transition probabilities to R, V, and $k+1$ are scaled down accordingly. It is straightforward to show that this modification ensures that $\mathbb{E}[\hat{A}^I_{\hat{\tau}^I}] = \lambda'_I(\rho_I)$ and $\mathbb{E}[A^C_{\tau^C}] = \lambda'_C(\rho_C)$. Furthermore, for any list length distribution, it remains possible to couple $(\hat{X}^I_t, \hat{A}^I_t)$ and (X^C_t, A^C_t) such that (10) holds: whenever $\hat{A}^I_{\hat{t}} = A^C_t$, it follows that $\hat{X}^I_{\hat{t}} \leq X^C_t$. If the hazard rate is increasing, this coupling implies that the chain corresponding to idiosyncratic priorities is less likely to reach state E (which corresponds to a student reaching the end of her list), and more likely to reach state V (which corresponds to a vacancy being filled). If the hazard rate is decreasing, the opposite conclusion holds.

Examining the data from New York City presented by Abdulkadiroglu et al. (2009), the list length distribution has a hazard rate that is essentially increasing (see Figure 5), and thus Theorem 5 suggests that a common tie-breaking procedure should result in fewer assigned students (consistent with the simulation results presented by Abdulkadiroglu et al. (2009)).

Note that even if a common priority rule assigns more students to their first choice and more students overall, it does not follow that the common priority rule rank dominates idiosyncratic priorities.[19] However, an analogue of the single crossing result from Theorem 3 continues to hold, if we consider only students who have applied to the same number of schools. Define

$$F_I(k|\ell) = 1 - (1 - \alpha(\lambda_I))^{\min(\ell,k)},$$

[19] For one example, suppose that schools each have a single seat and are equally popular, and $\rho = 1$. If student lists are equally likely to be of length 2 or 12, then a common priority rule assigns more students to their first choice and assigns more students overall, but assigns fewer students to one of their top four choices.

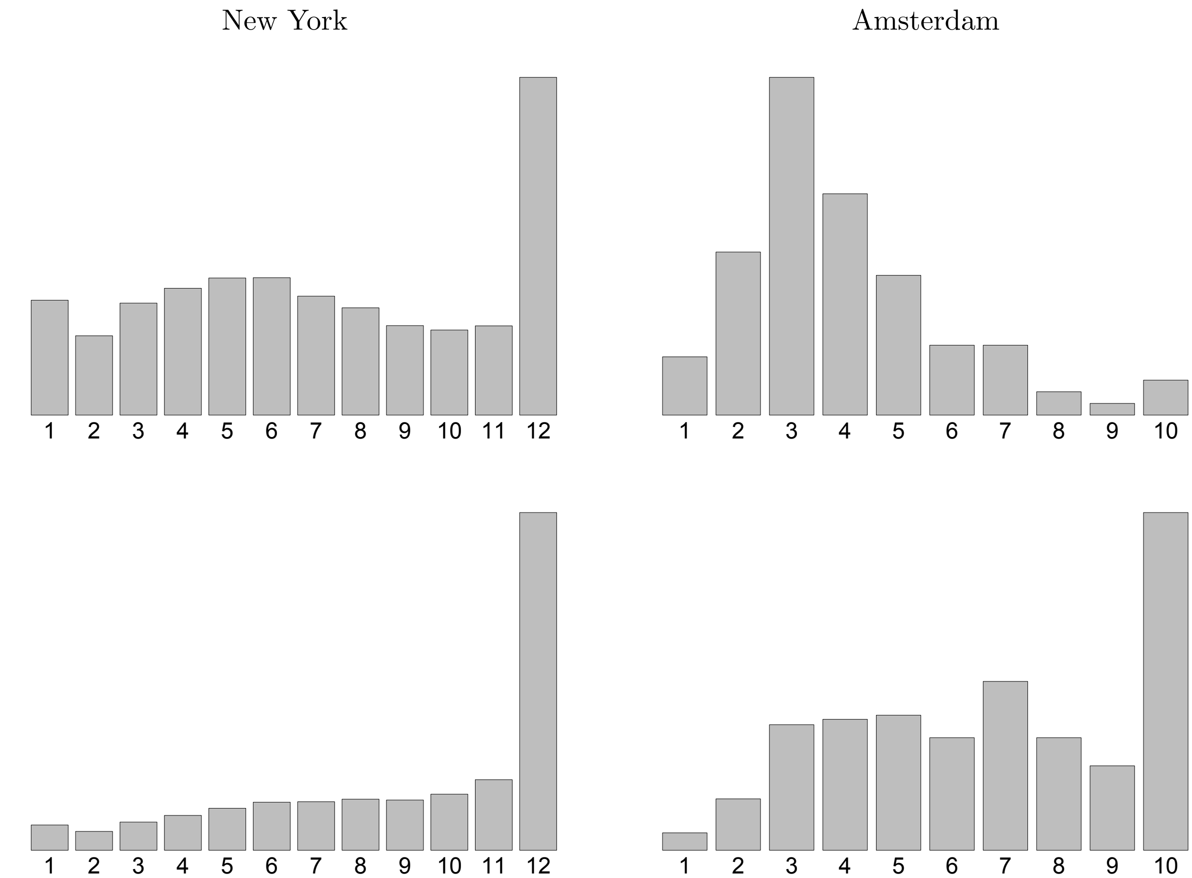

FIGURE 5. Top: the list length distributions in New York City and Amsterdam. Bottom: the corresponding hazard rates. Data from Abdulkadiroglu et al. (2009) and de Haan et al. (2015). Theorem 5 states that if the hazard rate is increasing, idiosyncratic priorities result in more matches than common priorities, while the reverse holds if the hazard rate is decreasing. In both cities, the hazard rate of the list length distribution is approximately increasing (very few students submit lists of length 8, 9, and 10 in Amsterdam, and thus the hazard rate at these extreme values is nearly irrelevant). This data, combined with Theorem 5, suggests that breaking ties using a common lottery should match fewer students in both cities. This prediction is consistent with the empirical findings from Abdulkadiroglu et al. (2009) and de Haan et al. (2015).

where λ_I is the solution to (5). This represents the probability that a student who lists ℓ schools will get one of her top$-k$ choices, when priorities are idiosyncratic. Analogously, define

$$F_C(k|\ell) = 1 - \int_0^1 (1 - \mathcal{V}(\Lambda_C(t)))^{\min(\ell,k)} dt.$$

Theorem 6. *Given any $\rho, \mathcal{L}, \mathcal{D}$, there exists $k' \in \mathbb{N}$ with $\overline{\mathcal{L}}(k'-1) > 0$ such that:*

I. If $\ell < k'$, then $F_I(k|\ell) \le F_C(k|\ell)$ for all k.
II. If $\ell \ge k'$, then $F_I(k|\ell) \le F_C(k|\ell)$ for $k < k'$, and $F_I(k|\ell) > F_C(k|\ell)$ for $k \ge k'$.

This means that students who list fewer than k' schools are unambiguously better off (in aggregate) under common priorities, but the (positive fraction of) students who list at least k' schools face the previously identified tradeoff: more of these students get one of their top choices if priorities are common, but more are assigned if priorities are idiosyncratic.

8. Conclusion

This chapter provides insight into the operation of centralized clearinghouses, by examining the ways in which match outcomes depend on market primitives. My first result gives exact asymptotic expressions for the fraction of students who get a top-k choice (for any value of k) as a function of market imbalance, student list length, the popularities of each school, and the method for ranking students. Among other things, these expressions provide the first systematic study of unmatched agents in centralized clearinghouses.

I use this result to address the operational question of how to determine school priorities. This question has generated discussion in cities across the United States and Europe, including a recent lawsuit in Amsterdam. Despite its importance, the effect of different priority rules on aggregate match outcomes remains poorly understood. This chapter demonstrates that very generally, there is a tradeoff between the objectives of assigning as many students as possible, and assigning many students to their top choice. Relative to the case of independent school lotteries, introducing common elements to school priorities and aligning priorities with student preferences both increase the number of students who receive a high quality match, at the cost of increasing the number of students who are unassigned. This result explains observational data on single vs. multiple tie-breaking from New York, Boston, and Amsterdam, and suggests that district administrators should weigh the relative importance of different objectives when determining school priorities. Furthermore, the intuition from Section 4.2 suggests new priority rules that may align with the district's objectives. If, for example, minimizing the number of unassigned students is important, then *negatively* correlated school lotteries might be appropriate.

Another factor that significantly affects the comparison between procedures (and has not received attention in previous work) is the *distribution* of student list lengths. In particular, when the hazard rate of the list length distribution is increasing, a common priority rule simultaneously assigns more students to their top choice *and* more students overall, when compared to idiosyncratic priorities.

Despite the progress offered by this work, it leaves many questions unanswered. Two such questions are,

- Is a stable matching algorithm a good way to assign students to schools?
- Would such a system be appropriate in other markets?

In order to answer these questions, it becomes important to consider the *reasons* that participants in these marketplaces submit short preference lists. One possibility is that they truly find all but a small number of potential matches to be unacceptable. In many cases, however, short lists are a byproduct of agents being uncertain about their preferences, or unable to accurately report them.

For example, parents in cities with many high schools may not be aware of all of their options, and might be unwilling to list schools about which they know very little.[20]

Even if students are certain about their preferences, they may be unable to report them accurately. For example, many cities, including New York, New Orleans, and Chicago, limit the number of schools that each student can list. In this case, students may have an incentive to drop very competitive schools from their list. School districts often provide students with information about the competition at various schools,[21] but despite this fact, many students incorrectly estimate their enrollment chances and list the "wrong" schools.[22]

Similar frictions cause many real-world matching settings to differ from the idealized cases considered in most models. For example, prior to submitting their preferences, participants in the National Residency Matching Program conduct a series of interviews, which are both costly and time-consuming. Short lists arise because hospitals only rank candidates whom they have interviewed. Furthermore, because interview slots are limited, both sides have an incentive to target interview partners strategically (for one theoretical analysis of this phenomenon, see Kadam (2015)). Such targeting is imperfect, and as a result, agents who (in hindsight) should have interviewed one another go unmatched.[23] While it may be possible to centrally coordinate interviews (see Lee and Schwarz (2012)), there are also less heavy-handed approaches to this problem.

Markets that use the deferred acceptance algorithm separate information acquisition and match formation into two distinct phases. This has the (well-studied) benefit of resolving miscoordination in the offer stage, but also converts the process of information acquisition into a simultaneous-move game, thereby *increasing* miscoordination in this stage. In particular, assking participants to simultaneously rank many options forces them to either (wastefully) research more possibilities than necessary, or to economize on screening and guess about their own preferences.

In some settings, wasteful information acquisition is minimized by interleaving information acquisition and matching. In most labor markets, both sides start with limited information (such as a resume or job description), and proceed through a series of increasingly costly screening stages. This reduces the time that agents spend investigating possibilities that are unlikely to lead to a successful match. A similar story applies to dating and romantic relationships.

Another interesting case is the market for federal law clerks, in which judges often make take-it-or-leave-it "exploding" offers upon completion of the interview. While this has several obvious drawbacks, it also allows judges to close their search as soon as they've found a suitable candidate. This, in turn, allows law students who might otherwise have interviewed for these positions to direct their search to openings that are more likely to be available. One open and very interesting

[20]Many cities try to rectify this by providing information about each school. Some go further by providing interactive tools that allow parents to search for schools by geography, availability of specific academic programs, and other criteria. For one example, see `https://www.noodle.com/nyc-choice`.

[21]See, for example, information from New Orleans at `http://enrollnola.org/enrollnola-annual-report/`, and from New York at `http://schools.nyc.gov/ChoicesEnrollment/High/Resources/default.htm`.

[22]Calsamiglia et al. (2010) use a lab experiment to further explore this topic.

[23]In 2015, of the 1,306 residency positions that did not fill, 1,193 participated in a supplemental matching procedure, and 1,129 (94.6%) were filled in this manner. This indicates that these positions were initially unfilled due to miscoordination, rather than because nobody was willing to take them.

research direction is to more formally characterize the tradeoff between simultaneous and sequential mechanisms in a setting where preferences are initially unknown.

Any analysis of this tradeoff, however, requires a model of centralized matching that incorporates the friction of miscoordinated interviews. I believe that the model presented here, with student lists reinterpreted as the set of positions for which they interview, provides an excellent starting point for this investigation.

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A. Proof of Theorem 1

The proof of Theorem 1 uses general martingale-based techniques to show that with high probability, a given random process "stays close" to the solution to a corresponding set of differential equations. This technique has been developed fairly generally by Wormald (1999), and I leverage his work in the proof below.

Consider a discrete state space $\mathcal{X}$, and a random process $\{X(t)\}_{t\in\mathbb{N}}$ on $\mathcal{X}$. Let H_t be the *history* of the process up to time t; that is, $H_t = (X(0), X(1), \ldots, X(t)) \in \mathcal{X}^{t+1}$.

We are interested in a set of summary statistics $Y(t) = (Y_1(t), \ldots, Y_m(t))$, where for $i \in \{1, \ldots, m\}$, $Y_i(t) = y_i(X(t))$ for functions $y_i : \mathcal{X} \to \mathbb{R}$. We aim to approximate $Y(t)$ using a system of differential equations; that is, to come up with a set of differential equations that have a unique solution $\hat{y} : \mathbb{R} \to \mathbb{R}^m$ such that $Y(t) \approx n \cdot \hat{y}(t/n)$ with high probability.

In order for this approximation to be possible, it must be the case that the low-dimensional state $Y(t)$ captures "enough information" about the process to predict $Y(t+1)$ with reasonable accuracy (we do not need, however, for $Y(t)$ to be Markovian).

In the notation that follows, I consider a sequence of processes $X^n(t)$, with $X^n \in \mathcal{X}^n$, and a sequence of functions y_i^n mapping the state space to one-dimensional summary variables.

Theorem 7 (Theorem 5.1 from Wormald (1999))**.**

For $1 \le i \le m$, suppose that there exists $\tilde{C} > 0$ and $y_i^n : \mathcal{X}^n \to \mathbb{R}$ such that $|y_i^n(x^n)| < \tilde{C}n$ for all i, n, and all $x^n \in \mathcal{X}^n$. Let $Y^n(t) = (Y_1^n(t), \ldots, Y_m^n(t))$, where $Y_i^n(t) = y_i^n(X^n(t))$. Let $D \subset \mathbb{R}^m$ be a bounded connected open set containing the points $\bigcup_n (Y_1^n(0)/n, \ldots Y_m^n(0)/n)$. Suppose that $f_i : \mathbb{R}^m \to \mathbb{R}$ satisfy the following: Assume that the following conditions hold:

(i) (Boundedness) There exists $\beta \ge 1$ such that for $1 \le i \le m$,

$$\max_{y\in D} |f_i(y)| \le \beta, \qquad \max_{n,t} |Y_i^n(t+1) - Y_i^n(t)| \le \beta.$$

(ii) (Trend hypothesis) There exists δ' such that for $1 \le i \le m$,

$$|\mathbb{E}[Y_i^n(t+1) - Y_i^n(t) - f_i(Y^n(t)/n)|H_t]| \le \delta'.$$

(iii) (Lipschitz hypothesis) Each function f_i is continuous, and there exists L such that for all i,

$$\left|f_i(y) - f_i(y')\right| \le L||y - y'||_\infty \text{ for all } y, y' \in D.$$

Then the following are true:

(a) (Unique solution) For $(0, \hat{y}_0) \in D$, the system of differential equations

$$\frac{d\hat{y}_i}{dx} = f_i(\hat{y}), \tag{12}$$

with initial condition $\hat{y}(0) = \hat{y}$ has a unique solution in D.

(b) (Convergence) Let $\hat{y}_i^n$ be the solution to (12) with initial conditions $\hat{y}_i^n(0) = Y^n(0)/n$. Let

$$\sigma = \sup\{x \ge 0 : ||(x, y(x)) - (\hat{x}, \hat{y})||_\infty > C\lambda \text{ for all } (\hat{x}, \hat{y}) \notin D\}.$$

Then for any $x < \sigma$, any i, and any $\epsilon > 0$,

$$\mathbb{P}(|Y_i^n(\lfloor nx \rfloor)/n - \hat{y}_i^n(x)| > \epsilon) \to 0.$$

This theorem and its proof are somewhat opaque and technical, and a complete proof is provided in Wormald (1999). Below, I briefly discuss the conditions in the Theorem, and how they are used in its proof.

The most fundamental condition is (ii), which states that the expected change in Y_i is well-approximated by an function $f : \mathbb{R}^m \to \mathbb{R}$. Of course, the fact $\mathbb{E}[Y_i(t+1)|H_t] \approx Y_i(t) + f_i(Y(t)/n)$ does not imply that $Y_i(t+1) \approx Y_i(t) + f_i(Y(t)/n)$, due to random fluctuation. However, over a large number of steps, this randomness should cancel out; we might hope that

$$Y_i(t+w) - Y_i(t) \approx \sum_{k=0}^{w-1} f_i(Y(t+k)/n) \approx w f_i(Y(t)/n),$$

provided that two conditions hold:

1. $Y_i(t+1) - Y_i(t)$ is unlikely to be large. This implies that
   - $Y_i(t+w) - Y_i(t) = \sum_{k=0}^{w-1} Y_i(t+k+1) - Y_i(t+k)$ is not dominated by any large jumps (so that the "law of large numbers" might apply)
   - $Y_i(t+k)/n \approx Y_i(t)/n$, so long as k is small relative to n.
2. f_i is sufficiently smooth, so that $f_i(Y(t+k)/n) \approx f_i(Y(t)/n)$.

These are exactly the conditions (i) and (iii). The proof of Theorem 7 follows this reasoning, with $w = n^\alpha$ for some $\alpha \in (0,1)$, so that:

- The sum $\sum_{k=0}^{w-1} Y_i(t+k+1) - Y_i(t+k)$ has a large number of terms
- The value $Y_i(t+w)/n$ cannot be "too far" from $Y_i(t)/n$.

Jointly, these allow for the application of a well-known Chernoff-style large deviation bounds, which establish that the sum of these many small terms is unlikely to be far from its expected value.

In what follows, I use the notation $\mathbf{1}_X$ to be the value 1 if X is true, and 0 if X is false. I also make use of the following:

Fact 3.

I. $\frac{d}{d\lambda}\mathbb{P}(\mathbf{Po}(\lambda) = C) = \mathbb{P}(\mathbf{Po}(\lambda) = C - 1) - \mathbb{P}(\mathbf{Po}(\lambda) = C)$.

II. $\frac{d}{d\lambda}\mathbb{P}(\mathbf{Po}(\lambda) > C) = \mathbb{P}(\mathbf{Po}(\lambda) = C)$

III. $\frac{d}{d\lambda}\mathbb{E}[\min(\mathbf{Po}(\lambda), C)] = \mathbb{P}(\mathbf{Po}(\lambda) < C)$

Each of these points are straightforward (and in fact were shown in the proof of Fact 2).

A.1. **Common Priorities.**

I start with the case of common priorities, as this is the simplest to analyze. Rank the students in order of decreasing priority. Step t of the process will correspond to having the t^{th}-highest-priority student apply. Because this student has lower priority than all students who preceded her, she applies until she finds a school with a vacancy, or exhausts the set of schools on her list. After t steps, the full state X consists of, for each $h \in \mathcal{H}$, the set of students matched to $\mathcal{H}$.

Of course, the identity of these students is irrelevant; it suffices to track, for each $h \in \mathcal{H}$, the number of vacancies at h (call this $V_h(t)$). This state can be further reduced by pooling high schools of similar popularity. Fix $k \in \mathbb{N}$, and for $1 \leq j \leq k$, define

$$R_{ij}(t) = \left| \left\{ h \in \mathcal{H} : \mathcal{D}(p_h, \infty) \in \left(\frac{j-1}{k}, \frac{j}{k} \right), V_h(t) = i \right\} \right|.$$

If k is large, then this is nearly sufficient to determine the future evolution of the system. For simplicity, the proof below assumes that $p_h = 1$ for all h, so that no binning is necessary. The general proof bins schools by popularity, letting the number of bins grow with n.

Throughout, track $R(t) = (R_0(t), R_1(t), R_2(t), \ldots) \in \mathbb{N}^{\mathbb{N}}$, where $R_i(t)$ is the number of schools with i remaining seats, after the t^{th} student has selected.[24] Define $M(t) = M_1(t), \ldots, M_\ell(t)$ such that $M_k(t)$ is the number of students (among the first t) who fail to get a top-k choice. These will be the "summary" random variables to which we apply Theorem 7.

Condition (i) from Theorem 7 clearly holds: because each student fills (at most) one seat, $|R_i(t+1) - R_i(t)| \leq 1$ for all i, t. Similarly, $0 \leq M_j(t+1) - M_j(t) \leq 1$ for all j, t.

Next, turn to condition (ii), which requires an evaluation of the expected change in R and M. Note that the compressed state $R(t), M(t)$ is Markovian. The $(t+1)^{st}$ student fails to match to one of her top k choices if and only if none of them have vacant seats. The probability that this occurs, given $R(t)$, is $\Delta_k(R_0(t))$, where we define $\Delta_k : \{0, 1, \ldots, n\} \to [0,1]$ by

$$\Delta_k(R_0) = \prod_{j=0}^{k-1} \frac{R_0 - j}{n - j}.$$

Thus,

$$\mathbb{E}[M_k(t+1) - M_k(t) | R(t)] = \Delta_k(R_0(t)). \tag{13}$$

In particular, the probability that the $(t+1)^{st}$ student fails to match is $\Delta_\ell(R_0(t))$. Conditioned on matching, the school to which this student matches is selected uniformly at random from the set of schools with vacancies (because $p_h = 1$ for all h). Thus, given $R(t)$, the probability that the $(t+1)^{st}$ student fills a seat at a school that previously had i vacancies is $(1 - \Delta_\ell(R_0(t)))\frac{R_i(t)}{n - R_0(t)}$ (using the fact that $\sum R_i(t) = n$ for all t). Note that

- $R_i(t+1) = R_i(t) - 1$ if the $(t+1)^{st}$ student fills a seat at a school that had i vacancies,
- $R_i(t+1) = R_i(t) + 1$ if she fills a seat that previously had $i+1$ vacancies, and
- $R_i(t+1) = R_i(t)$ otherwise.

From this, conclude that

$$\mathbb{E}[R_i(t+1) - R_i(t) | R(t)] = (1 - \Delta_\ell(R_0(t))) \frac{R_{i+1}(t) - R_i(t)\mathbf{1}_{i>0}}{n - R_0(t)}. \tag{14}$$

The expressions in (13) and (14) are exact, but complex. Next, we approximate these expressions with much simpler ones. It is clear that when n is large, the $\Delta_k(R_0) \approx \left(\frac{R_0}{n}\right)^k$. More formally, for

[24] To the reader concerned about infinite-dimensional systems, note that although the notation suggests that $R(t)$ is infinite-dimensional, n is fixed throughout the analysis. Thus, the actual system has dimension $\overline{C} = \max_{h \in \mathcal{H}} C_h < \infty$.

$1 \le k \le \ell$,

$$\left(\frac{(R_0-\ell)_+}{n}\right)^k \le \Delta_k(R_0) \le \left(\frac{R_0}{n}\right)^k.$$

It follows that

$$\begin{aligned}\left|\mathbb{E}[M_k(t+1)-M_k(t)|R(t)] - \left(\frac{R_0(t)}{n}\right)^k\right| &= \left(\frac{R_0(t)}{n}\right)^k - \Delta_k(R_0(t)) \\ &\le \left(\frac{R_0(t)}{n}\right)^k - \left(\frac{(R_0(t)-\ell)_+}{n}\right)^k \\ &\le k\left|\frac{R_0(t)}{n} - \frac{(R_0(t)-\ell)_+}{n}\right| \\ &\le \frac{\ell^2}{n}, \end{aligned} \tag{15}$$

where the second-to-last inequality uses the fact that for $a, b \in [0,1]$, $\left|a^k - b^k\right| \le k\,|a-b|$ - see (17).

Next, we approximate (14). For $i \in \mathbb{N}$, define the functions $f_i : [0,1]^{\mathbb{N}} \to [0,1]$, as follows:

$$f_i(r) = \mu(1-r_0)\left(r_{i+1} - r_i \mathbf{1}_{i>0}\right).$$

Note that $\mu(1-r_0) = \sum_0^{\ell-1} r_0^k = \frac{1-r_0^\ell}{1-r_0}$, from which it follows that

$$f_i(R(t)/n) = \left(1 - \left(\frac{R_0(t)}{n}\right)^\ell\right)\frac{R_{i+1}(t) - R_i(t)\mathbf{1}_{i>0}}{n - R_0(t)}$$

Therefore,

$$\begin{aligned}&\left|\mathbb{E}\left[R_i(t+1) - R_i(t)|R(t)\right] - f_i(R(t)/n)\right| \\ &\quad= \left|(1-\Delta_\ell(R_0))\frac{R_{i+1}(t) - R_i(t)\mathbf{1}_{i>0}}{n-R_0(t)} - \left(1 - \left(\frac{R_0(t)}{n}\right)^\ell\right)\frac{R_{i+1}(t) - R_i(t)\mathbf{1}_{i>0}}{n-R_0(t)}\right| \\ &\quad= \left|\Delta_\ell(R_0(t)) - (R_0(t)/n)^\ell\right|\left|\frac{R_{i+1}(t) - R_i(t)\mathbf{1}_{i>0}}{n-R_0(t)}\right| \\ &\quad\le \frac{\ell^2}{n}. \end{aligned} \tag{16}$$

where the final line follows because $\max(R_i(t), R_{i+1}(t)) \le n - R_0(t)$, and

$$\left|\Delta_\ell(R_0) - (R_0/n)^\ell\right| \le \ell^2/n,$$

as established in (15). Jointly, (15) and (16) establish that condition (ii) in Theorem 7 holds.

Next, we show that (iii) from Theorem 7 holds, that is, that the functions approximating the expected change in R and M are Lipschitz on $[0,1]^{\mathbb{N}}$. Note that for $a, b \in [0,1]$ with $a > b$,

$$0 \le a^k - b^k = (a-b)\left(\sum_{j=0}^{k-1} a^j b^{k-j-1}\right) \le k(a-b), \tag{17}$$

so the functions $\hat{\Delta}_k(r_0) = r_0^k$ are Lipschitz, with Lipschitz constant ℓ. Turning attention to the f_i, note that

$$f_i(r) - f_i(s) = \mu(1-r_0)\left((r_{i+1}-s_{i+1}) - (r_i - s_i)\mathbf{1}_{i>0}\right) - (\mu(1-s_0) - \mu(1-r_0))(s_{i+1} - s_i\mathbf{1}_{i>0})$$

Because $r_i, s_i \le 1$ and $\mu(\cdot) \le \ell$, it follows that

$$|f_i(r) - f_i(s)| \le \mu(1-r_0)(|r_{i+1}-s_{i+1}| + |(r_i - s_i)|) + \sup_{\alpha\in[0,1]} \left|\mu'(\alpha)\right| |s_0 - r_0|$$

$$\le 2\ell||r-s||_\infty + \frac{\ell(\ell-1)}{2}||r-s||_\infty = \frac{\ell^2+3\ell}{2}||r-s||_\infty, \tag{18}$$

where the second inequality uses the fact that

$$0 \ge \mu'(\alpha) = -\sum_{k=1}^{\ell-1} k(1-\alpha)^{k-1} \ge -\frac{\ell(\ell-1)}{2}.$$

In summary, condition (i) holds with $\beta = 1$, while (15), (16), establish that condition (ii) holds with $\delta' = \ell^2/n$ and (17) and (18) establish that condition (iii) holds with $L = \frac{\ell^2+3\ell}{2}$. Therefore, we may apply Theorem 7. The first conclusion states that for any initial state $(r(0), m(0))$, there is a unique solution $(r(x), a(x))$ to the system

$$\frac{dr_i}{dx} = f_i(r) = \mu(1-r_0)\left(r_{i+1} - r_i\mathbf{1}_{i>0}\right), \qquad \frac{dm_k}{dx} = r_0(x)^k. \tag{19}$$

Let $r^n(x), m^n(x)$ be this solution when the initial conditions are

$$r_i^n(0) = \mathcal{D}^n(\infty, i) - \mathcal{D}^n(\infty, i-1), \qquad m_k^n(0) = 0.$$

The second conclusion from Theorem 1 states that

$$\max_{i,t}\{|R_i(t)/n - r_i(t/n)|, |N(t)/n - \Lambda(t/n)|\} \xrightarrow{p} 0. \tag{20}$$

Because $\mathcal{D}^n \to \mathcal{D}$, $|\mathcal{S}^n|/n \to \rho$, and the equations (19) are continuous, we know that for all x,

$$(r^n(x), m^n(x)) \to (r(x), m(x)), \tag{21}$$

where $r(\cdot)$ and $m(\cdot)$ are the solution to (19) with initial conditions

$$r_i(0) = \mathcal{D}(\infty, i) - \mathcal{D}(\infty, i-1), \qquad m_k(0) = 0.$$

To complete the proof for common priorities, I give the solution $r(x), m(x)$ in terms of the function Λ_C from (6), reproduced below:

$$\Lambda_C(0) = 0, \qquad \Lambda_C'(t) = -\rho\,\mu(\mathcal{V}(\Lambda_C(t))). \tag{22}$$

Define

$$r_0(x) = 1 - \mathcal{V}(\Lambda_C(1 - x/\sigma)), \tag{23}$$

$$r_i(x) = \sum_{k=1}^{\infty} (\mathcal{D}(\infty,k) - \mathcal{D}(\infty,k-1))\,\mathbb{P}(\mathbf{Po}(\Lambda_C(1-x/\rho)) = k-i). \tag{24}$$

To verify that these r_i satisfy (19), note that by Fact 3,

$$\mathcal{V}'(\lambda) = \frac{d}{d\lambda}\sum_{k=1}^{\infty}(\mathcal{D}(\infty,k)-\mathcal{D}(\infty,k-1))\mathbb{P}(\mathbf{Po}(\lambda)<k)$$

$$= -\sum_{k=1}^{\infty}(\mathcal{D}(\infty,k)-\mathcal{D}(\infty,k-1))\mathbb{P}(\mathbf{Po}(\lambda)=k-1). \tag{25}$$

Thus, we have

$$\begin{aligned}\frac{dr_0}{dx} &= \frac{\Lambda_C'(1-x/\rho)}{\rho}\ \mathcal{V}'(\Lambda_C(1-x/\rho)) \\ &= -\mu(1-r_0)\ \mathcal{V}'(\Lambda_C(1-x/\rho)) \\ &= \mu(1-r_0)\sum_{k=1}^{\infty}(\mathcal{D}(\infty,k)-\mathcal{D}(\infty,k-1))\mathbb{P}(\mathbf{Po}(\Lambda_C(1-x/\rho))=k-1) \\ &= \mu(1-r_0)r_1 = f_0(r),\end{aligned}$$

where I have used (22) and (23), (25), and (24) in turn.

Similarly, for $i \geq 1$, we apply Fact 3, along with (22), (23), and (24), to conclude that

$$\frac{dr_i}{dx} = -\frac{\Lambda_C'(1-x/\rho)}{\rho}\,(r_{i+1}-r_i) = \mu(1-r_0)\,(r_{i+1}-r_i) = f_i(r).$$

Thus, the expressions for r_i given in (24) satisfy the system (19). Furthermore, we can express $m_k(x)$ in terms of Λ_C as follows:

$$m_k(x) = \int_0^x r_0(y)^k dy = \int_0^x (1-\mathcal{V}(\Lambda_C(1-y/\rho)))^k\,dy = \rho\int_{1-x/\rho}^1 (1-\mathcal{V}(\Lambda_C(t)))^k\,dt.$$

From this, we see that $m_k(\rho) = \rho\,(1-F_C(k))$. Note that by the definitions of $F_C^n(k)$ and M_k,

$$F_C^n(k) = 1 - \frac{M_k(|\mathcal{S}^n|)}{|\mathcal{S}^n|}$$

It follows that

$$F_C^n(k) - F_C(k) = \frac{m_k(\rho)}{\rho} - \frac{M_k(|\mathcal{S}^n|)/n}{|\mathcal{S}^n|/n}.$$

Applying (20), (21), and the fact that $|\mathcal{S}^n|/n \to \rho$ by assumption, it follows that $F_C^n(k) \xrightarrow{p} F_C(k)$, as claimed. The expressions for G_C follow.from the expressions for F_C and the fact that $U_s(h)$ and $U_h(s)$ are independent.

A.2. **Aligned Priorities.**

Let the *quality* of a match between s and h be $U_s(h)$. It is clear that under aligned priorities, the match of highest quality must be in any stable matching. After forming this match, this principle can be applied iteratively to the remaining potential matches to greedily construct the unique stable matching.

Correspondingly, the proof below studies a process in which students apply to schools in order of decreasing (global) match quality. Each step of the process corresponds to one application by a student. We let $s(t)$ denote the student who applies in round t, and $h(t)$ denote the school to which the application was sent.The state of the process after t proposals is as follows:

- for each student s, $\mathcal{H}_s(t)$ is the set of schools to which s has applied
- for each school h, $\mathcal{S}_h(t)$ is the set of students that have matched to h

As in the case of common priorities, we will seek a more compact representation. To simplify notation, we provide a complete proof for the case where $p_h = 1$ for all h; the general case proceeds by binning schools according to popularity (as discussed in the section on common priorities), and tracking state for each bin.

Rather than tracking the complete state, it suffices to track low-dimensional representation of this state. For $i \in \mathbb{N}$, let $\mathcal{R}_i(t)$ be the set of schools with i seats remaining after t proposals, and let $R_i(t)$ be the number of such schools. That is,

$$\mathcal{R}_i(t) = \{h : C_h - |\mathcal{S}_h(t)| = i\}, \qquad R_i(t) = |\mathcal{R}_i(t)|.$$

In addition, for $0 \le j \le \ell$, let $\mathcal{A}_j(t)$ be the set of students who are unmatched after t applications have been sent, and have been rejected from $\ell - j$ schools so far (i.e. have j schools "remaining" on their list). Let $A_j(t)$ denote the number of such students. That is,

$$\mathcal{A}_j(t) = \{s \notin \cup_h \mathcal{S}_h(t) : \mathcal{H}_s(t) = \ell - j\}, \qquad A_j(t) = |\mathcal{A}_j(t)|.$$

Finally, we track $U(t) \in [0, 1]$, which represents the quality of the match proposed in stage t. At step t of the process, we realize (in order) the values $s(t)$, $h(t)$, and $U(t) = U_{s(t)}(h(t))$. It turns out that the process $(R(t), A(t), U(t))$ is Markovian; I discuss transition probabilities in detail below.

We warm up by considering the first step of the process. Initially, $\mathcal{R}_i(0) = \{h \in \mathcal{H} : C_h = i\}$, with $\mathcal{A}_\ell(0) = \mathcal{S}$ and $\mathcal{A}_i(0) = \emptyset$ for $i < \ell$. Define $U(0) = 1$. At step $t = 1$, the highest quality match is proposed (and accepted). One student has now matched, and no others have applied, so $A_\ell(1) = |\mathcal{S}| - 1$, $A_j(1) = 0$ for $j < \ell$. Furthermore, one school seat is filled, so exactly one of the values R_i is decremented by one, and the value R_{i-1} is incremented by one. Because the school $h(1)$ to which the application was sent is drawn uniformly at random, the probability that R_i is decremented is $R_i(0)/n$. In other words, $\mathbb{P}(h(1) \in \mathcal{R}_i(0)|R(0)) = R_i(0)/n$. Finally, the value of this match is distributed as the maximum of $|\mathcal{S}|\,\ell$ independent $U[0, 1]$ random variables; draw $U(1)$ accordingly.

In general, if the application in round $t + 1$ is sent to a school with no vacancies, it is rejected, and $R(t+1) = R(t)$. Otherwise, the application is sent to a school with $i > 0$ vacancies, is accepted, and R_i is decremented by one and R_{i-1} incremented by one. In other words,

$$R_i(t + 1) = R_i(t) - \mathbf{1}_{h(t+1)\in\mathcal{R}_i(t)}\mathbf{1}_{i>0} + \mathbf{1}_{h(t+1)\in\mathcal{R}_{i+1}(t)}. \tag{26}$$

Turning our attention to the student side, if the student selected to apply in round $t + 1$ has k schools remaining on her list, then A_k will be decremented. If this student applies to a school with a vacancy, her application is accepted, and A_j remains unchanged for $j \neq k$. If this student applies to a school with no vacancy, her application is rejected, and A_{j-1} is incremented. In other words,

$$A_j(t + 1) = A_j(t) - \mathbf{1}_{s(t+1)\in\mathcal{A}_j(t)} + \mathbf{1}_{s(t+1)\in\mathcal{A}_{j+1}(t)}\mathbf{1}_{h(t+1)\in\mathcal{R}_0(t)}. \tag{27}$$

The probability that the application is sent to a school with a vacancy depends (very weakly) on the identity of the student sending it. If the applying student has been rejected $\ell - k$ times (i.e. has k remaining schools), then because each of these rejections was from a school with no vacancies, the probability that this student applies to a school with i vacancies is $\frac{R_i(t)}{n-\ell+k}$ for $i > 0$, and is $\frac{R_0(t)-\ell+k}{n-\ell+k}$ for $i = 0$. For notational convenience, given $r \in \mathbb{N}^{\overline{C}+1}$, define $p_{ik}(R) = \frac{R_i-(\ell-k)\mathbf{1}_{i=0}}{n-(\ell-k)}$. Then we have concluded that

$$\mathbb{P}(h(t+1) \in \mathcal{R}_i(t) | s(t+1) \in \mathcal{A}_k(t), R(t)) = \frac{R_i(t) - (\ell - k)\mathbf{1}_{i=0}}{n - (\ell - k)} = p_{ik}(R(t)). \tag{28}$$

The probability that the best remaining potential match involves student s (that is, the probability that $s(t+1) = s$) is proportional to the number of schools remaining on s's list. It follows that the probability that the application in step $t+1$ is from a student with k remaining schools is $\frac{kA_k(t)}{\sum jA_j(t)}$. Given a vector $a \in \mathbb{R}_+^{\ell+1}$ and $0 \le k \le \ell$, define

$$q_k(a) = \frac{ka_k}{\sum ja_j}.$$

For convenience, define $q_{\ell+1}(\cdot)$ to be identically zero. Thus, we have that

$$\mathbb{P}(s(t+1) \in \mathcal{A}_k(t) | A(t), R(t)) = \frac{kA_k(t)}{\sum jA_j(t)} = q_k(A(t)). \tag{29}$$

Finally, because values of remaining potential matches are distributed uniformly and independently on $[0, U(t)]$, we have

$$\mathbb{P}(U(t+1) \le u | R(t), A(t), U(t)) = \left(\frac{u}{U(t)}\right)^{\sum jA_j(t)}.$$

Combining (26), (28) and (29), we get that

$$\mathbb{E}[R_i(t+1) - R_i(t) | R(t), A(t)] = \sum_{k=1}^{\ell} q_k(A(t)) \cdot \left(p_{(i+1)k}(R(t)) - p_{ik}(R(t))\mathbf{1}_{i>0}\right). \tag{30}$$

The positive terms correspond to the probability that $s(t+1)$ applies to a school with $i+1$ vacancies (in which case R_i increases), and the negative terms are the probability that $s(t+1)$ applies to a school with i vacancies (in which case R_i decreases, unless $i = 0$).

Similarly, combining (27), (28) and (29), we see that

$$\mathbb{E}[A_j(t+1) - A_j(t) | R(t), A(t)] = q_{j+1}(A(t))p_{0(j+1)}(R(t)) - q_j(A(t)). \tag{31}$$

The first term is the probability that $s(t+1)$ had $j+1$ remaining applications and applies to a school with no vacancy (so that A_j increases by one), and the second (negative) term is the probability that $s(t+1)$ had j remaining applications (in which case A_j decreases by one).

We now seek to apply Theorem 7. The boundedness condition (i) clearly applies: (26) and (27) imply that $|R_i(t+1) - R_i(t)| \le 1$ and $|A_j(t+1) - A_j(t)| \le 1$.

Turning to the trend condition (ii), note that $p_{ik}(R) \approx R_i/n$. More precisely, for any i, k, R,

$$\frac{(R_i - \ell)_+}{n - \ell} \le \min(p_{ik}(R), R_i/n) \le \max(p_{ik}(R), R_i/n) \le \frac{R_i}{n - \ell},$$

from which it follows that $|p_{ik}(R) - R_i/n| \leq \ell/(n-\ell)$. Because $\sum q_k(A(t)) = 1$, (30) implies that

$$\left|\mathbb{E}[R_i(t+1) - R_i(t)|R(t), A(t)] - \left(\frac{R_{i+1}(t)}{n} - \frac{R_i(t)}{n}\mathbf{1}_{i>0}\right)\right| \leq \frac{2\ell}{n-\ell}. \tag{32}$$

Similarly, because $q_k(A(t)) = q_k(A(t)/n)$, (31) implies that

$$\left|\mathbb{E}[A_j(t+1) - A_j(t)|R(t), A(t)] - \left(q_{j+1}\left(\frac{A(t)}{n}\right)\frac{R_0(t)}{n} - q_j\left(\frac{A(t)}{n}\right)\right)\right| \leq \frac{\ell}{n-\ell}. \tag{33}$$

Finally, we turn to condition (iii). The functions $q_k(\cdot)$ are continuous, but not Lipschitz in the neighborhood of zero. Straightforward algebra reveals that

$$q_i(a) - q_i(a') = \frac{i(a_i - a_i')}{\sum ja_j} + q_i(a')\frac{\sum j(a_j' - a_j)}{\sum ja_j}.$$

Because $q_i(a') \in [0,1]$, it follows that

$$\left|q_i(a) - q_i(a')\right| \leq \frac{\sum j|a_j' - a_j|}{\sum ja_j} \leq \frac{\ell(\ell+1)}{2}\frac{||a-a'||_\infty}{\sum ja_j}.$$

For $\eta > 0$ (to be chosen later), define $D = \{a \in \mathbb{R}_+^{\ell+1} : \sum ja_j > \eta\}$. Then we have established that inside of D, each q_i is Lipschitz with constant $\frac{\ell(\ell+1)}{2\eta}$.

Applying Theorem 7, we conclude that for any initial state $(r(0), a(0))$ in D, there is a unique solution $(r(x), a(x))$ to the system of equations

$$\frac{dr_i}{dx} = r_{i+1} - r_i\mathbf{1}_{i>0}, \qquad \frac{da_j}{dx} = q_{j+1}(a)r_0 - q_j(a). \tag{34}$$

Let $r^n(x), a^n(x)$ be this solution when the initial conditions are

$$r_i^n(0) = \mathcal{D}^n(\infty, i) - \mathcal{D}^n(\infty, i-1), \quad a_\ell^n(0) = |\mathcal{S}^n|/n, \quad a_j^n(0) = 0 \text{ for } j < \ell.$$

Then Theorem 7 states that

$$\max_{i,j,t\leq t_D}\{|R_i(t)/n - r_i^n(t/n)|, \left|A_j(t)/n - a_j^n(t/n)\right|, |U(t) - u(t/n)|\} \xrightarrow{p} 0. \tag{35}$$

I claim that the functions $r(\cdot)$ and $a(\cdot)$ given in (36), (37), (38) satisfy (34) with initial conditions

$$r_i(0) = \mathcal{D}(\infty, i) - \mathcal{D}(\infty, i-1), \quad a_\ell(0) = \rho, \quad a_j(0) = 0 \text{ for } j < \ell.$$

$$r_0(x) = 1 - \mathcal{V}(x) \tag{36}$$

$$r_i(x) = \sum_{k=1}^{\infty}(\mathcal{D}(\infty, k) - \mathcal{D}(\infty, k-1))\,\mathbb{P}(\mathbf{Po}(x) = k - i) \tag{37}$$

$$a_j(x) = \rho \cdot G_j(u(x)), \tag{38}$$

where $G_j : [0,1] \to \mathbb{R}_+$ and $u : [0, \Lambda_A(0)] \to [0,1]$ are defined by

$$G_j(u) = \binom{\ell}{j}u^j\left(\int_u^1 1 - \mathcal{V}(\Lambda_A(s))ds\right)^{\ell-j} \tag{39}$$
$$u(x) = \Lambda_A^{-1}(x),$$

and $\Lambda_A : [0,1] \to \mathbb{R}_+$ is defined as in Theorem 1:

$$\Lambda_A(1) = 1, \qquad \Lambda_A'(u) = -\rho\,\ell\left(1 - \int_u^1 \mathcal{V}(\Lambda_A(s))ds\right)^{\ell-1}. \tag{40}$$

Verifying that the initial conditions are correct and that $\frac{dr_i}{dx} = r_{i+1} - r_i\mathbf{1}_{i>0}$ is straightforward (and analogous to the calculations done for common priorities above). Next, I verify that $a(\cdot)$ satisfies (34). Note that

$$\begin{aligned}
G_j'(u) &= \binom{\ell}{j} j \cdot u^{j-1}\left(\int_u^1 1 - \mathcal{V}(\Lambda_A(s))ds\right)^{\ell-j} \\
&\qquad - \binom{\ell}{j} u^j(\ell-j)\left(\int_u^1 1 - \mathcal{V}(\Lambda_A(s))ds\right)^{\ell-j-1}(1-\mathcal{V}(\Lambda_A(u))) \\
&= \frac{1}{u}\Bigg(j \cdot \binom{\ell}{j} u^j \left(\int_u^1 1 - \mathcal{V}(\Lambda_A(s))ds\right)^{\ell-j} \\
&\qquad -(1-\mathcal{V}(\Lambda_A(u)))\cdot(j+1)\cdot\binom{\ell}{j+1}u^{j+1}\left(\int_u^1 1 - \mathcal{V}(\Lambda_A(s))ds\right)^{\ell-j-1}\Bigg) \\
&= \frac{1}{u}\left(j \cdot G_j(u) - (1-\mathcal{V}(\Lambda_A(u)))\cdot(j+1)\cdot G_{j+1}(u)\right).
\end{aligned}$$

From this, and the fact that $1 - \mathcal{V}(\Lambda_A(u(x))) = 1 - \mathcal{V}(x) = r_0(x)$, it follows that

$$a_j'(x) = \rho \cdot G_j'(u(x))u'(x) = \rho\frac{u'(x)}{u(x)}\left(jG_j(u(x)) - r_0(x)(j+1)G_{j+1}(u(x))\right). \tag{41}$$

I claim that

$$\rho\frac{u'(x)}{u(x)} = -1/\sum_{k=0}^{\ell} kG_k(u(x)). \tag{42}$$

Jointly with (41), this implies that

$$a_j'(x) = \frac{(j+1)G_{j+1}(u(x))}{\sum kG_k(u(x))}r_0(x) - \frac{jG_j(u(x))}{\sum kG_k(u(x))} = q_{j+1}(a(x))r_0(x) - q_j(a(x)).$$

To establish (42), note that by implicit differentiation, $u'(x) = 1/\Lambda_A'(u(x))$. Applying the definition of Λ_A' from (40), and letting $u = u(x), u' = u'(x)$, we see that

$$
\begin{aligned}
-\frac{u}{\rho u'} &= -u \cdot \Lambda_A'(u)/\rho = u\ell \left(1 - \int_u^1 \mathcal{V}(\Lambda_A(s))ds\right)^{\ell-1} \\
&= u\ell \left(u + \int_u^1 (1 - \mathcal{V}(\Lambda_A(s)))ds\right)^{\ell-1} \\
&= u \cdot \ell \cdot \sum_{j=1}^{\ell} \binom{\ell-1}{j-1} u^{j-1} \left(\int_u^1 (1 - \mathcal{V}(\Lambda_A(s)))ds\right)^{\ell-j} \\
&= \sum_{j=1}^{\ell} j \binom{\ell}{j} u^j \left(\int_u^1 (1 - \mathcal{V}(\Lambda_A(s)))ds\right)^{\ell-j} \\
&= \sum_{j=1}^{\ell} jG_j(u),
\end{aligned}
$$

as claimed (note that the third line uses of the identity $(x+y)^{\ell-1} = \sum_{i=0}^{\ell-1} \binom{\ell-1}{i} x^i y^{\ell-1-i}$).

By definition of U and A, the number of students who fail to get a match of quality at least $U(t)$ is exactly $\sum_{j\geq 0} A_j(t)$. In symbols, this is expressed as

$$
G_A^n(U(t)) = \frac{n}{|\mathcal{S}^n|} \sum_{j\geq 0} \frac{A_j(t)}{n} \tag{43}
$$

Furthermore, the definition of a_j in (38) implies that for any $u \in [0,1]$, we have

$$
\begin{aligned}
\sum_{j=0}^{\ell} a_j(\Lambda_A(u)) &= \rho \sum_{j=0}^{\ell} G_j(u) \\
&= \rho \sum_{j=0}^{\ell} \binom{\ell}{j} u^j \left(\int_u^1 1 - \mathcal{V}(\Lambda_A(s))ds\right)^{\ell-j} \\
&= \rho \left(u + \int_u^1 (1 - \mathcal{V}(\Lambda_A(s)))ds\right)^{\ell} \\
&= \rho \left(1 - \int_u^1 \mathcal{V}(\Lambda_A(s))ds\right)^{\ell} \\
&= \rho \cdot G_A(u), \qquad (44)
\end{aligned}
$$

where $G_A(u)$ is defined in Theorem 1.

Fix $u \in [0,1]$, and let $t_n = \lfloor n\Lambda_A(u) \rfloor$. Then

$$
G_A^n(u) - G_A(u) = (G_A^n(u) - G_A^n(U(t_n))) + (G_A^n(U(t_n)) - G_A(u)) \tag{45}
$$

Because (35) implies that $U(t_n) \xrightarrow{p} u$, the first term converges in probability to zero. Turning attention to the term $G_A^n(U(t_n)) - G_A(u)$, note that by (43) and (44),

$$G_A^n(U(t_n)) - G_A(u) = \left(\frac{n}{|\mathcal{S}^n|} \sum_{j \geq 0} \frac{A_j(t_n)}{n} - \frac{1}{\rho} \sum_{j \geq 0} a_j^n(t_n/n) \right) + \left(\frac{1}{\rho} \sum_{j \geq 0} a_j^n(t_n/n) - \frac{1}{\rho} \sum_{j \geq 0} a_j(\Lambda_A(u)) \right)$$

Because $n/|\mathcal{S}^n| \to 1/\rho$, (35) implies that the first difference above converges to zero in probability. Meanwhile, because $\mathcal{D}^n \to \mathcal{D}$, $t_n/n \to \Lambda_A(u)$, and the differential equations (34) are continuous, it follows that $a_j^n(t_n/n) \to a_j(\Lambda_A(u))$. Hence, (45) implies that $G_A^n(u) \xrightarrow{p} G_A(u)$, as claimed.

To establish convergence of $F_A^n(k)$ to $F_A(k)$, define $M_k(t)$ to be the number of students who have been rejected at least k times in the first t proposals. Then

$$\mathbb{E}[M_k(t+1) - M_k(t) | R(t), A(t)] = q_{\ell-k+1}(A(t)) \frac{R_0(t) - (k-1)}{n - (k-1)},$$

from which it follows that

$$\left| \mathbb{E}[M_k(t+1) - M_k(t) | R(t), A(t)] - q_{\ell-k+1}(A(t)) \frac{R_0(t)}{n} \right| \leq \frac{\ell}{n}. \tag{46}$$

If τ is the time at which the process stops, then

$$F_A^n(k) = 1 - M_k(\tau)/|\mathcal{S}^n|.$$

From (46), an application of Theorem 7 reveals that for $\lambda < \lambda_A$,

$$M_k(\lambda n)/n \xrightarrow{p} \int_0^\lambda q_{\ell-k+1}(a(x)) r_0(x) dx.$$

Because $\tau/n \xrightarrow{p} \lambda_A$ by (35) and $|\mathcal{S}^n|/n \to \rho$, it follows that

$$\begin{aligned}
M_k(\tau)/|\mathcal{S}^n| &\xrightarrow{p} \frac{1}{\rho} \int_0^{\lambda_A} q_{\ell-k+1}(a(x)) r_0(x) dx \\
&= \int_0^{\lambda_A} (\ell - k + 1) G_{\ell-k+1}(u(x))(1 - \mathcal{V}(x)) u'(x)/u(x) dx \\
&= \int_0^1 (\ell - k + 1) G_{\ell-k+1}(t)(1 - \mathcal{V}(\Lambda_A(t))) t^{-1} dt, \\
&= \ell \binom{\ell-1}{k-1} \int_0^1 t^{\ell-k} \left(\int_t^1 1 - \mathcal{V}(\Lambda_A(s)) ds \right)^{k-1} (1 - \mathcal{V}(\Lambda_A(t))) dt.
\end{aligned}$$

where the second line follows from (36), (38) and (42), the third from substituting $t = u(x)$, and the fourth from the definition of G_j in (39).

B. Proof of Theorem 2

I prove Theorem 2 in several parts:

1. $G_I(0) < G_C(0)$.
2. $G_C(0) < G_A(0)$.
3. $G'_I(1) < G'_C(1) < G'_A(1)$.

Note that this third claim, along with the fact that $G_I(1) = G_C(1) = G_A(1) = 1$, immediately implies that $G_I(u) > G_C(u) > G_A(u)$ for all sufficiently large $u < 1$.

B.1. **Proof that $G_I(0) \leq G_C(0)$.** It is intuitively clear (and simple to show formally) that if $\ell = 1$, $G_I(\cdot) = G_C(\cdot)$. I thus restrict attention to the case $\ell > 1$ and prove that $G_I(0) < G_C(0)$.

The equations from Theorem 1 imply that $G_I(0) = 1 - \mathcal{E}(\lambda_I)/\rho$. It is intuitively clear that $G_C(0) = 1 - \mathcal{E}(\lambda_C)/\rho$ (as both represent the fraction of students left unassigned under common priorities), and this can be formally established by noting that

$$\begin{aligned} G_C(0) &= \int_0^1 (1 - \mathcal{V}(\Lambda_C(t)))^\ell dt \\ &= 1 + \frac{1}{\rho} \int_0^1 \Lambda'_C(t)\mathcal{V}(\Lambda_C(t))dt \\ &= 1 - \frac{1}{\rho} \int_0^{\lambda_C} \mathcal{V}(\lambda)d\lambda \\ &= 1 - \mathcal{E}(\lambda_C)/\rho, \end{aligned}$$

where the second equality follows from the characterizing differential equation for Λ_C, the third from the u-substitution $\lambda = \Lambda_C(t)$, and the last from Fact 2, which states that $\mathcal{E}'(\lambda) = \mathcal{V}(\lambda)$.

Because the function $\mathcal{E}(\cdot)$ is increasing, to show that $G_I(0) < G_C(0)$, it suffices to show that $\lambda_I > \lambda_C$. In Appendix E, I analyze the case of a general list length distribution. Applying the result from Theorem 8, in order to conclude that $\lambda_I > \lambda_C$, it suffices to show that the function $f(\alpha) = \frac{\alpha}{1-(1-\alpha)^\ell}$ is convex. This is straightforward, though tedious; I give the complete argument below.

It is equivalent to show that $g(x) = \frac{1-x}{1-x^\ell}$ is convex. Differentiating twice, we see that $g''(x) > 0$ if and only if

$$(1 - x^\ell)\left(\ell x^{\ell-1} + \ell(\ell-1)x^{\ell-2} - \ell^2 x^{\ell-1}\right) + 2\ell x^{\ell-1}\left((1-x)\ell x^{\ell-1} - (1-x^\ell)\right) > 0.$$

The term on the left is equal to

$$\ell(\ell-1)x^{\ell-2} - \ell(\ell+1)x^{\ell-1} + \ell(\ell+1)x^{2\ell-2} - \ell(\ell-1)x^{2\ell-1}$$

Dividing by $\ell x^{\ell-2}$, we see that the sign of the above expression is equal to the sign of

$$(\ell-1) - (\ell+1)x + (\ell+1)x^\ell - (\ell-1)x^{\ell+1}$$

I wish to show that this function is non-negative. Because is clearly zero at $x = 1$, it is enough to show that it is decreasing. To show this, differentiate to obtain

$$(\ell+1)(\ell x^{\ell-1} - 1) - (\ell-1)(\ell+1)x^\ell$$

I wish to show that this is negative; because it is zero at $x = 1$, it is enough to show that it is increasing. To show this, divide by $\ell + 1$ and differentiate again to get

$$\ell(\ell-1)x^{\ell-2} - \ell(\ell-1)x^{\ell-1} > 0.$$

B.2. **Proof that $G_C(0) < G_A(0)$.**

Recall that under both common and aligned priorities, there is a unique stable matching, which can be constructed through a simple greedy procedure.

P1 Common priorities: the top-ranked student must receive her most preferred school. Form this match, and apply this principle iteratively.

P2 Aligned priorities: the match of highest quality must be in any stable matching. Form this match, and apply this principle iteratively.

The proof that $G_C(0) < G_A(0)$ proceeds by coupling variants of the chains used in A.1 and A.2 to prove Theorem 1. The chains as originally described are not directly comparable, because their time periods differ: in the chain corresponding to common priorities, a step involves having a student apply until she is matched or exhausts her list, while in the chain corresponding to aligned priorities, a step involves the sending of a single application. To allow for a direct comparison, I modify both chains as follows:

- When the last seat at a school is filled, immediately reveal which students listed this school.
- A step of each chain corresponds to an application being sent to a school with a vacancy.

More formally, the state of each chain at time t is $(\mathcal{R}(t), \mathcal{A}(t))$, where

- $\mu(t)$ is a matching
- $\mathcal{R}_i(t) = \{h : V_h(t) = i\}$ is the set of schools with i vacancies remaining,
- $\mathcal{A}_j(t) = \{s : \mu(s) = \emptyset, |\mathcal{H}_s \backslash \mathcal{R}_0(t)| = j\}$ is the set of students who are unmatched and listed j schools that still have at least one vacant seat.

The initial state of both chains is $\mathcal{R}_i(0) = \{h : C_h = i\}$, $\mathcal{A}_\ell(0) = \mathcal{S}$, and $\mathcal{A}_j(0) = \emptyset$ for $j < \ell$. Define

$$R_i(t) = |\mathcal{R}_i(t)|, \qquad A_j(t) = |\mathcal{A}_j(t)|.$$

For simplicity, I first give the argument for the case where each school has popularity $p_h = 1$ (so that student lists are drawn uniformly at random). The case of general popularity distributions is discussed at the end of the proof.

When priorities are common, one step of the chain corresponds to the following procedure:

I Select the highest-priority student who remains unmatched and has listed at least one school with a vacancy.

II Determine the highest-ranked school on this students list that still has a vacancy.

III Form the corresponding match. If this fills the final seat in the school, reveal all other students that listed this school.

Somewhat more formally, the step from time t to $t + 1$ proceeds as follows:

C1 Select $k \in \{1, \ldots, \ell\}$ with

$$\mathbb{P}(k = j' | \mathcal{A}(t)) = A_{j'}(t) / \sum_{j \geq 1} A_j(t).$$

Select $s(t+1) \in \mathcal{A}_k(t)$ uniformly at random.

C2 Select $h(t+1) \in \mathcal{H} \backslash \mathcal{R}_0(t)$ uniformly at random.

C3 Let i be such that $h(t+1) \in \mathcal{R}_i(t)$.

- Set $\mathcal{R}_i(t+1) = \mathcal{R}_i(t) \backslash \{h(t+1)\}$ and $\mathcal{R}_{i-1}(t+1) = \mathcal{R}_{i-1}(t) \cup \{h(t+1)\}$. For $i' \notin \{i-1, i\}$, $\mathcal{R}_{i'}(t+1) = \mathcal{R}_{i'}(t)$.
- If $h(t+1) \notin \mathcal{R}_1(t)$, $\mathcal{A}_k(t+1) = \mathcal{A}_k(t) \backslash \{s(t+1)\}$ and $\mathcal{A}_j(t+1) = \mathcal{A}_j(t)$ for $j \neq k$.
- If $h(t+1) \in \mathcal{R}_1(t)$, for each $j \leq \ell$, and each $s \in \mathcal{A}_j(t) \backslash \{s(t+1)\}$, draw $\mathcal{B}_s(t) \in \{0,1\}$ as iid Bernoulli variables with success probability $j/(n - R_0(t))$. For $j \leq \ell$, set

$$\mathcal{A}_j(t+1) = \{s \in \mathcal{A}_j(t) : \mathcal{B}_s(t) = 0\} \cup \{s \in \mathcal{A}_{j+1}(t) : \mathcal{B}_s(t) = 1\}.$$

If we define

$$B_j(t) = \sum_{s \in \mathcal{A}_j(t)} \mathcal{B}_s(t), \tag{47}$$

then it follows that

$$A_j(t+1) = A_j(t) - \mathbf{1}_{k=j} + (B_{j+1}(t) - B_j(t))\mathbf{1}_{h(t+1) \in \mathcal{R}_1(t)}. \tag{48}$$

The chain for aligned priorities is identical, except that step C1 is replaced with

A1 Select $k \in \{1, \ldots, \ell\}$ with

$$\mathbb{P}(k = j' | \mathcal{A}(t)) = j' A_{j'}(t) / \sum_{j \geq 1} j A_j(t) = q_{j'}(A(t)).$$

Select $s(t+1) \in \mathcal{A}_k(t)$ uniformly at random.

In other words, when priorities are aligned, students are selected *in proportion to the number of listed schools with vacancies*, rather than uniformly at random. The number of matches formed under each priority rule is simply the number of steps required to reach the set $\{\mathcal{A} : \sum j A_j = 0.\}$. Let τ^C, τ^A be the times at which the chains for common and aligned priorities reach this set.

To argue that the first chain produces more matches in expectation, I couple the chains such that if $(\mathcal{R}^C, \mathcal{A}^C)$ and $(\mathcal{R}^A, \mathcal{A}^A)$ are the states of the two chains, then for all $t < \tau^A$, it holds that

$$\mathcal{R}^C(t) = \mathcal{R}^A(t), \text{ and } \qquad \forall\, k \in \{1, \ldots, \ell\}, \sum_{j \geq k} A^C_j(t) \geq \sum_{j \geq k} A^A_j(t). \tag{49}$$

From this, it immediately follows that $\tau^C \geq \tau^A$.

Because the initial states of the two chains are identical, it is enough to argue that if (49) holds at t, then the chains can be coupled such that it continues to hold at time $t+1$.

The coupling proceeds as follows. First, it is trivially possible to couple the chains such that for all $t < \tau^A$, $h^C(t) = h^A(t)$, and therefore $\mathcal{R}^C(t) = \mathcal{R}^A(t)$. Second, if k^C is the value selected in step C1 and k^A is the value selected in step A1, then I claim that k^C and k^A can be coupled such that

$$\mathbb{P}(\{\sum_{j \geq k^C} A^C_j(t) = \sum_{j \geq k^C} A^A_j(t)\} \cap \{k^A < k^C\}) = 0. \tag{50}$$

In other words, whenever the constraint that A^C dominates A^A is "tight" at k^C, it must be that $k^A \geq k^C$. This is possible because if $\sum_{j \geq k} A_j^C = \sum_{j \geq k} A_j^A$, then

$$\mathbb{P}(k^A \geq k) = \frac{\sum_{j \geq k} jA_j^A}{\sum_j jA_j^A} \geq \frac{\sum_{j \geq k} A_j^A}{\sum_j A_j^A} = \frac{\sum_{j \geq k} A_j^C}{\sum_j A_j^A} \geq \frac{\sum_{j \geq k} A_j^C}{\sum_j A_j^C} = \mathbb{P}(k^C \geq k).$$

Note that the second inequality follows because dominance implies that $\sum_j A_j^C \geq \sum_j A_j^A$. It follows from (50) and the inductive assumption (49) that

$$\sum_{j \geq k} A_j^C(t) - \mathbf{1}_{k^C \geq k} \geq \sum_{j \geq k} A_j^A(t) - \mathbf{1}_{k^A \geq k} \quad \text{for } 1 \leq k \leq \ell. \tag{51}$$

Applying (48), we see that this establishes (49) for $t+1$ so long as $h(t+1) \notin \mathcal{R}_1(t)$.

If $h(t+1) \in \mathcal{R}_1(t)$, there is more work to be done. Define B^C and B^A as in (47) to correspond to the chains for common and aligned priorities. I now couple B^C and B^A. For each j, let

$$M_j = (A_j^A(t) - \mathbf{1}_{(k^A = j)}) - (A_j^C(t) - \mathbf{1}_{(k^C = j)}). \tag{52}$$

Generate B_j^C and B_j^A as follows:

- If $M_j \geq 0$, generate B_j^C and let $B_j^A = B_j^C + \text{Binom}(M_j, j/(n - R_0))$
- If $M_j < 0$, generate B_j^A and let $B_j^C = B_j^A + \text{Binom}(|M_j|, j/(n - R_0))$

It is clear that the marginal distributions of B_j^C and B_j^A are correct. Furthermore, I claim that under this coupling, (49) holds for $t+1$, meaning that for any $1 \leq k \leq \ell$,

$$\sum_{j \geq k} A_j^C(t+1) = \sum_{j \geq k} A_j^C(t) - \mathbf{1}_{k^C \geq k} - B_k^C(t) \geq \sum_{j \geq k} A_j^A(t) - \mathbf{1}_{k^A \geq k} - B_k^A(t) = \sum_{j \geq k} A_j^A(t+1). \tag{53}$$

Note that the equalities above follow from (48); thus, all that remains is to show the inequality.

First consider the case when $M_k \geq 0$. In this case, we have $B_k^C \leq B_k^A$, and thus (53) follows immediately from (51). When $M_k < 0$, the coupling of B_k^C and B_k^A implies that $B_k^C + M_k \leq B_k^A$. Furthermore, (51) and the definition of M_k in (52) imply that

$$\sum_{j > k} A_j^C - \mathbf{1}_{(k^C > k)} \geq \sum_{j > k} A_j^A - \mathbf{1}_{(k^A > k)} - M_k.$$

Combining these facts establishes (53).

Finally, a note on the case where schools are not uniformly popular. To analyze this case, I first consider a model in which student lists are generated by sampling ℓ schools iid in proportion to popularity (i.e. sampling *with replacement*, so that some students list a given school multiple times). In this case, the only required modifications are that in step C2, the school $h(t+1)$ is selected in proportion to its popularity, and in the final part of step C3, the values $B_s(t) \in \{0, \ldots, j\}$ are independent binomial random variables with j' trials and success probability $p_{h(t+1)} / \sum_{h \in \mathcal{H} \setminus R_0(t)} p_h$, and $\mathcal{A}$ is updated as follows:

$$\mathcal{A}_{j'}(t+1) = \bigcup_j \{s \in \mathcal{A}_j(t) : j - B_s(t) = j'\}.$$

An analogous coupling argument applies to the chains corresponding to common and aligned priorities. The statement $G^C(0) \geq G^A(0)$ follows from this coupling, along with the fact that the difference between sampling schools with and without replacement vanishes as the size of the market grows.

B.3. **Proof that $G'_I(1) < G'_C(1) < G'_A(1)$.**

Recall that

$$G_I(u) = (1-(1-u)\alpha(\lambda_I))^{\ell}.$$

$$G_C(u) = \int_0^1 (1-(1-u)\mathcal{V}(\Lambda_C(t)))^{\ell}\,dt.$$

$$G_A(u) = \left(1 - \int_u^1 \mathcal{V}(\Lambda_A(t))dt\right)^{\ell}$$

From this it follows that $G'_I(1) = \alpha(\lambda_I)\ell$, $G'_C(1) = \ell\int_0^1 \mathcal{V}(\Lambda_C(t))dt$, and $G'_A(1) = \ell$. Because $\alpha(\cdot)$ and $\mathcal{V}(\cdot)$ are strictly less than one on $(0,\infty)$, it follows that $G'_A(1) > \max(G'_I(u), G'_C(u))$

All that remains to show is that

$$\alpha(\lambda_I) < \int_0^1 \mathcal{V}(\Lambda_C(t))dt. \tag{54}$$

Note that for any $\lambda > 0$,

$$\alpha(\lambda) = \mathcal{E}(\lambda)/\lambda = \int_0^1 \mathcal{V}((1-t)\lambda)dt.$$

The first equality is simply the definition of the function $\alpha(\cdot)$, and the second follows by the substitution $u = (1-t)\lambda$ and Fact 2. Next, I claim that $\Lambda_C(t) \le (1-t)\lambda_I$ for all t. Because $\mathcal{V}(\cdot)$ is decreasing, this immediately implies that (54) holds.

To prove that $\Lambda_C(t) \le (1-t)\lambda_I$, I use two facts. First, $\Lambda_C(0) < \Lambda_I(0) = \lambda_I$. This follows because (as shown in Section 6.2.1)

$$\mathcal{E}(\lambda_I) = \rho(1-G_I(0)) > \rho(1-G_C(0)) = \mathcal{E}(\lambda_C),$$

and $\mathcal{E}(\cdot)$ is increasing. Second, I claim that $\Lambda_C(\cdot)$ is convex, implying that $\Lambda_C(t) \le (1-t)\Lambda_C(0)$. To see that $\Lambda_C(\cdot)$ is convex, examine its characterizing differential equation (6):

$$\Lambda'_C(t) = -\rho\cdot\mu(\mathcal{V}(\Lambda_C(t)))$$

Convexity of Λ_C follows from the fact that μ, $\mathcal{V}$, and Λ_C are all decreasing functions.

C. Proof of Theorem 3

The proof of Theorem 3 makes use of the following fact.

Fact 4. *Let $f, g : \mathbb{R}_+ \to \mathbb{R}_+$, and let $A \subseteq \mathbb{R}_+$. Then*

- *If g is convex, $\int_A g(f(t))dt \ge g\left(\int_A f(t)dt\right)$.*
- *If g is concave, $\int_A g(f(t))dt \le g\left(\int_A f(t)dt\right)$.*

Theorem 2 establishes that $G_I(u) < G_C(u) < G_A(u)$ for all sufficiently small u, and $G_I(u) > G_C(u) > G_A(u)$ for all sufficiently large $u < 1$. Because the functions G_I, G_C, G_A are continuous, this implies that each pair of functions crosses at least once in the interval $(0,1)$.

Having established that G_I and G_C cross at least once, I now show that they cross only once. I do this by showing that if $G_I(u) = G_C(u)$ for $u < 1$, then G_I is "steeper" at u; that is, $G'_I(u) > G'_C(u)$.

Suppose that $G_I(u) = G_C(u)$ for some $u < 1$. Then (letting $\alpha = \alpha(\lambda_I)$)

$$\begin{aligned}
\frac{1-u}{\ell} G_I'(u) &= (1-u)\,\alpha\,(1-(1-u)\alpha)^{\ell-1} \\
&= (1-(1-u)\alpha)^{\ell-1} - (1-(1-u)\alpha)^{\ell} \\
&= G_I(u)^{\frac{\ell-1}{\ell}} - G_I(u) \\
&= G_C(u)^{\frac{\ell-1}{\ell}} - G_C(u) \\
&= \left(\int_0^1 (1-(1-u)\mathcal{V}(\Lambda_C(t)))^{\ell}\,dt.\right)^{\frac{\ell-1}{\ell}} - \int_0^1 (1-(1-u)\mathcal{V}(\Lambda_C(t)))^{\ell}\,dt. \\
&> \int_0^1 (1-(1-u)\mathcal{V}(\Lambda_C(t)))^{\ell-1}\,dt - \int_0^1 (1-(1-u)\mathcal{V}(\Lambda_C(t)))^{\ell}\,dt \\
&= \int_0^1 (1-u)\mathcal{V}(\Lambda_C(t))\,(1-(1-u)\mathcal{V}(\Lambda_C(t)))^{\ell-1}\,dt \\
&= \frac{1-u}{\ell} G_C'(u),
\end{aligned}$$

where the inequality follows from concavity of the function $g(x) = x^{\frac{\ell-1}{\ell}}$ (see Fact 4).

Similarly, having established that G_I and G_A cross at least once in $(0,1)$, I show that at any such crossing point u, it must be that $G_I'(u) > G_A'(u)$. Note that

$$\begin{aligned}
\frac{1-u}{\ell} G_I'(u) &= (1-u)\alpha(1-(1-u)\alpha)^{\ell-1} \\
&= (1-(1-u)\alpha)^{\ell-1} - (1-(1-u)\alpha)^{\ell} \\
&= G_I(u)^{\frac{\ell-1}{\ell}} - G_I(u) \\
&= G_A(u)^{\frac{\ell-1}{\ell}} - G_A(u) \\
&= \left(1 - \int_u^1 \mathcal{V}(\Lambda_A(t))dt\right)^{\ell-1} - \left(1 - \int_u^1 \mathcal{V}(\Lambda_A(t))dt\right)^{\ell} \\
&= \int_u^1 \mathcal{V}(\Lambda_A(t))dt \left(1 - \int_u^1 \mathcal{V}(\Lambda_A(t))dt\right)^{\ell-1} \\
&> (1-u)\mathcal{V}(\Lambda_A(u)) \left(1 - \int_u^1 \mathcal{V}(\Lambda_A(t))dt\right)^{\ell-1} \\
&= \frac{1-u}{\ell} G_A'(u),
\end{aligned}$$

where the inequality follows from the fact that both $\mathcal{V}$ and Λ_A are decreasing functions.

D. Proof of Theorem 4

Note that $F_I(\ell) = 1 - G_I(0)$, and $F_C(\ell) = 1 - G_C(0)$. Thus, Theorem 2 establishes that $F_I(\ell) > F_C(\ell)$ (that is, more students are unassigned when priorities are common). Furthermore, by Theorem 1, $F_I(1) = \alpha(\lambda_I) = G_I'(1)/\ell$ and $F_C(1) = \int_0^1 \mathcal{V}(\Lambda_C(t))dt = G_C'(1)/\ell$. Thus, it follows from Theorem 2 that $F_I(1) < F_C(1)$ (more students get their top choice under common priorities).

All that remains is to show that if $F_I(k) > F_C(k)$, then $F_I(k') > F_C(k')$ for all $k \le k' \le \ell$. Recall that Theorem 1 gives the following equations:

$$F_I(k) = 1 - (1-\alpha(\lambda_I))^k, \qquad F_C(k) = 1 - \int_0^1 (1-\mathcal{V}(\Lambda_C(t)))^k dt.$$

But then for any $k' \ge k$,

$$\begin{aligned} 1 - F_I(k') &= (1-F_I(k))^{k'/k} \\ &< (1-F_C(k))^{k'/k} \\ &= \left(\int_0^1 (1-\mathcal{V}(\Lambda_C(t)))^k dt\right)^{k'/k} \\ &\le \int_0^1 (1-\mathcal{V}(\Lambda_C(t)))^{k'} \\ &= 1 - F_C(k'), \end{aligned}$$

where the first inequality follows because $F_I(k) > F_C(k)$, and the second from Fact 4.

E. Proofs of Theorem 5

I begin by introducing some additional notation. Given a list length distribution with cdf $\mathcal{L}$ on $\mathbb{N}$ such that $\mathcal{L}(0) = 0$, let $\overline{\mathcal{L}}(k) = 1 - \mathcal{L}(k)$. Define $\mu(\cdot) : [0,1] \to \mathbb{R}_+$ by

$$\mu(\alpha) = \sum_{k=0}^{\infty} (1-\alpha)^k \overline{\mathcal{L}}(k). \tag{55}$$

Note that this generalizes (4); this expression continues to represent the expected number of applications sent by a student for whom each application is accepted with probability α.

With this generalization of μ, the logic from Section 5 suggests that the equations (5) and (6) for λ_I and Λ_C continue to hold, and that the quantities $F_I^n(k)$ and $F_C^n(k)$ converge to

$$\begin{aligned} F_I(k) &= 1 - \sum_{\ell=1}^{\infty} (\overline{\mathcal{L}}(\ell-1) - \overline{\mathcal{L}}(\ell))(1-\alpha(\lambda_I))^{\min(k,\ell)} \\ &= \sum_{\ell=1}^{\infty} (\overline{\mathcal{L}}(\ell-1) - \overline{\mathcal{L}}(\ell)) F_I(k|\ell). \end{aligned} \tag{56}$$

$$\begin{aligned} F_C(k) &= 1 - \sum_{\ell=1}^{\infty} (\overline{\mathcal{L}}(\ell-1) - \overline{\mathcal{L}}(\ell)) \int_0^1 (1-\mathcal{V}(\Lambda_C(t)))^{\min(k,\ell)} dt \\ &= \sum_{\ell=1}^{\infty} (\overline{\mathcal{L}}(\ell-1) - \overline{\mathcal{L}}(\ell)) F_C(k|\ell), \end{aligned} \tag{57}$$

where $F_I(k|\ell)$ and $F_C(k|\ell)$ are as defined in Section 7. Minor modifications to the proofs from Appendix A show that this convergence does indeed occur. Below, I use the notation $F_I(\infty)$ and $F_C(\infty)$ to denote $\lim_{k\to\infty} F_I(k)$ and $\lim_{k\to\infty} F_C(k)$, respectively.

For completeness, I show that (5) has a unique solution for any $\rho, \mathcal{L}, \mathcal{D}$. To prove this, multiply both sides by $\alpha(\lambda_I)$ to get

$$\alpha(\lambda_I)\lambda_I = \mathcal{E}(\lambda_I) = \rho \cdot \alpha(\lambda_I) \cdot \mu(\alpha(\lambda_I)).$$

It is clear that $\mathcal{E}(\cdot)$ is increasing, and (61) states that $\alpha(\cdot)$ is decreasing. I claim that $\alpha\mu(\alpha)$ is increasing in α, from which it follows that the right side above is increasing (and therefore the equation has a unique solution).

Fact 5. *For any $\mathcal{L}$,*

- *$\mu(\cdot)$ is decreasing and convex.*
- *$\alpha \cdot \mu(\alpha)$ is increasing in α.*

Monotonicity and convexity of μ follows immediately from differentiating the expression in (55).

Monotonicity of $\alpha \cdot \mu(\alpha)$ holds because

$$\begin{aligned}
\alpha \cdot \mu(\alpha) &= \sum_{k=0}^{\infty}(1-\alpha)^k\overline{\mathcal{L}}(k) - \sum_{k=0}^{\infty}(1-\alpha)^{k+1}\overline{\mathcal{L}}(k) \\
&= \sum_{k=0}^{\infty}(1-\alpha)^k\overline{\mathcal{L}}(k) - \sum_{k=1}^{\infty}(1-\alpha)^k\overline{\mathcal{L}}(k-1) \\
&= 1 - \sum_{k=1}^{\infty}(\overline{\mathcal{L}}(k-1) - \overline{\mathcal{L}}(k))(1-\alpha)^k, \qquad (58)
\end{aligned}$$

which is clearly increasing in α.

The fact that $\alpha\,\mu(\alpha)$ is increasing is very intuitive, as this quantity can be interpreted as the probability that a student who finds each application accepted with probability α eventually matches. Indeed, from (56) and (58), we immediately see that

$$F_I(\infty) = \alpha(\lambda_I)\mu(\alpha(\lambda_I)) = \mathcal{E}(\lambda_I)/\rho, \qquad (59)$$

where the final equality follows from $\alpha(\lambda) = \mathcal{E}(\lambda)/\lambda$ and the characterizing equation (5) for λ_I.

Similarly, (57) and (58) imply that

$$F_C(\infty) = \int_0^1 \mathcal{V}(\Lambda_C(t))\mu(\mathcal{V}(\Lambda_C(t)))dt = \frac{1}{\rho}\int_0^{\lambda_C}\mathcal{V}(\lambda)d\lambda = \mathcal{E}(\lambda_C)/\rho, \qquad (60)$$

where the second equality follows from the substitution $\lambda = \Lambda_C(t)$ and the characterizing equation (6) for Λ_C, and the final equality follows from Fact 2.

Having derived expressions for F_I and F_C, we turn to Theorem 5. By (59) and (60), we see that comparing the number of matched students under idiosyncratic and common priorities is equivalent to comparing λ_I and λ_C.

Define the function $f : [0,1] \to [0,1]$ by $f(\alpha) = 1/\mu(\alpha)$. Theorem 8 states that $\lambda_I > \lambda_C$ if f is convex, and $\lambda_I < \lambda_C$ if f is concave. Lemma 1 states that f is convex if $\mathcal{L}$ has an increasing hazard rate, and is concave if $\mathcal{L}$ has a decreasing hazard rate. Thus, Theorem 8 and Lemma 1 jointly imply Theorem 5.

In the proof of Theorem 8, I make use of the following:

$$\alpha'(\lambda) = \frac{\mathcal{V}(\lambda) - \alpha(\lambda)}{\lambda} < 0. \tag{61}$$

The equality follows because

$$\alpha'(\lambda) = \frac{d}{d\lambda}\frac{\mathcal{E}(\lambda)}{\lambda} = \frac{\lambda\mathcal{V}(\lambda) - \mathcal{E}(\lambda)}{\lambda^2} = \frac{\mathcal{V}(\lambda) - \alpha(\lambda)}{\lambda}.$$

The inequality follows because

$$\alpha(\lambda) = \mathcal{E}(\lambda)/\lambda = \int_0^1 \mathcal{V}((1-t)\lambda)dt > \int_0^1 \mathcal{V}(\lambda)dt = \mathcal{V}(\lambda).$$

(Note that the first equality holds by definition, the second follows by the substitution $u = (1-t)\lambda$ and Fact 2, and the inequality follows because $\mathcal{V}(\cdot)$ is decreasing.)

Theorem 8. *Fix ρ, $\mathcal{D}$ and $\mathcal{L}$, and define λ_I and $\lambda_C = \Lambda_C(0)$, by (5) and (6). If $\mathcal{L}$ is such that the function $f(\alpha) = 1/\mu(\alpha)$ is convex, then $\lambda_I \geq \lambda_C$. If $f(\cdot)$ is concave, then $\lambda_C \geq \lambda_I$.*

Proof. As before, the approach is to fix $\mathcal{L}$ and $\mathcal{D}$, and to consider the quantities λ_I and λ_C as functions of ρ. Because $\lambda_I(0) = \lambda_C(0) = 0$, to show that $\lambda_I(\rho) \geq \lambda_C(\rho)$ for all ρ, it suffices to show

$$\lambda_I(\rho_I) = \lambda_C(\rho_C) \Rightarrow \lambda_I'(\rho_I) > \lambda_C'(\rho_C).$$

Similarly, to show that $\lambda_I(\rho) \leq \lambda_C(\rho)$ for all ρ, it suffices to show that $\lambda_I'(\rho_I) < \lambda_C'(\rho_C)$.

Next, we compute λ_I' and λ_C'. Differentiating (5) with respect to ρ, we get

$$\lambda_I' = \mu(\alpha(\lambda_I)) + \rho\,\mu'(\alpha(\lambda_I))\,\alpha'(\lambda_I)\,\lambda_I'. \tag{62}$$

From (61) and (5), it follows that that

$$\rho \cdot \alpha'(\lambda_I) = \rho\,\frac{\mathcal{V}(\lambda_I) - \alpha(\lambda_I)}{\lambda_I} = \frac{\mathcal{V}(\lambda_I) - \alpha(\lambda_I)}{\mu(\alpha(\lambda_I))}.$$

For notational convenience, let $\alpha = \alpha(\lambda_I)$ and $\mathcal{V} = \mathcal{V}(\lambda_I)$. Substituting the above equality into (62) and solving for λ_I', we see that the derivative of λ_I with respect to ρ is

$$\lambda_I' = \frac{\mu(\alpha)}{1 - (\mathcal{V} - \alpha)\frac{\mu'(\alpha)}{\mu(\alpha)}}. \tag{63}$$

Meanwhile, λ_C' intuitively corresponds to the expected number of additional applications when a single student is added to the market. The logic from Section 5.3 suggests (and a generalization of the proof in Appendix A.1 shows formally) that

$$\lambda_C' = -\frac{1}{\rho}\Lambda_C'(t) = \mu(\mathcal{V}(\lambda_C)). \tag{64}$$

It follows from (63) and (64) that if $\lambda_I(\rho_I) = \lambda_C(\rho_C) = \lambda$, then (letting $\mathcal{V} = \mathcal{V}(\lambda), \alpha = \alpha(\lambda)$)

$$\begin{aligned}
\lambda_C'(\rho_C) < \lambda_I'(\rho_I) &\Leftrightarrow \mu(\mathcal{V}) < \frac{\mu(\alpha)}{1 - (\mathcal{V} - \alpha)\frac{\mu'(\alpha)}{\mu(\alpha)}} \\
&\Leftrightarrow \mu(\mathcal{V}) - \mu(\alpha) < (\mathcal{V} - \alpha)\mu'(\alpha)\frac{\mu(\mathcal{V})}{\mu(\alpha)} \\
&\Leftrightarrow \frac{\mu(\mathcal{V}) - \mu(\alpha)}{\mu(\mathcal{V})\mu(\alpha)} < (\alpha - \mathcal{V})f'(\alpha). \\
&\Leftrightarrow \frac{f(\alpha) - f(\mathcal{V})}{\alpha - \mathcal{V}} < f'(\alpha),
\end{aligned}$$

where the second line follows from the fact that $f'(x) = -\mu'(x)/\mu(x)^2$. Because $\mathcal{V} < \alpha$ by (61), this holds if f is convex; if f is concave, the inequality reverses.

□

F. Proofs of Theorem 6

The proof of Theorem 6 makes use of the following Lemma.

Lemma 1. *Given cdf $\mathcal{L}$ on $\mathbb{N}$, define μ as in (55). Then $1/\mu(\alpha)$ is convex in α if and only if $\mathcal{L}$ has an increasing hazard rate, and is concave in α if and only if $\mathcal{L}$ has a decreasing hazard rate.*

To prove Theorem 6, I will show that there exists $k' \geq 2$ with $\overline{\mathcal{L}}(k'-1) > 0$ such that

I. $F_I(k|k) \leq F_C(k|k)$ for $k < k'$
II. $F_I(k|k) \geq F_C(k|k)$ for $k \geq k'$.

Because $F_I(k|\ell)$ and $F_C(k|\ell)$ depend only on the minimum of k and ℓ, the first point above implies part I. of Theorem 6, and the two points above jointly imply part II. of Theorem 6.

The initial argument is identical to the case with a common list length: for any fixed λ_I and Λ_C,

$$(65) \qquad (1-\alpha(\lambda_I))^k < \int_0^1 (1-\mathcal{V}(\Lambda_C(t)))^k dt \Rightarrow (1-\alpha(\lambda_I))^{k+1} < \int_0^1 (1-\mathcal{V}(\Lambda_C(t)))^{k+1} dt.$$

Because $(1-\alpha(\lambda_I))^k = 1 - F_I(k|k)$ and $\int_0^1 (1-\mathcal{V}(\Lambda_C(t)))^k dt = 1 - F_C(k|k)$, it follows that $F_I(k|k)$ and $F_C(k|k)$ either cross once or not at all.

Next, I show that they do cross, meaning that neither procedure dominates the other. The intuitive argument is that if (say) common priorities rank dominated idiosyncratic priorities, then students would (in aggregate) send fewer applications under common priorities. Because the total number of matches in a large market is a known function of the number of applications sent, this implies that fewer students would match under common priorities, contradicting rank dominance.

More formally, first suppose that $F_I(1|1) \geq F_C(1|1)$, meaning that $1-\alpha(\lambda_I) \leq \int_0^1 (1-\mathcal{V}(\Lambda_C(t)))dt$. From this, we reach two conclusions:

- Conclude from (65) that $F_I(k|k) \geq F_C(k|k)$ for all k, and thus by (56), (57) that $F_I(k) \geq F_C(k)$ for all k, and in particular that $F_I(\infty) \geq F_C(\infty)$
- Fact 5 states that μ is convex and decreasing. Apply Fact 4 to conclude that

$$\lambda_I = \rho \cdot \mu(\alpha(\lambda_I)) \leq \rho \cdot \mu\left(\int_0^1 \mathcal{V}(\Lambda_C(t))dt\right) \leq \rho \cdot \int_0^1 \mu(\mathcal{V}(\Lambda_C(t)))dt = \lambda_C,$$

with the second inequality strict unless μ is constant (i.e. $\overline{\mathcal{L}}(1) = 0$). It follows from (59) and (60) that $F_I(\infty) < F_C(\infty)$.

Because these conclusions contradict each other, it must be that $F_I(1|1) < F_C(1|1)$.

Conversely, suppose that $F_I(k|k) \leq F_C(k|k)$ for all k such that $\overline{\mathcal{L}}(k-1) > 0$, with strict inequality for $k = 1$. By (56) and (57), we conclude that $F_I(\infty) < F_C(\infty)$. On the other hand, we must have

$$\begin{aligned}\lambda_I &= \rho \cdot \mu(\alpha(\lambda_I)) \\ &= \rho \cdot \sum_{k=0}^{\infty} \overline{\mathcal{L}}(k)(1-\alpha(\lambda_I))^k \\ &> \rho \cdot \sum_{k=0}^{\infty} \overline{\mathcal{L}}(k) \int_0^1 (1-\mathcal{V}(\Lambda_C(t)))^k dt \\ &= \int_0^1 \rho \cdot \mu(\mathcal{V}(\Lambda_C(t)))dt \\ &= -\int_0^1 \Lambda_C'(t)dt \\ &= \lambda_C,\end{aligned}$$

where the first equality follows from the definition of λ_I in (5), the second and fourth lines follow from the definition of μ in (55), the third line follows by the assumption (because $(1-\alpha(\lambda_I))^k = 1 - F_I(k|k)$ and $\int_0^1 (1-\mathcal{V}(\Lambda_C(t)))^k dt = 1 - F_C(k|k)$), and the fifth line follows from the equation (6) defining Λ_C.

But if $\lambda_I > \lambda_C$, then (59) and (60) imply that $F_I(\infty) > F_C(\infty)$, which contradicts our previous conclusion. Therefore, there must exist $k' > 1$ such that $\overline{\mathcal{L}}(k'-1) > 0$ and $F_I(k'|k') > F_C(k'|k')$.

G. Analysis as $\ell \to \infty$

Define $\overline{C}$ to be the average school size. That is,

$$\mathcal{E}(\infty) = \int_0^{\infty} \mathcal{V}(\lambda)d\lambda = \sum_{k\geq 0}(1-\mathcal{D}(\infty,k)) \triangleq \overline{C}.$$

Similarly, define $\bar{\ell}$ to be the average list length. That is,

$$\bar{\ell} = \sum_{k\geq 0} \overline{\mathcal{L}}(k).$$

Note that

$$\mu(\alpha) = \sum_{k\geq 0}(1-\alpha)^k \overline{\mathcal{L}}(k) \leq \min(\bar{\ell}, 1/\alpha), \tag{66}$$

where the inequalities follow from taking $\alpha = 0$ and $\overline{\mathcal{L}}(k) = 1$, respectively.

G.1. **Analysis of Unmatched Students.**

G.1.1. *Idiosyncratic Priorities.* In the case of ER random graphs ($\mathcal{L}$ is Poisson with mean ℓ) when $p_h = C_h = 1$, we can derive exact closed-form expressions .

If $\rho = 1$, we get that

$$G_I(0) = e^{-\sqrt{\ell(1-G_I(0))}} \sim e^{-\sqrt{\ell}}, \qquad G_C(0) = \log(1-e^{-\ell})/\ell \sim \log(2)/\ell, \qquad G_A(0) = 1/(\ell+1).$$

If $\rho < 1$, we get that

G.1.2. *Common Priorities.*

G.2. **Analysis of Top Choices.**

G.2.1. *Idiosyncratic Priorities.* The fraction of students who receive their top choice is

$$F_I(1) = \alpha(\lambda_I) = \mathcal{E}(\lambda_I)/\lambda_I,$$

where λ_I is the solution to

$$\rho \cdot \mu(\alpha(\lambda_I)) = \lambda_I.$$

Because the fraction who are assigned at all is

$$F_I(\infty) = \mathcal{E}(\lambda_I)/\rho,$$

it follows that

$$\frac{F_I(1)}{F_I(\infty)} = \rho \cdot \alpha(\lambda_I)/\mathcal{E}(\lambda_I) = \rho/\lambda_I.$$

Suppose that $\rho > \overline{C}$. Then as list lengths grow,

Suppose that $\rho < \overline{C}$, meaning that there are enough school seats for every student. Then as list lengths grow, λ_I converges to the solution to $\mathcal{E}(\lambda) = \rho$.

If $C_h = \overline{C}$ for all h, then intuitively, things are best when popularities are uniform. In the case of $C_h = 1$, the solution is $\lambda = -\log(1-\rho)$, and $\alpha(\lambda) = -\rho/\log(1-\rho) \geq \sqrt{1-\rho}$,

Because $\mu(\cdot) \geq 1$, it follows that $\lambda_I \geq \rho$, so as $\rho \to \infty$, we have that $\lambda_I \to \infty$ and thus

$$\alpha(\lambda_I) = \mathcal{E}(\lambda_I)/\lambda_I \leq \overline{C}/\lambda_I \to 0.$$

From this, we see that for large ρ, $\lambda_I \approx \rho\overline{\ell}$, and thus, $\frac{F_I(1)}{F_I(\infty)} \approx 1/\overline{\ell}$.

G.2.2. *Common Priorities.* The fraction of students who receive their top choice under common priorities is

$$F_C(1) = \int_0^1 \mathcal{V}(\Lambda_C(t))dt = \frac{1}{\rho}\int_0^{\lambda_C} \frac{\mathcal{V}(\lambda)}{\mu(\mathcal{V}(\lambda))}d\lambda \geq \frac{1}{\rho}\int_0^{\lambda_C} \mathcal{V}(\lambda)^2 d\lambda,$$

where we have used (66). The fraction of students who are assigned is

$$F_C(\infty) = \mathcal{E}(\lambda_C)/\rho = \frac{1}{\rho}\int_0^{\lambda_C} \mathcal{V}(\lambda)d\lambda.$$

Thus, the fraction of assigned students who receive their top choice is at least

$$\frac{F_C(1)}{F_C(\infty)} \geq \frac{\int_0^{\lambda_C} \mathcal{V}(\lambda)^2 d\lambda}{\int_0^{\lambda_C} \mathcal{V}(\lambda)d\lambda} \geq \frac{\int_0^{\infty} \mathcal{V}(\lambda)^2 d\lambda}{\int_0^{\infty} \mathcal{V}(\lambda)d\lambda}, \tag{67}$$

where we have used the fact that

$$\frac{d}{d\lambda_C}\frac{\int_0^{\lambda_C} \mathcal{V}(\lambda)^2 d\lambda}{\int_0^{\lambda_C} \mathcal{V}(\lambda)d\lambda} < 0 \Leftrightarrow \mathcal{V}(\lambda_C)^2\int_0^{\lambda_C} \mathcal{V}(\lambda)d\lambda < \mathcal{V}(\lambda_C)\int_0^{\lambda_C} \mathcal{V}(\lambda)^2 d\lambda,$$

and the inequality on the right holds because $\mathcal{V}(\cdot)$ is decreasing. Thus, (67) gives a lower bound that holds uniformly in $\rho, \mathcal{L}$ (for a given $\mathcal{D}$). This bound cannot be uniformly bounded (in $\mathcal{D}$); if $C_h = 1$ for all h and

$$D_n(p) = \begin{cases} 0 & p < 1/(n-1) \\ \frac{n-1}{n} & 1/(n-1) \leq p < n-1 \\ 1 & n-1 \leq p \end{cases},$$

then the above bound is approximately $1/n$.

This set of examples has some school popularities approaching zero, and others diverging. Suppose that $p_h \in [u, v]$, and $C_h = 1$ for all h. Then $\mathcal{D}(p) = \beta$ for $p \in [u, v)$ and $\mathcal{D}(v) = 1$, where β is such that $\beta u + (1-\beta)v = 1$. In this case,

$$\mathcal{V}(\lambda) = \beta u e^{-u\lambda} + (1-\beta) v e^{-v\lambda}$$

$$\begin{aligned} \int_0^\infty \mathcal{V}(\lambda)^2 d\lambda &= \frac{\beta^2 u^2}{2u} + \frac{(1-\beta)^2 v^2}{2v} + \frac{2\beta(1-\beta)uv}{u+v} \\ &= \frac{1+uv}{2(u+v)}. \end{aligned}$$

Because $\int_0^\infty \mathcal{V}(\lambda) d\lambda = \overline{C} = 1$, this implies that among assigned students, at least a fraction $\frac{1+uv}{2(u+v)}$ get their top choice.

One easy way to think of this is to assume that the ratio of popularities between schools is at most $r > 1$, i.e. $r = v/u$. It is straightforward to show that the choices of u, v that minimize the bound, given $v/u = r$, is $v = \sqrt{r}, u = 1/\sqrt{r}$ in which case we get that

$$\frac{F_C(1)}{F_C(\infty)} \geq \frac{1}{\sqrt{r} + 1/\sqrt{r}}.$$

So if, for example, no school is more than four times more popular than any other, this implies that at least 40% of assigned students get their top choice. If no school is more than nine times more popular than any other, this implies that at least 30% of assigned students get their top choice.

For arbitrary C (constant across schools), I conjecture that the same choice of u and v is optimal, given r. As $C \to \infty$, this bound approaches $2\frac{\sqrt{r}-1}{r-1}$. Could some of these results hold when schools have different capacities, assuming that capacity and popularity are independent?