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Pricing and Optimization in Shared Vehicle Systems: An Approximation Framework

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Optimizing shared vehicle systems (bike-sharing/car-sharing/ride-sharing) is more challenging compared to traditional resource allocation settings due to the presence of complex network externalities – changes in the demand/supply at any location affect future supply throughout the system within short timescales. These externalities are well captured by steady-state Markovian models, which are therefore widely used to analyze such systems. However, using such models to design pricing/control policies is computationally difficult since the resulting optimization problems are high-dimensional and non-convex.

To this end, we develop a general approximation framework for designing pricing policies in shared vehicle systems, based on a novel convex relaxation which we term elevated flow relaxation. Our approach provides the first efficient algorithms with rigorous approximation guarantees for a wide range of objective functions (throughput, revenue, welfare). For any shared vehicle system with $n$ stations and $m$ vehicles, our framework provides a pricing policy with an approximation ratio of $1 + (n - 1)/m$. This guarantee is particularly meaningful when $m/n$, the average number of vehicles per station is large, as is often the case in practice.

Further, the simplicity of our approach allows us to extend it to more complex settings: rebalancing empty vehicles, redirecting riders to nearby vehicles, multi-objective settings (such as Ramsey pricing), incorporating travel-times, etc. Our approach yields efficient algorithms with the same approximation guarantees for all these problems, and in the process, obtains as special cases several existing heuristics and asymptotic guarantees.

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History:
1. Introduction

Shared vehicle systems, such as those for bike-sharing (e.g., Citi Bike in NYC, Vélib in Paris), car-sharing (e.g., car2go, Zipcar) and ride-sharing (e.g., Uber, Lyft), are fast becoming essential components of the urban transit infrastructure. In such systems, customers have access to a collection of personal transportation vehicles, which can be engaged anytime (subject to vehicle availability) and between a large number of source-destination locations. In bike- and car-sharing, the vehicles are operated by the customers themselves, while in ride-sharing they are operated by independent drivers.

All vehicle sharing systems experience inefficiencies due to limited supply (vehicles) and demand heterogeneity across time and space. These inefficiencies, however, can often be greatly reduced by rebalancing the demand and/or supply. Pricing has traditionally been the main tool to balance demand and supply in settings with limited supply and heterogeneous demand: for instance, in limited-item auctions or airplane/hotel reservations [12, 7]. Ride-sharing platforms have also long utilized dynamic pricing, and today, many bike- and car-sharing platforms are experimenting with point-to-point prices and other location-based incentive schemes [14]. Shared vehicle platforms also enable other means of rebalancing, such as repositioning empty vehicles, or redirecting customers to nearby stations (cf. Section 1.3). Moreover, these tools can be used towards achieving different objectives – examples include revenue/welfare/throughput maximization, multi-objective settings, etc.

In contrast to other limited supply settings, however, shared vehicle settings are more challenging due to two unique features. The first is the presence of spatial and temporal supply externalities: whenever a customer engages a vehicle, this not only decreases the instantaneous availability at the source location, but also affects the future availability at all other locations in the system. The other distinguishing feature is the high frequency of events (passenger arrivals/rides) in such settings. This necessitates treating the problem as an infinite-horizon control problem, as opposed to finite-horizon dynamic programming approaches used in traditional transportation and revenue management settings [7, 1]. The high frequency of events tends to drive such systems into operating under a dynamic equilibrium state, jointly determined by the demand and supply characteristics as well as the chosen controls; the performance of any pricing/control policy is determined by this equilibrium.

All the above features (limited supply, demand heterogeneity, supply externalities and fast operational timescales) are well captured by closed queueing network models [22, 16], which are thus widely adopted in the literature on shared vehicle systems [8, 26, 23]. These models use a Markov chain to track the number of vehicles across locations. Each location experiences a stream of arriving customers, who engage available vehicles and take them to their desired destination. Increasing the price for a ride between a pair of stations decreases the number of customers willing to take that ride, which over time affects the distribution of vehicles across all stations. Even though such models can be well-calibrated, based on demand rates and price elasticities estimated from historical data, the problem of designing good pricing/control policies under such a model is complex due to a combination of high dimensionality and intrinsic non-concavity of the optimization problem (cf. Section 2.4). Consequently, previous work has focused only on a narrow set of objectives (typically, weighted throughput) and is largely based on heuristics or simulation/numerical techniques, with few provable guarantees (cf. Section 1.3 for a discussion). Algorithms for other objectives (e.g. revenue and welfare) as well as more complex constrained settings, have yet to be addressed.

In this context, our work develops the first efficient algorithms for designing pricing and control policies in closed queueing networks, with approximation guarantees for a large class of objectives. More generally, we provide a unified framework for designing rebalancing policies, which can incorporate a variety of controls and constraints, including multi-objective settings and incorporating...
travel time distributions. In all settings, we obtain parameterized performance guarantees, which improve with the number of vehicles in the system, and are near-optimal in the parameter regimes of real systems. Moreover, our guarantees also provide an elementary proof of the asymptotic optimality of our policies under the so-called large-market scaling \[3, 19\], without necessitating the derivation of the associated fluid limits. Our approach leverages techniques from convex optimization and approximation algorithms, and combines these with a novel infinite projection and pullback technique. Given the widespread use of closed queueing models for a variety of other applications, we anticipate that our framework can prove useful in other areas as well.

1.1. Outline of our Contributions
We model a shared vehicle system with \(m\) vehicles and \(n\) stations as a continuous-time finite-state Markov chain that tracks the number of vehicles (units) at each station (node), and use this to study a variety of pricing and control problems. We use pricing as a primary example to illustrate our methodology and present extensions to other controls in Section 5.

A brief description of our model is as follows (cf. Section 2 for details). Each station in the system observes a Poisson arrival of customers. Arriving customers draw a value and a destination from some known distribution. Upon arrival at a station, the customer is quoted a price and one of three scenarios occurs: i) the customer is not willing to pay the price, i.e. the price exceeds her value, and she leaves the system, ii) the customer is willing to pay the price but no unit is available at the node; therefore she again leaves the system, or iii) the customer is willing to pay the price, and a vehicle is available. A ride occurs only in the final case with the vehicle moving to the customer’s destination; the number of vehicles at the origin is decremented instantaneously, while the state at the destination is incremented either instantaneously (cf. Section 3), or more generally, after some random time interval (cf. Section 5.3). This describes the basic dynamics under which we aim to maximize the long-run average performance, measured by the throughput, the social welfare, or the revenue obtained in steady-state. The prices can in general depend on the instantaneous state of the system. Thus, the resulting optimization problem is high-dimensional, and moreover, is non-convex even in basic settings (cf. Section 2.4).

In Section 3, we propose a simple pricing policy, based on optimizing over a novel convex relaxation, which we term \textit{elevated flow relaxation}. We adapt the objective by identifying a concave pointwise upper bound which we call \textit{elevated objective}. Furthermore, we introduce additional flow-conservation constraints to capture the network externalities. As the elevated objective is bounded below by the original objective, optimal solutions in the elevated optimization problem are bounded below in value by optimal solutions in the original optimization problem.

In Section 4, we present our main result: we show that the elevated flow relaxation can be efficiently solved to derive a pricing policy which has an approximation ratio of \(1 + (n - 1)/m\) under a large class of objectives. Even though we consider general state-dependent pricing policies, the policy that achieves our guarantee surprisingly turns out to be state-independent, i.e. the prices do not differ based on the configuration of units across nodes.

In Section 5, we demonstrate how the above framework, comprising of a policy derived via an appropriate elevated flow relaxation, and the three-step process to prove its guarantees, can be extended and applied to other settings.

• In Section 5.1 we study two other rebalancing controls considered in the literature, and obtain \(1 + (n - 1)/m\) approximation guarantees for the respective optimization problems. In the first, units can move to a new location after ending a trip; in the second, customers can be matched to units at neighboring nodes. In both cases, we recover and strengthen the previous results.

• In Section 5.2 we turn our attention to multiobjective settings, where the goal is to maximize one objective subject to a lower bound on another, the so-called Ramsey pricing problem \([20]\): designing a pricing policy to maximize system revenue subject to a lower bound on the system
welfare. For many shared vehicle systems this is the most relevant objective, since they are operated by private companies in close partnership with city governments. For instance, the Citi Bike system in New York City is run by Motivate, a private company, under service-level agreements with the NYC Department of Transportation. Note that the complementary problem (maximizing welfare subject to revenue constraints) is of interest when such systems are managed by non-profit organizations (e.g., Pronto in Seattle) and is considered in other paradigms such as the FCC spectrum auction [18]. In this context we demonstrate how our approach can be used to obtain a \((\gamma, \gamma)\) bicriteria approximation guarantee with 
\[
\gamma = 1 + \frac{n-1}{m}.
\]

- In Section 5.3 we show that similar approximation guarantees continue to hold when rides do not occur instantaneously but instead require some delay (travel time). Our results in this section provide an elementary proof of the so-called large-market optimality [3] of our algorithms.
- In Section 5.4, we consider a special case of the basic pricing problem where the customers’ value distributions depend only on their source node and the platform is limited to point prices, that are based only on the origin of a trip, but not on the destination. The latter assumption is motivated by contemporary schemes like surge pricing. We show that in this case the optimization problem then collapses to a one-dimensional concave maximization, allowing us to incorporate additional constraints.

Our results recover and unify many existing results in this area, and provide a general framework for deriving approximation algorithms for many other settings. Moreover, the guarantees we obtain are close to 1 for realistic system parameters. For instance, for the parameters \((m = 10000, n = 600)\) of New York City’s Citi Bike system, we obtain an approximation ratio of 1.06.

1.2. Technical contributions

Besides their applications to shared vehicle systems, our work also provides a novel approach for deriving approximately optimal control policies for closed queueing networks (and, more generally, finite-population Markovian interacting particle systems) in equilibrium. We now briefly describe our approach; cf. Section 4 for more details.

Our approach towards deriving approximate control policies is based on the following three steps:

1. First, we derive an efficiently computable upper bound for the performance of any control policy. In particular, we observe that in our setting, the elevated flow relaxation is an upper bound on the performance of any state-dependent policy in the \(m\)-unit system, and hence, an upper bound for the optimal state-dependent policy in the \(m\)-unit system. The main idea here is to construct bounds which encode essential ‘conservation laws’ of the system, yet are easily computable; cf. Section 5.3 for a more complex example.

2. Next, we show that for a particular class of control policies in an appropriate ‘large-population’ limit, the achievable objective values match the set of achievable upper bounds. In particular, we consider a restricted subset of state-independent pricing policies, under which the resulting Markov chain has the structure of a so-called closed Jackson network. We prove that as the number of vehicles grows to infinity, the set of ‘flows’ achievable under such policies exactly matches the polytope considered under the elevated flow relaxation. Moreover, for these policies, the elevated objective collapses to the original objective.

3. Finally, we use the above restricted subset of control policies to derive an approximation guarantee for the performance of the policy provided by our upper bound in the finite-population setting. In our setting, we show that the performance of any policy in the \(m\)-unit setting approximates its performance in the infinite-supply setting within a factor of \(1 + (n-1)/m\). Though the intuition behind this pullback step can be observed via stochastic coupling arguments, we provide a fully algebraic proof based on a combinatorial construction of a biregular graph that relates the state spaces of the \(m\)- and \((m-1)\)-unit systems.
An interesting feature of our technique is that it demonstrates the optimality of our control policies in the so-called ‘large market’ regimes \[19\], where both number of vehicles and arrival rates of customers jointly scale to infinity. Compared to other work, however, our proof is elementary in that we do not need to characterize the limiting processes.

1.3. Related work
There is a large literature on characterizing open and closed queueing network models, building on seminal work of Jackson \[13\], Gordon and Newell \[10\], and Basket et al. \[2\]; the books by Kelly \[16\] and Serfozo \[22\] provide an excellent summary. Optimal resource allocation in open queueing networks also has a long history, going back to the work of Whittle \[25\]. However, there is much less work for closed networks, in part due to the presence of a normalization constant for which there is no closed-form (though it is computable in \(O(nm)\) time via iterative techniques \[5, 21\]). Most existing work on optimizing closed queueing networks use heuristics, with limited or no guarantees. In contrast, our work focuses on obtaining algorithms with provable guarantees for a wide range of problems.

Three popular approaches for closed queueing network optimization in the literature are: (i) using open queueing network approximations, (ii) heuristically imposing a ‘fairness’ property, which we refer to as the demand circulation constraint (cf. Section 3.3), and (iii) characterizing the fluid limits of closed queueing networks, and obtaining solutions that are optimal in these scaling regimes. We now briefly describe each approach.

The first approach was formalized by Whitt \[24\], via the fixed-population-mean (FPM) method, where exogenous arrival rates are chosen to ensure the mean population is \(m\). It has since been used in many applications; for example, Brooks et al. \[4\] use it to derive policies for matching debris removal vehicles to routes following natural disasters. Performance guarantees however are available only in restricted settings.

Another line of work is based on heuristics that enforce the demand circulation property (variously referred to as the demand rebalancing, the fairness, or the bottleneck property). In transportation settings, George et al. used these to optimize weighted throughput \[8\], Zhang et al. to minimize rebalancing costs \[26\]. Most works typically only provide asymptotic guarantees \[9\].

More recently, Ozkan and Ward \[19\] and Braverman et al. \[3\] characterized appropriate fluid (or large-market) limits for closed queueing networks, and used it to study the operations of ride-sharing systems. In contrast to our work, which focuses on optimizing a given finite-\(m\) system, these works consider a regime where \(m\) and the arrival rates of passengers together scale to \(\infty\), and characterize the optimal policy in the limit. Within this limit, the former studied the assignment of customers to nearby drivers, whereas the latter considered directing drivers at the end of each trip to under-served locations. Our extensions to settings beyond pricing (cf. Section 5.1) are inspired by these works; in particular, we show that similar scaling results can be derived within our framework. Moreover, our work provides guarantees for the resulting policies in the finite case (i.e., before taking the limit), and also against a much more general class of state-dependent policies.

The closest work to ours is that of Waserhole and Jost \[23\], who provide a pricing policy for maximizing throughput in closed queueing networks, with the same approximation ratio we obtain. They do this via a different argument wherein they observe that, under the demand circulation property, the Markov chain is doubly stochastic, and hence has a uniform distribution (this was also noted earlier by Whitt \[24\]). A simple counting argument then implies that the probability of a station having a vehicle is \(m/(m+n-1)\). Moreover, since the maximum throughput under any policy is bounded by the maximum demand circulation, the maximum throughput under demand circulation is within a \(m/(m+n-1)\) factor of the optimum. This argument is finely tuned to this particular setting (maximizing throughput via pricing with no delays). In contrast, our approach can accommodate several objectives and rebalancing controls as well as delays.
Finally, we note that there is a parallel line of work which tackles settings with dynamic arrivals and pricing, using techniques from approximate dynamic programming [1, 17, 11]. These typically can deal only with small systems, as their dimensionality scales rapidly with the number of stations; moreover, many of the techniques have no provable guarantees.

2. Preliminaries

In this section, we first formally define our model of shared vehicle systems and formulate the optimal pricing problem. To capture the complex network externalities of the system, we define a probabilistic model of customer arrivals, which we analyze in steady state. Subsequently, we introduce known results from the queuing literature that provide the technical background upon which our analysis relies. Finally, we present an example that shows that even in the restricted sets of pricing policies, that are independent of the configuration of vehicles across the system, the optimization problems we consider are non-convex.

2.1. Basic setting

We consider a system with \( m \) units (corresponding to vehicles) and \( n \) nodes (corresponding to stations). Customers traveling between nodes \( i \) and \( j \) arrive at node \( i \) according to a Poisson process of rate \( \phi_{ij} \). Each customer traveling from \( i \) to \( j \) has a value drawn independently from a distribution \( F_{ij}(\cdot) \). We assume that \( F_{ij} \) has a density and that all values are positive with some probability, i.e. \( F_{ij}(0) < 1 \). Upon arrival at \( i \), a customer is quoted a price \( p_{ij} \), and engages a unit if her value exceeds this price, i.e. with probability \( 1 - F_{ij}(p_{ij}) \), and at least one unit is available at node \( i \); else she leaves the system.

As is common with pricing, the related optimization problems are often more easily framed in terms of the inverse demand (or quantile) function associated with the user as \( q_{ij} = 1 - F_{ij}(p_{ij}) \). For ease of presentation we assume that the density of \( F_{ij} \) is positive everywhere in its domain, implying that there is a 1-1 mapping between prices and quantiles. As \( F_{ij} \) is therefore invertible, we can write \( p_{ij} = F_{ij}^{-1}(1 - q_{ij}) \). This allows us to abuse notation throughout the paper by using prices and quantiles interchangeably.

A continuous-time Markov chain tracks the number of units across nodes. At time \( t \geq 0 \), the state of the Markov chain \( \mathbf{X}(t) = (X_1(t), \ldots, X_n(t)) \) contains the number of units \( X_i(t) \) present at each node \( i \). The state space of the system is denoted by \( \mathcal{S}_{n,m} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{N}_0^n | \sum_i x_i = m \} \).

Throughout the paper we use \( \mathbf{X}(t) \), \( X_i(t) \) to indicate random variables, and \( \mathbf{x}, x_i \) to denote specific elements of the state space. Note that the state-space is finite; moreover, \( |\mathcal{S}_{n,m}| = (m+n-1) = \Omega(m^n) \).

Since our focus is on the long-run average performance, i.e. system performance under the steady state of the Markov chain, we henceforth suppress the dependence on \( t \) for ease of notation.

For ease of presentation, we assume that rides between nodes occur without delay. In the context of our model, this translates into an instantaneous state transition from \( \mathbf{X} \) to \( \mathbf{X} - e_i + e_j \) when a customer engages a unit to travel from \( i \) to \( j \) (where \( e_i \) denotes the \( i \)th canonical unit vector). We relax this assumption in Section 5.3.

2.2. Pricing Policies and Objectives

We consider pricing policies that select point-to-point prices \( p_{ij} \) as a function of the overall state \( \mathbf{X} \). Formally, given arrival rates and demand elasticities \( \{\phi_{ij}, F_{ij}(\cdot)\} \), we want to design a pricing policy \( \mathbf{p}(\cdot) = \{p_{ij}(\cdot)\} \), where each \( p_{ij} : \mathcal{S}_{n,m} \to \mathbb{R} \cup \{\pm \infty\} \) maps the state to a price for a ride between \( i \) and \( j \). Equivalently, we want to select quantiles \( \mathbf{q}(\cdot) = \{q_{ij}\} \) where each \( q_{ij} : \mathcal{S}_{n,m} \to [0,1] \). For a fixed pricing policy \( \mathbf{p} \) with corresponding quantiles \( \mathbf{q} \), the effective demand stream from \( i \) to \( j \) (i.e. customers traveling from \( i \) to \( j \) with value exceeding \( p_{ij} \)) thus follows a state-dependent Poisson process with rate \( \phi_{ij} q_{ij}(\mathbf{X}) \). This follows from the notion of probabilistic thinning of a Poisson process – the rate of customers wanting to travel from \( i \) to \( j \) is a Poisson process of rate \( \phi_{ij} \), and
each customer is independently willing to pay $p_{ij}$ with probability $q_{ij} = 1 - F_{ij}(p_{ij})$. State-dependent prices also allow us to capture unavailability by defining $q_{ij}(x) = 0$ if $x = 0$ (i.e., a customer with origin $i$ is always turned away if there are no units at that station; recall we defined $F_{ij}(\infty) = 1$). Thus, a pricing policy $p$, along with arrival rates and demand elasticities $\{\phi_{ij}, F_{ij}(\cdot)\}$, determines the transitions of the Markov chain. Note that this is a finite-state Markov chain, and furthermore, is irreducible under weak assumptions on the prices and the demand (cf. Appendix A); hence, it has a unique steady-state distribution $\pi(\cdot)$ with $\pi(x) \geq 0 \forall x \in S_{n,m}$ and $\sum_{x \in S_{n,m}} \pi(x) = 1$.

Our goal is to design a pricing policy $p$ to maximize the steady-state performance under various objectives. In particular, we consider objective functions that decompose into per-ride reward functions $I_{ij} : \mathbb{R} \to \mathbb{R}$, which correspond to the reward obtained from a customer engaging a ride between stations $i$ and $j$ at price $p$. The per-ride rewards corresponding to the three canonical objective functions are:

- Throughput: the total rate of rides in the system; for this, we set $I_{ij}^T(p) = 1$.
- Social welfare: the per-ride contribution to welfare is given by $I_{ij}^W(p) = \mathbb{E}_{V \sim F_{ij}}[V | V \geq p]$.
- Revenue: to find the system’s revenue rate, we can set $I_{ij}^R(p) = p$.

We abuse notation to define $I_{ij}(q) \triangleq I_{ij}(F_{ij}^{-1}(1 - q))$ as a function of the quantile instead of the price. We also define the reward curves $R_{ij}(q) \coloneqq q \cdot I_{ij}(q)$ (analogous to the notion of revenue curves; cf. [12]). Our results require the technical condition that $R_{ij}(q)$ are concave in $q$, which implies that $I_{ij}(q)$ are non-increasing in $q$ (equivalently $I_{ij}(p)$ are non-decreasing in $p$). We note that this assumption holds for throughput and welfare under all considered distributions, and revenue for regular distributions. For completeness, we prove these observations in Appendix B.

For a given objective, our aim is to select a pricing policy $p$, equivalently quantiles $q$, that maximizes the steady-state rate of reward accumulation, given by

$$
\text{OBJ}_m(q) = \sum_{x \in S_{n,m}} \pi(x) \cdot \left( \sum_{i,j} \phi_{ij} \cdot q_{ij}(x) \cdot I_{ij}(q_{ij}(x)) \right) = \sum_{x \in S_{n,m}} \pi(x) \cdot \left( \sum_{i,j} \phi_{ij} \cdot R_{ij}(q_{ij}(x)) \right).
$$

Intuitively, Equation (1) captures that at any node $i$, customers destined for $j$ arrive via a Poisson process with rate $\phi_{ij}$, and find the system in state $x \in S_{n,m}$ with probability $\pi(x)$. They are then quoted a price $p_{ij}(x)$ (corresponding to quantile $q_{ij}(x)$), and engage a ride with probability $q_{ij}(x)$. The resulting ride then contributes in expectation $I_{ij}(q_{ij}(x))$ to the objective function. Recall that unavailability of units is captured by our assumption that $q_{ij}(x) = 0$ whenever $x_i = 0$.

### 2.3. State-Independent Pricing and Closed Queueing Models

The Markov chain described in Section 2.1 has the structure of a closed queueing network (cf. [22, 16]), a well-studied class of models in applied probability (closed refers to the fact that the number of units remains constant; in open networks, units may arrive and depart from the system). Our analysis crucially relies on some classical results from the queuing theory literature, which we review in this section. Our presentation here closely resembles that of Serfozo [22]. One particular class of pricing policies is that of state-independent policies, wherein we set point-to-point prices $\{p_{ij}\}$ which do not react to the state of the system. As a consequence, the rate of units departing from any node $i$ at any time $t$ when $X_i(t) > 0$ is a constant, independent of the state of the network. The resulting model is a special case of a closed queueing model proposed by Gordon and Newell [10].

**Definition 1.** A Gordon-Newell network is a continuous-time Markov chain on states $x \in S_{n,m}$, in which for any state $x$ and any $i,j \in [n]$, the chain transitions from $x$ to $x - e_i + e_j$ at a rate $\lambda_{ij} \mu_i 1_{\{x_i(t) > 0\}}$, where $\mu_i > 0$ is referred to as the service rate at node $i$, and $\lambda_{ij} \geq 0$ as the routing probabilities satisfying $\sum_j \lambda_{ij} = 1$. 
In other words, if units are present at a node \( i \) in state \( x \), then departures from that node occur according to a Poisson distribution with rate \( \mu_i > 0 \); conditioning on a departure, the destination \( j \) is chosen according to state-independent routing probabilities \( \lambda_{ij} \).

The Markovian dynamics resulting from state-independent pricing policies fulfill the conditions of Gordon-Newell networks: fixing a price \( p_{ij} \) (with corresponding \( q_{ij} \)) results in a Poisson process with rate \( \phi_{ij}q_{ij} \) of arriving customers willing to pay price \( p_{ij} \). These customers engage a unit only if one is available, else leave the system. Thus, given quantiles \( q \), the time to a departure from node \( i \) is distributed exponentially with rate \( \mu_i = \sum_j \phi_{ij}q_{ij} \) when \( X_i > 0 \) and with rate 0 otherwise. Further, conditioned on an arriving customer having value at least equal to the quoted price, the probability that the customer’s destination is \( j \), is \( \lambda_{ij} = \phi_{ij}q_{ij}/\sum_k \phi_{ik}q_{ik} \), independent of system state.

One advantage of considering state-independent policies (and drawing connections with Gordon-Newell networks) is that the resulting steady-state distribution \( \{\pi_{p,m}(x)\}_{x \in S_{n,m}} \) can be expressed in product form, as established by the Gordon-Newell theorem.

**Theorem 2 (Gordon-Newell Theorem [10])** Consider an \( m \)-unit \( n \)-node Gordon-Newell network with transition rates \( \mu_i \) and routing probabilities \( \lambda_{ij} \). Let \( \{v_i\}_{i \in [n]} \) denote the invariant distribution associated with the routing probability matrix \( \{\lambda_{ij}\}_{i,j \in [n]} \), and define the traffic intensity at node \( i \) as \( r_i = w_i/\sum_j \phi_{ij} \). Then the stationary distribution is given by:

\[
\pi(x) = \frac{1}{G_m} \prod_{j=1}^{n} (r_j)^{x_j},
\]

where the Gordon-Newell normalization constant is given by \( G_m = \sum_{x \in S_{n,m}} \prod_{j=1}^{n} (r_j)^{x_j} \).

We now show how the Gordon-Newell theorem can be used to simplify the objective function in Equation (1). Recall that for an \( m \)-unit system with state-independent policy \( p \) (with corresponding quantiles \( q \)), we obtain a Gordon-Newell network with service rate \( \sum_j \phi_{ij}q_{ij} \) and routing probabilities \( \phi_{ij}q_{ij}/\sum_k \phi_{ik}q_{ik} \) at node \( i \). Let \( \{\pi(x)\}_{x \in S_{n,m}} \) be the corresponding steady-state distribution. Since \( q \) is no longer a function of the system state, we can no longer set \( q_i = 0 \) when \( X_i = 0 \). Instead, we define \( A_{i,m}(q) = \sum_{x \in S_{n,m}} \pi(x)I_{\{x_i > 0\}} \) as the steady-state availability of units at node \( i \) (i.e. the probability in steady-state that at least one unit is present at node \( i \)), and \( f_{ij,m}(q) = A_{i,m}(q) \cdot \phi_{ij}q_{ij} \) to be the steady-state rate of units moving from node \( i \) to \( j \). Then, from Equation (2), one can derive (see e.g. Proposition 1.33 and Equation 1.31 in [22])

\[
A_{i,m}(q) = (G_{m-1}(q)/G_m(q)) \cdot r_i(q).
\]

Notice that \( r_i(q) \) denotes the traffic intensity as defined above. Now, the objective in Equation (1) can be written as

\[
\text{OBJ}_m(q) = \sum_i A_{i,m}(q) \cdot \left( \sum_j \phi_{ij}q_{ij} \cdot I_{ij}(q_{ij}) \right) = \sum_i f_{ij,m}(q)I_{ij}(q).
\]

For ease of notation, we omit the explicit dependence on \( m \) when clear from context.

**The infinite-unit limit:** The stationary distribution described above (for state-independent pricing policies) holds for any finite \( m \); moreover, it can also be used to obtain the limiting distribution when the number of units tends to infinity. This infinite-unit limit is described in detail in Section 3.7 in [22] (and we provide more details in Appendix C). For the purposes of our results, we rely on one particular fact, which we state in the proposition below. Recall first that given \( p = \{p_{ij}\} \), the quantities \( w_i(p) \) and \( r_i(p) \) are independent of \( m \).
Proposition 3 Given a policy with quantiles \( q \), in the infinite-unit limit, the steady-state availability of each node \( i \) is given by \( r_i(q)/\max_j r_j(q) \); in particular, there exists at least one node \( i \) with \( A_i(q) = 1 \).

The existence of a node with availability 1 essentially captures the fact that in an infinite-unit system, at least one node must have an infinite number of units. For a formal proof of this result, cf. Section 3.7 in [22].

2.4. Non-concavity of objective under state-independent pricing

Directly optimizing the finite-unit system is non-trivial as the objective function is not concave in prices (or quantiles); we now demonstrate this in a simple network \((m = 1 \text{ and } n = 3)\), using throughput as the objective. Our example is presented in Figure 1. The network comprises of three nodes \((A, B, C)\); the labels on the edges show the effective demand rate \( \phi_{ij}(q) \) with which people wanting to move from node \( i \) to node \( j \) arrive for the corresponding pricing policies \( p \) (and corresponding quantiles \( q \)). In particular, the first figure corresponds to setting all prices to 0 (quantiles to 1), while in the second and third figures, we increase the price between \( B \) and \( C \) to set quantile \( q_{BC} = (1 + \epsilon)/2 \) in figure II, and \( q_{BC} = \epsilon \) in figure III. Note that the demand in network II is the average of the demands in networks I and III. To prove that this is non-concave with respect to the demand rates we now demonstrate that the throughput in network II is less than half of the sum of its value in networks I and III. To compute the throughput in each network, note that the expected waiting time at a node is inversely proportional to the total effective demand at each node. Furthermore, the unit makes exactly two rides between consecutive visits to node \( B \). Thus, the expected throughput is twice the expected rate of return to node \( B \). This holds because, starting from node \( B \), the expected time for the first \( 2k \) rides (for any positive integer \( k \)) is \( k \) times the expected return time to node \( B \). The expected return-time to \( B \) in the three networks can be computed as follows, where we use that the total expected waiting time can be computed as the sum of the expected waiting time at \( B \), the probability of waiting at \( A \) times the expected waiting time at \( A \) (1), plus the probability of waiting at \( C \) times the expected waiting time at \( C \) (\( \epsilon \)).

\[
\begin{align*}
\text{Network I:} & \quad 1 \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{\epsilon} = \frac{1+2\epsilon}{2\epsilon} = \Omega\left(\frac{1}{\epsilon}\right) \\
\text{Network II:} & \quad 1 \cdot \left(1 + \frac{1+\epsilon}{2}\right) + \left(\frac{2}{3+\epsilon}\right) \cdot 1 + \frac{1+\epsilon}{3+\epsilon} \cdot \frac{1}{\epsilon} = \frac{5\epsilon+1}{\epsilon \cdot (3+\epsilon)} = \Omega\left(\frac{1}{\epsilon}\right) \\
\text{Network III:} & \quad 1 \cdot \left(1 + \frac{1}{\epsilon}\right) + \left(\frac{1}{1+\epsilon}\right) \cdot 1 + \left(\frac{\epsilon}{1+\epsilon}\right) \cdot \frac{1}{\epsilon} = \frac{3}{1+\epsilon} = O(1).
\end{align*}
\]

Thus, the throughput in I and II is \( O(\epsilon) \), whereas it is constant in III, so the throughput is non-concave in the demand-rates (quantiles).

3. Pricing via the Elevated Flow Relaxation

In this section, we present our algorithm for the pricing problem. Section 2.4 demonstrates that the state-independent pricing problem is non-convex; moreover, this non-convexity appears in both the objective and the constraints. We circumvent this via a novel convex relaxation, based on two separate interventions, that alleviates the technical hurdles. Surprisingly, the resulting pricing policy has strong performance guarantees even with respect to state-dependent policies, as we prove in Section 4.
3.1. Elevated Objective Function

Recall from Equation (4) that our objective can be written as

$$\text{OBJ}_m(q) = \sum_{i,j} (f_{ij,m}(q) \cdot I_{ij}(q_{ij})).$$

Let $\hat{q}_{ij} = f_{ij,m}(q)/\phi_{ij} = A_{i,m}(q) \cdot q_{ij}$; note that $\hat{q}_{ij} \leq q_{ij}$, and moreover, unlike the quantiles $q_{ij}$ which are in one-to-one correspondence to prices, there is no straightforward way to derive $\hat{q}_{ij}$ from prices. Since we assume that the per-ride rewards $I_{ij}(\cdot)$ are non-increasing on the quantile space, we have $I_{ij}(q_{ij}) \leq I_{ij}(\hat{q}_{ij})$. We now define the elevated objective function as

$$\text{OBJ}(\hat{q}) = \sum_{i,j} \phi_{ij} \hat{q}_{ij} I_{ij}(\hat{q}_{ij}) = \sum_{i,j} \phi_{ij} R_{ij}(\hat{q}_{ij}).$$

(5)

The elevated objective has two useful properties: i) for all $m$ and $q$, the elevated objective upper bounds the true objective function, i.e. $\text{OBJ}(\hat{q}) \geq \text{OBJ}_m(q)$, and ii) it is a concave function of $\hat{q}$ (since we focus on objectives corresponding to concave reward curves $R_{ij}(\cdot)$).

3.2. The Flow Polytope

We now turn our attention to the constraints of our pricing problem. As we discussed above, each pricing policy (with corresponding quantiles $q$) realizes steady-state flows (steady-state rates of units) $f_{ij,m}(q) = A_{i,m}(q) \phi_{ij} q_{ij}$. As before, we define the change of variables $\hat{q}_{ij} = f_{ij,m}(q)/\phi_{ij}$. Note that while it is not the case that all flows obeying natural flow constraints can be realized as steady-state flows $\{f_{ij,m}(q)\}$ under some policy $q$, all realized flows do have to obey flow conservation and capacity constraints. This motivates the following relaxation $\{\hat{q}_{ij}\}$ of the set of possible steady-state flows under any policy $q$ and for any number of units $m$.

A natural capacity constraint arises since prices only decrease demand; the steady-state flow of units between a pair of nodes is thus bounded above by the rate of customers wanting to travel between the nodes. We refer to this constraint as demand bounding. Formally, for every pair $(i,j)$, we have $f_{ij,m}(q) \leq \phi_{ij}$ and hence $\hat{q}_{ij} \in [0,1]$.

Next, any steady-state flow must obey a natural flow conservation constraint, wherein the rate of incoming units at each node must equal the rate of outgoing units. We refer to this constraint as supply circulation. Formally, at any node $i$, we have $\sum_k f_{ki,m}(q) = \sum_j f_{ij,m}(q)$, and hence

$$\sum_k \phi_{ki} \hat{q}_{ki} = \sum_j \phi_{ij} \hat{q}_{ij}.$$
Note that the above two constraints hold for every finite \( m \) and every \( \mathbf{q} \); indeed, if they did not hold and the rate of incoming units to node \( i \) was larger than the rate of outgoing units then after letting the system run in steady-state for long enough, the number of units in \( i \) would be larger than \( m \). Moreover, the constraints are also true for the infinite-unit limit (cf. Appendix C). We refer to the set of flows defined by the above (linear) constraints as the flow polytope.

### 3.3. Pricing via the Elevated Flow Relaxation

Combining the elevated objective and the flow polytope, we obtain the elevated flow relaxation program (cf. Algorithm 1). Note that this is a convex optimization problem since the objective function is concave while the polytope is linear; hence it can be efficiently maximized.

**Algorithm 1** The Elevated Flow Relaxation Program

Require: arrival rates \( \phi_{ij} \), value distributions \( F_{ij} \), reward curves \( R_{ij} \).

1: Find \( \{q_{ij}\} \) that solves the following relaxation:

\[
\begin{align*}
\text{Maximize} & \quad \sum_{i,j} \phi_{ij} R_{ij}(\hat{q}_{ij}) \\
\text{Subject to} & \quad \sum_k \phi_{ki} \hat{q}_{ki} = \sum_j \phi_{ij} \hat{q}_{ij} \quad \forall i \\
& \quad \hat{q}_{ij} \in [0, 1] \quad \forall i, j.
\end{align*}
\]

2: Output state-independent prices \( p_{ij} = F_{ij}^{-1}(1 - q_{ij}) \).

Note that the prices (quantiles) returned by Algorithm 1 impose the flow conservation not only on the units (supply) but also on the customers (demand); we henceforth refer to this property as demand circulation.

### 4. Main approximation guarantee

In this section, we prove the main approximation guarantee of the paper, which bounds the performance of Algorithm 1 with respect to the optimal state-dependent pricing policy. This is formalized in the following theorem.

**Theorem 4** Consider any objective function \( \text{OBJ}_m \) for the \( m \)-unit system with concave reward curves \( R_{ij}(\cdot) \). Let \( \bar{p} \) be the pricing policy returned by Algorithm 1 and \( \text{OPT}_m \) be the value of the objective function for the optimal state-dependent pricing policy in the \( m \)-unit system. Then

\[
\text{OBJ}_m(\bar{p}) \geq \frac{m}{m + n - 1} \text{OPT}_m. \tag{6}
\]

**Proof.** The proof is based on the following three lemmas. First (Lemma 5), we show that the objective of the optimal state-independent policy is upper bounded by the elevated objective of the policy \( \bar{p} \) returned by the Elevated Flow relaxation (Algorithm 1). Next (Lemma 6), we show that the elevated objective of \( \bar{p} \) is equal to its objective in the infinite-unit system. Finally (Lemma 7), we show that for any pricing policy (and so in particular for \( \bar{p} \)), the objective in the \( m \)-unit system is within a factor of \( \frac{m}{m + n - 1} \) of the objective in the infinite-unit system.

**Lemma 5** For objectives with concave reward curves \( R_{ij}(\cdot) \), the value of the objective function of the optimal state-dependent policy is upper bounded by the value of the elevated objective function of the pricing policy \( \bar{p} \) returned by Algorithm 1

\[
\widehat{\text{OBJ}}(\bar{p}) \geq \text{OPT}_m.
\]
Lemma 6 The value of the elevated objective function of the pricing policy \( \tilde{p} \) returned by Algorithm 1 is equal to the value of its objective function in the infinite-unit system

\[
\text{OBJ}_\infty(\tilde{p}) = \hat{\text{OBJ}}(\tilde{p}).
\]

Lemma 7 For any state-independent pricing policy \( p \), the value of the objective of the policy \( p \) in the \( m \)-unit system is at least \( m / (m + n - 1) \) times the value of the objective of the same policy in the infinite-unit system.

\[
\text{OBJ}_m(p) \geq \frac{m}{m + n - 1} \text{OBJ}_\infty(p).
\]

In the remainder of this section, we prove these three lemmas.

4.1. From finite-unit state-dependent to the elevated flow relaxation

Lemma 8 (Lemma 5 restated) For objectives with concave reward curves \( R_{ij}(\cdot) \), the value of the objective function of the optimal state-dependent policy is upper bounded by the value of the elevated objective function of the pricing policy \( \tilde{p} \) returned by Algorithm 1

\[
\hat{\text{OBJ}}(\tilde{p}) \geq \text{OPT}_m.
\]

Proof. Our proof applies Jensen’s inequality to show that \( \text{OPT}_m \) is bounded above by the elevated objective value of some quantiles \( \hat{q} \) that form a feasible solution of the elevated flow relaxation program. Since the pricing policy \( \tilde{p} \) maximizes this mathematical program, the lemma follows.

Let \( q^*(X) \) denote the quantiles of the optimal state dependent policy and \( \pi^*(X) \) denote the steady-state distribution it induces. Then \( \text{OPT}_m \) can be written as

\[
\sum_{X \in S_{n,m}} \pi^*(X) \sum_{i,j} \phi_{ij} R_{ij}(q^*_i(X)).
\]

We define \( \hat{q} \) via

\[
\hat{q}_{ij} = \sum_{X \in S_{n,m}} \pi^*(X) q^*_i(X).
\]

Since the price-setting reward curve is concave, Jensen’s inequality implies that

\[
\text{OPT}_m \leq \sum_{i,j} \phi_{ij} R_{ij}(\hat{q}_{ij}).
\]

Also, note that, by definition \( q^*_i(X) = 0 \) when \( X_i = 0 \). Therefore demand circulation and demand bounding constraints of the elevated flow relaxation program are satisfied as convex combinations of the state-dependent supply circulation and demand bounding properties.

Hence \( \hat{q} \) is a feasible solution to the elevated flow relaxation program and the result follows. □

4.2. From the elevated flow relaxation to infinite-unit state-independent

Lemma 9 (Lemma 6 restated) The value of the elevated objective function of the pricing policy \( \tilde{p} \) returned by Algorithm 1 is equal to the value of its objective function in the infinite-unit system

\[
\text{OBJ}_\infty(\tilde{p}) = \hat{\text{OBJ}}(\tilde{p}).
\]

Proof. The pricing policy \( \tilde{p} \) satisfies the demand circulation property since it is a feasible solution to the elevated flow relaxation program. By Lemmas 10 and Proposition 3, the availabilities at all nodes is equal to 1. This means that (i) the value of the objective function in the infinite-unit limit for pricing policy \( \tilde{p} \) is equal to its elevated value (since no term was increased), and (ii) the flow of customers on each edge is equal to \( \phi_{ij} \hat{q}_{ij} \). □
Lemma 10 For any \( m \) (including \( \infty \)) if state-independent quantiles \( q \) satisfies the demand circulation property then, at all nodes \( i \), the availabilities \( A_{i,m}(q) \) are equal.

Proof. Consider \( i^* \in \arg \max A_{i,m}(q) \). Then the demand circulation and supply circulation properties imply

\[
A_{i^*,m}(q) \sum_j \phi_{j,i^*} q_{j,i^*} = A_{i^*,m}(q) \sum_j \phi_{i^*,j} q_{i^*,j} = \sum_j A_{j,m}(q) \phi_{j,i^*} q_{j,i^*}
\]

and thus \( \sum_j (A_{i^*,m}(q) - A_{j,m}(q)) \phi_{j,i^*} q_{j,i^*} = 0 \). By choice of \( i^* \), each summand is nonnegative, so for each \( j \) such that \( \phi_{i^*,j} > 0 \) we obtain \( A_{j,m}(q) = A_{i^*,m}(q) \). All availabilities being equal then follows inductively using connectivity of the underlying graph. \( \square \)

4.3. From finite-unit to infinite-unit state-independent

Lemma 11 (Lemma 7 restated) For any state-independent pricing policy \( p \), the value of the objective of the policy \( p \) in the \( m \)-unit system is at least \( m/(m+n-1) \) times the value of the objective of the same policy in the infinite-unit system.

\[
\text{OBJ}_m(p) \geq \frac{m}{m+n-1} \text{OBJ}_\infty(p).
\]

Proof. By Lemma 12, we have:

\[
\frac{\text{OBJ}_m(p)}{\text{OBJ}_\infty(p)} = r_{\max}(p) \cdot \frac{G_{m-1}(p)}{G_m(p)}.
\]

In order to uniformly bound the above expression, the essential ingredient is the construction of a particular weighted biregular graph between the states in \( S_{n,m-1} \) and the states in \( S_{n,m} \). In this graph, non-zero edges only exist between neighboring states, i.e., between states \( y, e_i \in S_{n,m-1} \) and \( y + e_i \in S_{n,m} \); further, the total weight of edges incident to any state in \( S_{n,m} \) is equal to 1, and the total weight of edges incident to any state in \( S_{n,m-1} \) is equal to \( \frac{m+n-1}{m} \). We construct such a graph in Lemma 13.

Throughout this proof, we use \( s \) for a state in \( S_{n,m-1} \) and \( t \) for one in \( S_{n,m} \). The weight of the edge \((s,t)\) in the bipartite graph constructed in Lemma 13 is denoted by \( \omega_{st} \).

\[
\frac{\text{OBJ}_m(p)}{\text{OBJ}_\infty(p)} = r_{\max}(p) \cdot \frac{G_{m-1}(p)}{G_m(p)} = r_{\max}(p) \cdot \frac{\sum_{s \in S_{n,m-1}} \prod_{j=1}^n (r_j(p))^{s_j}}{\sum_{t \in S_{n,m}} \prod_{j=1}^n (r_j(p))^{t_j}}
\]

\[
= r_{\max}(p) \cdot \frac{\sum_{(s,t) \in S_{n,m-1} \times S_{n,m}} \omega_{st} \prod_{j=1}^n (r_j(p))^{s_j+(t_j-s_j)}}{\sum_{s \in S_{n,m-1}} \prod_{j=1}^n (r_j(p))^{s_j}}
\]

\[
\geq \frac{\sum_{s \in S_{n,m-1}} \prod_{j=1}^n (r_j(p))^{s_j}}{(m+n-1) \sum_{s \in S_{n,m-1}} \prod_{j=1}^n (r_j(p))^{s_j}} = \frac{m}{m+n-1}
\]

The third equality holds as \( \sum_s \omega_{st} = 1 \), while the second-to-last follows from \( \sum_t \omega_{st} = \frac{m+n-1}{m} \). Crucially, \( \omega_{st} > 0 \) only holds for neighboring states \( s \) and \( t \), which implies the inequality. \( \square \)
Lemma 12 For any state-independent pricing policy \( p \), let \( A_m(p) = \max_i (A_{i,m}(p)) \) denote the maximum steady-state availability across all nodes. Then the objective function of \( p \) in the \( m \)-unit system is related to the infinite-limit objective as

\[
\frac{\text{OBJ}_m(p)}{\text{OBJ}_\infty(p)} = r_{\max}(p) \cdot \frac{G_{m-1}(p)}{G_m(p)} = A_m(p).
\]

Proof. Let \( B_i(p) = \sum_j \phi_{ij} q_{ij} \cdot I_{ij}(q_{ij}) \) denote the contribution of node \( i \) to the objective per unit of time in which station \( i \) is available. By substituting \( A_{i,m}(p) = (G_{m-1}(p)/G_m(p)) \cdot r_i(p) \), \( A_{i,\infty}(p) = r_i(p)/r_{\max}(p) \), and \( B_i \) into the definition of the objectives in Equation 4, we obtain

\[
\frac{\text{OBJ}_m(p)}{\text{OBJ}_\infty(p)} = \sum_i A_{i,m}(p) B_i(p) = \sum_i A_{i,\infty}(p) B_i(p) = \frac{G_{m-1}(p)}{G_m(p)} \frac{1}{r_{\max}(p)} \sum_i r_i(p) B_i(p) = r_{\max}(p) \cdot \frac{G_{m-1}(p)}{G_m(p)} = A_m(p),
\]

where the last equality follows from the characterization of the availabilities in Equation (3). Note that the argument relies on \( \text{OBJ}_\infty(p) > 0 \) which is the case for all policies/settings we consider.

\( \square \)

Lemma 13 We call \( y \in S_{n,m-1} \) a neighbor of \( y + e_i \in S_{n,m} \forall i \). There exists a weighted biregular graph on \( S_{n,m-1} \cup S_{n,m} \) such that i) an edge has non-zero weight only if it is connecting neighboring states, ii) for any vertex corresponding to a state in \( S_{n,m-1} \) the total weight of incident edges is equal to \( \frac{m+n-1}{m} \), and iii) for any vertex corresponding to a state in \( S_{n,m} \) the total weight of incident edges is equal to 1.

Proof. Our construction is shown in figure 2. Each state \( x \in S_{n,m} \) is adjacent to \( x - e_i \in S_{n,m-1} \) for all \( i \) with \( x_i > 0 \). On these edges, the weight is \( \frac{x_i}{m} \). Thus, the total weight incident to \( x \) is \( \sum_i \frac{x_i}{m} = 1 \). On the other hand, each state \( y \in S_{n,m-1} \) is adjacent to the states \( y + e_i \forall i \in [n] \). The respective weight on these edges is \( \sum_i y_i+1 = \frac{m-1+n}{m} \). Finally, there is only weight on edges between neighboring states. This concludes the proof of the lemma.

\( \square \)

![Graph between \( S_{2,3}, S_{2,2} \) and \( S_{2,1} \)](image)

(a) Graph between \( S_{2,3}, S_{2,2} \) and \( S_{2,1} \)

![Construction for general \( n, m \)](image)

(b) Construction for general \( n, m \)

Figure 2 Construction of biregular graph between states in \( S_{n,m} \) and \( S_{n,m-1} \), as described in Lemma 13. Fig. 2(a) shows the construction for \( (S_{2,3}, S_{2,2}) \) and \( (S_{2,2}, S_{2,1}) \). Fig. 2(b) shows the general construction. Note that the sum of weights of incident edges for any node on the left (i.e. any state in \( S_{n,m} \)) is 1, while it is \( (m+n-1)/m \) for nodes on the right (i.e. states in \( S_{n,m-1} \)).

To prove Theorem 4 it suffices to show that Lemma 7 holds when \( p \) is a demand circulation. This has been known since the 1980s [24] and has been used in similar settings [23]. However, in the next section (cf. Sections 5.1, 5.4) we consider scenarios under which no demand circulation is optimal/feasible. For these, the stronger statement of Lemma 7 is required.
5. Extensions

In this section, we relax some restrictions we previously imposed. All of the algorithms and proofs for these extensions make use of our elevated flow relaxation framework of Section 3, demonstrating its generality. First, in Section 5.1, we allow the designer to have additional rebalancing controls beyond pricing by redirecting supply and demand. Second, in Section 5.2, we consider multi-objective settings where the goal is to maximize some objective subject to a constraint on another. Next, in Section 5.3, we relax our assumption that changes in the state should be instantaneous by allowing travel-times for the trips. For each of these results, the proof follows from the three steps of Section 4; coincidentally, in each case, two steps are easily extended whereas one is more evolved. Further, the more evolved lemma is different for the three settings, hinting that each of the lemmas captures some inherent structure of the problem. Last, in Section 5.4, we consider constrained settings where prices can only depend on the source and where the prices should come from a discrete set.

5.1. Beyond pricing

Pricing is just one of several control levers in shared vehicle systems for balancing supply and demand; we now investigate two other levers, which we refer to as supply redirection and demand redirection, and show how they fit into our approximation framework. In the former we make a decision at the end of every trip on whether the unit remains at the destination of the trip or moves elsewhere whilst incurring a cost. In the latter, we redirect passengers arriving at a node to take units from nearby nodes. In practice, this would be achieved by pulling units from nearby nodes; for example in ridesharing services, the platform can dispatch a driver from a nearby node. Mathematically, the two are equivalent.

Supply Redirection We consider a state-dependent policy \( r(X) \) which, for each trip ending at a node \( i \), chooses to redirect the unit to some other node \( j \) (leading to state \( X - e_i + e_j \)), else allows the unit to stay at \( i \). For a state-independent policy, let \( r_{ij} \in [0, 1] \) be the probability that an arriving unit at \( i \) is redirected to \( j \). We assume that each redirection from \( i \) to \( j \) has associated cost \( c_{ij} \), and that units arriving empty (redirected) are not redirected again.

With \( m \) units, a fixed pricing policy \( p \) (with corresponding quantiles \( q \)), and a fixed redirection policy \( r \), we observe a rate \( f_{ij,m}(q,r) \) of customers traveling from \( i \) to \( j \), and a rate of redirected vehicles \( z_{ij,m}(q,r) \) from \( i \) to \( j \), i.e. trips with destination \( i \) which are redirected to \( j \). For a state-independent policy, since each unit arriving at \( i \) is redirected to \( j \) with probability \( r_{ij} \), it holds that

\[ z_{ij,m}(q,r) = r_{ij} \sum_k f_{ki,m}(q,r). \]

Similarly to the correspondence between \( q_{ij} \) and \( f_{ij,m} \), we observe a correspondence between \( r_{ij} \) and \( z_{ij,m} \), wherein the former are the controls and induce the latter in the objective via the steady-state dynamics. As a result, the objective can be written as

\[ \text{OBJ}_m(q,r) = \sum_{i,j} f_{ij,m}(q,r) I_{ij}(q) - c_{ij} z_{ij,m}(q,r). \]

In order to define the constraints of the elevated flow relaxation, we write (as in Section 3) \( \hat{q}_{ij} = f_{ij,m}(q,r)/\phi_{ij} \) and \( \hat{z}_{ij} = z_{ij,m}(q,r) \). We can now write the following relaxed flow polytope:

\[ \text{(1) } \hat{q}_{ij} \in [0,1], \quad \text{(2) } \sum_k (\phi_{ki} \hat{q}_{ki} + \hat{z}_{ki}) = \sum_j (\phi_{ij} \hat{q}_{ij} + \hat{z}_{ij}), \quad \text{(3) } \sum_k \hat{z}_{ik} \leq \sum_j \phi_{ji} \hat{q}_{ji} \forall i. \]

The first constraint is demand bounding, exactly as explained in Section 3. The second is a variant of the supply circulation in Section 3 to incorporate redirected vehicles. Finally, the third reflects that only units that are dropping off customers at a node, but not empty ones, can be
curves $R_{ij}(\cdot)$ defined in Section 3, we obtain an upper bound $\widehat{\text{OBJ}}(q, r)$ on our desired objective via the Elevated Flow Relaxation with the above constraints; through this, we obtain prices and redirection probabilities in Algorithm 2. Note that the redirection probabilities $r_{ij}$ returned by the algorithm correspond to the rate of redirected units $z_{ij}$ returned by the relaxation over the total incoming rate of (non-empty) units at node $i$, i.e. $\sum_k \phi_{kj}q_{kj}$. We now derive the equivalent of Theorem 4 to bound the performance of this algorithm.

**Theorem 14** Consider any objective function $\text{OBJ}_m$ for the $m$-unit system with concave reward curves $R_{ij}(\cdot)$. Let $\hat{p}$ and $\hat{r}$ be the pricing and redirection policies returned by Algorithm 2, and $\text{OPT}_m$ be the objective of the optimal state-dependent policies in the $m$-unit system. Then

$$\text{OBJ}_m(\hat{p}, \hat{r}) \geq \frac{m}{m + n - 1} \text{OPT}_m.$$ 

**Algorithm 2** The Elevated Flow Relaxation Program with Supply Redirection

**Require:** arrival rates $\phi_{ij}$, value distributions $\hat{F}_{ij}$, reward curves $R_{ij}$, rerouting costs $c_{ij}$.

1. Find $\{q_{ij}, z_{ij}\}$ that solves the following relaxation:

$$\begin{align*}
\text{Maximize} & \quad \sum_{i,j} (\phi_{ij} R_{ij}(\hat{q}_{ij}) - c_{ij} \hat{z}_{ij}) \\
\text{subject to} & \quad \sum_k (\phi_{ki} \hat{q}_{ki} + \hat{z}_{ki}) = \sum_j (\phi_{ij} \hat{q}_{ij} + \hat{z}_{ij}) \quad \forall i \\
& \quad \sum_k \hat{z}_{ik} \leq \sum_j \phi_{ji} \hat{q}_{ji} \quad \forall i \\
& \quad \hat{q}_{ij} \in [0, 1] \\
& \quad \hat{z}_{ij} \in [0, 1] \\
& \quad \hat{q}_{ij} = \left(1 - q_{ij}\right).
\end{align*}$$

2. Output state-independent prices $p_{ij} = F_{ij}^{-1}(1 - q_{ij})$ and redirection probabilities $r_{ij} = z_{ij}/\sum_k \phi_{ki}q_{ki}$

**Proof.** The proof closely resembles that of Theorem 4. As before, we show the inequality through three intermediate steps: (i) $\text{OBJ}(\hat{p}, \hat{r}) \geq \text{OPT}_m$, (ii) $\text{OBJ}(\hat{p}, \hat{r}) = \text{OBJ}_\infty(\hat{p}, \hat{r})$, and (iii) $\text{OBJ}_m(\hat{p}, \hat{r}) \geq \frac{m}{m + n - 1} \text{OBJ}_\infty(\hat{p}, \hat{r})$. The proof of the first inequality is the same as in Lemma 5, with the relaxation defined in Algorithm 1 replaced by the relaxation defined in Algorithm 2. The second step relies on Lemma 15, which uses Lemma 10 to prove that in the infinite-unit system all availabilities are 1. Based on this claim, similarly to the proof of Lemma 6, we observe that the flow of customers on each edge is $\phi_{ij}\hat{q}_{ij}$. The definition of the redirection probabilities in Algorithm 2 then immediately implies that $z_{ij,\infty}(\hat{p}, \hat{r}) = z_{ij}$, i.e. the flow of redirected units from $i$ to $j$ is also equal to the value of $z_{ij}$ in the solution of the relaxation. Finally, for the third step, we apply the same proof as in Lemma 7 with just one small modification. In Lemma 7, $B_i(p)$ denotes the contribution of node $i$ per unit of time in which a unit is present at $i$. Previously, this just captured rides leaving node $i$. Now, we also charge $B_i(p)$ for the cost incurred through the possible redirection of vehicles traveling from $i$ to $j$ that are redirected to $k$. Replacing $B_i(p)$ by $\sum_j \phi_{ij} q_{ij} (I_{ij}(q_{ij}) - \sum_k r_{jk}p_{jk})$ formalizes this charging argument – the remainder of the proof is equivalent to that of the Lemma 7. This concludes the proof of the theorem. □

**Lemma 15** With $\hat{p}$ and $\hat{r}$ as returned by Algorithm 2, all availabilities are equal to one in the infinite-unit system.
Proof. Denote by \( \tilde{q} \) the quantiles corresponding to \( \tilde{p} \). We consider a closed queueing network with the same transition probabilities between states as the one resulting from \( \tilde{q} \) and \( \tilde{r} \). In our hypothetical network, quantiles are all one, there is no redirection, and the demand circulation property holds. Since the hypothetical network does not have redirection and satisfies the demand circulation property, Lemma 10 implies that there the availabilities at all nodes are equal. However, the two networks have the same transition probabilities so they also have the same steady-state distribution. As a result, in the original network all availabilities are also equal and thus, equal to 1 in the infinite-unit limit. We define the demand in the hypothetical network as

\[
\phi_{ij} = \phi_{ij}\tilde{q}_{ij}(1 - \sum_k \tilde{r}_{jk}) + \sum_k \phi_{ik}\tilde{q}_{ik}\tilde{r}_{kj}.
\]

Observe that transitions occur at the same rate in this network as in the one with \( \tilde{q} \) and \( \tilde{r} \). Since quantiles are equal to 1, the demand circulation property says that \( \sum_j \phi_{ij} = \sum_k \phi_{ki} \). To show this property, notice first that the demand at node \( i \)

\[
\sum_j \tilde{r}_{ij} = \sum_j \phi_{ij}\tilde{q}_{ij} - \sum_j \phi_{ij}\tilde{q}_{ij}\left(\sum_k \tilde{r}_{jk}\right) + \sum_j \sum_k \phi_{ik}\tilde{q}_{ik}\tilde{r}_{kj} = \sum_j \phi_{ij}\tilde{q}_{ij}.
\]

On the other hand, due to the definition of \( \tilde{r}_{ij} \) (first equality), the definition of \( \tilde{r}_{ij} \) in Algorithm 2 (third equality), and the supply circulation constraint in Algorithm 2 (last equality), the demand of customers traveling to \( i \)

\[
\sum_k \phi_{ki} = \sum_k \phi_{ki}\tilde{q}_{ki} - \sum_j \phi_{ij}\tilde{q}_{ij} - \sum_j \phi_{ij}\tilde{q}_{ij}\tilde{r}_{ji} + \sum_k \phi_{kj}\tilde{q}_{kj}\tilde{r}_{ji} + \sum_j \tilde{r}_{ij}\left(\sum_k \phi_{ki}\tilde{q}_{ki}\right) = \sum_k \phi_{ki}\tilde{q}_{ki} + \sum_j \left(z_{ji} - z_{ij}\right) = \sum_j \phi_{ij}\tilde{q}_{ij}.
\]

Demand Redirection For the control defined in this section, we assume that there exists a graph \( G = (V,E) \) on the set of nodes with edges between nodes that are so close that a customer arriving at one node can be served through a vehicle at an adjacent node. We consider a state-dependent policy \( \mu(X) \) which, for each customer arriving at node \( i \) willing to pay the price quoted, decides from which node in \( \{i\} \cup \{j : (i,j) \in E\} \), the customer is served. With \( m \) units, fixed quantiles \( q(X) \), and a fixed matching policy \( \mu(X) \), we observe a rate \( f_{ij,m}(q,\mu) \) of customers arriving at \( i \) that travel to \( j \) potentially after being matched to a unit at \( k \), and a rate \( z_{ik,m}(q,\mu) \) of customers that arrived to travel from \( i \) but have been matched to a unit at \( k \). We can write the objective in this setting as \( \text{Ob} = \sum_{i,j} f_{ij,m}(q,\mu)I_{ij}(q) \). We again write \( \tilde{q}_{ij} = f_{ij,m}(q,\mu)/\phi_{ij} \) and \( \tilde{z}_{ij} = z_{ij,m}(q,\mu) \) to define the following relaxed flow polytope:

1. \( \tilde{q}_{ij} \in [0,1] \)
2. \( \sum_k \tilde{q}_{ki}\phi_{ki} + \tilde{z}_{ik} = \sum_j \tilde{q}_{ij}\phi_{ij} + \tilde{z}_{ji} \forall i \)
3. \( \sum_k \tilde{z}_{ki} \leq \sum_j \tilde{q}_{ji}\phi_{ji} \forall i \)

The first constraint is again demand bounding. The second is a variant of the supply circulation to incorporate matchings to nearby nodes. In particular, the left hand side accounts for the total number of units arriving at node \( i \), which equals all users arriving at \( i \) together with all units arriving due to matching from nearby nodes \( k \). Similarly, the right hand side accounts for the total number of units leaving \( i \), which are the users leaving from \( i \) together with users from other
We now discuss how to derive bicriterion approximations in multi-objective optimization settings. Formally, the problem is as follows: we are given a $m$-unit system, a requirement $c \geq 0$, and objectives $\Phi_m(\cdot)$ and $\Psi_m(\cdot)$; the goal is to maximize $\Phi_m(q)$ subject to $\Psi_m(q) \geq c$. We again assume that both objectives can be decomposed into per-ride rewards with associated concave reward curves $\{R_{ij}^\Phi\}$ and $\{R_{ij}^\Psi\}$.

Similarly to Equation (5), we first elevate both objectives to obtain $\hat{\Phi}(\hat{q}) = \sum_{i,j} \phi_{ij} R_{ij}^\Phi(\hat{q}_{ij})$ and $\hat{\Psi}(\hat{q}) = \sum_{i,j} \phi_{ij} R_{ij}^\Psi(\hat{q}_{ij})$. Since per-ride rewards are non-increasing on the quantiles, this can only increase the values of the objectives. We then impose the supply circulation and demand bounding constraints to create the flow polytope constraints. This mathematical program (Algorithm 4) is the elevated flow relaxation for our multi-objective setting; we argue below that this is indeed a relaxation. It can be efficiently optimized since the objective is concave while the polytope is convex: the convex combination of any two feasible quantiles is feasible since $\Psi(\cdot)$ is concave.

**Theorem 17** Let $\Phi_m$ and $\Psi_m$ be objectives for the $m$-unit system with concave reward curves. Then the solution $\hat{q}$ returned by Algorithm 4 is a $(\gamma, \gamma)$ bicriterion approximation for the multi-objective pricing problem where $\gamma = m/(m+n-1)$, i.e. $\Phi_m(q^*) \geq \gamma \text{OPT}_m$ and $\Psi_m(q^*) \geq \gamma \cdot c$.

**Proof.** Let $q'$ denote the optimal solution of an auxiliary program where we only elevate objective $\Phi$, i.e. we maximize $\hat{\Phi}(\cdot)$ subject to $\Psi_m(\cdot) \geq c$ as well as the demand circulation and demand conditions.

---

**Algorithm 3** The Elevated Flow Relaxation Program With Matching

**Require:** arrival rates $\phi_{ij}$, value distributions $F_{ij}$, reward-curves $R_{ij}$, edges $E$.  
1: Find $\{q_{ij}, z_{ij}\}$ that solves the the following relaxation:

\[
\begin{align*}
\text{Maximize} & \quad \sum_{i,j} \phi_{ij} R_{ij}(\hat{q}_{ij}) \\
\sum_k (\phi_{ki} \hat{q}_{ki} + \hat{z}_{ik}) & = \sum_j (\phi_{ij} \hat{q}_{ij} + \hat{z}_{ji}) \quad \forall i \in [n] \\
\sum_k \hat{z}_{ki} & \leq \sum_j \phi_{ji} \hat{q}_{ji} \quad \forall i \in [n] \\
\hat{q}_{ij} & \in [0,1] \quad \forall i, j \in [n] \\
\hat{z}_{ij} & = 0 \quad \forall (i, j) \notin E
\end{align*}
\]

2: Output state-independent prices $p_{ij} = F_{ij}^{-1}(1 - q_{ij})$ and matching probabilities $\mu_{ij} = z_{ij}/\sum_k \phi_{ik} q_{ik}$

---

nodes $j$ that use supply at $i$. Finally, the third ensures that customers are matched only to units arriving at nearby nodes. Maximizing the elevated objectives over these constraints again yields a $m/(m+n-1)$ approximation algorithm. We omit the proof, because of its similarity to the one of Theorem 14.

**Theorem 16** Solving for the elevated objective under the constraints defined above yields a $m/(m+n-1)$ approximation algorithm for pricing and matching.

In Appendix D we show that the results obtained in this section continue to hold in settings, in which matching and/or redirecting is allowed, but pricing is not. In such scenarios, the optimal solution may not have the demand circulation property. Nevertheless, the same techniques yield $m/(m+n-1)$ approximation algorithms.

**5.2. Multi-objective optimization**

We now discuss how to derive bicriterion approximations in multi-objective optimization settings, in which one objective is maximized subject to a lower bound on another. For ease of presentation, we restrict ourselves to pricing. Formally, the problem is as follows: we are given a $m$-unit system, a requirement $c \geq 0$, and objectives $\Phi_m(\cdot)$ and $\Psi_m(\cdot)$; the goal is to maximize $\Phi_m(q)$ subject to $\Psi_m(q) \geq c$. We again assume that both objectives can be decomposed into per-ride rewards with associated concave reward curves $\{R_{ij}^\Phi\}$ and $\{R_{ij}^\Psi\}$.

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Similarly to Equation (5), we first elevate both objectives to obtain $\hat{\Phi}(\hat{q}) = \sum_{i,j} \phi_{ij} R_{ij}^\Phi(\hat{q}_{ij})$ and $\hat{\Psi}(\hat{q}) = \sum_{i,j} \phi_{ij} R_{ij}^\Psi(\hat{q}_{ij})$. Since per-ride rewards are non-increasing on the quantiles, this can only increase the values of the objectives. We then impose the supply circulation and demand bounding constraints to create the flow polytope constraints. This mathematical program (Algorithm 4) is the elevated flow relaxation for our multi-objective setting; we argue below that this is indeed a relaxation. It can be efficiently optimized since the objective is concave while the polytope is convex: the convex combination of any two feasible quantiles is feasible since $\Psi(\cdot)$ is concave.

**Theorem 17** Let $\Phi_m$ and $\Psi_m$ be objectives for the $m$-unit system with concave reward curves. Then the solution $\hat{q}$ returned by Algorithm 4 is a $(\gamma, \gamma)$ bicriterion approximation for the multi-objective pricing problem where $\gamma = m/(m+n-1)$, i.e. $\Phi_m(q^*) \geq \gamma \text{OPT}_m$ and $\Psi_m(q^*) \geq \gamma \cdot c$.

**Proof.** Let $q'$ denote the optimal solution of an auxiliary program where we only elevate objective $\Phi$, i.e. we maximize $\hat{\Phi}(\cdot)$ subject to $\Psi_m(\cdot) \geq c$ as well as the demand circulation and demand conditions.
When the unit reaches its destination, the state changes to $X$. We now discuss how to remove the assumption that units move instantaneously by adding travel-times between nodes. We state our result only for pricing; however, our arguments below only depend on properties of the Markov chain, and hence can incorporate the other controls we consider.

A standard way to model travel-times is to assume that each unit takes an i.i.d. random time to travel from node $i$ to $j$. Formally, we expand the network state to $X = \{X_i(t), X_{ij}(t)\}$, where node queues $X_i(t)$ track the number of available units at node $i$, and link queues $X_{ij}(t)$ track the number of units in transition between nodes $i$ and $j$. When a customer engages a unit to travel from $i$ to $j$, the state changes to $X - e_i + e_{ij}$ (i.e., $X_i \rightarrow X_i - 1$ and $X_{ij} \rightarrow X_{ij} + 1$). The unit remains in transit for an i.i.d. random time, distributed exponentially with mean $\tau_{ij}$ (this is primarily for ease of notation; our results extend if the travel time is distributed according to some general $G_{ij}(\cdot)$). When the unit reaches its destination, the state changes to $X - e_{ij} + e_j$. Finally, we assume that pricing policies and passenger-side dynamics remain the same as before; in particular, we assume that the demand characteristics $\{\phi_{ij}, F_{ij}\}$ and reward-functions $\{I_{ij}\}$ are independent of the actual transit times (dependence on average transit times $\tau_{ij}$ can be embedded in the functions).

The system described above is a generalization of the Gordon-Newell network (Definition 1) referred to as a BCMP network (introduced by [2]; cf. [22], Section 3.3; also see [26] for the use of such a model for vehicle sharing). It is also a special case of a closed migration process; our presentation here follows Kelly and Yudovina [15] (Chapter 2).
Definition 18 A closed migration process on states $S_{n^2,m}$ is a continuous-time Markov chain in which transitions from state $X$ to state $X - e_i + e_j$ occur at rate $\lambda_{ij}\mu_i(X_i)$ when $X_i > 0$ and at rate 0 otherwise. The $\lambda_{ij}$ again form routing probabilities with $\sum_k \lambda_{ik} = 1, \lambda_{ij} \geq 0 \forall i, j$. Notice that $\mu_i(X_i)$ is a function of $X_i$ only, whereas $\lambda_{ij}$ are independent of the state alltogether.

Given quantiles $q$, the above-described process is a closed migration process with $\lambda_{i,ij} = \phi_{ij}q_{ij}/\sum_k \phi_{ik}q_{ik}$ and $\lambda_{i,j} = 1$ for every $i$ and $j$. Further, the service rate $\mu_i(X_i) = \sum_k \phi_{ik}q_{ik}$ when $X_i > 0$ for node queues and $\mu_{ij}(X_{ij}) = X_{ij}/\tau_{ij}$ for link queues. Intuitively, the latter captures the idea that each of the $X_{ij}$ units has an exponential rate of $1/\tau_{ij}$ and therefore the rate until the first is removed from the link queue is $X_{ij}/\tau_{ij}$. The stationary distribution can then be obtained as follows.

Theorem 19 (Theorem 2.4 in [15]) For a closed migration process as described in Definition 18, let $\{w_i\}_{i \in [n^2]}$ denote the invariant distribution associated with the routing probability matrix $\{\lambda_{ij}\}_{i,j \in [n]}$. Then the equilibrium distribution for a closed migration process is

$$
\pi(x) = \frac{1}{G_m} \prod_{i=1}^{n^2} w_{i}^{x_{i}} \prod_{y=1}^{\phi_{i,y}} \phi_{i,y},
$$

where $G_m = \sum_{x} \prod_{i=1}^{n^2} \frac{w_{i}^{x_{i}}}{\prod_{y=1}^{\phi_{i,y}} \phi_{i,y}}$ is a normalizing constant.

This implies for our setting, with $w$ denoting again the invariant distribution of the routing matrix.

$$
\pi_{x,m}(q) = \frac{1}{G_m(q)} \left[ \prod_{i \in [n]} \left( \frac{w_{i}(q)}{\sum_k \phi_{ik}q_{ik}} \right)^{x_{i}} \right] \left[ \prod_{i,j \in [n]^2} \frac{(\tau_{ij}w_{ij}(q))^{x_{ij}}}{x_{ij}!} \right].
$$

We remark that in comparison to the invariant distribution $w'$ when rides occur instantaneously, $w_D$ with delays would be $w_{i}^{D} = w_{i}'/2$ for node queues and $w_{ij}^{D} = w_{ij}' \phi_{ij}q_{ij}$ for link queues.

One consequence of the above characterization is that the resulting flows $f_{i,j,m}(q)$ continue to satisfy demand bounding and supply circulation – consequently, the Elevated Flow Relaxation (cf. Algorithm 1) continues to provide an upper bound. Moreover, adding link queues does not affect the optimization problems we consider in the infinite-unit system; in particular, Lemma 6 also continues to hold in this setting. Finally, from Lemma 12, we know that the ratio of objectives between the infinite-unit system and the finite-unit system equals the maximum availability, among all nodes, in the finite-unit system, i.e. $\frac{G_{\text{inf}}(q)}{G_{\text{fin}}(q)} = \max A_{i,m}(q)$. In order to obtain an approximation ratio, we now need to understand how $\max A_{i,m}(q)$ changes when link queues are added.

Let $M$ denote the random variable corresponding to the steady-state number of available (i.e. not in transit) units across all nodes, and define $A_{m}(q|M) \triangleq \max_{i \in [n]} \mathbb{E}[1_{\{X_i > 0\}}|M], A_{m}(q) = \max_{i \in [n]} A_{i,m}(q)$. Now we have the following

Lemma 20 Conditioned on $M$, the distribution of $\{X_i\}_{i \in [n]}$ in the network with travel-times is identical to an $n$-node $M$-unit Gordon-Newell network with the same quantiles and arrival rates.

This follows directly from the product-form nature of the steady-state distribution in Equation (7). Using this, we now obtain the following bound for the $m$-unit system availability.
Lemma 21 For any network with parameters \( \{\phi_{ij}, F_{ij}(\cdot), \tau_{ij}\} \) if \( m \geq 100 \) and quantiles \( q \) satisfy
\[
\sum_{ij} \phi_{ij} \tau_{ij} q_{ij} \leq m - 2\sqrt{m \ln(m)}
\]
then
\[
A_m(q) \geq \left(1 - \frac{3}{\sqrt{m}}\right) \left(\frac{\sqrt{m \ln m}}{\sqrt{m \ln m + n - 1}}\right).
\]

Note that the above converges to 1 as \( m \to \infty \).

Proof. First, for any given policy \( q \), as before we have the realized flows \( f_{ij,m}(q) = q_{ij} \phi_{ij} A_{i,m}(q) \); moreover, this is the expected rate of units entering link queue \( X_{ij} \). Let \( D = m - M \) be the number of units which are in transit. Now, by Little’s law (cf. [16] or [22]), we have that the expected number of units in link queues is given by \( \sum_{i,j} A_{i,m}(q) \phi_{ij} q_{ij} \tau_{ij} \).

Note that the link queues \( \{X_{ij}\} \) are stochastically dominated by independent \( M/M/\infty \) queues with input rate \( \phi_{ij} q_{ij} \) and average transition time \( \tau_{ij} \). This follows from a simple coupling argument, where incoming customers follow an independent Poisson process of rate \( \phi_{ij} q_{ij} \) and enter the link queue with a virtual unit, irrespective of whether the customer engages a unit or not in the real system. Thus \( D \) is stochastically dominated by \( \hat{D} = Pois(\sum_{i,j} \phi_{ij} q_{ij} \tau_{ij}) \). Further, since \( D \) is bounded above by \( m \), \( D \) is also stochastically dominated by \( \hat{D} = \min\{\hat{D}, m\} \).

Next, from Lemma 20, we know that conditioned on there being \( m \) available units in the steady-state system, the distribution of units in node queues is identical to that of an \( n \)-node \( M \)-unit Gordon-Newell network; moreover, from Lemma 7, we have that for any \( n \)-node, \( m \)-unit Gordon-Newell network, \( A_m(q|M) \geq M/(M+n-1) \). Since \( M = m - D \) and \( (m - x)/(m + n - 1 - x) \) is decreasing in \( x \) for \( x \leq m \), it follows that
\[
A_m(q) \geq \mathbb{E}\left[\frac{m - D}{m + n - 1 - D}\right] \geq \mathbb{E}\left[\frac{m - \hat{D}}{m + n - 1 - \hat{D}}\right].
\]

Further, by definition of \( \hat{D} \) we observe that \( \mathbb{P}[\hat{D} > m\left(1 - \sqrt[4]{\ln m/m}\right)] = \mathbb{P}[\hat{D} > m\left(1 - \sqrt[4]{\ln m/m}\right)] \). We can now apply a standard Chernoff bound for the Poisson random variable \( \hat{D} \) (cf. from Lemma 36 in Appendix F), using the assumption that \( m - 2\sqrt{m \ln(m)} \geq \sum_{i,j} \phi_{ij} \tau_{ij} q_{ij} = \mathbb{E}[\hat{D}] \). In particular, we may bound \( \mathbb{P}[\hat{D} > m\left(1 - \sqrt[4]{\ln m/m}\right)] \) by
\[
\mathbb{P}\left[\hat{D} > m\left(1 - \sqrt[4]{\ln m/m}\right)\right] \leq \exp\left(-\frac{m \ln m (m - 3\sqrt{m \ln m})}{2(m - 2\sqrt{m \ln m})^2}\right) \leq \exp\left(-\frac{-\ln m \cdot (1 - 3\sqrt{\ln m/m})}{2\left(1 - 4\sqrt{\ln m/m}(1 - \sqrt{\ln m/m})\right)}\right) \leq \exp\left(-\ln m \left(1 - 3\sqrt{\ln m/m}\right)\right) \leq \exp\left(-\frac{-\ln m}{2}\left(1 - \frac{3\sqrt{\ln m/m}}{2 - 3\sqrt{\ln m/m}}\right)\right) \leq \frac{1}{\sqrt{m}} \exp\left(-\frac{3\ln m}{4\sqrt{m} \ln m - 6}\right) \leq \frac{3}{\sqrt{m}} \quad \text{for } m \geq 100.
\]
We can use the above to bound the availability in Inequality (8) as
\[
A_m(q) \geq \left(1 - \frac{3}{\sqrt{m}}\right) \left(\frac{m - (m - \sqrt{m \ln m})}{m - (m - \sqrt{m \ln m}) + n - 1}\right) + \frac{3}{\sqrt{m}} \cdot 0.
\]
Simplifying, we obtain the result. □

We are now ready to extend our pricing/control policies to the setting with transit delays. In order to do so, we need to first extend the elevated flow relaxation by adding an extra constraint. The main observation is that in an \(m\)-unit system with transit delays, there is an additional conservation constraint induced by the fact that the number of units in the link queue cannot exceed \(m\). As before, let \(f_{ij}^m(q) = \tilde{q}_{ij} \phi_{ij}\) denote the expected rate of units entering link queue \(X_{ij}\); then by Little’s law (cf. [16] or [22]), we have that the expected number of units in link queues is given by \(\sum_{i,j} \phi_{ij} \tilde{q}_{ij} \tau_{ij}\), which, in an \(m\)-unit system, must be bounded by \(m\). To incorporate this, we need to add an additional rate-limiting constraint to the elevated flow relaxation wherein we ensure that \(\sum_{i,j} \phi_{ij} \tilde{q}_{ij} \tau_{ij} \leq m\). This gives us the Rate-Limited Elevated Flow Relaxation Program in Algorithm 5.

**Algorithm 5** The Rate-Limited Elevated Flow Relaxation Program

**Require:** arrival rates \(\phi_{ij}\), value distributions \(F_{ij}\), reward curves \(R_{ij}\), scaling parameter \(\varepsilon_m\), travel-times \(\tau_{ij}\).

1: Find \(\{q_{ij}\}\) that solves the following relaxation:

\[
\begin{align*}
\text{Maximize} & \quad \sum_{(i,j)} \phi_{ij} R_{ij}(\tilde{q}_{ij}) \\
\sum_k \phi_{ki} \tilde{q}_{ki} & = \sum_j \phi_{ij} \tilde{q}_{ij} \quad \forall i \\
\sum_{i,j} \phi_{ij} \tilde{q}_{ij} & \leq m \\
\tilde{q}_{ij} & \in [0, 1] \\
\end{align*}
\]

2: Set \(\tilde{q}_{ij} = q_{ij} \cdot (1 - \varepsilon_m)\)

3: Output state-independent prices \(\tilde{p}_{ij} = F_{ij}^{-1}(1 - \tilde{q}_{ij})\).

**Theorem 22** For any objective function \(\text{OBJ}_m\) with concave reward curves \(R_{ij}(\cdot)\) in the \(m\)-unit system, let quantiles \(\tilde{q}\) be the output of Algorithm 5 with input \(\varepsilon_m = 2\sqrt{\ln m/m}\), \(\text{OPT}_m\) be the value of the objective function for the optimal state-dependent pricing policy, and \(m \geq 100\). Then

\[
\frac{\text{OBJ}_m(\tilde{q})}{\text{OPT}_m} \geq (1 - \varepsilon_m) \left(\frac{\sqrt{m \ln m}}{m \ln m + n - 1} - \frac{3}{\sqrt{m \ln m}}\right).
\]

**Proof.** The proof follows a similar roadmap as that of Theorem 4. In particular, we argue that
1. the rate-limited elevated flow relaxation provides an upper bound for any state-dependent policy,
2. the rate-limited elevated flow relaxation solution is achieved by a state-independent policy in the infinite-unit system, and
3. the ratio of the performance of any state-independent policy \(q\) in the infinite-unit and \(m\)-unit system is equal to the maximum availability \(A_m(q)\).

First, similar to Lemma 5, note that since the realized flows in the \(m\) unit system must obey the conservation laws encoded by the rate-limited elevated flow relaxation, hence \(\text{OPT}_m\) is bounded by the solution to the rate-limited elevated flow relaxation \(\sum_{(i,j)} \phi_{ij} R_{ij}(q_{ij})\). Moreover, since per-ride
rewards $I_j(\cdot)$ are non-increasing in $q$, therefore scaling the $q_{ij}$ by $(1 - \varepsilon_m)$ results in an elevated objective value that obeys

\[
(1 - \varepsilon_m) \sum_{(i,j)} \phi_{ij} R_{ij}(q_{ij}) \leq \sum_{(i,j)} \phi_{ij} R_{ij}(\tilde{q}_{ij}),
\]

and moreover, $\sum_{i,j} \phi_{ij} \tilde{q}_{ij} \tau_{ij} \leq m \cdot (1 - \varepsilon_m)$. Now, using similar arguments as in Lemma 6, we can show that using a state-independent policy $\tilde{q}$ in the infinite-unit limit gives $\text{OBJ}_\infty(\tilde{q}) = \sum_{(i,j)} \phi_{ij} R_{ij}(\tilde{q}_{ij})$ (note that we use the same $\tilde{q}$ as derived from the $m$ unit rate-limited elevated flow relaxation in the infinite unit limit; in other words, we scale the number of units to infinite, but retain the constraint $\sum_{i,j} \phi_{ij} \tilde{q}_{ij} \leq m$ for a fixed $m$). Next, from Lemma 12, we get that $\text{OBJ}_m(\tilde{q}) = A_m(\tilde{q}) \text{OBJ}_\infty(\tilde{q})$. Finally, using Lemma 21, we get the desired bound

\[
\frac{\text{OBJ}_m(\tilde{q})}{\text{OPT}_m} \geq (1 - \varepsilon_m) \left( \frac{\sqrt{m \ln m}}{\sqrt{m \ln m + n - 1}} - \frac{3}{\sqrt{m}} \right).
\]

\[\square\]

Note that for any fixed $n$, the theorem shows that the policy returned by the rate-limited elevated flow relaxation is asymptotically optimal as $m \to \infty$ for any demand rates and transit delays $\{\phi_{ij}, \tau_{ij}\}$. In Appendix D we use this to recover and give a finite-$m$ characterization for the asymptotic results in [3, 19].

5.4. Constrained point pricing

In this section, we focus on a special case of the vanilla pricing problem wherein the platform is only allowed to set point prices, i.e. prices based on the origin node, and the value distributions of all customers arriving at a node are identical (i.e. $p_{ij} = p_i$, respectively $q_{ij} = q_i$, and $F_{ij}(\cdot) = F_i(\cdot)$ for all $i, j$). We provide a simple optimal pricing policy for the infinite-unit system, which involves just one eigenvector computation (for throughput/social welfare) or a concave maximization over a single variable (for revenue).

We then consider the additional constraint that prices are only allowed to come from a discrete price set. Using our infinite-to-finite unit reduction, all our results are then translated back to the finite unit setting. We emphasize that in the latter restricted settings, there may not be a feasible solution satisfying demand circulation.

**Unrestricted price set**: We begin by providing the point pricing equivalent to Algorithm 1 and Theorem 4.

**Algorithm 6** The Point Pricing Elevated Flow Relaxation Program

**Require**: arrival rates $\phi_{ij}$, value distributions $F_i$, reward curves $R_{ij}$.

1. Find $\{q_i\}$ that solves the following point price relaxation:

   \[
   \begin{align*}
   \text{Maximize} & \quad \sum_{(i,j)} \phi_{ij} R_{ij}(\tilde{q}_{ij}) \\
   \text{s.t.} & \quad \sum_k \phi_{ki} \tilde{q}_k = \sum_j \tilde{q}_{ij} \tilde{q}_i, \quad \forall i \\
   & \quad \tilde{q}_i \in [0,1] \\
   & \quad \forall i.
   \end{align*}
   \]

2. Output *state-independent* prices $p_i = F_i^{-1}(1 - q_i)$. 

Theorem 23 Consider any objective function $\text{OBJ}_m$ for the $m$-unit system with concave reward curves $R_{ij}(\cdot)$. Let $\hat{p}$ be the pricing policy returned by Algorithm 6, $\text{OPT}_m$ be the value of the objective function for the optimal state-dependent point pricing policy in the $m$-unit system. Then

$$\text{OBJ}_m(\hat{p}) = \frac{m}{m + n - 1} \text{OPT}_m$$

Proof. The proof is again based on three steps that compare $\text{OBJ}_m(\hat{p})$ with $\text{OBJ}_\infty(\hat{p})$, $\text{OBJ}_\infty(\hat{p})$ with $\hat{\text{OBJ}}(\hat{p})$, and $\hat{\text{OBJ}}(\hat{p})$ with $\text{OPT}_m$. The application of Jensen’s inequality to prove $\hat{\text{OBJ}}(\hat{p}) \geq \text{OPT}_m$ is the same as in Lemma 5, with the polytope in Algorithm 1 replaced by the one in Algorithm 6. Lemma 6 applies since its proof only relies on $\hat{p}$ fulfilling the demand circulation property, which it does (cf. Algorithm 6). Thus, $\text{OBJ}_\infty(\hat{p}) \geq \hat{\text{OBJ}}(\hat{p})$. Finally, Lemma 7 implies that $\text{OBJ}_m(\hat{p}) \geq \frac{m}{m + n - 1} \text{OBJ}_\infty(\hat{p})$, which concludes the proof of the theorem. □

Notice that the optimization problem in Algorithm 6 has the demand circulation property as a constraint; thus, with the resulting pricing policy, the availability is equal at every node (cf. Lemma 10). Recall from Section 2.3 that the availability at each node in the infinite-unit system depends on the traffic intensity at that particular node and the maximum traffic intensity among all nodes. Further, the traffic intensity at each node $i$ depends on (i) the $i$th coordinate of the eigenvector $w(q)$ of the routing matrix $(\phi_{ij}(q_i)/\sum_k \phi_{ik}(q_i))_{i,j \in [n]^2}$, and (ii) the rate of arrivals $\sum_k \phi_{ik}$ at $i$. In particular, $r_i(q) = w_i(q)/\sum_j \phi_{ij}$. In the setting of point prices however, $w$ is unaffected by the prices and $r_i(q) = r_i(q)p_i$ implies that $w_i \sum_k \phi_{jk}q_j = w_j \sum_k \phi_{ik}q_i$ for all $i,j$. Substituting in the optimization problem for every $j$

$q_j = w_j \sum_k \phi_{ik}q_i / w_i \sum_k \phi_{ik}$,

we find that the convex optimization problem can actually be written in just one variable. Further, in the case of social welfare, and revenue, it is always the case that $\max_i q_i = 1$ for an optimal solution in the infinite-unit system. Hence, in these cases only one eigenvector computation is needed.

Discrete price set: We now show how the pricing policy from Algorithm 6 can be modified when there is a discrete set of available prices for each node. We handle this case with an extra loss in the objective that depends on how well the prices represent each part of the distribution. In particular, there is a discrete set of available prices for each node. We handle this case with an extra loss in the objective that depends on how well the prices represent each part of the distribution. In particular, we obtain the pricing policy $\hat{p}$ by solving for the unconstrained case as in Algorithm 6 to obtain prices $p$ and then setting each $\hat{p}_i$ to be the lowest available price greater or equal to $p_i$. We now prove the performance guarantee for $\hat{p}$.

Theorem 24 Let $\{p_1^1, \ldots, p_l^k\}$ be the set of available prices for node $i$ in increasing order, $\{q_1^1, \ldots, q_l^k\}$ be the corresponding quantiles (in decreasing order), and $p, \hat{p}$ be defined as above. Suppose that for all $i$ there exists an available price $p^l_i$ such that $q_i^l \leq q_i$, and that there exists $\alpha$ such that for all $i$ and all $s$, $\alpha \cdot q_i^s \geq q_i^{s+1}$. Then,

$$\alpha \text{OBJ}_m(\hat{p}) \geq \frac{m}{m + n - 1} \text{OPT}_m,$$

where $\text{OPT}_m$ is the objective of the optimal state-dependent policy for discrete prices in the $m$-unit system.

Proof. Since $\hat{\text{OBJ}}(\hat{p})$ is an upper bound on the unrestricted point pricing problem (cf. Theorem 23), it is also an upper bound on $\text{OPT}_m$. Lemma 6 implies that $\hat{\text{OBJ}}(\hat{p}) = \text{OBJ}_\infty(\hat{p})$, since $p$ fulfills the demand circulation property (cf. Algorithm 6). Further, by Lemma 7, $\text{OBJ}_m(\hat{p}) \geq \frac{m}{m + n - 1} \text{OBJ}_\infty(\hat{p})$. Thus, what remains is to bound $\text{OBJ}_\infty(\hat{p})$ with respect to $\text{OBJ}_\infty(p)$. Since $\hat{q}_i \leq q_i$ for all $i$ and the per-ride rewards $I_{ij}(\cdot)$ are assumed to be non-decreasing in the quantiles, we only need to bound the changes in the availability of the infinite-unit system for each $i$. Since the $w_i$ are constant...
under point-pricing, the availabilities are only affected by prices in the denominator, where the change is equal to \( \tilde{a}_i/q_i \). Thus, no traffic intensity changes by more than a factor of \( \alpha \) and the result follows. \( \square \)

The assumption that the value distributions at each node are identical may seem too restrictive. Notice though that the same analysis also applies to the following setting: for each \( i,j \) there exists a base price \( d_{ij} \) (e.g., based on geographic distance). This price is multiplied by the (state-dependent) control \( p_i \), which is the same for all \( j \). The behavioral assumption is now that customers react the same way to the control, regardless of their destination.

6. Conclusions

We have studied pricing and optimization in shared vehicle systems for various objectives. Our work parallels existing work through our use of a closed-queueing network model to capture network externalities. It distinguishes itself, however, through the rigorous guarantees in finite settings and the generality of controls/objectives considered. In that sense, it unifies and extends several results from the literature. In particular, our main technical contribution (the elevated flow relaxation), has the potential to apply to other settings. Given the widespread use of fluid limits in the queuing theory literature, our framework may yield provable guarantees for finite instances in these settings as well.

Further, it would be interesting to study how our framework can be extended to constrained settings beyond multi-objective and discrete prices. For instance, in recent events, Uber was exposed to bad publicity when turning off surge pricing for trips originating at JFK airport. While the details of these events are not mathematical in nature, it demonstrates the significance of studying settings where prices (in some locations) are bounded above. Additionally, our pricing policies do not impose triangle inequality, potentially creating incentives for customers to reach their destination via an extra stop. Addressing such strategic considerations opens up an intriguing avenue for future research.

Finally, although our work suggests that state-independent prices have strong performance, this is under the steady-state assumption with complete knowledge of the system parameters. Relaxing either of these assumptions is a compelling extension of our work.

Appendix A: Irreducibility of the Priced System

We justify here our assumption from Section 2 that the infinite-unit solutions we obtain induce a connected graph; to do so, we first need to assume that the graph created by edges \((i,j)\) on which \( \phi_{ij} > 0 \) is strongly connected. We then prove that given any solution to the infinite-unit pricing problem, there exists a solution with arbitrarily close objective that also induces a connected graph. Throughout this section we work with the flow \( f_{ij,\infty}(p) \) induced by the demands in the infinite-unit system, but suppress all dependencies on \( \infty \) in the notation.

**Theorem 25** Let \( \epsilon > 0 \). For any non-decreasing objective and any pricing policy \( p \) that induces a supply circulation \( f_{ij} \) on \( k \) components in the infinite-unit system, there exists a policy \( p' \) inducing a supply circulation \( f'_{ij} \) in the infinite-unit system such that the graph with edge-set \( E = \{(i,j) : f'_{ij} > 0\} \) is strongly connected and the objective with \( p' \) is at least \((1 - \epsilon)\) times that of \( p \).

**Proof.** To prove the theorem we repeatedly add flow to edges \((i,j)\) with \( f_{ij} = 0 \), but also take flow away from edges \((i,j)\) with \( f_{ij} > 0 \). To ensure that edges of the second kind do not have their flow reduced by too much, we set

\[
\delta = \frac{\epsilon}{k} \times \min \left\{ \min_{ij} \{ f_{ij} : f_{ij} > 0 \}, \min_{ij} \{ \phi_{ij} : \phi_{ij} > 0 \} \right\}.
\]

Whenever we decrease flow on an edge, this is done by an additive \( \delta \) amount. Reducing flow at most \( k \) times to obtain \( f'_{ij} \) we guarantee that \( f'_{ij} \geq (1 - \epsilon)f_{ij} \) holds.

As we assume our underlying graph with edge-set \( \{(i,j) : \phi_{ij} > 0\} \) to be strongly connected, it must be the case that there exists a minimal sequence of components \( C_1, C_2, \ldots, C_d = C_1, d > 2 \), and nodes \( u_\ell, v_\ell \in C_\ell \)
such that $\lambda_{u_{t}v_{t+1}} > 0$, but $f_{u_{t}v_{t+1}} = 0$. In particular, it being minimal implies that no component other than the first appears repeatedly.

Since each $u_{t}, v_{t}$ are in the same strongly connected component of the graph with edge-set $E$, we know that for each $\ell$ there exists a simple path from $u_{t}$ to $v_{t}$ with positive flow on it. We change flows as follows: for all pairs $(u_{t}, v_{t+1})$ we increase flow by $\epsilon$ and for each edge along the path from $u_{t}$ to $v_{t}$ we decrease flow by $\delta$. At all other edges the flow remains unchanged.

We need to first argue that the new circulation is feasible. Each node along a path within a component has its in-flow and out-flow reduced by $\delta$, whereas at the nodes $u_{t}, v_{t}$ both the sum of in-flows and the sum of out-flows has remained the same. At all other nodes, nothing is altered. Thus, flow conservation continues to hold. By choice of $\delta$ none of the edge-capacities are violated. Thus, the resulting flow is a circulation with at most $k - 1$ distinct components. Applying this procedure $k - 1$ times, we obtain a single strongly connected component.

Finally, since $I_{ij}(\cdot)$ are nondecreasing with price and decreasing flow is equivalent to increasing prices, the choice of $\delta$ guarantees that the objective on paths from $u_{t}$ to $v_{t}$ has been reduced by at most a factor of $(1 - \epsilon)$. Since $I_{ij}(\cdot)$ are non-negative, the additional flow on edges from $u_{t}$ to $v_{t+1}$ only increases the total objective. Thus, the pricing policy $p'$ that induces the circulation $f'_{ij}$ has the desired properties. □

Appendix B: Concave Reward Curves

In this section, we investigate conditions under which throughput, social welfare and revenue satisfy the conditions of theorem 4. In particular, we first show that the respective reward curves $R(q) = q I(q)$ are concave. We then prove that the concave reward curves assumption implies the non-increasing (quantiles) per-ride rewards assumption.

**Lemma 26** Revenue (i) satisfies the assumptions of Theorem 4 under regular value distributions, Throughput (ii) and Social Welfare (iii) satisfy the assumptions under any value distribution.

**Proof.** We drop the subscripts throughout this proof to simplify notation. We begin by considering (i) revenue, for which the result holds due to the fact that the reward curve is concave if and only if the distribution is regular (cf. Proposition 3.10 in [12]). For (ii) throughput, $R(q) = q \cdot I(q) = q$ is a linear function of $q$ for any value distribution and thus concave.

Lastly, for (iii) social welfare, we use the so-called hazard rate $h(y) = \frac{f(y)}{1 - F(y)}$ of a distribution $F$ with density $f$. Given $F$, denote by $p(q)$ and $q(p)$ a price as a function of its corresponding quantile and vice-versa. Then, by the definition of hazard rate:

$$q(p) = \exp \left( - \int_{0}^{p(q)} h(y) dy \right)$$  \hspace{1cm} (12)

Taking logarithms and differentiating, we obtain:

$$- \frac{1}{q(p)} = h(p(q)) \frac{dp(q)}{dq}$$  \hspace{1cm} (13)

Hence, as $R(q(p)) = q(p) \cdot I(q(p))$ and $f(p) = (1 - F(p))h(p) = q(p)h(p)$ we have

$$R(q) = \int_{p(q)}^{\infty} v f(v) dv = \int_{p(q)}^{\infty} v h(v) \exp \left( - \int_{0}^{v} h(y) dy \right) dv$$

The first derivative $\frac{dR(q)}{dq}$ of $R(q)$ is equal to

$$-p(q)h(p(q)) \exp \left( - \int_{y=0}^{p(q)} h(y) dy \right) \frac{dp(q)}{dq} = \frac{p(q) \exp \left( - \int_{y=0}^{p(q)} h(y) dy \right)}{q(p)} = p(q),$$

where the first equality comes from Equation (13), the second from (12).

The second derivative is then given by

$$\frac{d^2 R(q)}{dq^2} = \frac{dp(q)}{dq} = - \frac{1}{qh(p(q))} = - \frac{1 - F(p(q))}{f(p(q))q(p)} < 0,$$

which concludes the proof of the Lemma. □
Lemma 27 If some objective satisfies the concave reward curves assumption, it also satisfies the non-increasing (in quantiles) per-ride rewards assumption.

Proof. Suppose an objective has concave reward curves, but does not have non-increasing (in quantiles) per-ride rewards. Then there must exist \(i, j, q_1, q_2\) with \(0 < q_1 < q_2\) such that \(I_{ij}(q_1) < I_{ij}(q_2)\). Let \(A = \frac{q_1}{q_2}\). Then
\[
q_1 I_{ij}(q_2) = A \cdot q_2 I_{ij}(q_2) = A \cdot q_2 I_{ij}(q_2) + (1 - A) \cdot I_{ij}(0) \\
\leq (A \cdot q_2 + (1 - A) \cdot 0) I(A \cdot q_2 + (1 - A) \cdot 0) = q_1 I_{ij}(q_1),
\]
where the inequality follows from Jensen’s inequality on since the rewards curve \(q I_{ij}(q)\) is a concave function.

As \(q_1 > 0\), it follows that \(I_{ij}(q_2) \leq I_{ij}(q_1)\) and we therefore arrive at a contradiction. \(\Box\)

Appendix C: Infinite-unit Limit

In Section 2.1 we briefly introduced the infinite-unit limit of the Gordon-Newell network, i.e., the characterization of the limiting Markov chain wherein we keep all system parameters (\(\phi_{ij}, F_{ij}, \text{etc.}\)) constant, and scale \(m \to \infty\). We also mentioned that the primary result we use from this characterization is that the steady-state availability of each node \(i\) is given by \(A_i, \infty(p) = r_i(p) / \max_j r_j(p)\), and that there exists at least one node \(i\) with \(A_i, \infty(p) = 1\) (cf. Proposition 3). We now describe this limit in a little more detail. Our presentation follows closely that of [22], Section 3.7, which we refer the reader to for more details.

Recall first that given \(p = \{r_j\}\), we can compute quantities \(w_i(p)\) and \(r_i(p)\), which are independent of \(m\). We define \(r_{\text{max}} = \max_i r_i(p)\) and \(\tilde{r}_i(p) = r_i(p) / r_{\text{max}}\). We also define \(J = \{i \in [n] | \tilde{r}_i(p) = 1\}\) to be the set of bottleneck nodes in the network (note that \(J\) has at least one element), and \(K = [n] \setminus J\) be the remaining nodes. Then as \(m \to \infty\), the stationary distribution of the \(m\)-unit system (as specified in Equation (2)) converges to a limiting distribution (cf. [22] for the specific technical sense in which the steady-state distributions converge to the limit) as \(m \to \infty\), with the following properties:

- The bottleneck nodes, i.e., nodes in set \(J\) with \(\tilde{r}_i(p) = 1\), all have \(A_i(p) = 1\).
- The bottleneck nodes feed the non-bottleneck nodes in set \(K\), which together form an open Jackson network, with each node behaving as a stable \(M/M/1\) queue.
- For all \(i \in K\), we have \(A_i(p) = \tilde{r}_i(p) < 1\).

The above description has the following physical interpretation: in the infinite-unit limit, the bottleneck nodes have an infinite queue of units, and hence always have availability 1. Moreover, the rate of units traveling from one of these nodes \(i\) to a non-bottleneck node \(j\) is exactly \(\phi_{ij}(p)\). Thus from the perspective of a non-bottleneck node \(j\), it appears as if a steady-stream of units (with total rate \(\phi_j(p)\)) arrive from (and depart to), an external node; the number of units in node \(j\) therefore behaves according to the dynamics of a stable \(M/M/1\) queue.

Lemma 28 The objective of the elevated flow relaxation for the policy returned by Algorithm 1 upper bounds the objective of any state-independent policy \(p\) in the infinite-unit system.

Proof. This follows if we show that the flows in the infinite-unit limit satisfy supply circulation and demand bounding. The latter is clear from the dynamics of the system (the flow out of a node can not exceed the rate of arriving customers). To see that the former follows from the above listed properties, note that \(w_i(p)\) is defined to be the leading left eigenvector of \(\{\lambda_{ij}(p)\}_{i,j}\), where \(\lambda_{ij}(p) = \phi_{ij}(p) / \phi_i(p)\). From this we get for all \(i\):
\[
\sum_j w_j \phi_{ij}(p) / \phi_j(p) = w_i \left( \sum_k \phi_{ik}(p) / \phi_i(p) \right) = \sum_j r_j(p) \phi_{ji}(p) = \sum_k r_i(p) \phi_{ik}(p)
\]
Dividing both sides by \(r_{\text{max}}(p)\) we get that for all nodes \(i\), we have \(\sum_j \tilde{r}_j(p) \phi_{ij}(p) = \sum_k \tilde{r}_i(p) \phi_{ik}(p)\). However, as we noted above, \(A_i, \infty(p) = \tilde{r}_i(p)\), and hence \(f_{ij}^*(p) = \tilde{r}_i(p) \phi_{ij}(p)\). Thus the \(f_{ij}^*(p)\) satisfy flow conservation. \(\Box\)

Combining with Lemma 6, we get that the elevated flow relaxation solution is tight in the infinite-unit limit.

Lemma 29 The objective of the elevated flow relaxation for the policy returned by Algorithm 1 is equal to the objective of the optimal state-independent policy in the infinite-unit system.
Appendix D: Settings without Prices

In Section 5.1 we discussed how two control levers, redirection of supply and of demand, can be combined with pricing to obtain the same guarantees we obtain for the pure pricing problem. We now show that our technique extends to settings in which only redirection of supply/demand is allowed, but pricing is not. Because demand cannot be modulated in these settings, one may assume that $I_{ij}$ is constant for each $i$ and $j$, because $I_{ij}$ is not a function of prices. Thus, the elevated objective, defined analogously to Section 3, is always equal to the objective now. Further, the interpretation of our results changes slightly.

Similarly to Algorithm 6, we introduce quantiles $q_i$: unlike Section 5.4 however, we cannot change prices to modulate demand according to these quantiles. We adopt the same notation as in Section 5.1, with the exception that we do not allow for pricing policies and thus everything is just a function of $r$. The quantiles $q$ now correspond to the induced availabilities, i.e., $q_i = A_{i,m}(r)$. Observe that the resulting flows are within the following polytope (as in Sections 5.1 and 5.4):

$$(1) \hat{q}_i \in [0, 1], \quad (2) \sum_k (\phi_{ik} \hat{q}_k + \hat{z}_{ki}) = \sum_j (\phi_{ij} \hat{q}_j + \hat{z}_{ij}), \quad (3) \sum_k \hat{z}_{ik} \leq \sum_j \phi_{ji} \hat{q}_j \forall i.$$

As in Section 5.1, these constraints stem from demand bounding, supply circulation, and the limitation that $\text{Obj}$ would not necessarily be the case. Thus, we bound the infinite unit performance of this policy compared to the value of the elevated flow relaxation.

Algorithm 7 The Elevated Flow Relaxation Program for Redirection without Prices

**Require**: arrival rates $\phi_{ij}$, per-ride rewards $I_{ij}$, rerouting costs $c_{ij}$.

1: Find $\{q_i, z_{ij}\}$ that solves the following relaxation:

$$\text{Maximize} \quad \sum_{i,j} (\phi_{ij} \hat{q}_i I_{ij} - c_{ij} \hat{z}_{ij})$$
$$\sum_k (\phi_{ik} \hat{q}_k + \hat{z}_{ki}) = \sum_j (\phi_{ij} \hat{q}_j + \hat{z}_{ij}) \quad \forall i$$
$$\sum_k \hat{z}_{ik} \leq \sum_j \phi_{ji} \hat{q}_j \quad \forall i$$
$$\hat{q}_i \in [0, 1] \quad \forall i$$

2: Output redirection probabilities $r_{ij} = z_{ij} / \sum_k \phi_{ki} q_{ki}$

Lemma 30 Denote by $\hat{q}$ the quantiles solved for in the relaxation of Algorithm 7 and by $\hat{r}$ the redirection probabilities returned. Then $\text{Obj}_{\infty}(\hat{r}) \geq \text{Obj}(\hat{q}, \hat{r})$.

**Proof.** Consider first $\text{Obj}_{\infty}(\hat{q}, \hat{r})$, the objective obtained when implementing both the redirection policy $\hat{r}$ and the quantiles $\hat{q}$ that Algorithm 7 solves for. By the same argument as in Lemma 15, all availabilities are equal to 1 (and all traffic intensities are equal) in this system, and thus its objective matches $\text{Obj}(\hat{q}, \hat{r})$. In order for us to compare $\text{Obj}_{\infty}(\hat{q}, \hat{r})$ with $\text{Obj}_{\infty}(\hat{r})$, consider a node $v \in \arg\max_j \hat{q}_j$. Increasing each quantile by a factor of $1/\hat{q}_v$, we obtain quantiles $\hat{q}$. Notice that in the system with quantiles $\hat{q}$, the traffic intensity at each node is changed by the same factor, so the traffic intensities are still equal and the availabilities are still equal at every node. In fact, for the relaxation in Algorithm 7, there exists at least one $i$ such that $q_i = 1$, so no quantile changes. Allowing for delays and scaling demand with the number of units, this would not necessarily be the case. Thus, $\text{Obj}_{\infty}(\hat{q}, \hat{r}) \geq \text{Obj}_{\infty}(\hat{q}, \hat{r})$. Thereafter, for each node $j \neq v$, we increase its quantile to 1. Notice that each such change only decreases the traffic intensity at $j$, so the maximum traffic intensity remains unchanged. The lemma follows because the decrease in the traffic intensity (and thus availability) at each node $j \neq v$ is exactly balanced by the increased rate of arrivals at $j$. Formally, we have that $f_{j,\infty}(\hat{q}, \hat{r})$ remains unchanged when the $j$th coordinate of the quantiles is set to 1. Therefore, $\text{Obj}_{\infty}(\hat{r}) = \text{Obj}_{\infty}(\hat{q}, \hat{r}) \geq \text{Obj}(\hat{q}, \hat{r})$. □

Now, using Lemma 30 in place of Lemma 15 in the proof of Theorem 14, we get the following.

Theorem 31 With $\hat{r}$ defined as above, $\text{Obj}_{\infty}(\hat{r}) \geq \frac{m}{m+n-1} \text{OPT}_m$. 
D.1. Delays without prices

Accommodating settings in which we are not allowed pricing, but do have delays, requires an additional idea. This is because the argument in Section 5.3 explicitly relied on pricing to ensure that (on average) not too many units are in transit simultaneously, thereby enabling a lower bound on the maximum availability. Without prices to regulate demand, we can no longer control the maximum availability. Instead, we use the following stochastic dominance characterization for closed-queueing networks.

**Lemma 32** (cf. Theorem 3.8 in Chen and Yao [6]) In a closed Jackson network, with state-independent service rates, increasing the service rate functions, in a pointwise sense, at any subset of nodes will increase throughput.

In our context, this is equivalent to saying that increasing quantiles at a subset of nodes only increases throughput. In fact, one can show that throughput also increases locally, i.e., increasing quantiles at one node (which we henceforth refer to as point quantiles) does not decrease the rate of units on any edge.

**Lemma 33** Let \( q = \{q_i\} \) be a vector of point quantiles, and \( \bar{q} \) be a vector of point quantiles with \( \bar{q}_k \geq q_k \forall k \). Then for any pair \((i,j)\), we have \( f_{ij,m}(q) \leq f_{ij,m}(\bar{q}) \), i.e., the rate of realized trips from \( i \) to \( j \) does not decrease when point quantiles are increased.

**Proof.** The proof relies on two observations. Note first for \( q \) and \( \bar{q} \), we have

\[
\frac{\phi_{ij}q_i}{\sum_k \phi_{ik}q_i} = \frac{\phi_{ij}\bar{q}_i}{\sum_k \phi_{ik}\bar{q}_i} \quad \forall i,j,
\]

and therefore, letting \( w(q') \) denote the eigenvector of the routing matrix \( \{\phi_{ij}(q')/\sum_k \phi_{ik}(q')\}\) (cf. Section 5.4), we obtain \( w_i(q) = w_i(q') \). Define \( \Gamma_m(q) \triangleq G_m(q)/G_{m-1}(q) \). We now have that the ratio of the rates \( f_{ij,m}(q)/f_{ij,m}(\bar{q}) \) is equal to

\[
\frac{f_{ij,m}(q)}{f_{ij,m}(\bar{q})} = \frac{A_{i,m}(q)/\phi_{ij}}{A_{i,m}(\bar{q})/\phi_{ij}} = \frac{\Gamma_m(q)\sum_k w_k(q)}{\sum_k w_k(\bar{q})} = \frac{\Gamma_m(q)}{\Gamma_m(\bar{q})}.
\]

Note that the ratio of \( f_{ij,m}(q) \) and \( f_{ij,m}(\bar{q}) \) does not depend on \( i \) and \( j \). Moreover, from Theorem 32 we have \( \sum_{i,j} f_{ij,m}(q) \leq \sum_{i,j} f_{ij,m}(\bar{q}) \). Combining the two, we get \( f_{ij,m}(q) \leq f_{ij,m}(\bar{q}) \). \( \square \)

This allows us to prove the guarantee of Theorem 22 for settings in which prices cannot be used to provide a lower on the maximum availability within the system.

**Algorithm 8** The Rate-Limited Elevated Flow Relaxation Program for Redirection w/o Prices

**Require:** scaling parameter \( \epsilon_m \), arrival rates \( \phi_{ij} \), rewards \( I_{ij} \), rerouting costs \( c_{ij} \), travel-times \( \tau_{ij} \).

1. Find \( \{q_i, z_{ij}\} \) that solves the following relaxation:

\[
\begin{align*}
\text{Maximize} & \quad \sum_{i,j} (\phi_{ij}q_i I_{ij} - c_{ij}z_{ij}) \\
\text{subject to} & \quad \sum_{i,j} \phi_{ij} \tau_{ij} q_i + z_{ij} \leq m \\
& \quad \sum_k (\phi_{ki} q_k + z_{ki}) = \sum_j (\phi_{ij} \tilde{q}_i + \tilde{z}_{ij}) \quad \forall i \\
& \quad \sum_k \tilde{z}_{ik} \leq \sum_j \phi_{ij} \tilde{q}_j \quad \forall i \\
& \quad \tilde{q}_i \in [0,1] \quad \forall i \\
& \quad \tilde{z}_{ij} \in [0,1] \quad \forall i,j.
\end{align*}
\]

2. Output redirection probabilities \( r_{ij} = z_{ij}/\sum_k \phi_{ki} q_k \)

**Theorem 34** Let \( \tilde{\text{Obj}} \) denote the output of Algorithm 8 with \( \epsilon_m := 2\sqrt{\ln m/m} \). \( \text{OPT}_m \) be the value of the objective function for the optimal state-dependent pricing policy, and \( m \geq 100 \). Then

\[
\frac{\text{Obj}_m(\tilde{\text{Obj}})}{\text{OPT}_m} \geq (1 - \epsilon_m) \left( \frac{\sqrt{m \ln m}}{\sqrt{m \ln m + n - 1}} - \frac{3}{\sqrt{m \ln m}} \right).
\]
Proof. The same proof as in Theorem 22 guarantees that using point prices as given by \( q(1 - \epsilon_m) \), where \( q \) comes from the solution of the relaxation in Algorithm 8 yields the required guarantee. Lemma 33 then guarantees that increasing all quantiles to one yields a solution no worse. \( \Box \)

We remark that with \( m \to \infty \), the above theorem recovers the result of Braverman et al [3].

Finally, we note that Lemma 33 also yields an alternate proof of Lemma 7: given quantiles \( q \) that do not induce a demand circulation, we consider a system with rates \( \delta_{ij} = \phi_i \max_q r_{ij}(q) \). We observe that (i) the objectives with rates \( \delta_{ij} \) and rates \( \phi_{ij} \) are the same in an infinite unit system and (ii) that the system with rates \( \delta_{ij} \) obeys the demand circulation property. Thus, the counting argument of [24] guarantees an objective within \( m/(m + n - 1) \) of the infinite unit system in a system with rates \( \delta_{ij} \). However, by (i) the latter was equal to the upper bound on \( \text{OPT}_m \). Since Lemma 33 implies that the \( m \)-unit system with rates \( \phi_{ij} \) has objective no worse than the \( m \)-unit system with rates \( \delta_{ij} \), the statement of the lemma follows.

Appendix E: Tightness Of Our Guarantees

In this section, we discuss an example of [23], that proves that the guarantees we prove for our algorithms are tight. Interestingly, this does not require the distinction between state-dependent and state-independent policies, i.e. the objectives obtained through our algorithms can be as far away from the optimal state-independent policy as from the optimal state-dependent policy.

Proposition 35([23]) For any number \( m \) of units and \( n \) of nodes, the objective of the solution returned in Algorithm 1 and the optimal objective may be arbitrarily close to the approximation guarantee \( \frac{m}{m + n - 1} \).

Proof. Consider a system of \( n \) nodes \( \{1, \ldots, n\} \) with demand only occurring from nodes \( i \) to \( i + 1 \) and from node \( n \) to node 1. In particular, suppose that for some \( k \) that is yet to be set, we have \( \phi_{12} = \phi_{23} = \ldots = \phi_{n-1,n} = k \), and \( \phi_{n,1} = 1 \). Further, suppose we are maximizing throughput, though the same construction works for revenue and social welfare. The policy returned by Algorithm 1 sets quantiles \( q_{12} = q_{23} = \ldots = q_{n-1,n} = \frac{1}{k} \) and \( q_{n,1} = 1 \). Given that the availability of each node is then \( \frac{m}{m+n-1} \) (cf. Lemma 7 with all inequalities holding tightly) and that there are \( n \) nodes from which a ride can occur (at rate 1), the throughput is \( \frac{mn}{m+n-1} \). On the other hand, for the solution that sets all quantiles to 1, the throughput converges to \( n \) as \( k \to \infty \). Intuitively, this is because the expected time between an arrival at node \( n \) (triggering that unit to move to node 1) and the expected return time of that unit to node \( n \) converges to 0. Thus, for each arrival at node \( n \), occurring at rate 1, the system observes \( m \) rides. The details of this argument can be found in Proposition 3 of [23]. \( \Box \)

Appendix F: Auxiliary lemma

We present a basic Chernoff tail bound for Poisson random variables, which we use in Section 5.3

Lemma 36 For \( X \sim \text{Poisson}(\lambda) \), we have for any \( 0 \leq x \leq \lambda \):

\[
P[X > \lambda + x] \leq \exp \left( -\frac{x^2}{2\lambda} \left( 1 - \frac{x}{\lambda} \right) \right)
\]

Proof. Using a standard Chernoff bound argument, we have for any \( \theta \geq 0 \):

\[
P[X > \lambda + x] = \mathbb{P}[e^{\theta X} > e^{\theta(\lambda+x)}] \leq e^{-\theta(\lambda+x)} \mathbb{E}[e^{\theta X}] = e^{-\theta(\lambda+x) - \lambda(e^\theta - 1)}
\]

Now, optimizing over the choice of \( \theta \), we get

\[
P[X > \lambda + x] \leq \exp \left( \inf_{\theta} \left( \lambda(e^\theta - 1 - \theta) - x\theta \right) \right)
\]

\[
= \exp \left( x - (x + \lambda) \log(1 + x/\lambda) \right)
\]

\[
\leq \exp \left( x - (x + \lambda) \left( \frac{x}{\lambda} - \frac{x^2}{2\lambda^2} \right) \right) = \exp \left( -\frac{x^2}{2\lambda} \left( 1 - \frac{x}{\lambda} \right) \right)
\]

\( \Box \)

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References