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APPENDIX B: PROOFS AND DERIVATIONS

IN THIS SUPPLEMENT, we provide proofs for Lemma 1 and Proposition 1, the second part of Proposition 5, and Propositions 7, 8, and 9.

B.1. *Proof of Lemma 1 and Proposition 1*

We construct the proof in steps. In particular, we separate Proposition 1 into the following four lemmas. These four lemmas are sufficient to prove Proposition 1.

LEMMA B.1: *If the equation system (12)–(13), (7)–(9) has a solution where  $c_h^* < R_K$ , and both  $v(c)$  and  $q(c)$  are increasing in the range  $c \in [c_l^*, c_h^*]$ , then Proposition 1 holds.*

LEMMA B.2: *The system (12)–(13), (7)–(9) always has at least one solution.*

LEMMA B.3: *If  $h - l$  is sufficiently small, then  $c_h^* < R_K$ .*

LEMMA B.4:  *$q(c)$  is decreasing in  $c$ . If  $h - l$  is sufficiently small, then  $v(c)$  is increasing for  $c \in [c_l^*, c_h^*]$ .*

B.1.1. *Step 1: Proof of Lemma 1 and Lemma B.1*

Denote the dollar share of capital in the firm’s asset holdings by  $\psi_t^i$ , so that  $\psi_t^i = K_t^i p_t / w_t^i$ . According to our conjecture, the value function can be written as (recall the aggregate cash-to-capital ratio  $c = C/K$ )

$$J(K_t, C_t, K_t^i, C_t^i) = w_t^i \left[ (1 - \psi_t^i) q(c_t) + \frac{\psi_t^i}{p_t} v(c) \right] = J(K_t, C_t, w_t^i),$$

is linear in  $w_t$ . This is equivalent to  $J(C, K, K_t^i, C_t^i) = K_t^i v(c) + C_t^i q(c)$  stated in the lemma. Also, we have the wealth dynamics, expressed in terms of capital share  $\psi_t^i$ , as

$$dw_t^i = -d\alpha_t^i - \theta dK_t^i + \psi_t^i w_t^i \frac{1}{p_t} (dp_t + \sigma dZ_t).$$

And,  $q(c) \geq 1$  has to hold as firms can consume cash at the final date (and there is no discounting), which implies  $d\alpha_t^i = 0$ , that is, firms do not consume in the aggregate stage.

1 As the firm is choosing capital share  $\psi_t^i$ , and the capital to build or dismantle  
2  $dK_t^i$ , the Hamiltonian–Jacobi–Bellman (HJB) of problem (3) can be written as

$$3 \quad 0 = \max_{d\psi_t^i, dK_t^i} d\alpha_t^i + J_C \mathbb{E}_t[dC_t] + \frac{1}{2} J_{CC} \mathbb{E}_t[dC_t^2]$$

$$4 \quad + J_w \mathbb{E}_t(dw_t) + J'_K dK_t^i + J_{w,C} \mathbb{E}_t[dw_t dC_t].$$

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8 The endogenous price dynamics (using Ito’s lemma) is

$$9 \quad dp_t = \frac{1}{2} \sigma^2 p''(c) dt + \sigma p'(c) dZ_t + dB_t^p - dU_t^p,$$

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13 where  $dB_t^p$  ( $dU_t^p$ ) reflects  $p$  at  $p(c_l^*) = l$  ( $p(c_h^*) = h$ ). This is because, in any  
14 market equilibrium, firms will create (dismantle) capital if  $p_t = h$  ( $p_t = l$ ), and  
15 keep doing it until the price adjusts. We derived the boundary conditions in the  
16 main text. Also, by risk neutrality and the initial homogeneity of firms, before  
17 the final date the price of the capital has to make firms indifferent whether  
18 to hold capital or cash. Otherwise, markets could not clear. We also explained  
19 that  $\hat{p}_\tau = c_\tau$ .

20 Thus, inside the reflection boundary  $(c_l^*, c_h^*)$ , the above HJB equation is (we  
21 drop  $i$  from now on)

$$22 \quad 0 = \max_{\psi_t} \left\{ \frac{\sigma^2}{2} w_t q''(c_t) + q(c_t) \psi_t w_t \frac{1}{2} \frac{\sigma^2 p''(c_t)}{p_t} \right.$$

$$23 \quad + q'(c_t) \left( \left( \psi_t w_t \frac{\sigma}{p_t} (\sigma + p'(c_t) \sigma) \right) \right)$$

$$24 \quad + \xi w_t \left[ \frac{1}{2} \left( \frac{\psi_t R_K}{p_t} + (1 - \psi_t) \frac{R_K}{c_t} \right) \right.$$

$$25 \quad \left. \left. + \frac{1}{2} \left( \frac{\psi_t}{p_t} R_C c_t + (1 - \psi_t) R_C \right) - q(c_t) \right] \right\}.$$

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34 Since the problem is linear in  $\psi_t$ , in equilibrium firms must be indifferent in  
35 their choice of  $\psi_t$ . Thus, we can calculate the dynamics of the cash (capital)  
36 value by choosing  $\psi_t = 0$  ( $\psi = 1$ ). Setting  $\psi_t = 0$  directly implies (10). Choosing  
37  $\psi_t = 1$  gives

$$38 \quad 0 = \frac{\sigma^2}{2} q''(c) + q(c) \frac{1}{2} \frac{\sigma^2 p''(c)}{p} + q'(c) \left( \frac{1}{p} (\sigma + p' \sigma) \sigma \right)$$

$$39 \quad + \frac{1}{p} \left( \frac{\xi}{2} (R_K + R_C c) - q(c) p \right).$$

1 Since  $v(c) = p(c)q(c)$ ,  $v' = q'p + p'q$ , and  $v'' = q''p + 2p'q' + p''q$ , we can  
2 rewrite the above equation as (11). Given that the ODEs for  $v(c)$  and  $q(c)$   
3 were derived by substituting in  $\psi_t = 1$  and  $\psi_t = 0$ , it is easy to see that these  
4 functions can be interpreted as the value of a capital and that of a unit of cash.  
5 This implies that

$$J(C, K, w_t^i) = \left( w_t^i(1 - \psi_t^i)q(c) + \frac{\psi_t^i}{p_t} w_t^i v(c) \right) = q(c)w_t^i,$$

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9 verifying both Lemma 1 and our conjecture on the form of  $J(C, K, w_t^i)$ .

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11 **B.1.2. Step 2: Proof of Lemma B.2**

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13 First, note that for any arbitrary  $c_h$  and  $c_l$  from (9), we can express  $A_1$ – $A_4$   
14 in (12)–(13) as functions of  $c_h$  and  $c_l$  only. Substituting back to (12)–(13), we  
15 get our functions parameterized by  $c_h$  and  $c_l$  which we denote as  $v(c; c_l, c_h)$  and  
16  $q(c; c_l, c_h)$ . Evaluating these functions at  $c = c_l$  and  $c = c_h$ , we get the following  
17 expressions. Define

$$f_l(c_l, c_h) \equiv \frac{e^{-\gamma c_h} (\text{Ei}[c_h \gamma] - \text{Ei}[c_l \gamma]) + e^{\gamma c_h} (\text{Ei}[-c_h \gamma] - \text{Ei}[-c_l \gamma])}{e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}},$$

$$g_l(c_l, c_h) \equiv \frac{e^{-\gamma c_h} (\text{Ei}[c_h \gamma] - \text{Ei}[c_l \gamma]) + e^{\gamma c_h} (\text{Ei}[-\gamma c_l] - \text{Ei}[-\gamma c_h])}{e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}},$$

$$f_h(c_l, c_h) \equiv \frac{e^{-\gamma c_l} (\text{Ei}[c_h \gamma] - \text{Ei}[c_l \gamma]) + e^{\gamma c_l} (\text{Ei}[-\gamma c_h] - \text{Ei}[-\gamma c_l])}{e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}},$$

$$g_h(c_l, c_h) \equiv \frac{e^{-\gamma c_l} (\text{Ei}[c_h \gamma] - \text{Ei}[c_l \gamma]) + e^{\gamma c_l} (\text{Ei}[-\gamma c_l] - \text{Ei}[-\gamma c_h])}{e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}},$$

29 and

$$m(c_l, c_h) \equiv \frac{e^{\gamma(c_h - c_l)} - 1}{1 + e^{\gamma(c_h - c_l)}} \in (0, 1).$$

34 Then the cash and capital values can be rewritten as

$$q(c_l; c_l, c_h) = \frac{R_C}{2} + \frac{R_K \gamma}{2} f_l(c_l, c_h), \quad q(c_h; c_l, c_h)$$

$$= \frac{R_C}{2} + \frac{R_K \gamma}{2} f_h(c_l, c_h),$$

$$v(c_l; c_l, c_h) = R_K + \frac{c_l R_C}{2} + \frac{R_C}{2\gamma} m(c_l, c_h)$$

$$+ \frac{R_K \gamma}{2} \left( \frac{g_l(c_l, c_h)}{\gamma} - c_l f_l(c_l, c_h) \right),$$

1 and

$$2 \quad v(c_h; c_l, c_h) = R_K + \frac{c_h R_C}{2} - \frac{R_C}{2\gamma} m(c_l, c_h) \\ 3 \\ 4 \quad + \frac{R_K \gamma}{2} \left( \frac{g_h(c_l, c_h)}{\gamma} - c_h f_h(c_l, c_h) \right). \\ 5 \\ 6$$

7 For any  $c_h$ , define the function  $H(c_h)$  implicitly as the corresponding lower  
8 threshold  $c_l$  so that, at  $c = c_h$ , the market price is just  $h$ , that is,  
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$$10 \quad p(c_h; c_l = H(c_h), c_h) = \frac{v(c_h; c_l = H(c_h), c_h)}{q(c_h; c_l = H(c_h), c_h)} = h. \\ 11 \\ 12$$

13 Similarly, define  $L(c_h)$  is defined implicitly by

$$14 \quad p(c_l; c_l = L(c_h), c_h) \equiv \frac{v(c_l; c_l = L(c_h), c_h)}{q(c_l; c_l = L(c_h), c_h)} = l, \\ 15 \\ 16 \\ 17$$

18 which makes the market price to be  $l$  at  $c = c_l$ . Obviously, once we find such  $c_h$   
19 that  $H(c_h) = L(c_h)$ , then this particular  $c_h$  and the corresponding  $c_l = H(c_h) =$   
20  $L(c_h)$  is a solution of (7)–(9), (12)–(13). To show that this solution exists, we  
21 first establish properties of  $L(c_h)$ ; then we proceed to the properties of  $H(c_h)$ .  
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23 *Properties of  $L(c_h)$ .* It is useful to observe that

$$24 \quad \frac{\partial f_l}{\partial c_l} = \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} \left( \gamma f_l - \frac{1}{c_l} \right), \quad \frac{\partial f_l}{\partial c_h} = 2 \frac{\frac{1}{c_h} - \gamma f_h}{e^{\gamma(c_h - c_l)} - e^{\gamma(c_l - c_h)}}, \\ 25 \\ 26 \\ 27 \quad \frac{\partial g_l}{\partial c_l} = \frac{1}{c_l} + \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} \gamma g_l, \quad \frac{\partial g_l}{\partial c_h} = - \frac{2\gamma g_h}{e^{\gamma(c_h - c_l)} - e^{\gamma(c_l - c_h)}}, \\ 28 \\ 29 \\ 30 \\ 31 \quad \lim_{c_l \rightarrow c_h} f_l = \frac{1}{\gamma c_h}, \quad \lim_{c_l \rightarrow c_h} g_l = 0, \quad \lim_{c_l \rightarrow c_h} m = 0. \\ 32 \\ 33$$

34 1. We show that  $f_l(c_h, c_l)$  is monotonically decreasing in  $c_l$ . Its slope in  $c_l$  is

$$35 \quad (B.1) \quad \frac{\partial f_l}{\partial c_l} = \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} \left( \gamma f_l(c_h, c_l) - \frac{1}{c_l} \right), \\ 36 \\ 37$$

38 and the second derivative is

$$39 \quad \frac{\partial^2 f_l}{\partial c_l^2} = - \left( 4\gamma e^{2\gamma c_h} \frac{e^{2\gamma c_l}}{(e^{2\gamma c_h} - e^{2\gamma c_l})^2} - \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})^2}{(e^{2\gamma c_h} - e^{2\gamma c_l})^2} \gamma \right) \\ 40 \\ 41 \quad \times \left( \frac{1}{c_l} - \gamma f_l(c_h, c_l) \right) \\ 42 \\ 43 \\ 44$$

$$\begin{aligned}
 & - \frac{\left(-\frac{1}{c_l^2}\right)(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} \\
 & = \gamma \left(\frac{1}{c_l} - \gamma f_l(c_h, c_l)\right) + \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} \frac{1}{c_l^2}.
 \end{aligned}$$

Note that if the first derivative is zero, then the second derivative is positive, implying that  $f_l(c_h, c_l)$  can have only local minima, but no local maxima in  $c_l$ . At the limit, one can check that

$$\begin{aligned}
 \lim_{c_l \rightarrow c_h} \frac{\partial f_l}{\partial c_l} & = \lim_{c_l \rightarrow c_h} \left( \frac{1}{c_l} \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} (\gamma c_l f_l(c_h, c_l) - 1) \right) \\
 & = \frac{1}{c_h} \left( -\frac{1}{2\gamma c_h} \right) < 0.
 \end{aligned}$$

Thus,  $f_l(c_h, c_l)$  is decreasing at  $c_h = c_l$ . Suppose that it is not monotonic over the range of  $c_l < c_h$  in  $c_l$ . Then the largest  $\hat{c}_l$  where the first derivative is 0 would be a local maximum. But we have just ruled out the existence of a local maximum. Thus  $f_l(c_h, c_l)$  is monotonically decreasing over the whole range of  $c_l < c_h$  in  $c_l$ . This statement is equivalent to  $\gamma f_l(c_h, c_l) - \frac{1}{c_l} < 0$  for  $c_l < c_h$ , for any fixed  $c_h$ .

2. We show that  $X(c_l) \equiv f_l(c_h, c_l) - \frac{1}{\gamma c_l}$  is increasing in  $c_l$ . We would like to show that

$$\text{(B.2)} \quad X'(c_l) = \gamma \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} X(c_l) + \frac{1}{\gamma c_l^2} > 0.$$

Clearly, we have

$$X(c_l = c_h) = 0, \quad X'(c_l = c_h) = f_l'(c_h, c_h) + \frac{1}{\gamma c_h^2} = \frac{1}{2\gamma c_h^2} > 0.$$

We know that when  $c_l \rightarrow 0$ ,  $f(c_h, c_l)$  has the order of  $\text{Ei}(\gamma c_l)$  which is  $O(\ln c_l)$ ; this implies that  $X(c_l) \rightarrow -\infty$  when  $c_l \rightarrow 0$ . Then, if  $X(c_l)$  is not monotone, we must have two points  $x_1 < x_2$  closest to (but below)  $c_h$  so that

$$0 > X(x_1) > X(x_2), \quad X'(x_1) = X'(x_2) = 0.$$

Setting (B.2) to be zero, we have (because  $0 < x_1 < x_2$ )

$$X(x_1) = -\frac{(e^{2\gamma c_h} - e^{2\gamma x_1})}{\gamma^2 x_1^2 (e^{2\gamma c_h} + e^{2\gamma x_1})} < -\frac{(e^{2\gamma c_h} - e^{2\gamma x_2})}{\gamma^2 x_2^2 (e^{2\gamma c_h} + e^{2\gamma x_2})} = X(x_2),$$

in contradiction with  $X(x_1) > X(x_2)$ . Thus (B.2) holds always.

3. We show that the function  $\frac{g_l(c_h, c_l)}{\gamma} - c_l f_l(c_h, c_l)$  is monotonically increasing in  $c_l$ . Its first derivative is (all the derivatives in this part are with respect to  $c_l$ )

$$\begin{aligned} \left(\frac{g_l}{\gamma} - c_l f_l\right)' &= \frac{1}{\gamma c_l} + \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} g_l(c_l, c_h) \\ &\quad - \left(\frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} (c_l \gamma f_l(c_l, c_h) - 1) + f_l(c_l, c_h)\right) \\ &= \frac{1}{\gamma c_l} + \gamma \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} \left(\frac{g_l}{\gamma} - c_l f_l\right) \\ &\quad + \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} - f_l. \end{aligned}$$

Whenever the first derivative is zero, at that point we have

$$(B.3) \quad \frac{g_l}{\gamma} - c_l f_l = \frac{f_l - \frac{1}{\gamma c_l}}{\gamma \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})}} - \frac{1}{\gamma}.$$

We also know that

$$\lim_{c_l \rightarrow c_h} \left(\frac{g_l}{\gamma} - c_l f_l\right)' = 0$$

and

$$\lim_{c_l \rightarrow c_h} \left(\frac{g_l}{\gamma} - c_l f_l\right)'' = -\frac{1}{3\gamma c_h^2} < 0;$$

so for any fixed  $c_h$ ,  $c_l = c_h$  is a local maximum. Thus, to show that  $\frac{g_l}{\gamma} - c_l f_l$  is monotone, it suffices to rule out the case of a local minimum  $\hat{c}_l < c_h$  so that  $(\frac{g_l}{\gamma} - c_l f_l)' = 0$  and  $(\frac{g_l}{\gamma} - c_l f_l)'' > 0$ . In general,

$$\begin{aligned} \left(\frac{g_l}{\gamma} - c_l f_l\right)'' &= -\frac{1}{\gamma c_l^2} + \gamma \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} \left(\frac{g_l}{\gamma} - c_l f_l\right)' - f_l' \\ &\quad + \frac{4e^{2\gamma c_h} e^{2\gamma c_l}}{(e^{2\gamma c_h} - e^{2\gamma c_l})^2} \gamma^2 \left(\left(\frac{g_l}{\gamma} - c_l f_l\right) + \frac{1}{\gamma}\right). \end{aligned}$$

Thus, if there were a  $\hat{c}_l$  such that  $(\frac{g_l}{\gamma} - c_l f_l)' = 0$ , using (B.1) and (B.3) we have  $(\frac{g_l}{\gamma} - c_l f_l)''$  to be equal to

$$\begin{aligned} & -\frac{1}{\gamma \hat{c}_l^2} - f_l' + \frac{4\gamma^2 e^{2\gamma c_h} e^{2\gamma \hat{c}_l}}{(e^{2\gamma c_h} - e^{2\gamma \hat{c}_l})^2} \left( \frac{f_l - \frac{1}{\gamma \hat{c}_l}}{\gamma \frac{(e^{2\gamma c_h} + e^{2\gamma \hat{c}_l})}{(e^{2\gamma c_h} - e^{2\gamma \hat{c}_l})}} - \frac{1}{\gamma} + \frac{1}{\gamma} \right) \\ & = -\frac{1}{\gamma \hat{c}_l^2} - \gamma \frac{(e^{2\gamma c_h} - e^{2\gamma \hat{c}_l})}{e^{2\gamma c_h} + e^{2\gamma \hat{c}_l}} \left( f_l - \frac{1}{\gamma \hat{c}_l} \right). \end{aligned}$$

But from (B.2) we know the above term is strictly negative, which proves the contradiction.

4. We show that  $q(c_l; c_l, c_h)$  is also decreasing in  $c_l$  for any  $c_l < c_h$ . Given that  $(\frac{g_l}{\gamma} - c_l f_l)' > 0$  and  $\partial(\frac{c_l R_C}{2} + \frac{R_C(e^{-\gamma(c_h-c_l)} + e^{\gamma(c_h-c_l)} - 2)}{2\gamma(e^{\gamma(c_h-c_l)} - e^{-\gamma(c_h-c_l)})})/\partial c_l = \frac{1}{2} R_C \frac{e^{-2\gamma c_h + 2\gamma c_l + 1}}{(e^{-\gamma c_h + \gamma c_l + 1})^2} > 0$ ,  $v(c_l; c_l, c_h)$  is increasing in  $c_l$ . Thus,  $p(c_l; c_l, c_h)$  is increasing in  $c_l$  for any  $c_l < c_h$ . Also one can show that  $\lim_{c_l \downarrow 0} p(c_l; c_l, c_h) = -\frac{\tanh(\gamma c_h)}{\gamma} < 0$ , and

$$\begin{aligned} \lim_{c_l \rightarrow c_h} p(c_l; c_l, c_h) &= \frac{R_K + c_h \frac{R_C}{2} + \frac{R_K \gamma}{2} \left( -c_h \frac{1}{\gamma c_h} \right)}{\frac{R_C}{2} + \frac{R_K \xi}{\gamma \sigma^2} \frac{1}{\gamma c_h}} \\ &= \frac{R_K + c_h \frac{R_C}{2} - \frac{R_K}{2}}{\frac{R_C}{2} + \frac{R_K}{2c_h}}, \end{aligned}$$

which is larger than  $l$  as long as  $c_h > l$ . Thus, as long as  $c_h > l$ ,  $\lim_{c_l \rightarrow c_h} p(c_l; c_l, c_h) \geq l$  and there is a unique solution  $c_l$  for any  $c_h$  of  $p(c_l; c_l, c_h) = l$ . Therefore  $L(c_h)$  exist. From the monotonicity in  $c_l$ , and continuity of  $p(c_l; c_l, c_h)$ , we also know that  $L(c_h)$  is continuous.

*Properties of  $H(c_h)$ .* First, we show that for any  $c_h \in [l, R_K]$ ,  $H(c_h)$  is a continuous function and  $H(c_h) \in [0, c_h]$ . Again, the notation ' means we are taking the derivative with respect to  $c_l$ . We use the following facts:

$$\begin{aligned} \frac{\partial f_h}{\partial c_l} &= \frac{2 \left( \gamma f_l(c_h, c_l) - \frac{1}{c_l} \right)}{(e^{\gamma(c_h-c_l)} - e^{-\gamma(c_h-c_l)})}, & \frac{\partial g_h}{\partial c_l} &= \frac{2\gamma g_l(c_h, c_l)}{(e^{\gamma(c_h-c_l)} - e^{-\gamma(c_h-c_l)})}, \\ \frac{\partial f_h}{\partial c_h} &= \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} \left( \frac{1}{c_h} - \gamma f_h(c_h, c_l) \right), \end{aligned}$$

$$\frac{\partial g_h}{\partial c_h} = \frac{1}{c_h} - \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} \gamma g_h(c_l, c_h),$$

$$\lim_{c_l \rightarrow c_h} f_h = \frac{1}{\gamma c_h}, \quad \lim_{c_l \rightarrow c_h} g_h = 0.$$

1. The result of  $\frac{\partial f_h}{\partial c_l} = 2(\gamma f_l(c_h, c_l) - \frac{1}{c_l}) / (e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}) < 0$  follows from step 1 in the previous subsection.

2. We show  $(\frac{g_h}{\gamma} - f_h c_h)' > 0$  for  $c_l < c_h$ . We have

$$\left(\frac{g_h}{\gamma} - f_h c_h\right)' = 2 \frac{g_l - c_h \gamma f_l + c_h \frac{1}{c_l}}{e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}}$$

and

$$\begin{aligned} \frac{\partial^2 \left(\frac{g_h}{\gamma} - f_h c_h\right)}{\partial^2 c_l} &= \frac{2g_l' - c_h 2\gamma f_l' - 2\frac{c_h}{c_l^2}}{e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)}} \\ &\quad + \gamma e^{-\gamma(c_h - c_l)} \frac{e^{2(-\gamma(c_h - c_l))} + 1}{(e^{-2\gamma(c_h - c_l)} - 1)^2} \left(2g_l - c_h 2\gamma f_l + \frac{2c_h}{c_l}\right). \end{aligned}$$

If the first derivative is zero at a point  $c_h > c_l$ , then the second derivative is

$$\begin{aligned} &\frac{2\frac{1}{c_l} + 2\gamma \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} \left(g_l(c_l, c_h) - c_h \gamma f_l(c_h, c_l) + \frac{c_h}{c_l}\right) - c_h 2\frac{1}{c_l^2}}{(e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)})} \\ &= \frac{-2\frac{c_h - c_l}{c_l^2}}{(e^{\gamma(c_h - c_l)} - e^{-\gamma(c_h - c_l)})} < 0, \end{aligned}$$

for any  $c_h > c_l$ , which implies that it can have no minimum in that range. Also

$$\lim_{c_l \rightarrow c_h} \frac{\partial \left(\frac{g_h}{\gamma} - f_h c_h\right)}{\partial c_l} = 0, \quad \lim_{c_l \rightarrow c_h} \frac{\partial^2 \left(\frac{g_h}{\gamma} - f_h c_h\right)}{\partial^2 c_l} = -\frac{1}{3\gamma c_h^2},$$

so  $c_l = c_h$  must be the unique maximum in the range  $c_h \geq c_l$ , and the result follows.

3. Consequently,  $q(c_h; c_h, c_l)$  is monotonically decreasing and  $v(c_h; c_h, c_l)$  is monotonically increasing in  $c_l$ . Thus,  $p(c_h; c_h, c_l)$  is monotonically increasing in  $c_l$ .



4. Observe that the following hold:

$$\begin{aligned} \lim_{c_l \rightarrow c_h} p(c_h; c_l, c_h) &= \lim_{c_l \rightarrow c_h} \frac{v(c_h; c_l, c_h)}{q(c_h; c_l, c_h)} = \frac{R_K c_h + c_h^2 \frac{R_C}{2} - \frac{R_K}{2} c_h}{\frac{R_C}{2} c_h + \frac{R_K}{2}} \\ &= \frac{c_h^2 R_C + R_K c_h}{R_C c_h + R_K} = c_h. \end{aligned}$$

Because  $\lim_{c_l \rightarrow 0} p(c_h; c_l, c_h) = -c_h$ , hence we know that, for any  $c_h > h$ , there is a unique  $c_l \in [0, c_h]$  which solves  $p(c_h; c_l, c_h) = h$ . From the monotonicity of  $p(c_h; c_h, c_l)$  in  $c_l$  and the continuity in  $c_h$ , the resulting function  $H(c_h)$  is continuous in  $c_h$ .

*Intercept of  $H(c_h)$  and  $L(c_h)$ .*

1. Here we show that  $H(h) > L(h)$ . We know that  $H(h) = h$  because

$$\begin{aligned} \lim_{c_l \rightarrow h} \frac{v(c_h; c_l, c_h)}{q(c_h; c_l, c_h)} &= \frac{R_K + h \frac{R_C}{2} + \frac{R_K \xi}{\gamma \sigma^2} \left( -h \frac{1}{\gamma h} \right)}{\frac{R_C}{2} + \frac{R_K \xi}{\gamma \sigma^2} \frac{1}{\gamma h}} \\ &= \frac{R_K + h \frac{R_C}{2} + \frac{R_K}{2} \gamma \left( -h \frac{1}{\gamma h} \right)}{\frac{R_C}{2} + \frac{R_K}{2} \gamma \frac{1}{\gamma h}} = h. \end{aligned}$$

However, note that

$$\lim_{c_l \rightarrow h} \frac{v(c_l; c_l, c_h)}{q(c_l; c_l, c_h)} = \frac{R_K + h \frac{R_C}{2} + \frac{R_K \gamma}{2} \left( -h \frac{1}{\gamma h} \right)}{\frac{R_C}{2} + \frac{R_K}{2h}} = h,$$

and  $\frac{v(c_l; c_l, c_h)}{q(c_l; c_l, c_h)}$  is increasing in  $c_l$ . Since  $L(h)$  is defined by  $\frac{v(c_l; L(h), h)}{q(c_l; L(h), h)} = l < h$ ,  $L(h) < h = H(h)$  must hold.

2. Now we show that  $\lim_{c_h \rightarrow \infty} H(c_h) = 0 < \lim_{c_h \rightarrow \infty} L(c_h)$ . It is easy to check that

$$\begin{aligned} \lim_{c_h \rightarrow \infty} f_l &= \frac{-\text{Ei}[-c_l \gamma]}{e^{\gamma(-c_l)}}, & \lim_{c_h \rightarrow \infty} g_l &= \frac{\text{Ei}[-\gamma c_l]}{e^{\gamma(-c_l)}}, \\ \lim_{c_h \rightarrow \infty} f_h &= 0, & \lim_{c_h \rightarrow \infty} g_h &= 0. \end{aligned}$$

Thus,  $\lim_{c_h \rightarrow \infty} \frac{v(c_l; c_l, c_h)}{q(c_l; c_l, c_h)}$  takes the value of

$$\begin{aligned} & \lim_{c_h \rightarrow \infty} \frac{R_K + \frac{c_l R_C}{2} + \frac{R_C m(c_l, c_h)}{2\gamma} + \frac{R_K \gamma}{2} \left( \frac{g_l(c_l, c_h)}{\gamma} - c_l f_l(c_l, c_h) \right)}{\frac{R_C}{2} + \frac{R_K \gamma}{2} f_l(c_l, c_h)} \\ &= \frac{R_K + \frac{c_l R_C}{2} + \frac{R_C}{2\gamma} + \frac{R_K \gamma}{2} \left( \frac{\text{Ei}[-\gamma c_l]}{\gamma e^{\gamma(-c_l)}} - c_l \frac{-\text{Ei}[-c_l \gamma]}{e^{\gamma(-c_l)}} \right)}{\frac{R_C}{2} - \frac{\text{Ei}[-c_l \gamma]}{e^{\gamma(-c_l)}}}. \end{aligned}$$

Thus,  $\lim_{c_h \rightarrow \infty} L(c_h)$  is the finite positive solution of

$$\frac{R_K + \frac{c_l R_C}{2} + \frac{R_C}{2\gamma} + \frac{R_K \gamma}{2} \left( \frac{\text{Ei}[-\gamma c_l]}{\gamma e^{\gamma(-c_l)}} - c_l \frac{-\text{Ei}[-c_l \gamma]}{e^{\gamma(-c_l)}} \right)}{\frac{R_C}{2} - \frac{\text{Ei}[-c_l \gamma]}{e^{\gamma(-c_l)}}} = l.$$

In contrast,  $\lim_{c_h \rightarrow \infty} \frac{v(c_h; c_l, c_h)}{q(c_h; c_l, c_h)}$  takes the value of

$$\begin{aligned} & \lim_{c_h \rightarrow \infty} \frac{R_K + \frac{c_h R_C}{2} - \frac{R_C}{2\gamma} m(c_l, c_h) + \frac{R_K \gamma}{2} \left( \frac{g_h(c_l, c_h)}{\gamma} - c_h f_h(c_l, c_h) \right)}{\frac{R_C}{2} + \frac{R_K \gamma}{2} f_h(c_l, c_h)} \\ &= \lim_{c_h \rightarrow \infty} \frac{\frac{R_K}{c_h} + \frac{R_C}{2} - \frac{R_C}{c_h 2\gamma} + \frac{R_K \gamma}{2} \left( \frac{g_h(c_l, c_h)}{c_h \gamma} - f_h(c_l, c_h) \right)}{\frac{R_C}{2c_h} + \frac{R_K \gamma}{2} \frac{f_h(c_l, c_h)}{c_h}} \\ &= \lim_{c_h \rightarrow \infty} \frac{\frac{R_C}{2} + \frac{R_K \gamma}{2} \left( \frac{g_h(c_l, c_h)}{c_h \gamma} \right)}{\frac{R_K \gamma}{2} \frac{f_h(c_l, c_h)}{c_h}} = \infty. \end{aligned}$$

Hence,  $\frac{v(c_h; c_l, c_h)}{q(c_h; c_l, c_h)}$  grows without bound for any fixed  $c_l$ , and  $\frac{v(c_h; c_l, c_h)}{q(c_h; c_l, c_h)}$  is monotonically increasing in  $c_l$ . As a result, in order to have a solution of  $\lim_{c_h \rightarrow \infty} \frac{v(c_h; c_l, c_h)}{q(c_h; c_l, c_h)} = l$ ,  $c_l$  has to go to zero, implying  $\lim_{c_h \rightarrow \infty} H(c_h) = 0$ .

The two results imply that there is always an intercept  $c_h \in (h, \infty)$  such that  $H(c_h) = L(c_h)$ . This concludes the step proving that (7)–(9), (12)–(13) has a solution.

1 B.1.3. *Step 3: Proof of Lemma B.3* 1

2 We have shown that  $H(h) = h$ . Note also that if  $c_h = c_l$ , then  $\frac{v_h}{q_h} = \frac{v_l}{q_l}$ . This, 2  
3 and the continuity of  $H(\cdot)$  and  $L(\cdot)$  in  $l$ , implies that at the limit  $l \rightarrow h$ , there is 3  
4 a solution of the system (7)–(9), (12)–(13) such that  $c_l^* - c_h^* \rightarrow 0$  and  $c_h^*, c_l^* \rightarrow h$ . 4  
5 Then, the statement comes from  $h < hR_C < R_K$  (as  $R_C > 1$ ). 5  
6

7  
8 B.1.4. *Step 4: Proof of Lemma B.4* 8

9 First we show that  $q(c)$  is always decreasing, and there exists a critical value 9  
10  $\hat{c} \in (c_l, c_h)$  so that  $q''(c) < 0$  for  $c \in (c_l, \hat{c})$  and  $q''(c) > 0$  for  $c \in (\hat{c}, c_h)$ . More- 10  
11 over, for  $c \in (c_l, \hat{c})$  where  $q''(c) < 0$ , we have that  $q'''(c) > 0$ . 11  
12

13 1. To show that  $q' < 0$ , we differentiate the ODE  $0 = \frac{\sigma^2}{2}q'' + \frac{\xi}{2}(R_C + \frac{R_K}{c}) - \xi q$  13  
14 again to reach 14

$$15 \quad (B.4) \quad 0 = \frac{\sigma^2}{2}q''' - \frac{\xi R_K}{2c^2} - \xi q'. \quad 15$$

16 Due to boundary conditions, we have at both ends  $c_l^*$  and  $c_h^*$ , the function  $q'(c)$  16  
17 equals zero and its second derivative  $\frac{\sigma^2}{2}q''' = \frac{\xi R_K}{2c^2} > 0$ . Suppose to the contrary 17  
18 that  $q'(\tilde{c}) > 0$  for some point  $\tilde{c} \in (c_l, c_h)$ ; then we can pick  $\tilde{c}$  so that  $q'(\tilde{c}) >$  18  
19 0 and  $q'''(\tilde{c}) = 0$  (otherwise the function  $q'(\cdot)$  is zero at one end, is convex 19  
20 globally, and thus never comes back to zero at the other end). But because 20  
21  $\frac{\sigma^2}{2}q'''(\tilde{c}) = \frac{\xi R_K}{2\tilde{c}^2} + \xi q'(\tilde{c}) > 0$ , contradiction. This proves that  $q' < 0$ . 21  
22

23 2. We know that  $q''(c_l) < 0$  and  $q''(c_h) > 0$ , and therefore there exists  $\hat{c}$  so 23  
24 that  $q''(\hat{c}) = 0$ . We show this point is unique. Because  $0 = \frac{\sigma^2}{2}q'' + \frac{\xi}{2}(R_C + \frac{R_K}{c}) -$  24  
25  $\xi q$ , we have  $0 = \frac{\sigma^2}{2}q''' - \frac{\xi R_K}{2c^2} - \xi q'$ , and 25  
26

$$27 \quad (B.5) \quad 0 = \frac{\sigma^2}{2}q'''' + \frac{\xi R_K}{c^3} - \xi q''. \quad 27$$

28 Suppose we have multiple solutions for  $q''(\hat{c}) = 0$ . Clearly, it is impossible to 28  
29 have  $q''(\hat{c}) = 0$  but  $q''(\hat{c}-) > 0$  and  $q''(\hat{c}+) > 0$ ; otherwise  $q''''(\hat{c}) > 0$  which 29  
30 contradicts (B.5). Then there must exist two points  $c_1 > \hat{c}$  and  $c_2 > c_1 > \hat{c}$  such 30  
31 that  $q''(c_1) = 0$ ,  $q''(c_2) < 0$  and  $q''''(c_2) > 0$ , but  $q''(c) < 0$  for  $c \in (c_1, c_2)$ . This 31  
32 implies that  $\frac{\sigma^2}{2}q''''(c_1) = -\frac{\xi R_K}{c_1^3} + \xi q''(c_1) < 0$ . As a result, there exists another 32  
33 point  $c_3 \in (c_1, c_2)$  so that  $q''''(c_3) = 0$  with  $q''(c_3) < 0$ . But this contradicts (B.5). 33

34 3. Now we show that for  $c \in (c_l, \hat{c})$  with  $q''(c) < 0$ , we have  $q'''(c) > 0$ , that 34  
35 is,  $q''(c)$  is increasing. Suppose not. Since  $q''(c_l) > 0$  so that  $q''(c)$  is increasing 35  
36 at the beginning, there must exist some reflecting point  $c_4$  for the function  $q''$  36  
37 so that  $q''''(c_4) = 0$ . But because  $q''(c_4) < 0$ , it contradicts (B.5). 37  
38

39 Second, we show that  $v(c)$  is increasing if  $h - l$  is sufficiently small. 39  
40  
41  
42  
43  
44

1 1. We show that if  $v''(c_l) > 0$ , then  $v(c)$  is increasing in  $c$ . Let  $F(c) \equiv v'(c)$ ,  
2 so that

$$3 \quad 4 \quad 5 \quad 6 \quad 0 = q''\sigma^2 + \frac{\sigma^2}{2}F'' + \frac{\xi}{2}R_C - \xi F$$

7 with boundary conditions that  $F(c_l) = F(c_h) = 0$ . The assumption  $v''(c_l) > 0$   
8 implies that  $F'(c_l) > 0$ . Thus, if there are some points with  $F(c) < 0$  in the  
9 range of  $(c_l, c_h)$ , then we can find two points  $c_1$  and  $c_2$  (a maximum and a  
10 minimum) so that  $c_1 < c_2$  but  $F''(c_1) < 0$ ,  $F''(c_2) > 0$ ,  $F'(c_1) = F'(c_2) = 0$  and  
11  $F(c_1) > 0 > F(c_2)$ . We can apply the ODE to these two points:

$$12 \quad 13 \quad 14 \quad 15 \quad 0 = q''(c_1)\sigma^2 + \frac{\sigma^2}{2}F''(c_1) + \frac{\xi}{2}R_C - \xi F(c_1),$$

$$16 \quad 17 \quad 18 \quad 0 = q''(c_2)\sigma^2 + \frac{\sigma^2}{2}F''(c_2) + \frac{\xi}{2}R_C - \xi F(c_2).$$

19 The second equation implies that  $q''(c_2) < 0$ , which implies that  $c_1 < c_2 < \hat{c}$ .  
20 However, the above two equations also imply that

$$21 \quad 22 \quad 23 \quad 24 \quad q''(c_1)\sigma^2 > \frac{\xi}{2}R_C > q''(c_2)\sigma^2,$$

25 in contradiction with the previous lemma which shows that  $q''$  is increasing over  
26  $[c_l, \hat{c}]$ .

27 2. Now we show that if  $h - l$  is sufficiently small, then  $v''(c_l) > 0$ ; with the  
28 first result, we obtain our claim. From our ODE,

$$29 \quad 30 \quad 31 \quad 32 \quad 33 \quad 34 \quad 35 \quad v''(c_l) = -\frac{\xi}{\sigma^2}2\left(\frac{(R_C c_l + R_K)}{2} - v(c_l)\right) \\
= \frac{\xi}{\sigma^2}2\left(\frac{R_K}{2} + \frac{R_C}{2\gamma}h(c_l, c_h) + \frac{R_K \xi}{\gamma \sigma^2}\left(\frac{g_l(c_l, c_h)}{\gamma} - c_l f_l(c_l, c_h)\right)\right).$$

36 We know that as  $h - l \rightarrow 0$ ,  $c_h - c_l \rightarrow 0$ . We will prove the statement by show-  
37 ing that (1)  $\lim_{c_l \rightarrow c_h} \left(\frac{(R_C c_l + R_K)}{2} - v(c_l)\right) = 0$ , because  $\lim_{c_l \rightarrow c_h} \left(\frac{(R_C c_l + R_K)}{2} - v(c_l)\right)$   
38 equals

$$39 \quad 40 \quad 41 \quad 42 \quad 43 \quad 44 \quad \lim_{c_l \rightarrow c_h} \left(\frac{R_K}{2} + \frac{R_C}{2\gamma}h(c_l, c_h) + \frac{R_K \gamma}{2}\left(\frac{g_l(c_l, c_h)}{\gamma} - c_l f_l(c_l, c_h)\right)\right) \\
= \frac{R_K}{2} + 0 + \frac{R_K \xi}{\gamma \sigma^2}\left(0 - \frac{1}{\gamma}\right) = 0$$

1 and (2)  $\lim_{c_l \rightarrow c_h} \partial(\frac{R_C c_l + R_K}{2} - v(c_l)) / \partial c_l = \lim_{c_l \rightarrow c_h} \partial(\frac{R_C}{2\gamma} h(c_l, c_h) + \frac{R_K \gamma}{2} (\frac{g_l(c_l, c_h)}{\gamma} -$   
2  $c_l f_l(c_l, c_h)) / \partial c_l < 0$ , because it equals

$$\begin{aligned}
& \lim_{c_l \rightarrow c_h} \left( -\frac{R_C e^{\gamma(c_h - c_l)}}{(e^{\gamma(c_h - c_l)} + 1)^2} \right. \\
& \quad \left. + \frac{R_K \gamma}{2} \left( \frac{1}{\gamma c_l} + \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} g_l - \frac{(e^{2\gamma c_h} + e^{2\gamma c_l})}{(e^{2\gamma c_h} - e^{2\gamma c_l})} (c_l \gamma f_l - 1) \right) \right) \\
& = -R_C \frac{1}{(1+1)^2} + \frac{R_K \gamma}{2} \left( \frac{1}{\gamma c_h} - \frac{1}{2\gamma c_h} - \frac{1}{2\gamma c_h} \right) = -\frac{R_C}{4} < 0.
\end{aligned}$$

13 These two statements imply that if  $c_h - c_l$  is small enough, then  $v''(c_l) >$   
14  $\lim_{c_l \rightarrow c_h} v''(c_l) = 0$ .

## 17 B.2. Proof of the Second Part of Proposition 5

19 The result  $c_h^* > h$  is a consequence of the fact that we defined  $H(c_h)$  as the  
20 unique  $c_l$  solving  $\frac{v_h(c_l, c_h)}{q_h(c_l, c_h)} = h$  when  $c_h > h$  (see part 4 in Section B.1.2).

21 For the result  $c_l^* \leq l$ , consider the possibility that  $c_l^* > l$ . The following lemma  
22 states that, in this case,  $p''(c_l^*) < 0$ . This implies that this is not an equilibrium.  
23 To see this, we have  $p'(c_l^*) = 0$  by the boundary conditions  $v'(c_l^*) = q'(c_l^*) = 0$ .  
24 Thus  $p''(c_l^*) < 0$ , combined with  $p(c_l^*) = l$  and  $p'(c_l^*) = 0$ , would imply that  
25  $p(c) < l$  for  $c$  sufficiently close to  $c_l^*$ .

27 LEMMA B.5: *The sign of  $p''(c_l^*)$  is the same as that of  $l - c_l^*$ .*

30 PROOF: Simple algebra implies that

$$\begin{aligned}
p''(c_l^*) &= \left( \frac{v'q - q'v}{q^2} \right)' \\
&= \frac{(v''q + v'q' - (q''v + v'q'))}{q^2} - 2q^{-3}(v'q - q'v) \\
&= \frac{v''q - q''v}{q^2} \\
&= \left( \left( -\frac{\xi}{2}(R_C c_l^* + R_K) + \xi l q(c_l^*) \right) \frac{2}{\sigma^2} q \right. \\
& \quad \left. - \left( -\frac{\xi}{2}(R_C c_l^* + R_K) + \xi c_l^* q(c_l^*) \right) \frac{2}{\sigma^2} v \right) / q^2
\end{aligned}$$

$$\begin{aligned}
&= \left( \left( -\frac{\xi}{2}(R_C c_l^* + R_K) + \xi l q(c_l^*) + \xi c_l^* q(c_l^*) - \xi c_l^* q(c_l^*) \right) \frac{2}{\sigma^2} q \right. \\
&\quad \left. - \left( -\frac{\xi}{2}(R_C c_l^* + R_K) + \xi c_l^* q(c_l^*) \right) \frac{2}{\sigma^2 c_l^*} v \right) / q^2 \\
&= \left( \left( -\frac{\xi}{2}(R_C c_l^* + R_K) + \xi c_l^* q(c_l^*) \right) \frac{2}{\sigma^2} \left( q - \frac{v}{c_l^*} \right) \right. \\
&\quad \left. + (l - c_l^*) \xi q(c_l^*) \frac{2}{\sigma^2} q \right) / q^2 \\
&= (l - c_l^*) \frac{\frac{1}{c_l^*} \left( \frac{\xi}{2}(R_C c_l^* + R_K) - \xi c_l^* q(c_l^*) \right) \frac{2}{\sigma^2} + \xi q(c_l^*) \frac{2}{\sigma^2}}{q},
\end{aligned}$$

which gives the lemma by noticing that  $q$  is decreasing in  $c$  and the boundary  $q'(c_l^*) = 0$  implies that

$$-\frac{\xi}{2}(R_C c_l^* + R_K) + \xi c_l^* q(c_l^*) \propto q''(c_l^*) < 0. \quad \text{Q.E.D.}$$

The third statement is a consequence of the following lemma.

LEMMA B.6: *We have the following limiting results:*

$$\begin{aligned}
\lim_{\gamma \rightarrow \infty} \gamma f_l &= \frac{1}{c_l}, & \lim_{\gamma \rightarrow \infty} \gamma f_h &= \frac{1}{c_h}, \\
\lim_{\gamma \rightarrow \infty} g_h &= 0, & \lim_{\gamma \rightarrow \infty} g_l &= 0,
\end{aligned}$$

and

$$\lim_{\gamma \rightarrow \infty} c_h^* = h, \quad \lim_{\gamma \rightarrow \infty} c_l^* = l.$$

PROOF: The first four results are based on L'Hôpital's rule. Take the first result for illustration:

$$\begin{aligned}
\lim_{\gamma \rightarrow \infty} \gamma f_l &= \lim_{\gamma \rightarrow \infty} \frac{\gamma(\text{Ei}[-c_h \gamma] - \text{Ei}[-c_l \gamma])}{e^{\gamma(-c_l)}} = \lim_{\gamma \rightarrow \infty} \frac{\text{Ei}[-c_h \gamma] - \text{Ei}[-c_l \gamma]}{\frac{1}{\gamma} e^{\gamma(-c_l)}} \\
&= \lim_{\gamma \rightarrow \infty} \frac{\frac{e^{-c_h \gamma}}{\gamma} - \frac{e^{-c_l \gamma}}{\gamma}}{-\frac{1}{\gamma^2} e^{\gamma(-c_l)} + \frac{(-c_l)}{\gamma} e^{\gamma(-c_l)}} = \lim_{\gamma \rightarrow \infty} \frac{-e^{-c_l \gamma} / \gamma}{\frac{(-c_l)}{\gamma} e^{\gamma(-c_l)}} = \frac{1}{c_l}.
\end{aligned}$$

1 These four results imply that

$$\begin{aligned}
 \lim_{\gamma \rightarrow \infty} \frac{v_h}{q_h} &= \lim_{\gamma \rightarrow \infty} \left( R_K + \frac{c_h R_C}{2} - \frac{R_C}{2\gamma} m(c_l, c_h) \right. \\
 &\quad \left. + R_K \frac{\gamma}{2} \left( \frac{g_h(c_l, c_h)}{\gamma} - c_h f_h(c_l, c_h) \right) \right) \\
 &\quad / \left( \frac{R_C}{2} + R_K \frac{\gamma}{2} f_h(c_l, c_h) \right) \\
 &= \frac{R_K + \frac{c_h R_C}{2} - R_K \frac{1}{2}}{\frac{R_C}{2} + R_K \frac{1}{2c_h}}.
 \end{aligned}$$

16 Thus, in the limit, the solution of  $\frac{v_h}{q_h} = h$  is the solution for the equation of

$$\frac{R_K + \frac{c_h R_C}{2} - R_K \frac{1}{2}}{\frac{R_C}{2} + R_K \frac{1}{2c_h}} = h,$$

24 which gives  $\lim_{\gamma \rightarrow \infty} c_h^* = h$ . Similarly, the following calculation implies that  
25  $\lim_{\gamma \rightarrow \infty} c_l^* = l$ :

$$\begin{aligned}
 \lim_{\gamma \rightarrow \infty} \frac{v_l}{q_l} &= \lim_{\gamma \rightarrow \infty} \left( R_K + \frac{c_l R_C}{2} + \frac{R_C}{2\gamma} m(c_l, c_h) \right. \\
 &\quad \left. + R_K \frac{\gamma}{2} \left( \frac{g_l(c_l, c_h)}{\gamma} - c_l f_l(c_l, c_h) \right) \right) \\
 &\quad / \left( \frac{R_C}{2} + R_K \frac{\gamma}{2} f_l(c_l, c_h) \right) \\
 &= \frac{R_K + \frac{c_l R_C}{2} + R_K \frac{1}{2}}{\frac{R_C}{2} + R_K \frac{1}{2c_l}}.
 \end{aligned}$$

38 *Q.E.D.*

### 40 B.3. Proof of Proposition 7

42 The proofs of the two statements follow the same logic. Thus, we prove the  
43 first statement in detail and explain the necessary modifications for the second  
44 statement at the end of the proof.

1 Consider the functions  $\tilde{q}(c; q_0, v_0, c_h)$  and  $\tilde{v}(c; q_0, v_0, c_h)$  of  $c$  parameterized  
2 by  $q_0, v_0$ , and  $c_h$ : 2

$$3 \quad (B.6) \quad 0 = \frac{\sigma^2}{2} \tilde{q}''(c) + \frac{\xi}{2} (R_C - \tilde{q}(c)) + \frac{\xi}{2} \left( \frac{R_K}{c} - \tilde{q}(c) \right), \quad 3$$

$$4 \quad (B.7) \quad 0 = \tilde{q}'(c) \sigma^2 + \frac{\sigma^2}{2} \tilde{v}''(c) + \frac{\xi}{2} (R_C c - \tilde{v}(c)) + \frac{\xi}{2} (R_K - \tilde{v}(c)), \quad 4$$

5 and the boundary conditions 5

$$6 \quad (B.8) \quad \tilde{v}'(c_h) = \tilde{q}'(c_h) = 0, \quad 6$$

$$7 \quad (B.9) \quad \tilde{q}(c_0) = q_0, \quad \tilde{v}(c_0) = v_0. \quad 7$$

8 The general solution is 8

$$9 \quad (B.10) \quad \tilde{q}(c) = \frac{R_C}{2} + e^{-c\gamma} A_1 + e^{c\gamma} A_2 \quad 9$$

$$10 \quad + \frac{R_K \gamma - e^{c\gamma} \text{Ei}(-\gamma c) + e^{-c\gamma} \text{Ei}(c\gamma)}{2}, \quad 10$$

$$11 \quad (B.11) \quad \tilde{v}(c) = R_K + \frac{c R_C}{2} + e^{c\gamma} (A_3 - c A_2) - e^{-c\gamma} (A_4 + c A_1) \quad 11$$

$$12 \quad + \frac{c R_K \gamma e^{\gamma c} \text{Ei}(-\gamma c) - e^{-c\gamma} \text{Ei}(\gamma c)}{2} \quad 12$$

$$13 \quad (B.12) \quad = R_K + R_C c + e^{c\gamma} A_3 - e^{-c\gamma} A_4 - c \tilde{q}(c), \quad 13$$

14 where  $A_1$ – $A_4$  (may differ from those in (12) and (13)) are pinned down by  
15 (B.8)–(B.9). We have 15

$$16 \quad \tilde{q}'(c) = -\gamma e^{-c\gamma} A_1 + \gamma e^{c\gamma} A_2 - \frac{R_K \gamma^2 (e^{-c\gamma} \text{Ei}[c\gamma] + e^{c\gamma} \text{Ei}[-c\gamma])}{2}, \quad 16$$

$$17 \quad \tilde{v}'(c) = \frac{R_C}{2} + \frac{R_K \gamma (-e^{-c\gamma} \text{Ei}[c\gamma] + e^{c\gamma} \text{Ei}[-c\gamma])}{2} \quad 17$$

$$18 \quad + \frac{R_K c \gamma^2 (e^{-c\gamma} \text{Ei}[c\gamma] + e^{c\gamma} \text{Ei}[-c\gamma])}{2} \quad 18$$

$$19 \quad + e^{c\gamma} ((-\gamma c - 1) A_2 + \gamma A_3) + e^{-c\gamma} ((\gamma c - 1) A_1 + \gamma A_4). \quad 19$$

20 Define the function  $c_h(q_0, v_0)$  implicitly by  $\tilde{v}(c_h; q_0, v_0, c_h) = h \tilde{q}(c_h;$   
21  $q_0, v_0, c_h)$ , and we are interested in the derivatives 21

$$22 \quad \frac{\partial c_h}{\partial q_0} = -\frac{\tilde{v}'_{q_0} - h \tilde{q}'_{q_0}}{\tilde{v}'_{c_h} - h \tilde{q}'_{c_h}}, \quad \frac{\partial c_h}{\partial v_0} = -\frac{\tilde{v}'_{v_0} - h \tilde{q}'_{v_0}}{\tilde{v}'_{c_h} - h \tilde{q}'_{c_h}}. \quad 22$$



We proceed as follows. First we show that  $\frac{\partial c_h}{\partial q_0} > 0$  and  $\frac{\partial c_h}{\partial v_0} < 0$ . This proves that if  $q_\pi(c_0) \leq q(c_0)$  and  $v_\pi(c_0) \geq v(c_0)$ , then  $c_h^\pi < c_h^*$ , that is, such policies make the overinvestment problem worse. Then we show that this is true even if  $q_\pi(c_0) \leq q(c_0)$  and  $v_\pi(c_0) \leq v(c_0)$ , as long as the policy increases the price at  $c_0$ , that is,  $\frac{v_\pi(c_0)}{q_\pi(c_0)} > \frac{v(c_0)}{q(c_0)}$ .

We start with the following lemmas.

LEMMA B.7: *We have*

$$(B.13) \quad \frac{\partial \tilde{q}(c_h; q_0, v_0, c_h)}{\partial q_0} = \frac{2}{e^{c_h \gamma} e^{-c_0 \gamma} + e^{-c_h \gamma} e^{\gamma c_0}} > 0,$$

$$(B.14) \quad \frac{\partial \tilde{v}(c_h; q_0, v_0, c_h)}{\partial v_0} = \frac{2}{e^{-\gamma(c_h - c_0)} + e^{\gamma(c_h - c_0)}} > 0, \quad \frac{\partial \tilde{q}(c_h; q_0, v_0, c_h)}{\partial v_0} = 0.$$

PROOF: We show (B.13) first. We know that  $\tilde{q}(c_0) = q_0$ , which, based on (B.10), can be written as  $e^{-c_0 \gamma} A_1 + e^{\gamma c_0} A_2 + l_q = q_0$  (where  $l_q$  is independent of  $q_0$ ), which implies

$$(B.15) \quad A_1 = \frac{-l_q - e^{\gamma c_0} A_2 + q_0}{e^{-c_0 \gamma}},$$

and  $\tilde{q}'(c_h) = 0$ , which can be rewritten as  $-e^{-c_h \gamma} \gamma A_1 + e^{c_h \gamma} \gamma A_2 + s_q = 0$  (where  $s_q$  is independent of  $q_0$ ), which implies

$$(B.16) \quad A_2 = \frac{e^{-c_h \gamma} \gamma A_1 - s_q}{e^{c_h \gamma} \gamma} = \frac{e^{-c_h \gamma} \gamma \frac{-l_q - e^{\gamma c_0} A_2 + q_0}{e^{-c_0 \gamma}} - s_q}{e^{c_h \gamma} \gamma}$$

$$\Rightarrow A_2 = \frac{e^{-c_h \gamma} \gamma \frac{-l_q + q_0}{e^{-c_0 \gamma}} - s_q}{(1 + e^{-2c_h \gamma} e^{\gamma 2c_0}) e^{c_h \gamma} \gamma}.$$

Thus, (B.16) and (B.15) imply that

$$(B.17) \quad \frac{\partial A_2}{\partial q_0} = \frac{e^{-c_h \gamma}}{e^{c_h \gamma} e^{-c_0 \gamma} + e^{-c_h \gamma} e^{\gamma c_0}},$$

$$(B.18) \quad \frac{\partial A_1}{\partial q_0} = \frac{1}{e^{-c_0 \gamma}} - e^{\gamma 2c_0} \frac{e^{-c_h \gamma}}{e^{c_h \gamma} e^{-c_0 \gamma} + e^{-c_h \gamma} e^{\gamma c_0}} = \frac{e^{c_h \gamma}}{e^{c_h \gamma} e^{-c_0 \gamma} + e^{-c_h \gamma} e^{\gamma c_0}}.$$

Using (B.10), we obtain our result.

The first result in (B.14) follows similarly. The second result  $\frac{\partial \tilde{q}(c_h; q_0, v_0, c_h)}{\partial v_0} = 0$  comes from the fact that (B.6) and the boundary conditions  $\tilde{q}'(c_h) = 0$  and  $\tilde{q}(c_0) = q_0$  are independent of  $v_0$ . *Q.E.D.*

1 LEMMA B.8: *We have*

$$\begin{aligned}
& \frac{\partial \tilde{v}(c_h; q_0, v_0, c_h)}{\partial q_0} \\
&= 2 \frac{e^{\gamma(c_h - c_0)} - e^{-\gamma(c_h - c_0)} - \gamma(c_h - c_0)(e^{-\gamma(c_h - c_0)} + e^{\gamma(c_h - c_0)})}{\gamma(e^{\gamma c_0} e^{-\gamma c_h} + e^{-\gamma c_0} e^{\gamma c_h})^2} < 0, \\
& \frac{\partial \tilde{v}(c_h; q_0, v_0, c_h)}{\partial q_0} - h \frac{\partial \tilde{q}(c_h; q_0, v_0, c_h)}{\partial q_0} \\
&= 2 \frac{e^{\gamma(c_h - c_0)} - e^{-\gamma(c_h - c_0)} - \gamma(c_h + h - c_0)(e^{-\gamma(c_h - c_0)} + e^{\gamma(c_h - c_0)})}{\gamma(e^{\gamma c_0} e^{-\gamma c_h} + e^{-\gamma c_0} e^{\gamma c_h})^2} < 0.
\end{aligned}$$

14 PROOF: We show the first result. We rewrite  $\tilde{v}(c_0)$  and  $\tilde{v}'(c_h)$  as (as before,  
15 here  $l_{vq}$  and  $s_{vq}$  are independent of  $q_0$ )

$$\begin{aligned}
\tilde{v}(c_0) &= e^{c_0 \gamma} (A_3 - c_0 A_2) - e^{-c_0 \gamma} (A_4 + c_0 A_1) + l_{vq}, \\
\tilde{v}'(c_h) &= s_{vq} + e^{c_h \gamma} ((-\gamma c_h - 1) A_2 + \gamma A_3) \\
&\quad + e^{-c_h \gamma} ((\gamma c_h - 1) A_1 + \gamma A_4).
\end{aligned}$$

22 Thus, the boundary conditions  $\tilde{v}(c_0) = v_0$  and  $\tilde{v}'(c_h) = 0$  imply that

$$\begin{aligned}
A_3 &= c_0 A_2 + e^{-c_0 \gamma} v_0 - e^{-c_0 \gamma} l_{vq} + e^{-2c_0 \gamma} (A_4 + c_0 A_1), \\
A_4 &= - \left( (-e^{\gamma c_h} (\gamma c_h - \gamma c_0 + 1)) A_2 \right. \\
&\quad \left. + (e^{-\gamma c_h} (\gamma c_h - 1) + \gamma c_0 e^{-2\gamma c_0} e^{\gamma c_h}) A_1 \right. \\
&\quad \left. + (\gamma e^{-\gamma c_0} e^{\gamma c_h}) v_0 + (s_{vq} - \gamma e^{-\gamma c_0} e^{\gamma c_h} l_{vq}) \right) \\
&\quad / (\gamma e^{-\gamma c_h} + \gamma e^{-2\gamma c_0} e^{\gamma c_h}).
\end{aligned}$$

33 Thus, using the result in (B.17) and (B.18), one can derive that

$$\frac{\partial A_4}{\partial q_0} = e^{\gamma c_h} \frac{2e^{\gamma c_0} e^{-\gamma c_h} - \gamma c_0 (e^{\gamma c_0} e^{-\gamma c_h} + e^{-\gamma c_0} e^{\gamma c_h})}{\gamma (e^{\gamma c_0} e^{-\gamma c_h} + e^{-\gamma c_0} e^{\gamma c_h})^2}.$$

38 Similarly, it implies that

$$\begin{aligned}
\frac{\partial A_3}{q_0} &= \frac{\partial A_1}{q_0} e^{-2c_0 \gamma} c_0 + \frac{\partial A_2}{q_0} c_0 + \frac{\partial A_4}{q_0} e^{-2c_0 \gamma} \\
&= \frac{2e^{-\gamma c_0} + \gamma c_0 (e^{\gamma c_0} e^{-2\gamma c_h} + e^{-\gamma c_0})}{\gamma (e^{\gamma c_0} e^{-\gamma c_h} + e^{-\gamma c_0} e^{\gamma c_h})^2}.
\end{aligned}$$

Consequently, using (B.12), we have (where we have used (B.13))

$$\begin{aligned} \frac{\partial \tilde{v}(c_h)}{\partial q_0} &= e^{c_h \gamma} \frac{\partial A_3}{q_0} - e^{-c_h \gamma} \frac{\partial A_4}{q_0} - c_h \frac{\partial \tilde{q}(c_h)}{\partial q_0} \\ &= 2 \frac{e^{\gamma(c_h - c_0)} - e^{-\gamma(c_h - c_0)} - \gamma(c_h - c_0)(e^{-\gamma(c_h - c_0)} + e^{\gamma(c_h - c_0)})}{\gamma(e^{\gamma c_0} e^{-\gamma c_h} + e^{-\gamma c_0} e^{\gamma c_h})^2} < 0. \end{aligned}$$

The last inequality comes from the fact that the function  $e^x - e^{-x} - x(e^{-x} + e^x)$  is negative and monotonically decreasing for all  $x > 0$ . The second statement comes directly from the expression for  $\frac{\partial \tilde{q}(c_h)}{\partial q_0}$ . *Q.E.D.*

LEMMA B.9: If  $\frac{v_0}{q_0} < h$ , then  $\tilde{v}(y; q_0, v_0, y) - h\tilde{q}(y; q_0, v_0, y) > 0$ .

PROOF: We parameterize  $c_h$  by  $y$ . The idea is that if the function  $\tilde{v}(y; q_0, v_0, y) - h\tilde{q}(y; q_0, v_0, y)$  is negative at  $y = c_0$  and positive as  $y \rightarrow \infty$ , then there is a  $y = c_h$  so that this function is zero (satisfying the definition of  $c_h$ ) and where the slope of this function is positive, which is the claim of our lemma.

The function  $\tilde{v}(y; q_0, v_0, y) - h\tilde{q}(y; q_0, v_0, y)$  can be solved by imposing the boundary conditions

$$(B.19) \quad \tilde{v}'(y) = \tilde{q}'(y) = 0, \quad \tilde{q}(c_0) = q_0, \quad \tilde{v}(c_0) = v_0,$$

for all  $y \geq c_0$ . Thus, by setting  $y = c_0$ , we must have

$$\tilde{v}(c_0; q_0, v_0, c_0) - h\tilde{q}(c_0; q_0, v_0, c_0) = v_0 - hq_0 < 0,$$

by the condition of the proposition.

Now we show that  $\tilde{v}(y; q_0, v_0, y) - h\tilde{q}(y; q_0, v_0, y) \rightarrow \infty$  as  $y \rightarrow \infty$ . We first calculate  $\lim_{y \rightarrow \infty} \tilde{q}(y; q_0, v_0, y)$  in (B.10). For this, we solve for  $e^{-y\gamma} A_1$  and  $e^{y\gamma} A_2$  from (B.10)–(B.11) and (B.19):

$$\begin{aligned} e^{-y\gamma} A_1 &= \frac{q_0 - \frac{R_C}{2} + e^{(c_0 - y)\gamma} \frac{R_K M'(y)}{2} - \frac{R_K \gamma}{2} M(c_0)}{e^{(y - c_0)\gamma} + e^{\gamma(c_0 - y)}}, \\ e^{y\gamma} A_2 &= \frac{q_0 - \frac{R_C}{2} - e^{(y - c_0)\gamma} \frac{R_K \gamma M'(y)}{2} - \frac{R_K \gamma}{2} M(c_0)}{e^{(y - c_0)\gamma} + e^{\gamma(c_0 - y)}}, \end{aligned}$$

where  $M(y) \equiv -e^{\gamma y} \text{Ei}[-\gamma y] + e^{-\gamma y} \text{Ei}[\gamma y]$ . Using  $\lim_{y \rightarrow \infty} M'(y) = 0$ , it is easy to show that  $\lim_{y \rightarrow \infty} e^{y\gamma} A_2 = \lim_{y \rightarrow \infty} e^{-y\gamma} A_1 = 0$ , which implies that  $\lim_{y \rightarrow \infty} \tilde{q}(y; q_0, v_0, y) = \frac{R_C}{2}$  in (B.10). A similar argument implies that  $\lim_{c \rightarrow \infty} \tilde{v}(c; q_0, v_0, c) = \infty$ . Thus,  $\tilde{v}(c; q_0, v_0, c) - h\tilde{q}(c; q_0, v_0, c) = \infty$ . This proves the statement. *Q.E.D.*

Putting together the above three lemmas, we have

$$\frac{\partial c_h}{\partial q_0} = -\frac{\tilde{v}'_{q_0} - h\tilde{q}'_{q_0}}{\tilde{v}'_{c_h} - h\tilde{q}'_{c_h}} > 0 \quad \text{and} \quad \frac{\partial c_h}{\partial v_0} = -\frac{\tilde{v}'_{v_0} - h\tilde{q}'_{v_0}}{\tilde{v}'_{c_h} - h\tilde{q}'_{c_h}} < 0.$$

This implies that  $c_h^\pi < c_h^*$  whenever  $q_\pi(c_0) \leq q(c_0)$  and  $v_\pi(c_0) \geq v(c_0)$ .

For the last step, as  $\frac{\partial c_h}{\partial v_0} = -\frac{\tilde{v}'_{v_0} - h\tilde{q}'_{v_0}}{\tilde{v}'_{c_h} - h\tilde{q}'_{c_h}} < 0$ , it suffices to show that this result holds for the worst  $v_0$  drop to maintain  $p_0$ , that is,  $v_0$  and  $q_0$  decrease proportionally so  $v_0/q_0$  remains at constant.

To this end, we consider decreasing  $q_0$  to  $\bar{q}_0 = q_0 - \varepsilon$  where  $\varepsilon$  is very small. To make sure that  $\frac{\bar{v}_0}{\bar{q}_0} = \frac{v_0}{q_0}$ , we need that  $\bar{v}_0 = v_0 - a\varepsilon$  where  $a = \frac{v_0}{q_0}$ . Let us refer to all the objects after the change with the bar. Our goal is to show that  $\bar{v}(c_h)/\bar{q}(c_h)$  would increase; then  $\tilde{v}'_{c_h} - h\tilde{q}'_{c_h} > 0$  implies that  $c_h^\pi < c_h^*$ . Using the first two lemmas above, we have (denoting  $x \equiv (c_h - c_0)\gamma$ )

$$\begin{aligned} \bar{q}(c_h) &= \tilde{q}(c_h) - \varepsilon \frac{2}{e^x + e^{-x}}, \\ \bar{v}(c_h) &= \tilde{v}(c_h) - 2\varepsilon \frac{e^x - e^{-x} - x(e^{-x} + e^x)}{\gamma(e^x + e^{-x})^2} - \frac{v_0}{q_0} \frac{2\varepsilon}{e^x + e^{-x}}. \end{aligned}$$

Hence, for sufficiently small  $\varepsilon$ , we have (up to the first order)

$$\begin{aligned} \text{(B.20)} \quad \frac{\bar{v}(c_h)}{\bar{q}(c_h)} &= \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} - \frac{2\varepsilon}{\tilde{q}(c_h)} \left( \frac{e^x - e^{-x} - x(e^{-x} + e^x)}{\gamma(e^x + e^{-x})^2} + \frac{v_0}{q_0} \frac{1}{e^x + e^{-x}} \right) \\ &\quad + \frac{\tilde{v}(c_h)}{\tilde{q}^2(c_h)} \frac{2\varepsilon}{e^x + e^{-x}} \\ &= \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} - \frac{2\varepsilon}{\tilde{q}(c_h)} \left( \frac{e^x - e^{-x} - x(e^{-x} + e^x)}{\gamma(e^x + e^{-x})^2} + \frac{\left( \frac{v_0}{q_0} - \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} \right)}{e^x + e^{-x}} \right) \\ &> \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)}. \end{aligned}$$

Here, the third inequality in (B.20) is because the term  $e^x - e^{-x} - x(e^{-x} + e^x) < 0$  for all  $x > 0$  and  $\frac{v_0}{q_0} - \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)}$  is strictly negative because  $\frac{v_0}{q_0} < \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} = h$ ; hence, the first-order impact of decreasing  $q_0$  is an increase in  $\bar{v}(c_h)/\bar{q}(c_h)$ . Because the above argument holds for any  $v_0$  and  $q_0$ , tracing out the first-order effect implies that any intervention which lowers cash value but keeps capital price unchanged will lower  $\frac{\bar{v}(c_h)}{\bar{q}(c_h)}$ . Compared to that change, an increase in  $v_0$  just decreases  $c_h^\pi$  further. That concludes our proof.

1 The second statement follows the same steps with the following modifica- 1  
2 tions. Each  $c_h$  has to be changed to  $c_l$  and each  $h$  has to be changed to  $l$  2  
3 at every point of the proof. Then the first lemma remains the same, the first 3  
4 statement in the second lemma changes to  $\frac{\partial \tilde{v}(c_l; q_0, v_0, c_l)}{\partial q_0} > 0$ , while the second 4  
5 statement does not change. Also, in the proof of the first statement, we use 5  
6 that  $e^x - e^{-x} - x(e^{-x} + e^x) > 0$  for all  $x < 0$ , and in the proof of the second 6  
7 statement, we use that  $e^x - e^{-x} - (x + y)(e^{-x} + e^x) < 0$  for all  $x < 0$  and  $y > 0$ . 7  
8 In the last part, we follow the same steps, but the inequality (B.20) in the mod- 8  
9 ified version is switched. This gives that  $c_l^\pi > c_l^*$  under the conditions of the 9  
10 statement. 10  
11

#### 12 B.4. Solution for Price-Floor Policy and Proof of Proposition 8 13

##### 14 B.4.1. Characterizing the Equilibrium With Price-Floor Policy 15

16 We first derive the solutions for price-floor policy. A price-floor policy  $\pi(c)$  16  
17 is defined as 17

$$18 \text{ (B.21) } 0 = q'(c)\sigma^2 + \frac{\sigma^2}{2}v''(c) - v(c) + \frac{\xi}{2}(R_C c + R_K) + c\pi(c), 19$$

$$20 \text{ (B.22) } 0 = \frac{\sigma^2}{2}q'' - q(c) + \frac{\xi}{2}\left(R_C + \frac{R_K}{c}\right) - \pi(c), 21$$

22 so that (1) for  $c \in (c_0, c_h^g]$ ,  $\pi(c) = 0$ , and at the upper investment threshold 24  
25  $p(c_h^g) = h$ ; and (2) for  $c \in [c_l^g, c_0]$ ,  $v(c) = (l + \delta)q(c)$  always. Here,  $v(c)$ ,  $q(c)$ , 25  
26  $\pi(c)$ ,  $c_0$ , and  $c_h^g$  are endogenous. We have the following lemma. 26  
27

28 LEMMA B.10: *Given the lower disinvestment threshold  $c_l^g$ , the solution to the 28  
29 price-floor policy can be calculated as follows. 29*

30 1. *Given the upper investment threshold  $c_h^g$ , first calculate the welfare function 30  
31  $j_g(c) = R_K + R_C c + D_1 e^{-\gamma c} + D_2 e^{\gamma c}$ , where the constants  $D_1$ – $D_2$  are given by the 31  
32 boundary conditions 32  
33*

$$34 j_g(c_l^g; c_h^g) = (c_l^g + l)j_g'(c_l^g) \quad \text{and} \quad j(c_h^g; c_h^g) = (c_h^g + h)j_g'(c_h^g). 34$$

35 2. *For  $c \in (c_0, c_h^g]$ , the capital price and cash price are given by 36  
37*

$$38 v(c) = R_K + \frac{R_C c}{2} + e^{c\gamma}(A_3 - cA_2) - e^{-c\gamma}(A_4 + cA_1) 38$$

$$39 + cR_K \frac{\gamma(e^{\gamma c} \text{Ei}(-\gamma c) - e^{-c\gamma} \text{Ei}(\gamma c))}{2}, 39$$

$$40 q(c) = \frac{R_C}{2} + e^{-c\gamma}A_1 + e^{c\gamma}A_2 + R_K \frac{\gamma - e^{c\gamma} \text{Ei}(-\gamma c) + e^{-c\gamma} \text{Ei}(c\gamma)}{2}. 40$$

1 Here,  $A_4 = -D_1$  and  $A_3 = D_2$ . The other four constants, that is,  $A_1$ – $A_2$ ,  $c_0$ , and  
2  $c_h^s$ , are determined by the following four boundary conditions:

$$3 \quad v'(c_h^s) = 0, \quad q'(c_h^s) = 0,$$

$$4 \quad v(c_0) = (l + \delta)q(c_0), \quad v'(c_0) = (l + \delta)q'(c_0).$$

7 3. For  $c \in [c_l^s, c_0]$ , we have

$$9 \text{ (B.23) } \quad q(c) = \frac{j_g(c)}{l+c} \quad \text{and} \quad v(c) = \frac{l+\delta}{l+c} j_g(c)$$

12 and the taxation is given by

$$14 \quad \pi(c) = \frac{\sigma^2}{2} q'' - \xi q(c) + \frac{\xi}{2} \left( R_C + \frac{R_K}{c} \right) > 0.$$

17 PROOF: The total welfare function  $j(c) = v(c) + cq(c)$  given in step 1 of  
18 Lemma B.10 only depends on the investment/disinvestment policies  $c_l^s$  and  $c_h^s$   
19 (see explanations around equations (18) and (19)). For  $c \in (c_0, c_h^s]$ , there is not  
20 taxation and the derivation is the same as before, except that at the endoge-  
21 nous intervention point  $c_0$ , we are value-matching and smooth-pasting so that  
22 the price is the implemented floor price  $l + \delta$ . Note that, by construction, we  
23 have  $v(c_h^s) = hq(c_h^s)$  (due to  $j(c_h^s) = (c_h^s + h)j'(c_h^s)$ ). For  $c \in [c_l^s, c_0]$ , notice that  
24  $v(c) = (l + \delta)q(c)$  always; (B.23) follows because of  $j_g(c) = v(c) + cq(c) =$   
25  $(l + c)q(c)$ . The endogenous taxation  $\pi(c)$  follows from (B.22). *Q.E.D.*

28 B.4.2. Proof of Proposition 8

29 Now we set  $\delta = 0$  and prove Proposition 8. There are three steps.

31 *Step 1. Rewrite the problem.* Clearly, for  $c \in (c_0, c_h^s]$ , the same structure solu-  
32 tion applies without policy, with the only difference at the lower end  $c_0$  so that  
33  $v'(c_0) = lq'(c_0)$  might not be zero. This allows us to draw a connection between  
34 the equilibrium with policy and the one without. We first show that for  $c_l^s < c_l^*$ ,  
35 the resulting slope at  $c_0$  has to be negative, that is,

$$37 \text{ (B.24) } \quad v'(c_0) = lq'(c_0) < 0.$$

39 To show this, focus on  $c \in [c_l^s, c_0]$ . By  $v(c) = \frac{l}{l+c} j_g(c)$  and boundary condition  
40 of  $j_g(c)$ , we have

$$42 \quad v'(c_l^s) = \frac{l[j_g'(c_l^s)(l + c_l^s) - j_g(c_l^s)]}{(l + c_l^s)^2} = 0.$$

Moreover, since  $j_g''(c) < 0$  (see Proposition 2 and its proof), we have  $[j_g'(c) \times (l + c) - j_g(c)]' = j_g''(c)(l + c) < 0$ . As a result, since  $c_0 > c_l^g$ , we have

$$\text{sign}[v'(c_0)] = \text{sign}[j_g'(c_0)(l + c_0) - j_g(c_0)] < 0.$$

This proves (B.24).

This suggests us to introduce  $\{v(\cdot), q(\cdot), c_0, c_h^g; x\}$  indexed by  $x$  as the solution to the ODE system (10) and (11), with modified boundary conditions

$$\begin{aligned} v'(c_h^g) &= q'(c_h^g) = 0, & v(c_h^g) &= hq(c_h^g), \\ v'(c_0) &= -xl, & q'(c_0) &= -x, & v(c_0) &= lq(c_0). \end{aligned}$$

Here, the parameter  $x > 0$  captures the negative slope of  $v'(c_0) = lq'(c_0) < 0$ . As shown shortly, our key result does not depend on the exact value of  $x$ , which will be determined by predetermined lower disinvestment threshold  $c_l^g$ .

It is easy to show that if  $c_l^g = c_l^*$ , that is, the policy sets the lower disinvestment threshold as the one in the market solution, then  $x = 0$  and we have  $c_0 = c_l^* = c_l^g$  and  $c_h^g = c_h^*$ . Given this result, the claim in Proposition 8 is equivalent to showing that

$$\lim_{\gamma \rightarrow \infty} \frac{\partial c_h}{\partial x} > 0.$$

*Step 2. Solve the new ODE system.* For simplicity, we denote  $c_h^g$  by  $c_h$ . Given  $c_0$  and  $c_h$ , the boundary conditions  $v'(c_h) = q'(c_h) = 0$  and  $v'(c_0) = -xl, q'(c_0) = -x$  imply that

$$\begin{aligned} q(c_0; c_0, x, c_h) &= q(c_l; c_l, c_h)|_{c_l=c_0} + \frac{x(e^{2\gamma c_0} + e^{2\gamma c_h})}{\gamma(e^{2\gamma c_h} - e^{2\gamma c_0})}, \\ q(c_h; c_0, x, c_h) &= q(c_h; c_l, c_h)|_{c_l=c_0} + \frac{2xe^{\gamma(c_0+c_h)}}{\gamma(e^{2\gamma c_h} - e^{2\gamma c_0})}, \\ v(c_0; c_0, x, c_h) &= v(c_l; c_l, c_h)|_{c_l=c_0} + \frac{x(e^{2\gamma c_0}(\gamma l + 1) + e^{2\gamma c_h}(\gamma l - 1))}{\gamma^2(e^{2\gamma c_h} - e^{2\gamma c_0})}, \\ v(c_h; c_0, x, c_h) &= v(c_h; c_l, c_h)|_{c_l=c_0} + \frac{2xe^{\gamma(c_0+c_h)}(c_0 - c_h + l)}{\gamma(e^{2\gamma c_h} - e^{2\gamma c_0})}, \end{aligned}$$

where  $q(c_l; c_l, c_h), q(c_h; c_l, c_h), v(c_l; c_l, c_h), v(c_h; c_l, c_h)$  have been defined above. Then,  $c_0$  and  $c_h$  solve  $F_h(c_0, x, c_h) = F_l(c_0, x, c_h) = 0$  where we define

$$\begin{aligned} F_h(c_0, x, c_h) & \\ & \equiv v(c_h; c_0, x, c_h) - hq(c_h; c_0, x, c_h) \end{aligned}$$

$$\begin{aligned}
&= R_K + \frac{(c_h - h)R_C}{2} - \frac{R_C}{2\gamma} m(c_0, c_h) \\
&\quad + \frac{R_K \gamma}{2} \left( \frac{g_h(c_0, c_h)}{\gamma} - (c_h + h) f_h(c_0, c_h) \right) \\
&\quad + \frac{2x e^{\gamma(c_0 + c_h)} (c_0 - c_h + l)}{\gamma(e^{2\gamma c_h} - e^{2\gamma c_0})} - h \frac{2x e^{\gamma(c_0 + c_h)}}{\gamma(e^{2\gamma c_h} - e^{2\gamma c_0})}, \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
&F_l(c_0, x, c_h) \\
&\equiv v(c_0; c_0, x, c_h) - lq(c_0; c_0, x, c_h) \\
&= R_K + \frac{(c_0 - l)R_C}{2} + \frac{R_C}{2\gamma} m(c_0, c_h) \\
&\quad + \frac{R_K \gamma}{2} \left( \frac{g_l(c_0, c_h)}{\gamma} - (c_0 + l) f_l(c_0, c_h) \right) \\
&\quad + \left( \frac{x(e^{2\gamma c_0}(\gamma l + 1) + e^{2\gamma c_h}(\gamma l - 1))}{\gamma^2(e^{2\gamma c_h} - e^{2\gamma c_0})} - l \frac{x(e^{2\gamma c_0} + e^{2\gamma c_h})}{\gamma(e^{2\gamma c_h} - e^{2\gamma c_0})} \right).
\end{aligned}$$

Simple derivation reveals

$$\begin{aligned}
\frac{\partial F_l}{\partial c_0} &= \frac{R_C}{2} - \frac{R_C}{2} \frac{2e^{(c_0 + c_h)\gamma}}{(e^{c_0\gamma} + e^{c_h\gamma})^2} + \frac{R_K \gamma}{2} \left( \frac{1}{\gamma c_0} + \frac{(e^{2\gamma c_h} + e^{2\gamma c_0})}{(e^{2\gamma c_h} - e^{2\gamma c_0})} g_l \right. \\
&\quad \left. - f_l - (c_0 + l) \frac{(e^{2\gamma c_h} + e^{2\gamma c_0})}{(e^{2\gamma c_h} - e^{2\gamma c_0})} \left( \gamma f_l - \frac{1}{c_0} \right) \right),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial F_h}{\partial c_0} &= \frac{R_C}{2} \frac{2e^{(c_0 + c_h)\gamma}}{(e^{c_0\gamma} + e^{c_h\gamma})^2} + \frac{R_K \gamma}{2} \left( \frac{2g_l(c_h, c_0)}{(e^{\gamma(c_h - c_0)} - e^{-\gamma(c_h - c_0)})} \right. \\
&\quad \left. - (c_h + h) \frac{2\left(\gamma f_l - \frac{1}{c_0}\right)}{(e^{\gamma(c_h - c_0)} - e^{-\gamma(c_h - c_0)})} \right) \\
&\quad + \frac{2x e^{\gamma(c_0 + c_h)}}{\gamma(e^{2\gamma c_h} - e^{2\gamma c_0})} \left( \frac{\gamma(c_0 - c_h + l - h)(e^{2\gamma c_0} + e^{2\gamma c_h})}{(e^{2\gamma c_h} - e^{2\gamma c_0})} + 1 \right),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial F_l}{\partial c_h} &= \frac{R_C}{2} \frac{2e^{(c_0 + c_h)\gamma}}{(e^{c_0\gamma} + e^{c_h\gamma})^2} + \frac{R_K \gamma}{2} \left( -\frac{2g_h}{e^{\gamma(c_h - c_0)} - e^{\gamma(c_0 - c_h)}} \right. \\
&\quad \left. - 2(c_0 + l) \frac{\frac{1}{c_h} - \gamma f_h}{e^{\gamma(c_h - c_0)} - e^{\gamma(c_0 - c_h)}} \right),
\end{aligned}$$



$$\begin{aligned}
\frac{\partial F_h}{\partial c_h} &= \frac{R_C}{2} - \frac{R_C}{2} \frac{2e^{(c_0+c_h)\gamma}}{(e^{c_0\gamma} + e^{c_h\gamma})^2} \\
&+ \frac{R_K\gamma}{2} \left( \frac{1}{\gamma c_h} - \frac{(e^{2\gamma c_h} + e^{2\gamma c_0})}{(e^{2\gamma c_h} - e^{2\gamma c_0})} g_h(c_0, c_h) \right. \\
&- (c_h + h) \frac{(e^{2\gamma c_h} + e^{2\gamma c_0})}{(e^{2\gamma c_h} - e^{2\gamma c_0})} \left( \frac{1}{c_h} - \gamma f_h(c_h, c_0) \right) - f_h(c_0, c_h) \Big) \\
&- \frac{2xe^{\gamma(c_0+c_h)}}{\gamma(e^{2\gamma c_h} - e^{2\gamma c_0})} \left( \frac{\gamma(c_0 - c_h + l - h)(e^{2\gamma c_0} + e^{2\gamma c_h})}{(e^{2\gamma c_h} - e^{2\gamma c_0})} + 1 \right).
\end{aligned}$$

*Step 3. Prove the claim.* Now we are ready to show our desired result  $\lim_{\gamma \rightarrow \infty} \frac{\partial c_h}{\partial x} > 0$ . First of all, it is easy to show that when  $\gamma \rightarrow \infty$ ,  $c_h \rightarrow h$  and  $c_0 \rightarrow l$  are bounded. Cramer's rule (or implicit function theorem) implies

$$\begin{aligned}
\lim_{\gamma \rightarrow \infty} \frac{\partial c_h}{\partial x} &= - \lim_{\gamma \rightarrow \infty} \left| \frac{\frac{\partial F_h}{\partial x}}{\frac{\partial F_l}{\partial x}} \quad \frac{\frac{\partial F_h}{\partial c_0}}{\frac{\partial F_l}{\partial c_0}} \right| \bigg/ \left| \frac{\frac{\partial F_h}{\partial c_h}}{\frac{\partial F_l}{\partial c_h}} \quad \frac{\frac{\partial F_h}{\partial c_0}}{\frac{\partial F_l}{\partial c_0}} \right| \\
&= \lim_{\gamma \rightarrow \infty} \frac{-\frac{\partial F_h}{\partial x} \frac{\partial F_l}{\partial c_0} + \frac{\partial F_h}{\partial c_0} \frac{\partial F_l}{\partial x}}{\frac{\partial F_h}{\partial c_h} \frac{\partial F_l}{\partial c_0} - \frac{\partial F_l}{\partial c_h} \frac{\partial F_h}{\partial c_0}}.
\end{aligned}$$

Focus on the denominator first. It is easy to show that

$$\lim_{\gamma \rightarrow \infty} \frac{\partial F_l}{\partial c_0} = \frac{R_C}{2} + \frac{R_K}{2} \frac{l}{c_0^2}, \quad \lim_{\gamma \rightarrow \infty} \frac{\partial F_h}{\partial c_h} = \frac{R_C}{2} + \frac{R_K h}{2c_h^2},$$

and

$$\lim_{\gamma \rightarrow \infty} \frac{\partial F_h}{\partial c_0} = \lim_{\gamma \rightarrow \infty} \frac{\partial F_l}{\partial c_h} = 0,$$

implying

$$\text{(B.25)} \quad \lim_{\gamma \rightarrow \infty} \frac{\partial c_h}{\partial x} = \frac{\lim_{\gamma \rightarrow \infty} \left( \frac{\partial F_h}{\partial c_0} \frac{\partial F_l}{\partial x} - \frac{\partial F_h}{\partial x} \frac{\partial F_l}{\partial c_0} \right)}{\left( \frac{R_C}{2} + \frac{R_K h}{2c_h^2} \right) \left( \frac{R_C}{2} + \frac{R_K l}{2c_0^2} \right)}.$$

1 For the numerator, since  $\frac{\partial F_l}{\partial x} = -\frac{1}{\gamma^2}$  and  $\frac{\partial F_h}{\partial x} = -\frac{2e^{(c_0+c_h)\gamma}(h-l+(c_h-c_0))}{(e^{\gamma^2 c_h} - e^{\gamma^2 c_0})^\gamma}$ , we can show  
2 the following two limiting results:  
3

$$\begin{aligned}
4 \text{ (B.26)} \quad & \lim_{\gamma \rightarrow \infty} \gamma(e^{\gamma(c_h-c_0)} - e^{-\gamma(c_h-c_0)}) \frac{\partial F_h}{\partial x} \frac{\partial F_l}{\partial c_0} \\
5 & = -2(h-l+(c_h-c_0)) \left( \frac{R_C}{2} + \frac{R_K}{2} \frac{l}{c_0^2} \right)
\end{aligned}$$

6  
7 and  
8

$$\begin{aligned}
9 \text{ (B.27)} \quad & \lim_{\gamma \rightarrow \infty} \gamma(e^{\gamma(c_h-c_0)} - e^{-\gamma(c_h-c_0)}) \frac{\partial F_h}{\partial c_0} \frac{\partial F_l}{\partial x} \\
10 & = \lim_{\gamma \rightarrow \infty} \frac{1}{\gamma} \left( \frac{R_C}{2} \frac{2(e^{\gamma c_h} - e^{\gamma c_0})}{(e^{c_0 \gamma} + e^{c_h \gamma})} \right. \\
11 & \quad + \frac{R_K \gamma}{2} \left( 2g_l(c_h, c_0) - (c_h + h) 2 \left( \gamma f_l - \frac{1}{c_0} \right) \right) \\
12 & \quad \left. + \frac{2x}{\gamma} \left( \frac{\gamma(c_0 - c_h + l - h)(e^{2\gamma c_0} + e^{2\gamma c_h})}{(e^{2\gamma c_h} - e^{2\gamma c_0})} + 1 \right) \right) = 0.
\end{aligned}$$

13 Hence, applying (B.26) and (B.27) to (B.25), we have  
14

$$\begin{aligned}
15 \quad & \lim_{\gamma \rightarrow \infty} \gamma(e^{\gamma(c_h-c_0)} - e^{-\gamma(c_h-c_0)}) \frac{\partial c_h}{\partial x} \\
16 & = \frac{2(h-l+(c_h-c_0)) \left( \frac{R_C}{2} + \frac{R_K}{2} \frac{l}{c_0^2} \right)}{\left( \frac{R_C}{2} + \frac{R_K h}{2c_h^2} \right) \left( \frac{R_C}{2} + \frac{R_K}{2} \frac{l}{c_0^2} \right)} > 0.
\end{aligned}$$

17  
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