Optimal Executive Compensation when Firm Size Follows Geometric Brownian Motion

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This paper studies a continuous-time agency model in which the agent controls the drift of the geometric Brownian motion firm size. The changing firm size generates partial incentives, analogous to awarding the agent equity shares according to her continuation payoff. When the agent is as patient as investors, performance-based stock grants implement the optimal contract. Our model generates a leverage effect on the equity returns, and implies that the agency problem is more severe for smaller firms. That the empirical evidence shows that grants compensation are largely based on the CEO's historical performance—rather than current performance—lends support to our model. (*JEL* G32, D82, E2)

This paper analyzes optimal executive compensation by studying a continuous-time moral hazard problem. The existing continuous-time agency models typically employ the less-appealing arithmetic Brownian motion (ABM) framework that essentially entails a constant firm size. However, the relevance of firm size in the context of agency problems is widely documented.[1] Our model represents a significant departure from the previous literature in that we allow firm size to be time-varying and follow a geometric Brownian motion (GBM). We address the following questions: (i) Does time-varying firm size affect incentive provisions in the optimal contract? (ii) Is the optimal contract under this environment different from the one under the ABM setting? and (iii) How can the resulting optimal contract be implemented?

A large literature studies dynamic contracting under moral hazard. Formally introduced in Spear and Srivastava (1987), the agent's *continuation payoff* has been acknowledged as a powerful tool to serve as the state variable in dynamic programming. However, this literature is reluctant to bring in another state variable to capture the time-varying technology, largely for the sake of

[1] For instance, Gertler and Gilchrist (1994) find that small firms are more constrained when the monetary policy is tightening.

tractability. For instance, typical continuous-time moral hazard models assume an ABM output process (e.g., Holmstrom and Milgrom, 1987; and Sannikov, 2006a). A divergence exists between this specification and the one employed in the standard finance literature (see, among others, Goldstein, Ju, and Leland, 2001). By adopting the GBM framework, this paper makes the first attempt to bridge the gap between the continuous-time agency model and the conventional continuous-time finance literature.

In our model, investors hire an agent for business operation. The firm size process follows a GBM, and the agent controls *firm size* growth through unobservable effort. In contrast, the existing literature (DeMarzo and Sannikov, 2006, DS; Biais et al., 2007, BMPR; DeMarzo and Fishman, 2007, DF) focuses on the setting with constant firm size, in which the agent controls the drift of instantaneous ABM *cash flows*.[2] Later we refer to these models as ABM, as opposed to our GBM model. Relative to the existing literature, this paper highlights how changing firm size affects the agency problem.

In addition, in the ABM models the cash flows are unbounded from below. Consequently, substantial losses can arise during any time interval and, therefore, the agent is always constrained. However, the GBM model has positive cash flows, and we show that in the optimal contract there are absorbing states in which the constraint disappears and the first-best outcome is achieved (see Section 2.3.2). These first-best absorbing states are attained when the agent has a long history of successes, or equivalently, when the firm has experienced rapid growth. Both the role of firm size, and the possibility that the agency issue may be resolved along the optimal path, are realistic features that are present in discrete-time models, but not in the existing continuous-time literature. Our modeling thus advances the continuous-time optimal contracting literature in important ways.

The key tradeoffs in this type of setting (DF, DS, BMPR, and this paper) are as follows. Implementing high effort requires sufficient incentives, which mandate that poor results be met with penalties. As the agent's limited liability precludes negative wages, these penalties will accumulate until inefficient termination is triggered. This implies that incentive provision is potentially costly, and hence the optimal contract provides just enough incentives to induce the agent to exert effort.

Different from ABM models, the time-varying firm size in our GBM setting generates a portion of incentives through the agent's continuation payoff. Intuitively, this mechanism works as if investors grant the agent a number of equity shares according to her current continuation payoff, and this hypothetical inside

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[2] DF study a discrete-time model; DS study a continuous-time model; and BMPR solve the discrete-time model first, then take the result to the continuous-time limit. In their main models, all three papers study the problem in which the agent can secretly divert cash from the current output for her own consumption. Under the ABM setup, the cash-diverting problem is isomorphic to the standard moral hazard problem with binary effort and binary outcome. In our GBM model, since cash flows are predetermined, there is no such equivalence. However, our model is equivalent to the agency problem in which the agent can "steal" the firm's assets (secretly sell part of the firm's plants and pocket the sale proceeds).

stake provides some incentives for the agent when the firm size is changing (see discussion in Section 2.3.3). However, along the optimal path, these incentives are not sufficient to motivate the agent. Therefore, additional incentives are provided in the optimal contract (e.g., through future performance-based stock grants).

Other than the tradeoff between incentive provision and inefficient termination, there is a wedge between two contracting parties: the agent is, at most, as patient as investors. Therefore, exchanging relative consumption timings between these two parties improves efficiency, and the optimal contract pays cash (wage) to the agent as early as possible. However, paying cash earlier to the agent, or setting a lower payment boundary in the employment contract, is potentially costly. The reason is that by reducing the agent's continuation payoff, this might make future inefficient liquidation more likely. As a result, the optimal contract calls for investors to set the optimal cash payment boundary such that the marginal benefit equals the marginal cost. Consistent with DS and BMPR, for the case of a strictly impatient agent where the marginal benefit of paying cash earlier is positive, the payment threshold is a reflecting barrier, and a positive marginal cost of paying cash earlier is maintained.

The novel result in this paper pertains to the case of an equally patient agent under the continuous-time framework. When the agent is equally patient, most discrete-time long-term agency models derive an optimal contract with a first-best absorbing state, as agency issue will be completely resolved when the agent's stake within the contractual relationship becomes sufficiently high (see DF, BMPR, and Albuquerque and Hopenhayn, 2004). However, in the *continuous-time* ABM setting (DS and BMPR), unbounded cash flows imply that future inefficient liquidation is always possible, and the first-best state obtained in the *discrete-time* model (DF and BMPR) disappears. In fact, because an earlier cash payment has zero marginal benefit due to the irrelevance of relative consumption timings, while the marginal cost brought on by future termination is always positive, in the ABM model DS and BMPR find that investors should delay the agent's wage indefinitely to minimize the probability of inefficient liquidation. Consequently, when the agent is as patient as investors, the optimal contract fails to exist in their ABM models (see Section 2.3.2).

In contrast, we derive an optimal contract for the equally patient agent case in our GBM model. When the agent's continuation payoff is sufficiently high, she is granted certain equity shares and works forever in the firm; and in this situation the positive cash flows in the GBM model preclude future inefficient liquidation. Therefore, our GBM setting recovers the interesting absorbing first-best state, but with a mechanism that is distinct from the discrete-time setup studied in DF or BMPR. Furthermore, in this equally patient agent case, we derive a new optimal contract even when it is suboptimal to implement working all the time. Under the latter contract, shirking becomes another absorbing state. This extends the results in DS, who study only the case of an impatient agent.

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Our optimal contract can be implemented through a performance-based compensation scheme: *Incentive Points Plan*. Under this plan, the points trace the agent's scaled (by firm size) continuation payoff, and the agent can redeem those points above a prespecified threshold. Interestingly, in the case of equally patient agents, this plan corresponds to *performance-based stock grants*: once the agent has accumulated enough points, she can convert them to a prescribed number of equity shares. This implementation resembles "performance shares" that are currently used in most long-term incentive plans (see, among others, Frydman and Saks, 2005).[3]

We discuss several interesting implications of our results. Larger firms that experience a better performance history suffer less severe agency problems. And, equity returns exhibit rising volatility when the firm's performance is poor. This "leverage" effect caused by agency problem is more compelling than the one obtained in BMPR, because in their ABM framework, a constant volatility in *levels* could lead to a leverage effect for *returns*, even without the agency problem. Using simulation, we contend that research on CEO pay–performance sensitivity should consider long-term incentives when analyzing executives' remuneration contracts. Empirical evidence that shows that for stock and option grants CEOs are primarily compensated based on their historical achievements rather than their current performance lends support to this paper.

The related literature on long-term agency models includes Sannikov (2006a), who considers an ABM environment with an equally patient risk-averse agent, and allows for a continuum of effort levels from the agent. There, the optimal contract features an upper absorbing retirement state without working, while, here, we find an upper absorbing state where the agent works voluntarily forever.[4] Williams (2006) develops a general theory about the principal–agent model that accommodates both hidden actions and hidden states. Tchistyi (2005) extends DF by allowing for correlated cash flows, and Sannikov (2006b) studies a mixture of moral hazard and adverse selection problem.

The theory of optimal dynamic lending contracts (Albuquerque and Hopenhayn, 2004; Hart and Moore, 1994; and Thomas and Worrall, 1994, etc.) is also related. This strand of literature focuses on the dynamic borrowing constraints caused by the possibility of strategic default from the borrower,[5]

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[3] For executives' long-term compensation components, a recent survey ("2005 CEO Compensation Survey and Trends" conducted by Mercer Human Resource Consulting) documents a trend toward performance shares. From 2003 to 2005, the use of performance shares increases from 18% to 21%, while that of stock options drops from 72% to 52% during the period 2002–2005.

[4] This difference stems from the agent's risk aversion and accompanying income effect, which imply that providing incentives becomes extremely costly when the agent's continuation payoff is sufficiently high. Holmstrom and Milgrom (1987) also analyze a risk-averse agent, where the effort cost is in terms of *monetary* units rather than the agent's *utility* units. This specification (under CARA utility) eliminates the income effect.

[5] In Thomas and Worrall (1994), default can be on both sides, and the more interesting binding constraint comes from the default of the host country rather than the transnational firm. As there are no other agency issues, in that model the host country behaves as if it is the "borrower" in Albuquerque and Hopenhayn (2004).

and there is no interperiod agency problem as modeled in DF or BMPR. For instance, Albuquerque and Hopenhayn (2004) relate the borrowing constraint to the endogenous equity value (the borrower's continuation payoff).[6]

We present the model in Section 1, and characterize the optimal contract in Section 2. Section 3 considers the model's extensions. Section 4 discusses implementations and implications, including an empirical study about long-term grant-performance sensitivity. Finally, Section 5 concludes. Proofs are in the Appendix.

1. The Model

Our basic framework is a continuous-time principal–agent model, in which risk-neutral investors of an infinitely lived firm hire a risk-neutral agent to operate the business. The firm produces cash flow δ_t per unit of time, which evolves according to a GBM

$$d\delta_t = a_t \delta_t dt + \sigma \delta_t dZ_t,$$

where $Z = \{Z_t, \mathcal{F}_t; 0 \le t < \infty\}$ is a standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and $a_t \in \{0, \mu\}$ is the agent's *binary* effort choice. Here, $a_t = \mu > 0$ stands for "working," while $a_t = 0$ stands for "shirking." Investors discount future cash flows at the market interest rate $r > \mu > 0$. Note that if the agent works all the time, then from the view of investors, the firm's first-best value at time t is $\mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} \delta_s ds \right] = \frac{1}{r-\mu} \delta_t$, which follows a GBM process as well.

We interpret the cash flow rate δ_t—which is proportional to the firm's first-best value—as the current firm size. Firm size process $\{\delta\}$ is observable and contractible, while the agent's effort choice a_t is not. The agent derives a positive nonpecuniary private benefit $\phi \delta_t dt$ from shirking, where ϕ is a positive constant. This benefit is proportional to the current firm size, because administering a larger firm requires more effort.[7]

The agent has no initial wealth, and negative wage is ruled out by limited liability. We assume that the agent's reservation value is zero, which ensures

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[6] Based on Holmstrom and Milgrom (1987), there is another active area on the continuous-time contracting problem where CARA utility and ABM processes are usually assumed (e.g., Ou-Yang (2005) with a constant volatility). Their framework differs fundamentally from that of this paper: instead of allowing for interim consumption and endogenous termination, the authors assume a lump-sum payment at the end of an exogenous employment horizon $[0, T]$. More importantly, in contrast to limited liability imposed in this paper where the negative wage is ruled out, that framework allows for a two-way transfer between the principal and the agent. See He (2007) who embeds the CARA agency model into the GBM framework to study the agency impact on the firm's capital structure decision.

[7] The shirking benefit (available only when the agent is hired in the firm) can be interpreted as the negation of the agent's effort cost. Note that this assumption is also consistent with the notion that the agent's private benefit is increasing with the firm size.

the scale invariance property of the model.[8,9] The agent has a discount rate $\gamma \geq r$—that is, the agent is (weakly) less patient than investors. Note that the ABM model in DS or BMPR requires $\gamma > r$ strictly.

Agent's employment starts at $t = 0$, and is terminated when the firm is liquidated. At the time of liquidation, investors recover a value $L\delta_t$ from the firm's assets, and fire the agent. We assume that $L < \frac{1}{r-\mu}$; that is, liquidation is inefficient. Later, we endogenize L by allowing the firm to replace the incumbent agent with a new identical agent (see Section 3.1).

Assume that investors can commit to an employment contract that specifies an endogenous stochastic liquidation time τ, and a right-continuous-left-limit nondecreasing *cumulative* wage process $\{U\} = \{U_t : 0 \leq t \leq \tau\}$. We denote such a contract by $\Pi \equiv \{\{U\}, \tau\}$, where both elements are δ-measurable, and τ could take the value ∞. We impose the usual square-integrable condition on Π as follows:

$$\mathbb{E}\left[\left(\int_0^\tau e^{-\gamma s} dU_s\right)^2\right] < \infty. \tag{1}$$

A contract Π is incentive-compatible if it motivates the agent to work until liquidation; in other words, if $\{a_t^* = \mu : 0 \leq t < \tau\}$ solves the following agent's problem:

$$\max_{a = \{a_t \in \{0,\mu\} : 0 \leq t < \tau\}} \mathbb{E}^a\left[\int_0^\tau e^{-\gamma t}\left(dU_t + \phi\left(1 - \frac{a_t}{\mu}\right)\delta_t dt\right)\right],$$

where $\mathbb{E}^a[\cdot]$ is the expectation operator under the probability measure over $\{\delta\}$ that is induced by any effort process $a = \{a_t \in \{0, \mu\} : 0 \leq t < \tau\}$. We assume that it is optimal to implement working all the time, and verify its optimality in Section 3.2. Therefore in this paper, unless otherwise stated, the expectation operator is under the measure induced by $\{a_t = \mu : 0 \leq t < \tau\}$.

Throughout, we assume that the firm possesses full bargaining power. Denote the set of incentive-compatible contracts as $\mathbb{IC}$, and the firm's problem is

$$\max_{\Pi \in \mathbb{IC}} \mathbb{E}\left[\int_0^\tau e^{-rt}\delta_t dt + e^{-r\tau}L\delta_\tau - \int_0^\tau e^{-rt} dU_t\right].$$

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[8] In the same spirit of Thomas and Worrall (1994); and Albuquerque and Hopenhayn (2004), we can assume that the agent is able to appropriate a fraction of the firm so that her reservation value is $\underline{k}\delta_t$, where $\underline{k}$ is a non-negative constant that is sufficiently small to ensure that "stealing-absconding" is inferior to "behaving" in the optimal contract. This specification can also be interpreted as that the agent with better performance records faces a more favorable outside option. The entire analysis can be conducted by replacing 0 with $\underline{k}$.

[9] This assumption is consistent with the notion of competitive labor markets. Besides, evidence suggests that failed managers are not as competent as other candidates, even if the previous corporation failure is viewed to be beyond the manager's control. Cannella, Fraser, and Lee (1995) find that these "innocent bystander" managers are 63% less likely to find banking posts compared to those at nonfailed banks.

There is no agent's participation constraint in this problem, as the agent enjoys a positive rent once she is hired. Denote the solution for this problem as $\Pi^* = \{\{U^*\}, \tau^*\}$.

2. Model Solution and Optimal Contracting

2.1 Continuation payoff and incentive compatibility

This section gives a key proposition for any incentive-compatible contract $\Pi \in \mathbb{IC}$. Fix the effort process $a = \{a_t = \mu : 0 \leq t < \tau\}$. For any contract Π, define the agent's continuation payoff at time t, as

$$W_t(\Pi) \equiv \mathbb{E}_t\left[\int_t^\tau e^{-\gamma(s-t)} dU_s\right].$$

In words, W_t is the agent's continuation value obtained under Π when she plans to work from t onward.

Define $\lambda \equiv \frac{\phi}{\mu}$, which relates to the minimum incentive required to motivate the agent.[10] Based on Martingale Representation Theorem, the following proposition expresses the evolution of W_t in terms of observable performance δ_t, and provides a necessary and sufficient condition for any contract Π to be incentive-compatible.

Proposition 1. *For any contract $\Pi = \{\{U\}, \tau\}$, there exists a progressively measurable process $\{\sigma_t^W : 0 \leq t < \tau\}$, such that, under working (i.e., $a_t = \mu$ always), the agent's continuation value W_t evolves according to*

$$dW_t = \gamma W_t dt - dU_t + \frac{\sigma_t^W}{\sigma}(d\delta_t - \mu\delta_t dt). \tag{2}$$

The contract $\Pi \in \mathbb{IC}$, i.e., is incentive-compatible, if and only if $\sigma_t^W \geq \lambda\sigma$ for $t \in [0, \tau)$.

Proposition 1 states that the agent's instantaneous compensation—the wage (dU_t) plus the change of continuation payoff (dW_t)—has a predetermined drift part $\gamma W_t dt$ that corresponds to the Promise-Keeping condition in the discrete-time formulation, and a diffusion part

$$\delta_t \sigma_t^W dZ_t = \frac{\sigma_t^W}{\sigma}(d\delta_t - \mu\delta_t dt),$$

which links to her effort choice and provides working incentive. To motivate the agent, the instantaneous volatility of continuation payoff, $\delta_t \sigma_t^W$, must be higher than $\lambda\sigma\delta_t$. To see this, if the agent chooses to shirk, she gains a private

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[10] It is clear that the larger the personal benefit ϕ, the more difficult it is to motivate the agent. But μ matters too; a higher drift makes it easier to detect shirking, hence less incentive is needed.
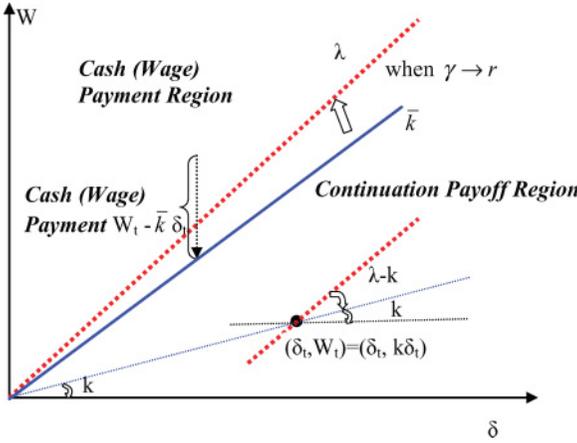
Figure 1
The optimal cash (wage) payment and incentive provision policy
There exists a positive constant $\bar{k} \leq \lambda$, so that it is optimal to pay the agent cash $W_t - \bar{k}\delta_t$ once $W_t > \bar{k}\delta_t$. The incentive compatibility constraint requires a slope λ for the local movement of $(\delta_t, W_t) = (\delta_t, \bar{k}\delta_t)$, while the agent's hypothetical inside stake contributes only a slope $k < \lambda$ due to the diffusion of δ. Consequently, the optimal contract provides additional incentives to fulfill the slope discrepancy $\lambda - k$ (see discussion in Section 2.3.3).

benefit $\phi\delta_t dt$, but loses $\frac{\mu\delta_t\sigma_t^W}{\sigma}dt$ in compensation ($d\delta_t$ becomes driftless under shirking). Thus, she will work if and only if $\frac{\mu\delta_t\sigma_t^W}{\sigma} \geq \phi\delta_t$, or $\sigma_t^W \geq \lambda\sigma$. Therefore, by putting her at risk, the volatility of $dW_t + dU_t$ pushes the agent to exert high effort. In other words, for enough punishing and rewarding, the volatility of the agent's continuation payoff must exceed a certain threshold $\lambda\sigma\delta_t$.

As we will see shortly, in the optimal contract we have $\sigma_t^W = \lambda\sigma$; that is, the incentive compatibility constraint always binds. Geometrically, on the (δ, W) plot in Figure 1, the (local) movement of the state-variable pair (δ, W) (which is determined by the diffusion term) must be as steep as λ.[11]

2.2 Optimality equation and its solution
2.2.1 Optimality equation and boundary conditions. There are two state variables in this model: firm size δ_t, and the agent's continuation payoff W_t. The investors' value function $b(\delta, W) \in \mathbb{C}^2$ (i.e., twice differentiable in both arguments) is the firm's highest expected future profit, given these two state variables. When the agent works all the time, the firm size δ_t evolves as

$$d\delta_t = \mu\delta_t dt + \sigma\delta_t dZ_t.$$

And the agent's continuation payoff W_t follows,

$$dW_t = \gamma W_t dt - dU_t + \delta_t\sigma_t^W dZ_t. \tag{3}$$

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[11] In the optimal contract, dW_t has a diffusion term $\lambda\sigma\delta_t dZ_t$. Therefore $\frac{dW_t}{d\delta_t} \simeq \frac{\lambda\sigma\delta_t dZ_t}{\sigma\delta_t dZ_t} = \lambda$.

As we will verify in Section 2.4, the concavity of the investors' value function implies that the optimal contract provides just enough incentives; i.e., $\sigma_t^W = \lambda\sigma$. Also, similar to DS, the optimal cash (wage) payment policy depends on $\frac{\partial b}{\partial W}$. If $\frac{\partial b}{\partial W} > -1$, then promising one dollar of continuation payoff to the agent costs the firm less than paying one dollar cash. As a result, in this case the firm should hold the cash and promise to pay later.

The tractability of our GBM model hinges on the scale invariance property, which implies that the optimal policy is homogeneous in firm size δ_t.[12] As a result, the investors' value function $b(\delta, W)$ must be of the form $\delta c(k)$, where the agent's scaled continuation payoff $k \equiv W/\delta$ is the only relevant state variable, and $c(\cdot) \in \mathbb{C}^2$ is a univariate smooth function. We call $c(\cdot)$ the investors' scaled value function.

In the Appendix, after writing down the Hamilton–Jacobi–Bellman equation, we find that $c(\cdot)$ must solve the following second-order ordinary differential equation (ODE) when there is no cash payment ($dU = 0$):

$$(r - \mu)c(k) = 1 + (\gamma - \mu)kc'(k) + \tfrac{1}{2}(\lambda - k)^2\sigma^2 c''(k). \tag{4}$$

This equation plays a key role in analyzing the optimal contract; we call it the *Optimality Equation.*

The optimality of cash payment yields two boundary conditions at the upper end. Scale invariance implies that the optimal cash payment barrier $\overline{W}_t$ is linear in δ—i.e., $\overline{W}_t \equiv \overline{k}\delta_t$, where $\overline{k}$ is a positive constant to be solved in the optimal contract. Once W_t sits above $\overline{W}_t$, investors will pay the agent $W_t - \overline{W}_t$ in cash to bring W_t back to $\overline{W}_t$ (see Figure 1). Because paying the agent cash to reduce her continuation payoff W is a barrier control with linear cost, we have the *Smooth-Pasting* condition $\frac{\partial b}{\partial W}\left(\delta_t, \overline{k}\delta_t\right) = -1$, and the *Super-Contact* condition $\frac{\partial^2 b}{\partial W^2}\left(\delta_t, \overline{k}\delta_t\right) = 0$ (see A. Dixit, 1993). In terms of $c(\cdot)$, the conditions are

$$c'\left(\overline{k}\right) = -1; \tag{5}$$
$$c''\left(\overline{k}\right) = 0. \tag{6}$$

Applying these two conditions to Equation (4), we find that at $\overline{k}$, $c(\cdot)$ attaches the function $\frac{1}{r-\mu} - \frac{\gamma-\mu}{r-\mu}k$ with slope -1. We extend $c(\cdot)$ linearly (with slope -1) for $k > \overline{k}$ based on the optimal wage policy (see Figure 2).

Termination delivers another boundary condition at the lower end. Let τ be the first hitting time at which $W_t = 0$. Once this occurs, the agent is fired, and investors liquidate the firm for a surrender value $L\delta_\tau$. Hence,

$$c(0) = L, \tag{7}$$

and $c(\cdot)$ solves Equation (4) with boundary conditions (5), (6), and (7).

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[12] Also, recall that both the shirking benefit $\phi\delta_t dt$ and liquidation value $L\delta_t$ are linear in the firm size, and that the agent's outside option is worth zero.
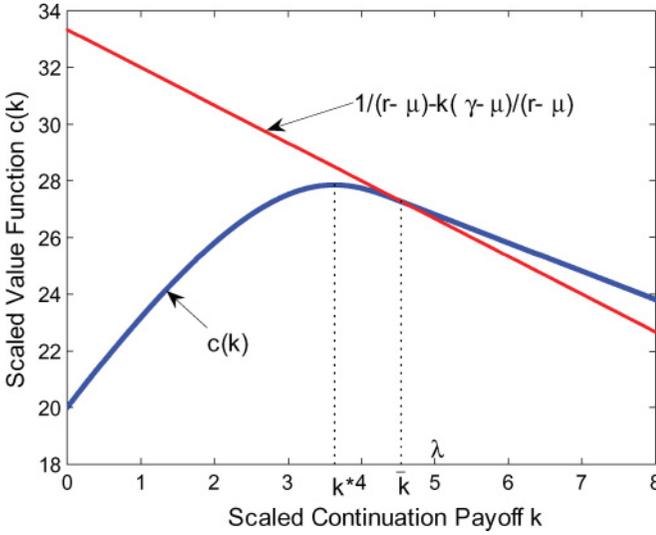
Figure 2
The scaled value function $c(\cdot)$ for the case $\gamma > r$ (an impatient agent)
Parameters are $r = 4\%$, $\gamma = 5\%$, $\mu = 1\%$, $\sigma^2 = 10\%$, $\lambda = 5$, $L = 20$. $\bar{k} < \lambda$ is a reflecting barrier. $c(\cdot)$ attaches $\frac{1}{r-\mu} - \frac{\gamma-\mu}{r-\mu}k$ with a slope -1, and is extended for $k > \bar{k}$ with a slope -1.

In light of the Feynman–Kac formula, $c(k)$ can be written in its probabilistic representation (see Lemma 2 in the Appendix)[13]

$$c(k) = \mathbb{E}^{k_0=k}\left[\int_0^\tau e^{-(r-\mu)t}dt + e^{-(r-\mu)\tau}L - \int_0^\tau e^{-(r-\mu)t}du_t\right],$$

where the process $\{k\}$ evolves according to

$$dk_t = (\gamma - \mu)k_t dt + (\lambda - k_t)\sigma dZ_t - du_t; \tag{8}$$

and u_t is a nondecreasing process that reflects k_t at $\bar{k}$.[14] Intuitively, the scaled value function $c(k)$ equals expected scaled cash flows $1dt$, plus the scaled liquidation value L, minus scaled wages, all discounted by the effective discount rate $r - \mu$.

We define the first-best scaled value function $c^{fb}(k) \equiv \frac{1}{r-\mu} - k$ for later references.

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[13] Note that this form does not require the Super-Contact condition (6), an important fact when we derive the comparative static results in Lemma 3.

[14] An interesting caveat exists regarding the evolution of process k. Similar to the difference between the risk-neutral and physical measures in asset pricing literature, the evolution (8) for k is under an auxiliary measure induced by Equation (4), which annihilates certain drift of k. Under the physical measure, without cash payment, $k_t = W_t/\delta_t$ evolves according to $dk_t = (\gamma - \mu)k_t dt + (k_t - \lambda)\sigma^2 dt + (\lambda - k_t)\sigma dZ_t$. This differs from Equation (8) by $(k_t - \lambda)\sigma^2 dt$ due to the scaling of δ_t (a quadratic covariation between k_t and δ_t). Nevertheless, because we focus on the diffusion part that provides incentives, the drift is of less importance.

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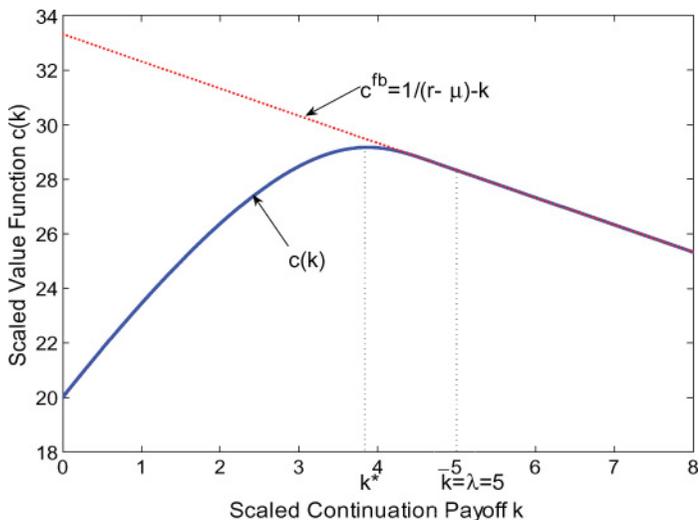
Figure 3
The scaled value function $c(\cdot)$ for the case $\gamma = r$ (an equally patient agent)
Parameters are $r = 4\%, \mu = 1\%, \sigma^2 = 10\%, \lambda = 5, L = 20$. The scaled value function $c(\cdot)$ attaches $c^{fb}(k) = \frac{1}{r-\mu} - k$ smoothly, and $\overline{k} = \lambda$ is an absorbing barrier.

2.2.2 Comparison to ABM setting in DS and BMPR.
As a comparison, under the ABM setting analyzed in DS and BMPR, the agent controls the instantaneous *cash flow dY_t*, which can be written as (when the agent is working)

$$dY_t = \mu dt + \sigma dZ_t.$$

In contrast, in the GBM model, the agent controls the change of *firm size (cash-flow rate) $d\delta_t$*, rather than the predetermined *cash flow $\delta_t dt$*. This distinction necessarily leads to different implementation mechanisms in Section 4.1. Also, in the GBM model the cash flow $\delta_t dt$ is positive, but in the ABM setting dY_t is unbounded from below. As we will see later, this divergence affects the existence of the first-best state in optimal contracting.

In the ABM model, once the agent (with a reservation utility R) shirks to enjoy the private benefit ϕdt, the drift of dY_t drops to 0. As before, define $\lambda = \phi/\mu$. Similar arguments as in Section 2.1 imply that the state variable, which is the agent's continuation payoff W (as opposed to $k = W/\delta$ in our GBM model), evolves according to

$$dW_t = \gamma W_t dt + \lambda \sigma dZ_t - dU_t. \tag{9}$$

Denote $\overline{W}$ as the payment boundary in the optimal contract. When $W \in [R, \overline{W}]$, cash payment $dU = 0$, and the unidimensional value function $b(W)$ satisfies

$$rb(W) = \mu + \gamma W b'(W) + \tfrac{1}{2}\lambda^2 \sigma^2 b''(W), \tag{10}$$

with similar boundary conditions $b'(\overline{W}) = -1, b''(\overline{W}) = 0$, and $b(R) = L$. The optimal contract pays out cash $dU_t > 0$ only when W_t exceeds the reflecting barrier $\overline{W}$. When $W_\tau = R$, the firm is liquidated. Comparing Equation (4) to Equation (10), we immediately discern a difference: because in the GBM setup the drift captures the firm's growth, the parameter μ enters Equation (4) by reducing both parties' discount rates.

The key difference, however, lies in the second-order term in these two equations: in Equation (4), the coefficient of the second-order term is $(\lambda - k_t)\sigma$, while in Equation (10) it is $\lambda\sigma$. Because the second-order term corresponds to the diffusion part of respective state variables, and the diffusion in turn captures incentives, two important implications ensue. First, note that according to Equation (1), the required incentives are $\lambda\sigma$, while only the $(\lambda - k_t)\sigma$ portion of incentives would lead to future inefficient liquidation—it is the diffusion that causes k_t to hit the liquidation boundary 0. This fact suggests that in the GBM model, the scaled continuation payoff k_t itself generates some "costless" incentives along the optimal path. In Section 2.3.3, we will see that this finding stems from the time-varying firm size in our GBM setting.

Second, this state-dependent diffusion $(\lambda - k_t)\sigma$ in Equation (4) leads to one significant result that contrasts drastically with the ABM model. Unlike Equation (10), Equation (4) involves a singular point (when $k_t = \lambda$, the diffusion of k_t dies), which corresponds to the absorbing state in which a sufficiently high inside stake drives the agent to work voluntarily (see Section 2.3.2). In fact, Section 2.3.2 shows that when the agent is equally patient, this absorbing state, as a part of the optimal contract, achieves the first-best result.

2.3 The optimal contract

Consistent with BMPR, we find that the optimal contract differs for the two cases $\gamma > r$ and $\gamma = r$. As we discussed in the introductory section, postponing the agent's consumption alleviates the agency problem, and thereby improves efficiency. However, if the agent is impatient, then postponing consumption will entail a cost, as the first-best result has the agent consume as early as possible. In contrast, for an equally patient agent, the payment delay is absolutely free. Therefore, whether the cost is present or not determines the structure of optimal contract.

2.3.1 When $\gamma > r$ (impatient agent). If the agent is impatient, earlier wage payments tend to be optimal, and the optimal payment boundary $\overline{k}$ is always below λ (see Figure 2), as stated in the next proposition.

Proposition 2. *When $\gamma > r$, we have $\overline{k} < \lambda$. There exists a unique solution $c(\cdot)$ to Equation (4) with boundary conditions (5), (6), and (7), and the solution is strictly concave on $[0, \overline{k}]$.*

Our resulting optimal contract can be described as follows. At $t = 0$, the firm hires an agent by offering her a continuation payoff $W_0 = k^* \delta_0$, and promises the evolution of her continuation payoff W_t to be

$$dW_t = \gamma W_t dt + \lambda \left(d\delta_t - \mu \delta_t dt \right). \tag{11}$$

When W_t achieves $\bar{k}\delta_t$, investors start paying the agent cash to maintain her continuation payoff W_t at $\bar{k}\delta_t$. When W hits zero at time τ, investors fire the agent and liquidate the firm.

This optimal contract is quite similar to that of DS and BMPR, except that the cash payment threshold $\bar{k}\delta_t$ is state-dependent, with an upper bound $\lambda\delta_t$. In addition, the result of $\bar{k} < \lambda$ implies a nondying diffusion of k_t, which suggests that along the optimal path, it is always possible to have k_t drop to zero if the agent's future performance is poor. This result is due to the gap between the two parties' patience levels. To see this, first note that the agent's impatience implies a strictly positive marginal benefit of paying cash to the agent earlier, or setting a lower payment threshold $\bar{k}$. However, in the Appendix (Lemma 4, part 3), we show that the *marginal cost* of setting a lower payment boundary (brought on by future inefficient liquidation) is zero at the absorbing state λ, and positive for $\bar{k} < \lambda$. To equate the marginal cost with the marginal benefit, the firm should pay the agent cash before k_t reaches λ. This tradeoff never exists for an equally patient agent, as we will discuss in the next section.

2.3.2 When $\gamma = r$ (equally patient agent). When the issue of relative consumption timing is absent, postponing cash payments has zero cost. As a result, $\bar{k} = \lambda$ is the optimal payment boundary, which is higher than the one obtained when $\gamma > r$ (see Figure 3). In fact, λ is the first-best absorbing state, and there will be no further chance of liquidation once k_t attains λ.

Proposition 3. *When $\gamma = r$, without loss of generality, we have $\bar{k} = \lambda$. There exists a unique solution $c(\cdot)$ to Equation (4) with boundary conditions (5), (6), and (7), and the solution is strictly concave on $[0, \lambda]$.*

Investors start the employment at $W_0 = k^* \delta_0$, and let the agent's continuation payoff evolve according to Equation (11). If W_t falls to zero, then investors liquidate the firm and fire the agent. However, once good fortune drives W_t to attain $\lambda\delta_t$, the agent receives cash payment $dU_s = \lambda (r - \mu) \delta_s ds$ for $s \geq t$, and, as an absorbing state, her continuation payoff W_s stays at $\lambda\delta_s > 0$ forever (so $k_s = \lambda$ from then on). Note that it is equivalent to granting $\lambda (r - \mu)$ shares to the agent, and these shares provide required incentives to motivate the agent.

We observe a key difference between our result and the one obtained in DS and BMPR who consider the impatient agent case only. Under their ABM setting, for however high the agent's continuation payoff, in any time interval, W_t can reach the agent's fixed outside option R due to unbounded Brownian

increments (check Equation (9)), and the marginal cost of setting a lower cash payment barrier $\overline{W}$ is always positive. In other words, in their ABM model the agent is always constrained, and there is always a gain from relaxing the constraint even further.[15] However, because the benefit of paying cash earlier is absent when $\gamma = r$, investors should postpone the agent's wage indefinitely, which renders the nonexistence of the optimal contract.[16] Under our GBM setup, because the firm's cash flows stay positive, we obtain an optimal contract with a first-best absorbing state $k = \lambda$ in which the marginal cost of paying cash early is zero. In this state, the agent with enough equity shares works voluntarily, and future liquidation never occurs.

Note that most discrete-time agency models, including those in DF and BMPR, feature a first-best absorbing state in the optimal contract—as agency issue will be completely resolved once the agent's continuation payoff becomes sufficiently high. The driving forces, however, are different. For instance, in the binomial model in BMPR, given the time-step size, the per-period loss is bounded. Therefore, there exists an upper first-best absorbing state in which the firm accumulates a large fund whose interest is sufficient to cover all potential future losses. When the time-step size goes to zero as the cash-flow process converges to an ABM, this absorbing state explodes. In contrast, we derive a bounded absorbing state (λ) in the GBM model.

2.3.3 Discussion: continuation payoff and inside stake.

This section provides economic intuition for the optimal contract. We first discuss the optimal incentive provision policy, and, for simplicity, we focus on the equally patient agent case ($\gamma = r$). The same argument applies to the $\gamma > r$ case.

It is interesting to note that due to the time-varying firm size in the GBM framework, the agent's continuation payoff can generate a portion of incentives. Consider the following thought experiment. Suppose that at time t, investors decide to reward the agent with equity shares according to her continuation payoff. Note that the agent values α fraction of the firm as $\frac{\alpha \delta_t}{r - \mu}$ (given that she is working all the time). Therefore, to fulfill W_t the agent is qualified to

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[15] In fact, the GBM model with positive cash flows also helps us disentangle the agency problem from the agent's limited-cash-reserve constraint. Note that in the ABM model, the costly termination is caused not only by the agent's moral hazard problem, but also by the fact that she only has a finitely "deep pocket." Specifically, even if the agent (given a fixed cash reserve) runs the firm as a proprietorship, unbounded cash flows—hence substantial losses—imply that future inefficient liquidation is always possible, and the probability of future liquidation is strictly decreasing in the level of the firm's cash reserve. Clearly, the latter differs from the inefficient punishment in the standard moral hazard literature.

[16] To see this, we have Equation (9) under the ABM setting. Recall $dU_t \geq 0$; therefore for a loss $dZ_t < 0$, W has to drop, and the size of the drop is independent of the level of W. This implies that within any time interval, there is always a positive probability for W to reach the termination boundary R. The higher the continuation payoff W, the lower the liquidation probability, and the higher the efficiency. It implies that the marginal cost of paying cash early is strictly positive. Given this fact, when $\gamma = r$ so that there is zero benefit to pay the agent cash early, the optimal contract should accumulate W as high as possible to approach (but never reaches) the first-best outcome. In other words, any contract given a payment boundary $\overline{W}$ could be improved by setting $\overline{W} + 1 > \overline{W}$, and the wage payments are further delayed. Therefore in the limit $dU_t = 0$ for $0 \leq t < \infty$, thus violating the Promise-Keeping condition ($W_t = 0$ always; investors' promise about future wages is actually void). Note that Sannikov (2006b) imposes a fixed finite life span for the firm, therefore this issue is absent.

own $\alpha = (r - \mu) k_t$ shares of this firm (recall $W_t = k_t \delta_t$), and these shares generate an instantaneous volatility of $\frac{\alpha}{r-\mu} \sigma \delta_t = k_t \sigma \delta_t$.[17] By Proposition 1, when $\alpha \geq \alpha^* \equiv \lambda (r - \mu)$, or $k_t \geq \lambda$, these incentives are sufficient to motivate the agent. Because the agent's continuation payoff obtained from these shares remains positive, there is no future liquidation and the first-best outcome is achieved.

Now, in the optimal contract, before reaching the absorbing state, the agent's scaled continuation payoff k_t is always lower than λ. This implies that only $\alpha_t = (r - \mu) k_t < \alpha^*$ shares can be awarded, if investors decide to do so. Also, as suggested by the optimal wage policy, investors should *wait to reward the agent later* (as $dU = 0$ before k_t reaches λ). Are the incentives described above still present in this scenario? The answer is yes. Imagine that these "hypothetical" shares (which are held by investors at time t) are promised to be delivered at $t + dt$, so that the agent cannot receive any portion of current dividends $\delta_t dt$ yet. Though hypothetical, these shares still generates incentives: because current dividends are in the lower order of dt, when firm size δ_t diffuses, the value of these hypothetical shares exhibits the same volatility $k_t \sigma \delta_t$ as actual shares.[18] Loosely speaking, these hypothetical shares represent the agent's *inside stake* in the firm, but in a *forward-looking* sense. Finally, as required by Proposition 1, the optimal contract Π^* imposes additional incentives $(\lambda - k_t) \sigma \delta_t$ to motivate the agent, and the above argument can be applied to any time s before k_s attains λ.

We illustrate the idea in the previous paragraph graphically on the (δ, W) plot in Figure 1. Fix k; given $W_t = k \delta_t$, the local movement of $(\delta_t, k \delta_t)$ is along a ray with slope k, due to the diffusion of δ_t. In fact, this slope k just captures those incentives generated by the agent's hypothetical inside stake when the firm size δ_t diffuses. The faster the firm grows, the higher the agent's continuation payoff, and the larger the incentives generated by these hypothetical shares.

In Section 2.1, we also observe that on the (δ, W) plot, the incentive compatibility constraint requires a slope $\lambda > k$ for the local movement of (δ_t, W_t), and the agent will shirk if these hypothetical shares are the only incentive scheme available. Therefore, to implement working, investors have to provide additional incentives $(\lambda - k) \sigma \delta_t$ to fill out the slope gap $\lambda - k$, and these incentives constitute the diffusion term of dk_t in Equation (8). Intuitively, these additional incentives are provided by promising larger future stock grants if her subsequent performance is superb, or liquidating the firm otherwise.

These observations lead to implications for the optimal wage policy. First, because the agent with α^* inside shares works voluntarily, when $\gamma = r$ (no relative consumption timing issue), it is the first-best absorbing state. Consequently,

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[17] Note that when $\gamma > r$, these $\alpha = (\gamma - \mu) k_t$ shares generate the same volatility level $k_t \sigma \delta_t$, and the similar argument about *hypothetical* shares (see below) can be applied to this case. Of course, the payment boundary $\overline{k}$ will be lower, as indicated by the optimal contract.

[18] In contrast, in the ABM model the firm size is fixed. Hence these hypothetical shares—without current dividends—cannot generate incentives as in our GBM setup.

for the case of an equally patient agent, granting α^* shares to the agent once W_t reaches $\lambda\delta_t$ must be part of the optimal contract. Second, when the agent is less patient than investors ($\gamma > r$), the first-best outcome not only avoids inefficient liquidation but also pays the agent as early as possible. Hence, it is never optimal for investors to wait until $W_t = \lambda\delta_t$ to award the agent with α^* shares. Accordingly, there exists a $\overline{\overline{k}} < \lambda$ so that the firm starts paying wage once W_t reaches $\overline{\overline{k}}\delta_t$. Both statements are exactly the optimal contracts derived in previous sections.

2.4 Justification for the optimal contract

Take any incentive-compatible contract $\Pi = \{\{U\}, \tau\}$, and, for any $t \leq \tau$, define its auxiliary gain process $\{G\}$ as

$$G_t(\Pi) = \int_0^t e^{-rs}(\delta_s ds - dU_s) + e^{-rt}b(\delta_t, W_t), \quad (12)$$

where the agent's continuation payoff W_t evolves according to Equation (3). Under the optimal contract Π^*, the associated optimal continuation payoff W_t^* has a volatility $\lambda\sigma\delta_t$, and $\{U^*\}$ reflects W_t^* at $\overline{W}_t^* = \overline{k}\delta_t$.

Recall that $k_t = W_t/\delta_t$ and $b(\delta_t, W_t) = \delta_t c(k_t)$. Ito's lemma implies that for $t < \tau$,

$$e^{rt}dG_t(\Pi) = \delta_t \left\{ \begin{array}{l} \left[\begin{array}{c} -(r-\mu)c(k_t) + 1 + (\gamma-\mu)k_t c'(k_t) \\ + \frac{1}{2}(\sigma_t^W - k_t\sigma)^2 c''(k_t) \end{array} \right] dt + \\ [-1 - c'(k_t)]dU_t + \\ \sigma\left[c(k_t) - k_t c'(k_t) + c'(k_t)\frac{\sigma_t^W}{\sigma} \right]dZ_t \end{array} \right\}.$$

Now, let us verify that, under any $\Pi \in \mathbb{IC}$, $e^{rt}dG_t(\Pi)$ has a nonpositive drift, and zero drift for the optimal contract. The first and second lines are our Equation (4), which, under the optimal contract, is always zero. Because we have $c''(k_t) < 0$ for $k_t < \overline{k}$, and since $\sigma_t^W \geq \lambda\sigma$ holds for any $\Pi \in \mathbb{IC}$, this term is nonpositive. The third line captures the optimality of the cash payment policy. It is nonpositive since $c'(k_t) \geq -1$, but equals zero under the optimal contract. Therefore, we have the following theorem.

Theorem 1. *Take the scaled value function $c(\cdot)$ and its corresponding payment threshold $\overline{k}$, and define $k^* \equiv \arg\max_{k \in [0,\overline{k}]} c(k)$. Under the optimal contract $\Pi^* = \{\{U^*\}, \tau^*\}$, we have*

$$dW_t^* = \gamma W_t^* dt - dU_t^* + \lambda(d\delta_t - \mu\delta_t dt),$$

where dU_t^* reflects W_t^* back to $\overline{k}\delta_t$, and $\tau^* = \inf\{t \geq 0 : W_t^* = 0\}$. Given δ_0, the firm initiates the employment by picking $W_0^* = k^*\delta_0$, and investors obtain an expected value $c(k^*)\delta_0$.

3. Extensions

3.1 Optimal contracting with costly replacements

As in DF and DS, in this section we endogenize the liquidation value (factor) L. Assume that the incumbent agent can be replaced with a new but identical agent. Replacement is costly (e.g., the entrenchment effect), and we formulate the cost as $l\delta_t$ (where l is a positive constant) in order to capture the underlying size effect. The form of a pure variable cost retains the scale invariance of this model.

The same analysis as in Section 2.2.1 goes through; the only difference is in the lower end boundary condition

$$c(k^*) - c(0) = l \text{ where } k^* = \arg\max_{k \in [0,\bar{k}]} c(k), \tag{13}$$

which embeds the optimal replacement policy. We solve Equation (4) with conditions (5), (6), and (13), and obtain the endogenous liquidation value L as $c(0)$.

The optimal contract with replacement is analogous to the previous liquidation case. When the agent is impatient, the incumbent agent receives some wage whenever her continuation payoff W_t exceeds $\bar{k}\delta_t$, and the firm replaces her once W_t falls to zero. In the equally patient agent case, poorly performing agents are fired, until a lucky one achieves $k_t = \lambda$ and henceforth continues to work for the firm forever.

Comparative static analyses.   Here, we carry out comparative static analyses for the replacement case. (To find the corresponding results for the liquidation case, simply replace ∞ with the liquidation time τ^*; and the results for L are listed as well.) We also examine the comparative statics for optimal policies— namely, the payment boundary $\bar{k}$, and the replacement point k^*. Two key conditions that we exploit here are $c(\bar{k}) = \frac{1}{r-\mu} - \frac{\gamma-\mu}{r-\mu}\bar{k}$, and $c'(k^*) = 0$.

As in DS, Lemma 3 in the Appendix expresses the marginal impact of any parameter on $c(k)$ in terms of the conditional expectation of a certain integral. Under the auxiliary measure induced by Equation (4), let $\{N_t\}$ be the counting process for replacements,[19] and define

$$d_1(k) = \mathbb{E}^{k_0=k}\left[\int_0^\infty e^{-(r-\mu)t} c(k_t)\, dt\right] > 0,$$

$$d_2(k) = \mathbb{E}^{k_0=k}\left[\int_0^\infty e^{-(r-\mu)t} k_t c'(k_t)\, dt\right],$$

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[19]   Under the auxiliary measure induced by Equation (4), k_t evolves as $dk_t = (\gamma - \mu)k_t dt + (\lambda - k_t)\sigma dZ_t - du_t + k^* dN_t$, where du_t reflects k_t at $\bar{k}$, and $dN_t \equiv \mathbf{1}_{\{k_t=0\}}$ is the counting process for replacements.

Table 1
Comparative static results

	$\partial c(k)/$	$\partial \bar{k}/$	$\partial k^*/$
$\partial \gamma$	$d_2(k), \pm$	$\left\{ \dfrac{-\bar{k}-(r-\mu)d_2(\bar{k})}{\gamma-r} < 0 \right\}$	$\left\{ \dfrac{d_2'(k^*)}{-c''(k^*)} < 0 \right\}$
$\partial \mu$	$\{d_1(k) - d_2(k) > 0\}$	$\left\{ \dfrac{\bar{k}+c(\bar{k})-(r-\mu)[d_1(\bar{k})-d_2(\bar{k})]}{\gamma-r} > 0 \right\}$	$\left\{ \dfrac{d_1'(k^*)-d_2'(k^*)}{-c''(k^*)} > 0 \right\}$
$\partial \sigma$	$d_3(k) < 0$	$\dfrac{-(r-\mu)d_3(\bar{k})}{\gamma-r} > 0$	$\dfrac{d_3'(k^*)}{-c''(k^*)}, \pm$
∂l	$-d_{rp}(k) < 0$	$\dfrac{(r-\mu)d_{rp}(\bar{k})}{\gamma-r} > 0$	$\left\{ \dfrac{-d_{rp}'(k^*)}{-c''(k^*)} > 0 \right\}$
∂L	$d_L(k) > 0$	$\dfrac{-(r-\mu)d_L(\bar{k})}{\gamma-r} < 0$	$\left\{ \dfrac{d_L'(k^*)}{-c''(k^*)} < 0 \right\}$

$$d_3(k) = \mathbb{E}^{k_0=k}\left[\int_0^\infty e^{-(r-\mu)t} \sigma(\lambda - k_t)^2 c''(k_t) \right] < 0,$$

$$d_{rp}(k) = \mathbb{E}^{k_0=k}\left[\int_0^\infty e^{-(r-\mu)t} dN_t \right] > 0, \quad \text{and}$$

$$d_L(k) = \mathbb{E}^{k_0=k}[e^{-(r-\mu)\tau^*}] > 0.$$

Table 1 summarizes our results. Note that since both λ and σ measure the degree of the agency problem in this model, their comparative static results share the same sign.[20]

Most signs follow easily from $d(k)$'s. The less obvious signs, especially those involving the derivative information of $d(k)$'s, are placed in $\{\cdot\}$ (see proofs in the Appendix). Two of the terms ($\frac{\partial c(k)}{\partial \gamma}$ and $\frac{\partial k^*}{\partial \sigma}$) could have either sign. Finally, when $\gamma = r$, the results still hold except for those regarding $\bar{k}$; recall that $\bar{k} = \lambda$ always in this case.

Most of the results are intuitive. For instance, we have $\partial c(k)/\partial \sigma < 0$—as if investors were risk averse—because costly termination is more likely with a riskier project. This result also implies that $\partial \bar{k}/\partial \sigma > 0$, since investors should accumulate more continuation payoff along the optimal path in order to increase the buffer capacity. The same intuition applies to $\partial \bar{k}/\partial l > 0$.

A number of interesting empirical predictions ensue. For instance, the more severe the agency problem (higher σ or λ), the later the agent will receive incentive payouts (larger $\bar{k}$). Also, for more profitable firms—with a higher μ—the new agent is offered more favorable terms (larger k^*), but will receive incentive payouts later (larger $\bar{k}$). A larger $\bar{k}$ follows from the fact that investors

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[20] Here, we do not show comparative static results with respect to r, because most of them are ambiguous. For instance, there are two offsetting effects on the payment boundary $\bar{k}$. When r increases, the benefit from exchanging the relative consumption timings is smaller; investors should pay cash later, i.e., a larger $\bar{k}$. However, the cost from future terminations is also reduced due to a larger discounting effect, which makes investors less worried about inefficient turnovers, and thus lowers $\bar{k}$. As a result, the overall effect is ambiguous.

avert to costly liquidations of highly profitable projects, and therefore they set a higher payment boundary $\bar{k}$ to reduce the chance of future terminations.

3.2 When is it optimal to allow shirking?

3.2.1 General analysis. When the agent is shirking, she enjoys a private benefit $\phi \delta_t dt$, and the firm size follows as $d\delta_t = \sigma \delta_t dZ_t$. Since no cash payment is needed, the agent's continuation payoff W_t evolves according to $dW_t = (\gamma W_t - \phi \delta_t) \, dt + \sigma_t^W \delta_t dZ_t$, where $\sigma_t^W \leq \lambda\sigma$. In other words, the working incentive must be lower than the level required by Proposition 1. For the auxiliary gain process $\{G\}$ in Section 2.4 to remain a supermartingale given this policy, we need that

$$-rc(k) + 1 + (\gamma k - \phi) c'(k) + \tfrac{1}{2} \left(\sigma_t^W - k\sigma \right)^2 c''(k) \leq 0 \ \text{ for } \ \forall k \in \left[0, \bar{k}\right].$$

Equivalently, we can rewrite the above condition as

$$-rc(k) + 1 + (\gamma k - \phi) c'(k) \leq 0 \ \text{ for } \ \forall k \in \left[0, \bar{k}\right], \tag{14}$$

because investors can set $\sigma_t^W = k_t \sigma \leq \lambda\sigma$ in order to remove the negative second-order term. This interesting fact implies that under our GBM setup, although incentives are superfluous when shirking is allowed, the optimal incentive provision is $k_t \sigma \delta_t$, rather than zero as in the ABM framework. In fact, they are merely incentives generated by the agent's hypothetical inside stake as discussed in Section 2.3.3.

Similar to DS, based on Equation (14) we find the following sufficient condition for the optimality of working all the time:

$$\gamma c \left(\frac{\phi}{\gamma} \right) + (r - \gamma) c \left(k^* \right) \geq 1. \tag{15}$$

DS also find that when the agent is impatient, if the shirking benefit ϕ is sufficiently high, then the contract with an absorbing shirking state is optimal. It transpires that for the equally patient agent case, this class of contracts is indeed optimal among all contracts that involve shirking along the history.[21]

3.2.2 The optimal contract with shirking when $\gamma = r$. When $\gamma = r$, one can check that $c(k) - (k - \frac{\phi}{r})c'(k)$ is quasi-concave and achieves its minimum at $\frac{\phi}{r}$. Therefore, Equation (14) implies that the necessary and sufficient condition for the optimality of working all the time is simply

$$c \left(\frac{\phi}{r} \right) \geq \frac{1}{r}.$$

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[21] Note that in the $\gamma > r$ case, if the scaled shirking benefit ϕ is only slightly higher than the level such that Equation (15) binds, then a more sophisticated contract will be optimal (see DS for detail).

Note that by "shirking all the time," the agent has a value $\frac{\phi}{r}\delta_t$, while investors obtain $\frac{1}{r}\delta_t$. Hence, it is just the optimality condition of working at the state $k = \frac{\phi}{r}$. The interesting point is, this necessary condition is also sufficient for the optimality of implementing high effort at all states.

We can go one step further. Suppose that $rc(\frac{\phi}{r}) < 1$—that is, the point $(\frac{\phi}{r}, \frac{1}{r})$ sits above $c(\cdot)$ in Figure 4—hence working all the time must be suboptimal. We show below that in the new optimal contract with shirking, $(\frac{\phi}{r}, \frac{1}{r})$ is another absorbing state in which the agent is shirking forever, and the agent works whenever her continuation payoff $W_t \neq \frac{\phi}{r}\delta_t$. Therefore, there are two absorbing states in this optimal contract: the upper working state where $W_t = \lambda\delta_t$ (the first-best result), and the middle shirking state where $W_t = \frac{\phi}{r}\delta_t$ (not the first-best result).

In the Appendix, based on this two-absorbing-state policy, we provide details about constructing $c^S(\cdot)$—that is, the scaled value function with shirking. Moreover, we show that $c_-^{S'}(\frac{\phi}{r}) > c'(\frac{\phi}{r}) > c_+^{S'}(\frac{\phi}{r})$, where $c_+^{S'}(\frac{\phi}{r})$ (or $c_-^{S'}(\frac{\phi}{r})$) denotes the right (or left) derivative of $c^S(\cdot)$ at $\frac{\phi}{r}$ (see Figure 4). Because $c^S(\cdot)$ exhibits a downward kink (relative to $c(\cdot)$) at $\frac{\phi}{r}$, the function remains strictly concave, and the similar verification argument as in Section 2.4 applies. We have the following proposition.

Proposition 4. *Suppose $\gamma = r$. When $rc(\frac{\phi}{r}) < 1$, it is suboptimal to induce working all the time. Given $c^S(\cdot)$, denote $k^{*S} = \arg\max c_{k\in[0,\lambda]}^S(k)$. Along the optimal path, investors initiate the employment contract at $W_0^* = k^{*S}\delta_0$, and their expected value is $c^S(k^{*S})\delta_0$. If $k^{*S} \neq \frac{\phi}{r}$, investors ask the agent to work by promising her $dW_t^* = rW_t^* dt + \lambda\sigma(d\delta_t - \mu\delta_t dt)$, until W_t hits $\lambda\delta_t$, where she works forever (with cash payments $\lambda(r - \mu)\delta_s ds$ for $s \geq t$), or reaches $\frac{\phi}{r}\delta_t$, where she begins shirking forever (without any wage). If $k^{*S} = \frac{\phi}{r}$, then shirking all the time is optimal.*

Figure 4 shows one example of scaled value function with shirking $c^S(\cdot)$, and the original scaled value function $c(\cdot)$. In this example, we assume that at termination, investors can either liquidate their assets at a surrender value $L\delta_t$, or replace the agent at a cost $l\delta_t$. Clearly, the optimal termination policy depends on the relative magnitude of l and L. Interestingly, because shirking reduces the possibility of future terminations, the specific optimal termination policy might depend on whether or not the employment contract allows for shirking. Figure 4 demonstrates that due to a relatively large replacement cost l, the optimal contract with working all the time stipulates liquidation as the optimal termination policy; however, if shirking is allowed, replacement becomes optimal.
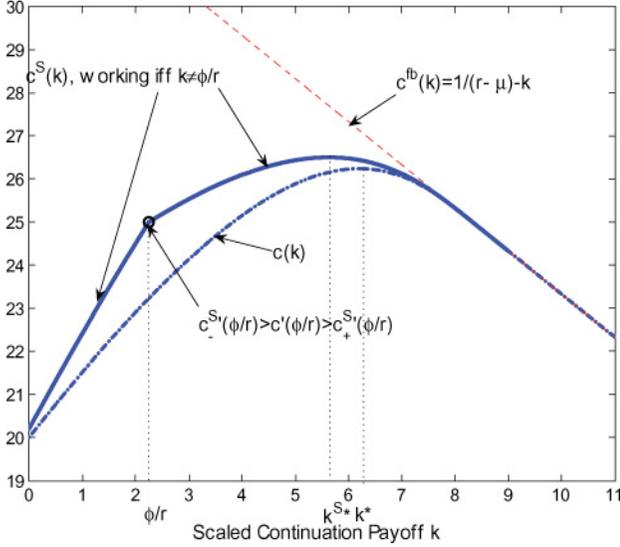
Figure 4
The scaled value function $c^S(\cdot)$ with shirking
Parameters are $r = 4\%$, $\mu = 1\%$, $\sigma^2 = 10\%$, $\phi = 0.09$, $\lambda = 9$, $L = 20$, $l = 6.3$. Since $c(\frac{\phi}{r}) = 23.23 < \frac{1}{r} = 25$, working all the time is suboptimal. The new scaled value function c^S (the solid curve on top of c) solves Equation (4) on both sides of $(\frac{\phi}{r}, \frac{1}{r})$, and $c^{S\prime}_-(\frac{\phi}{r}) > c'(\frac{\phi}{r}) > c^{S\prime}_+(\frac{\phi}{r})$. The agent shirks forever if and only if $k_t = \frac{\phi}{r}$. In this example, $c^S(0) = 20.21 > L$, while $c(k^*) - c(0) = 6.24 < l$. This implies that in the absence of shirking, the optimal termination policy is liquidation, while when shirking is allowed on the optimal path, replacement becomes optimal.

4. Implementation and Applications

4.1 Implementation

To implement the optimal contract in Theorem 1, we design an *Incentive Points Plan* where the points trace the agent's scaled continuation payoff k_t. Specifically, the agent starts with k^* points when she is hired by the firm. From then on, this plan rewards the agent with incentive points according to[22]

$$dk_t = [(\gamma - \mu)k_t + \sigma^2(k_t - \lambda)]dt + (\lambda - k_t)\left(\frac{d\delta_t}{\delta_t} - \mu dt\right). \quad (16)$$

Once k_t hits zero, she is fired. As featured by the payment boundary in the optimal contract, there is a redemption threshold $\bar{k}$ in this plan. When her points balance k_t exceeds $\bar{k}$, the agent can redeem $k_t - \bar{k}$ points for $(k_t - \bar{k})\delta_t$ amount of cash from the firm.

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[22] Check $dk_t = d(W_t^*/\delta_t)$. Note that this is under the physical measure (see discussion in note 14). Starting from k_0, before k is regenerated (at 0) or regulated (at $\bar{k}$), this linear SDE admits the solution $k_t = e^{\kappa t - \sigma Z_t}[\lambda\sigma \int_0^t e^{-\kappa s + \sigma Z_s} dZ_s + k_0]$, where $\kappa \equiv \gamma - \mu + \frac{\sigma^2}{2}$. This result is useful in simulating our model.

Performance-based stock grants.   Now, focus on the case of an equally pa-
tient agent. As suggested by Section 2.3.3, when $\gamma = r$, the optimal contract
can be easily implemented by the performance-based stock grants, where the
firm initially puts $\alpha^* = (r - \mu)\lambda$ incentive shares in the treasury. Under the
incentive points plan, once the agent accumulates sufficient points $(k_t = \lambda)$
after a long history of successes, she can redeem these points to obtain α^* in-
centive shares, and receive her portion of dividends $\alpha^* \delta_t dt$ onward. In contrast,
following poor performance, the agent will be fired if she depletes all her points
before receiving the stock grants.

Note that the performance-based stock grants are merely a variant of stock op-
tions, with a zero strike price, and a nonstandard exercise boundary.[23] Also, we
require those equity shares to be restricted shares, a feature consistent with what
we observe in practice. In fact, our implementation resembles "performance
shares" or "rights" in the long-term incentive plans of today's corporations.[24]

4.1.1 Implications.   There are several implications that follow from the evo-
lution of the scaled continuation payoff k in Equation (16). The incentive points,
which track the agent's scaled continuation payoff, have a positive drift that is
increasing in the level of k_t. This indicates the *positive feedback effect* of the
agent's performance on her future cash payouts. This effect also shows up in the
probability of the agent's future layoff: the higher the incentive points' balance,
the larger the drift, and also the lower the volatility of k_t. As discussed in Sec-
tion 2.3.3, less "additional" incentives are needed when the agent's continuation
payoff is higher.

Second, in this model the agent's scaled continuation payoff k_t measures
the extent of the firm's agency problem. Because k_t comoves positively with
firm size growth, cross-sectionally we expect that agency issues will tend to
be more severe in small firms. Note that the aforementioned positive feedback
effect in this model could potentially amplify this divergence. If, in addition,
the firm's value affects the firm's investment policy (not modeled here),[25] then
this amplification mechanism can be strengthened even further. Future work on
this cross-sectional divergence is worth pursuing.

4.1.2 Can we have similar implementations as in DS or BMPR?   We do
not propose implementations that are similar to those in DS or BMPR. In their

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[23] Under this framework we cannot implement the optimal contract using common stock options, because there
is no one-to-one relationship between k and the firm value. Besides, using American stock options leads to
another potentially interesting issue: once granted some shares of stock options, the agent will solve a doubly
stochastic-control problem: one is how to control the drift, and the other is the optimal exercising policy.

[24] For example, Citect—one of the top five technology companies in Australia—Long Term Incentive plan for 2005
states that the executive will receive a certain number of rights (to acquire an equivalent number of equity shares)
on the commencement of employment. If prespecified performance targets are achieved during the employment,
these rights become gradually vested and the exercise price is nil; if not, some rights lapse. Furthermore, the
executive can dispose of vested shares only after three years from the date of granting rights. This is very similar
to the optimal contract derived in this paper.

[25] See recent related work by DeMarzo et al. (2008) who investigate the agency impact on the q-theory of investment.

papers, a fund balance, which evolves according to the firm's cash flows, keeps track of the agent's continuation payoff. For instance, in DS, the combination of *long-term debt*, *equity, and credit line* implements the optimal contract, and the credit line balance traces the agent's continuation payoff.

In our GBM model, however, the agent's continuation payoff cannot be linked to actual cash flows. This difference is rooted in the fundamental control equation. In their model, the agent controls the *cash flow* dY_t, hence their optimal contract can use dY_t to trace the agent's continuation payoff. In our model, however, the agent controls $d\delta_t$, which is the change of *firm size* rather than the *predetermined cash flow* $\delta_t dt$. This implies that our optimal contract has to rely upon $d\delta_t$ to keep track of the agent's continuation payoff, and thus cannot be implemented by standard cash-flow contracts (e.g., credit lines) in which only $\delta_t dt$ matters.

4.2 Applications

4.2.1 Financial distress and the leverage effect. In our model, during financial distress the firm's equity return becomes more volatile: the well-known leverage effect. To study the equity return, we first exclude the agent's nontradable stake. To accommodate corporate debt within our setting, we assume that the firm maintains a short-term debt $\rho\delta_t$ outstanding, where $0 < \rho < c(k)$ for all $k \in [0, \overline{k}]$. That is to say, the firm simply rolls over and adjusts this amount of *riskless* short-term debt according to the firm size. Therefore, the equity value is $(c(k_t) - \rho)\delta_t$. Note that without agency problems, the presence of such short-term riskless debt does not affect the constant volatility of equity return (the equity value is $(\frac{1}{r-\mu} - \rho)\delta_t$).

Now, under the agency problem, one can verify that the instantaneous equity return is

$$
\frac{d\left[(c\,(k_t) - \rho)\,\delta_t\right]}{(c\,(k_t) - \rho)\,\delta_t} = \left[r - \frac{1 - \rho r}{c\,(k_t) - \rho} - \frac{\rho\mu}{c\,(k_t) - \rho}\right] dt
$$
$$
+ \left[1 + \frac{c'\,(k_t)\,(\lambda - k_t)}{c\,(k_t) - \rho}\right] \sigma dZ_t.
$$

The drift term comprises three parts: (i) discount rate r; (ii) dividend payout rate $\frac{1 - \rho r}{c(k_t) - \rho}$; and (iii) stock repurchase rate for new debt $\frac{\rho\mathbb{E}_t[d\delta_t]}{(c(k_t) - \rho)\delta_t} = \frac{\rho\mu}{c(k_t) - \rho}$. The diffusion term exhibits a stochastic volatility, and the volatility rises when $k \to 0$, as the firm is on the verge of liquidation. BMPR also derive the leverage-effect result under the ABM framework. However, without the agency problem, the GBM setting would result in constant volatility for the *return*, as opposed to the *level* in the ABM framework. Therefore, our predicted leverage effect is more compelling than the one obtained in BMPR.

4.2.2 Executive's pay–performance sensitivity. Jensen and Murphy (1990) show that a CEO obtains only \$3.25 per \$1,000 increase in the shareholders'

wealth; in the authors' terminology, this constitutes the sensitivity of CEO's "*expected wealth*" on his or her performance. Interestingly, the concept of expected wealth in Jensen and Murphy (1990) captures an idea similar to the continuation payoff in this paper.[26] However, by considering only the instant compensation (cash or new grants) and the capital gains from existing inside shares and options, most of the current empirical literature on CEO compensation might understate the long-term incentives generated by the continuation payoff in executives' remuneration contracts.[27]

What might be missing in the ongoing empirical work? To illustrate the importance of continuation payoff in measuring executive pay–performance responsiveness, we first simulate our model for the $\gamma = r$ case. The optimal contract is implemented by the performance-based stock grants. We choose $r = 4\%$, $\mu = 0.5\%$, $\sigma^2 = 6.25\%$ to match the calibration in Goldstein, Ju, and Leland (2001), and set $l = 0.2$ and $\lambda = 0.18$.[28] We set $\rho = 10$ to have a debt ratio of about 35%. Following Jensen and Murphy (1990), we perform the following OLS regression:[29]

$$\Delta Comp_{i,t} = \beta_c + \beta_0 \Delta S_{i,t} + \beta_1 \Delta S_{i,t-1} + \varepsilon_{i,t},$$

where $Comp_{i,t}$ includes the grants value, dividends, and capital gains, and $\Delta S_{i,t}$ is the change of shareholders' wealth (including dividends). We find that the mean (standard deviation) of β_0 is 0.40% (0.047%), and the mean (standard deviation) of β_1 is 0.12% (0.028%). Therefore, the estimated sensitivity is 0.51% ($0.40\% + 0.12\% \times e^{-0.04}$); this is about 15% lower than the theoretical value $(r - \mu)\lambda = 0.63\%$, which takes into account the continuation payoff.

Two reasons may exist for researchers showing little concern about this issue. First, in pay–performance regressions, Jensen and Murphy (1990); and Joskow and Rose (1994) find that the higher order lagged performances display insignificant coefficients. Second, long-term observations of firm–CEO pairs are not easily available.[30] However, our model advocates that we focus on

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[26] For instance, the authors include both the current and the lagged annual performances in their regression, assume that the change of salary and bonus are permanent, and also compute the change of probability of dismissal.

[27] For example, Hall and Liebman (1998) group the salary and bonus together with the option and stock grants, and classify the bundle as direct compensation. They document a higher pay-performance sensitivity than Jensen and Murphy (1990).

[28] Because the (scaled) first-best firm value is $\frac{1}{r-\mu} = 28.57$, the replacement cost is about 0.7% of the firm value, and the first-best inside holding $(r - \mu)\lambda$ is circa 0.6% to match the median CEO pay–performance measure obtained in Hall and Liebman (1998).

[29] There are 10 years and 200 firms in each simulation (with simulating time interval 0.01, or 3.65 days), and we repeat it 500 times. Each firm's performance is driven by an independent Brownian motion. The regression is performed on annual data.

[30] Based on a VAR analysis, Boschen and Smith (1995) study the dynamic responses of executive's performance today on their future compensation. However, the time-series regression could overlap from one CEO to another, violating the underlying assumption of a long-term agency model in which the *same* agent stays in the contractual relationship.

the *stock and option grants* when measuring the dynamic pay–performance relationship. In addition, as these grants have been growing dramatically in large companies since the late 1980s (Hall and Liebman, 1998), we expect more pronounced results from recent years.

Grants-performance sensitivity: An empirical study. To test whether CEO *future* grants provide incentives for him or her to work *now*, we carry out a *Tobit* regression and find that[31]

$$Grants^*_{i,t} = -275.049 + 0.0913 \times \Delta S_{i,t} + 0.2076 \times \Delta S_{i,t-1} + 0.1608$$
$$\underset{(836.78)}{} \quad \underset{(0.02)}{} \quad \underset{(0.0221)}{} \quad \underset{(0.0233)}{}$$
$$\times \Delta S_{i,t-2} + 0.2420 \times \Delta S_{i,t-3} + 0.2516 \times \Delta S_{i,t-4} + \varepsilon_{i,t}.$$
$$\underset{(0.0281)}{} \quad \underset{(0.0367)}{}$$

We combine the restricted stock and option grants together as our dependent variable $Grants_{i,t}$ (in *thousands*, and $Grants_{i,t} = \max(Grants^*_{i,t}, 0)$), which is the value of total grants received by CEO i at year t. The independent variables $\Delta S_{i,t-j}$'s are the changes of the shareholders' wealth (in *millions*) of firm i at year $t - j$. We add *years served by the CEO* (not reported here) to control for possible "promised" compensations in remuneration contracts. Due to the units difference between *Grants* and ΔS, the coefficients for ΔS (with standard deviation underneath) measure the dollar change of the agent's grants value given a \$1,000 change in the company's equity value.

A number of interesting findings arise from the above regression. First, the coefficient for contemporaneous performance is dominated by those for the CEO's past performances, showing that the CEO's grants are primarily driven by his or her historical achievements. Second, in contrast to Jensen and Murphy (1990); and Joskow and Rose (1994), all coefficients are significant, and even increase with lags.[32] Their magnitudes, however, are quite small. For instance, for a discount rate $r = 4\%$, the total incentives from current and future grants are, at most, \$0.868 for a \$1,000 change of shareholders' wealth.[33] This weak result might be due to our simple econometric specification (see, e.g., Aggarwal and Samwick, 1999).

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[31] We use the ExecuComp data set in Compustat, which covers S&P 500 companies from 1992 to 2004. We use only the CEO data, and all numbers are adjusted in terms of 1992 dollars. The five-consecutive-year restriction on service results in 5,040 CEO-year observations. We estimate Tobit regression because there are 1,051 observations with zero total grants. We compute $\Delta S_{i,t}$ by multiplying $TRS1YR/100$ and the company's market value in the previous year. The four-lag structure is chosen to match the median serving years of CEOs. To calculate the CEO's tenure, we count back to 1992, or the first year (after 1992) when the manager became CEO.

[32] Note that our model implies that prior lags should have larger impacts due to the discount effect. For instance, the agent who was working at $t - 4$ should discount the time-t compensation β_4, while β_0 compensation is in today's dollars.

[33] $0.0913 + 0.2076 \times e^{-0.04} + 0.1608 \times e^{-0.08} + 0.2420 \times e^{-0.12} + 0.2516 \times e^{-0.16} = 0.868$. There is a slight overestimation due to the Tobit model structure. Note that Jensen and Murphy (1990) find a CEO pay–performance sensitivity measure (mostly driven by inside shareholdings) as \$3.15/\$1,000.

5. Concluding Remarks

We study optimal contracting in a GBM firm size setting. In this model, growing firm size—as the agent's positive performance—increases the agent's inside stake within the firm, and thereby alleviates the agency problem. Along the optimal path, the agent requires stronger incentives than those she would have by holding equity according to her inside stake. Such incentives can be provided by future performance-based stock grants, and they implement the optimal contract when the agent is as patient as investors. In this case, if it is too costly to work all the time, we further derive a new optimal contract that features two absorbing states along the optimal path: one is shirking forever, and the other is working forever.

Distinct from the existing ABM model (BS and BMPR), which only studies the case of an impatient agent, under the GBM setup we derive an optimal contract with a first-best absorbing state for the case of an equally patient agent. Also, a time-varying firm size in the GBM model highlights the connection between the agent's continuation payoff and her inside stake in the firm, which provides a better understanding of the optimal incentive provision in dynamic contracting. These interesting findings advance the current continuous-time contracting literature.

This paper initiates the first step to connect recent research on dynamic contracting with the conventional continuous-time finance literature. This line of research awaits future work; for instance, it would be interesting to incorporate systematic agency-issue-related shocks into this framework. Also, this paper enables us to draw several insights for empirical studies on CEO's pay–performance relations. Empirical results provide support to our model, which predicts that for stock/option grants, past performance is of greater importance than contemporaneous performance. This suggests that today's executive remuneration contracts should be analyzed from a dynamic perspective.

Appendix A

Proof of Proposition 1 in Section 2.1
Given any contract $\Pi = \{\{U\}, \tau\}$, define the process $V_t \equiv \mathbb{E}_t[\int_0^\tau e^{-\gamma s} dU_s]$ for $t \in [0, \tau)$ as the value process of the agent's discounted wages. Under condition (1), $\{V_t : 0 \le t < \tau\}$ forms a square-integrable martingale until τ. According to the *Martingale Representation Theorem*, there exists a progressively measurable process $\{\sigma_t^W : 0 \le t < \tau\}$ s.t. $V_t = V_0 + \int_0^t e^{-\gamma s} \delta_s \sigma_s^W dZ_s$ for $\forall t \in [0, \tau)$. Hence under the presumption $\{a_t = \mu : 0 \le t < \tau\}$, we have

$$V_t = V_0 + \int_0^t \frac{e^{-\gamma s} \delta_s \sigma_s^W}{\sigma} \left(\frac{d\delta_s}{\delta_s} - \mu ds \right) \text{ for } \forall t \in [0, \tau),$$

by replacing the Brownian increment dZ_s with $\frac{1}{\sigma}(\frac{d\delta_s}{\delta_s} - \mu ds)$. Now since $W_t = \mathbb{E}_t$ $[\int_t^\tau e^{-\gamma(s-t)} dU_s]$, we have $V_t = \int_0^t e^{-\gamma s} dU_s + e^{-\gamma t} W_t$. By taking derivative on both sides, we obtain W's evolution.

We show that $\Pi \in \mathbb{IC}$ if and only if $\sigma_t^W \geq \lambda\sigma$ a.e. Consider any effort policy $a = \{a_t \in \{0, \mu\} : 0 \leq t < \tau\}$. For $t < \tau$, her associated value process is $V_t(a) = V_0 + \int_0^t \frac{e^{-\gamma s}\delta_s \sigma_s^W}{\sigma}(\frac{d\delta_s}{\delta_s}(a) - \mu ds) + \int_0^t e^{-\gamma s}\lambda\delta_s(\mu - a_s)ds$. We have

$$dV_t(a) = e^{-\gamma t}\delta_t \frac{\sigma_t^W}{\sigma}((a_t - \mu)dt + \sigma dZ_t) + e^{-\gamma t}\lambda\delta_t(\mu - a_t)dt$$

$$= e^{-\gamma t}\delta_t\left(\frac{\sigma_t^W}{\sigma} - \lambda\right)(a_t - \mu)dt + e^{-\gamma t}\delta_t\sigma_t^W dZ_t.$$

If $\sigma_t^W \geq \lambda\sigma$, then it has a nonpositive drift, and is a martingale if $\{a_t = \mu : 0 \leq t < \tau\}$. If there is a positive probability event that $\sigma_t^W < \lambda\sigma$ during $[0, \tau)$, the agent will deviate to $a_t = 0$, and $\{a_t = \mu : 0 \leq t < \tau\}$ is suboptimal. Therefore $\Pi \in \mathbb{IC}$ if and only if $\sigma_t^W \geq \lambda\sigma$ a.e.

From HJB Equation to Optimality Equation (Section 2.2.1)
Recall the evolutions of two state variables $d\delta_t = \mu\delta_t dt + \sigma\delta_t dZ_t$ and $dW_t = \gamma W_t dt - dU_t + \lambda\sigma\delta_t dZ_t$. Therefore $b(\delta_t, W_t)$ must satisfy the following Hamilton–Jacobi–Bellman equation:

$$rbdt = \sup_{dU_t \geq 0}\left\{\delta dt - dU_t + b_1\mu\delta dt + b_2(\gamma W dt - dU_t)\right.$$
$$\left. + \tfrac{1}{2}(\sigma^2\delta^2 b_{11} + 2\lambda\sigma^2\delta^2 b_{12} + \lambda^2\sigma^2\delta^2 b_{22})dt\right\},$$

where b_i and b_{ij} denote the first- and second-order partial derivatives, respectively. Immediately we see that the optimal wage policy satisfies $dU_t = 0$ when $b_2 > -1$. The optimality equation is derived by utilizing $b(\delta, W) = \delta c(k)$, where $k = \delta/W$, hence $b_2 = c'(k)$, $b_1 = c(k) - kc'(k)$, and $\delta b_{11} = -\delta kb_{12} = \delta k^2 b_{22} = k^2 c''(k)$.

Lemma 1. *Suppose that k_t evolves according to $dk_t = \beta k_t dt + (\lambda - k_t)\sigma dZ_t - du_t$, and stops at τ when k_t hits 0, where u_t is a nondecreasing process that reflects k_t at $\bar{k}$. Let $\theta \in \mathbb{R}$, and $g : [0, \bar{k}] \to \mathbb{R}$ is a bounded function. Then the function $F \in \mathbb{C}^2 : [0, \bar{k}] \to \mathbb{R}$ solves the second-order ODE*

$$rF(k) = g(k) + \beta kF'(k) + \tfrac{1}{2}(\lambda - k)^2\sigma^2 F''(k), \tag{A1}$$

with boundary conditions $F(0) = L$ and $F'(\bar{k}) = -\theta$, if and only if it satisfies

$$F(k_0) = \mathbb{E}^{k=k_0}\left[\int_0^\tau e^{-rt}g(k_t)dt - \theta\int_0^\tau e^{-rt}du_t + e^{-r\tau}L\right].$$

If k_t evolves according to $dk_t = \beta k_t dt + (\lambda - k_t)\sigma dZ_t - du_t + k^* dN_t$, where $dN_t \equiv \mathbf{1}_{\{k_t = 0\}}$ regenerates k_t back to k^*, then a function $F \in \mathbb{C}^2 : [0, \bar{k}] \to \mathbb{R}$ solves the second-order ODE in Equation (A1) with boundary conditions $F(k^*) - F(0) = l$ and $F'(\bar{k}) = -\theta$, if and only if it satisfies

$$F(k_0) = \mathbb{E}^{k=k_0}\left[\int_0^\infty e^{-rt}g(k_t)dt - \theta\int_0^\infty e^{-rt}du_t - l\int_0^\infty e^{-rt}dN_t\right].$$

Proof. The proof is similar to DS Lemma D. The result with jumps is a simple extension. ∎

A Lemma for the Homogenous Version of Equation (4)
The following lemma is repeatedly used in our later proofs.

Lemma 2. *Suppose $f(\cdot) \in \mathbb{C}^2[0, \bar{k}]$ where $\bar{k} \leq \lambda$, and it satisfies*

$$(r - \mu)f(k) = (\gamma - \mu)kf'(k) + \tfrac{1}{2}(\lambda - k)^2\sigma^2 f''(k).$$

We have the following results:

(1) For $k_1 \in (0, \lambda)$, if $f(k_1) < 0$ and $f'(k_1) \geq 0$, then $f(k) < 0$, $f'(k) > 0$ and $f''(k) < 0$ for $k \in [0, k_1)$.

(2) If $0 \leq k_1 < k_2 \leq \lambda$, and $f(k_1) = f(k_2) = 0$, then $f(k) = 0$ for all $k \in [0, \lambda]$.

(3) If $0 \leq k_1 < k_2 \leq \lambda$, and $f(k_1) < 0$ but $f(k_2) = 0$, then $f(k) < 0$, $f'(k) > 0$ and $f''(k) < 0$ for $k \in [0, k_2]$.

Proof. (1) First let us show $f'(k) > 0$ for $k \in [0, k_1)$. Note that $f'(k_1 - \epsilon) > 0$ for some small $\epsilon > 0$ (because even if $f'(k_1) = 0$, $f''(k_1) = \frac{2(r-\mu)}{(\lambda - k_1)^2 \sigma^2} f(k_1) < 0$). Suppose that $f' < 0$ for some points on $[0, k_1]$; then $x \equiv \sup\{k \in [0, k_1) : f'(k) \leq 0\} < k_1$ is well defined, and $f'(x) = 0$, $f(x) < 0$ and $f'(x + \epsilon) > 0$ for some small $\epsilon > 0$. In words, x is the local minimum points closest (from left) to k_1. But then $\frac{1}{2}(\lambda - x)^2 \sigma^2 f''(x) = (r - \mu) f(x) < 0$, contradicting with $f'(x + \epsilon) > 0$. Therefore f is increasing on $[0, k_1)$, which implies that $f(k) < 0$ for $k \in [0, k_1]$. Finally, suppose that $f'' \geq 0$ for some k, then define $y \equiv \sup\{k \in [0, k_1] : f''(k) \geq 0\}$, and $f''(y) = 0$. If $y = 0$, then $f(0) = 0$, contradiction; if $y > 0$, then $f'(y) = \frac{(r-\mu)f(y)}{(\gamma - \mu)y} < 0$, contradiction.

(2) It is sufficient to consider the case $0 < k_1 < k_2 < \lambda$. Without loss of generality, suppose there exists $x \in (k_1, k_2)$ such that $f(x) < 0$, and let $y \equiv \inf\{k \in [x, k_2] : f(k) \geq 0\}$ (which could be k_2). According to the intermediate value theorem, there exists $z \in (x, y)$ such that $f(z) < 0$ and $f'(z) > 0$. Result (1) implies that $f(k_1) < 0$, contradiction. Therefore we have $f(k) = 0$ for $k \in [k_1, k_2]$. Furthermore, on $[0, k_1]$ given the initial condition $f(k_1) = 0$ and $f'(k_1) = 0$, the solution $f = 0$ is unique. Similarly, for $k \in [k_2, \lambda - \frac{1}{n}]$, we have $f = 0$ for $n = 1, 2, \ldots$. Invoking continuity, we have $f(\lambda) = 0$.

(3) Similar arguments in (2) and the result in (1) show that $f(k) < 0$ for all $k \in (k_1, k_2)$. Again, the intermediate value theorem shows that there exists $x \in (0, \lambda)$ such that $f(x) < 0$ and $f'(x) > 0$, delivering our claim by the result in (1). ∎

Proof of Proposition 2 in Section 2.3.1

We first show that $\bar{k} \neq \lambda$. Suppose $\bar{k} = \lambda$, so $c(\lambda) = \frac{1}{r-\mu} - \frac{\gamma - \mu}{r - \mu}\lambda$. Taylor expansion gives us $c(\lambda - \epsilon) = c(\lambda) + \epsilon + \frac{1}{2}c''(\theta_1)\epsilon^2$, where $\theta_1 \in (\lambda - \epsilon, \lambda)$, and (Taylor expansion for $c'(\lambda - \epsilon)$),

$$(r - \mu)c(\lambda - \epsilon) = 1 + (\gamma - \mu)(\lambda - \epsilon)\left(-1 - c''(\theta_2)\epsilon\right) + \frac{\epsilon^2 \sigma^2}{2}c''(\lambda - \epsilon),$$

where $\theta_2 \in (\lambda - \epsilon, \lambda)$. It implies that

$$r - \gamma = -\frac{r - \mu}{2}c''(\theta_1)\epsilon - c''(\theta_2)(\gamma - \mu)(\lambda - \epsilon) + \frac{\epsilon \sigma^2}{2}c''(\lambda - \epsilon).$$

When $\epsilon \to 0$, $c''(\theta_i) \to 0$ for both θ_i's and $c''(\lambda - \epsilon) \to 0$ due to $c \in \mathbb{C}^2$, RHS goes to 0, inconsistent with $r - \gamma < 0$. Notice that this argument does not involve the information about $c'''(\lambda)$, which might not exist due to the singularity of second-order term in Equation (4).

Now we show that $c''(k) < 0$ for $\forall k \in [0, \bar{k})$. Suppose not. When $k = \bar{k} \neq \lambda$, $\frac{1}{2}(\lambda - \bar{k})^2 \sigma^2 c'''(\bar{k}) = \gamma - r > 0$ implies that $c''(\bar{k} - \epsilon) < 0$ for some small $\epsilon > 0$. (Note $c'''(\bar{k})$ always exists if $\bar{k} \neq \lambda$.) Let $x = \sup\{k \in [0, \bar{k}) : c''(k) \geq 0\}$; continuity implies $c''(x) = 0$ and $c''(k) < 0$ for $k \in (x, \bar{k})$. We have $c(x) = \frac{1}{r-\mu} + \frac{\gamma - \mu}{r-\mu}xc'(x)$. Because $c(x) < \frac{1}{r-\mu}$, $c'(x) < 0$. Hence $\frac{1}{2}(\lambda - x)^2 \sigma^2 c'''(x) = (r - \gamma)c'(x) > 0$, which implies that $c''(x + \epsilon) > 0$, contradiction. Therefore $c(k)$ is strictly concave on $[0, \bar{k}]$. Now suppose $\bar{k} > \lambda$; strict concavity implies that $c(\lambda) < c(\bar{k}) - (\lambda - \bar{k}) = \frac{1}{r-\mu} - \frac{\gamma - r}{r-\mu}\bar{k} - \lambda < \frac{1}{r-\mu} - \frac{\gamma - \mu}{r-\mu}\lambda$. But we know that $c(\lambda) \geq \frac{1}{r-\mu} - \frac{\gamma - \mu}{r-\mu}\lambda$, simply because it can be achieved by granting $\alpha^* = (\gamma - \mu)\lambda$ shares of stock and the agent is working forever. Therefore we have $\bar{k} < \lambda$.

Existence follows from the probabilistic representation. Now we show uniqueness. Take $\bar{k} \in [0, \lambda)$; use initial condition $c(\bar{k}) = \frac{1}{r-\mu} - \frac{\gamma - \mu}{r-\mu}\bar{k}$ and $c'(\bar{k}) = -1$, $c(\cdot)$ is unique on $[0, \bar{k}]$, and the solution $c(\cdot; \bar{k})$ is continuous in $\bar{k}$. We want to show that $c(0; \bar{k})$ is strictly increasing in $\bar{k}$. Suppose

that $c(\cdot; k_1)$ and $c(; k_2)$ solves Equation (4) while taking $\bar{k}_1 < \bar{k}_2$ as upper boundaries respectively, and define $f(k) \equiv c(k; \bar{k}_2) - c(k; \bar{k}_1)$ on $[0, \bar{k}_1]$. We have $f(\bar{k}_1) < 0$ and $f'(\bar{k}_1) > 0$. According to Lemma 2, $f(k) < 0$ for $k \in [0, \bar{k}_1]$, which implies that $f(0) < 0$. Therefore $c(0; \bar{k})$ is increasing in $\bar{k}$, and as a result there is a unique $\bar{k}$ s.t. $c(0; \bar{k}) = L$.

Proof of Proposition 3 in Section 2.3.2

Suppose $\bar{k} > \lambda$. Given $\bar{k}$, $c'(\bar{k}) = -1$, and $c''(\bar{k}) = 0$, the only solution to Equation (4) on $(\lambda, \bar{k}]$ is $c^{fb}(k) = \frac{1}{r-\mu} - k$. It implies that $\bar{k} - \epsilon$ can serve the same role as $\bar{k}$ satisfying Equations (5) and (6). Similarly, if $\bar{k} < \lambda$, then $c'(\bar{k}) = -1$ and $c''(\bar{k}) = 0$ imply that, on $[0, \bar{k}]$ the solution is uniquely determined as $c(k) = c^{fb}(k) = \frac{1}{r-\mu} - k$; then $c(0) = \frac{1}{r-\mu}$, contradicting with Equation (7). Therefore $\bar{k} = \lambda$. If $c''(\cdot) \geq 0$ for some point on $[0, \lambda)$, then we can pick the closest one to λ (call it $x < \lambda$), with $c''(x) = 0$ and $c'(x) > -1$. But it immediately implies $c(x) > c^{fb}(k)$, contradiction. We conclude that $c''(k) < 0$ for $\forall k \in [0, \lambda)$. Existence and uniqueness follow by the same argument as in proof of Proposition 2.

Proof of Theorem 1 in Section 2.4

Under any incentive-compatible contract, for the auxiliary gain process we have

$$dG_t(\Pi) = \mu_G(t)dt + e^{-rt}\delta_t\sigma\left[c(k_t) - k_tc'(k_t) + c'(k_t)\frac{\sigma_t^W}{\sigma}\right]dZ_t,$$

where $\mu_G(t) \leq 0$. Let $\varphi_t \equiv e^{-rt}\delta_t\sigma[c(k_t) - k_tc'(k_t) + c'(k_t)\frac{\sigma_t^W}{\sigma}]$. Recall that $c(k) = c(\bar{k}) + k - \bar{k}$ for $k > \bar{k}$, which says that $c'(k)$ and $c(k) - c'(k)k$ are bounded. Combining with condition (1) and the related argument in the proof for Proposition 1, we conclude that $\mathbb{E}[\int_0^T \varphi_t dZ_t] = 0$ for $\forall T > 0$. And, under Π the investors' expected payoff is

$$\widetilde{G}(\Pi) \equiv \mathbb{E}\left[\int_0^\tau e^{-rs}\delta_s ds - \int_0^\tau e^{-rs}dU_s + e^{-r\tau}L\delta_\tau\right],$$

where each integral, even if $\tau = \infty$ (where $e^{-r\tau_n}L\delta_{\tau_n} = 0$), is well defined since they are monotone. Moreover, because $\mathbb{E}[\int_0^\tau e^{-rs}\delta_s ds + e^{-r\tau}L\delta_\tau] < \frac{\delta_0}{r-\mu} < \infty$, the payoff $\widetilde{G}$ is well defined. Then, given any $t < \infty$,

$$
\begin{aligned}
\widetilde{G}(\Pi) &= \mathbb{E}[G_\tau(\Pi)] \\
&= \mathbb{E}\left[G_{t\wedge\tau}(\Pi) + \mathbf{1}_{t\leq\tau}\left[\int_t^\tau e^{-rs}(\delta_s ds - dU_s) + e^{-r\tau}L\delta_\tau - e^{-rt}b(\delta_t, W_t)\right]\right] \\
&= \mathbb{E}[G_{t\wedge\tau}(\Pi)] + e^{-rt}\mathbb{E}\left\{\left[\int_t^\tau e^{-r(s-t)}(\delta_s ds - dU_s) + e^{-r(\tau-t)}L\delta_\tau - b(\delta_t, W_t)\right]\mathbf{1}_{t\leq\tau}\right\} \\
&\leq G_0 + e^{-rt}\mathbb{E}\left[\int_t^\infty e^{-r(s-t)}\delta_s ds\right].
\end{aligned}
$$

The first term of third inequality follows from the negative drift of $dG_t(\Pi)$ and the martingale property of $\int_0^{t\wedge\tau} \varphi_s dZ_s$, and the second term is the first-best without any payment and termination (note that dU and $b(\delta, W)$ are positive, and $L < \frac{1}{r-\mu}$). But since $e^{-rt}\mathbb{E}[\int_t^\infty e^{-r(s-t)}\delta_s ds] = \frac{\delta_0 e^{-(r-\mu)t}}{r-\mu} \to 0$ as $t \to \infty$, we have $\widetilde{G} \leq G_0$ for all $\Pi \in \mathbb{IC}$. Finally, under the optimal contract Π^*, the investors' payoff $\widetilde{G}(\Pi^*)$ achieves G_0 because the above weak inequality holds in equality when $t \to \infty$.

Proofs for Comparative Static Results in Section 3.1

We provide the lemma only for the replacement case. The liquidation case is immediate (see DS).

Lemma 3. *For* $\theta \in \left\{ r, \gamma, \mu, l, \lambda, \sigma^2 \right\}$, *denote by* $c_\theta(k)$ *the scaled value function for that parameter value. We have*

$$
\frac{\partial c_\theta(k)}{\partial \theta} = \mathbb{E}^{k_0 = k} \left\{ \int_0^\infty e^{-(r-\mu)t} \left[\left(-\frac{\partial(r-\mu)}{\partial \theta} c_\theta(k_t) + \frac{\partial(\gamma-\mu)}{\partial \theta} k_t c_\theta'(k_t) \right. \right. \right.
$$

$$
\left. \left. \left. + \frac{1}{2} \frac{\partial \left(\sigma^2(\lambda - k_t)^2 \right)}{\partial \theta} c_\theta''(k_t) \right) dt - \frac{\partial l}{\partial \theta} dN_t \right] \right\}.
$$

Proof. The proof is similar to DS Lemma F. Given a policy $\mathbb{P} \equiv (\overline{k}, k^*)$ that simply sends out cash at $\overline{k}$ and replaces a new agent back to k^*, the investors' payoff $c_\theta(k; \mathbb{P})$ must solve the ODE

$$
(r - \mu) c_\theta(k; \mathbb{P}) = 1 + (\gamma - \mu) k c_\theta'(k; \mathbb{P}) + \tfrac{1}{2}(\lambda - k)^2 \sigma^2 c_\theta''(k; \mathbb{P}), \tag{A2}
$$

with boundary conditions $c_\theta'(\overline{k}; \mathbb{P}) = -1$ and $c_\theta(k^*; \mathbb{P}) - c_\theta(0; \mathbb{P}) = l$. Note that both conditions are independent of $\mathbb{P}$. It follows that $\frac{\partial}{\partial \theta} c_\theta'(\overline{k}; \mathbb{P}) = 0$ and $\frac{\partial}{\partial \theta}[c_\theta(k^*; \mathbb{P}) - c_\theta(0; \mathbb{P})] = \frac{\partial l}{\partial \theta}$ for any feasible $\mathbb{P}$ (so does the optimal policy). Denote $\mathbb{P}(\theta)$ as the optimal policy under θ; then by definition $c_\theta(k) = c_\theta(k; \mathbb{P}(\theta))$. Differentiate both sides of Equation (A2) with respect to θ and evaluate at $\mathbb{P} = \mathbb{P}(\theta)$,

$$
(r - \mu) \frac{\partial c_\theta(k)}{\partial \theta} = -\frac{\partial(r-\mu)}{\partial \theta} c_\theta(k) + \frac{\partial(\gamma-\mu)}{\partial \theta} k_t c_\theta'(k) + \frac{1}{2} \frac{\partial \left(\sigma^2(\lambda - k_t)^2 \right)}{\partial \theta} c_\theta''(k)
$$

$$
+ (\gamma - \mu) k \frac{d}{dk} \left[\frac{\partial c_\theta(k)}{\partial \theta} \right] + \frac{\sigma^2(\lambda - k_t)^2}{2} \frac{d^2}{dk^2} \left[\frac{\partial c_\theta(k)}{\partial \theta} \right],
$$

with boundary conditions $\frac{d}{dk} [\frac{\partial c_\theta(k)}{\partial \theta}] = \frac{\partial}{\partial \theta} c_\theta'(\overline{k}) = 0$ and $\frac{\partial c_\theta(k^*)}{\partial \theta} - \frac{\partial c_\theta(0)}{\partial \theta} = \frac{\partial l}{\partial \theta}$ evaluated at the optimal policy $\mathbb{P}(\theta)$. According to Lemma 1, we get the stated result. ∎

We now show the signs for three terms inside $\{\cdot\}$ without derivatives of $d(k)$'s. First, $\partial c(k)/\partial \mu = d_1(k) - d_2(k)$, and $d_1(k) - d_2(k) = \mathbb{E}^{k_0 = k} [\int_0^\infty e^{-(r-\mu)t} (c(k_t) - k_t c'(k_t)) dt] > 0$ because $c(k) - kc'(k) > 0$ (its derivative is $-kc''(k) > 0$ and $c(0) > 0$). Second, $\partial \overline{k}/\partial \gamma \propto -[\overline{k} + (r - \mu) d_2(\overline{k})]$, while

$$
(r - \mu) d_2(\overline{k}) = \mathbb{E}^{k_0 = k} \left[\int_0^\infty e^{-(r-\mu)t} (r - \mu) k_t c'(k_t) dt \right]
$$

$$
> \mathbb{E}^{k_0 = k} \left[\int_0^\infty e^{-(r-\mu)t} (r - \mu) \left(-\overline{k} \right) dt \right] = -\overline{k}.
$$

Third, for $\partial \overline{k}/\partial \mu \propto \overline{k} + c(\overline{k}) - (r - \mu)(d_1(\overline{k}) - d_2(\overline{k}))$, notice that $c(k) - kc'(k)$ is increasing in k; then applying the same argument we obtain the result $\partial \overline{k}/\partial \mu > 0$.

For those $\partial k^*/\partial \theta$'s listed in the third column, we have to invoke the following lemma.

Lemma 4. *Let* $\beta \geq r > 0$. *Suppose* $dk_t = \beta k_t dt + (\lambda - k_t) \sigma dZ_t - du_t$, *where* du_t *reflects* k_t *at* $\overline{k}$, *and* dN_t *regenerates the system back to* k^* *once* k_t *hits* 0. *We have*

(1) *Let* $Q(k) = r \mathbb{E}^{k_0 = k} [\int_0^\infty e^{-rt} f(k_t) dt]$, *where* $\overline{k} \leq \lambda$. *Suppose a smooth function* f *satisfies* $f(k) < f(k^*)$ *for* $\forall k \in [0, k^*)$ *and* $f'(k) > 0$ *for* $\forall k \in [k^*, \overline{k}]$; *then we have* $Q'(k) > 0$ *for* $\forall k \in [k^*, \overline{k}]$.

(2) *Let* $Q(k) = r \mathbb{E}^{k_0 = k} [\int_0^\infty e^{-rt} dN_t]$, *then* $Q(k) > 0$ *is decreasing and convex.* $Q(\lambda) = 0$ *when* $\overline{k} = \lambda$. *Similar results hold for* $Q(k) = r \mathbb{E}^{k_0 = k} [e^{-r\tau}]$, *if* $dk_t = \beta k_t dt + (\lambda - k_t) \sigma dZ_t - du_t$ *and stops at* 0 *at time* τ.

(3) *For the normalized future termination cost* $Q(k) = r \mathbb{E}^{k_0 = k} [\int_0^\infty e^{-rt} dN_t]$, *index the solution* $Q(\cdot; \overline{k})$ *by policy* $\overline{k}$. *Using subscript to indicate the partial derivative, we have* $Q_2(\overline{k}; \overline{k}) \equiv$

$\frac{\partial}{\partial \overline{k}} Q(\overline{k}; \overline{k}) < 0$ for $\overline{k} < \lambda$ and $Q_2(\lambda; \lambda) = 0$. *This implies that the marginal cost of reducing the cash-payment barrier from $\overline{k} = \lambda$ is zero, and positive when $\overline{k} < \lambda$.*

Proof. For (1) we use two facts. First, $Q(k)$ is the time-discounting average of $f(k_t)$ under the measure induced by $k_0 = k$. Second, for $k_1 > k_2$ where $k_1 \geq k^*$, it must be true that $f(k_1) > f(k_2)$. According to Lemma 1, Q must solve the following second-order ODE:

$$r Q(k) = r f(k) + \beta k Q'(k) + \frac{(\lambda - k)^2 \sigma^2}{2} Q''(k), \tag{A3}$$

with boundary conditions $Q(0) = Q(k^*)$ and $Q'(\overline{k}) = 0$. We first show $Q'(\overline{k} - \epsilon) > 0$ for small $\epsilon > 0$. To see this, if $\overline{k} < \lambda$, then since $r Q(\overline{k}) = r f(\overline{k}) + \frac{(\lambda - \overline{k})^2 \sigma^2}{2} Q''(\overline{k})$ and $f(\overline{k})$ is the maximum, we have $Q''(\overline{k}) < 0$. If $\overline{k} = \lambda$, $Q(\lambda) = f(\lambda)$ (as λ is the absorbing state) reaches the unique maximum, therefore $Q'(\lambda - \epsilon) < 0$. Suppose $Q'(x) < 0$ for some $x \in [k^*, \overline{k})$, then there must be a point $y > x$ such that $Q'(y) = 0$, $Q''(y) > 0$, and Q is decreasing on (x, y). (y is the locally minimum point closest to x). We know then $r Q(y) = r f(y) + \frac{(\lambda - k)^2 \sigma^2}{2} Q''(y)$, so $Q(y) > f(y)$. Now focus on $[0, y]$, which contains k^*. We claim that $Q(\cdot)$ must be convex on $[0, y]$. To see this, suppose we can find a reflecting point (closest to y) $z < y$ satisfying $Q''(z) = 0$, $Q'(z) < 0$, and it must be the case that $Q(z) > Q(y)$. However, we have $r Q(z) = r f(z) + \beta k Q'(z) < r f(z)$. Combining with the result $Q(y) > f(y)$, we have $f(z) > f(y)$ with $z < y$ and $y \geq k^*$, contradiction. But if $Q(\cdot)$ is convex on $[0, y]$, then $Q' < 0$ on (x, y) implies that $Q' < 0$ on $(0, y)$, contradicting the boundary condition $Q(0) = Q(k^*)$. Hence the original counterfactual assumption of the existence of x s.t. $Q'(x) < 0$ does not hold, and the conclusion follows.

For (2), it is the extreme case of (1) (with $k^* = 0$ and f as a Dirac delta function with the support $\{0\}$), and the results directly follow from Lemma 2. When $\overline{k} = \lambda$, $Q(\lambda) = 0$ as $k = \lambda$ is absorbing state and the probability to return to $[0, \lambda)$ is zero.

For (3), let $P(k; \overline{k}) \equiv Q_2(k; \overline{k})$. Note that it is different from differentiating w.r.t. parameter as we do in Lemma 3: we are now differentiating w.r.t policy. We still have that $P(k; \overline{k})$ solves $r P(k; \overline{k}) = \beta k P_1(k; \overline{k}) + \frac{(\lambda - k)^2 \sigma^2}{2} P_{11}(k; \overline{k})$, with condition $P(k^*; \overline{k}) - P(0; \overline{k}) = 0$, where P_i and P_{ij} denote partial derivatives (similarly for Q_i and Q_{ij}). To use Lemma 1, we have to pin down $P_1(\overline{k}; \overline{k}) = Q_{12}(\overline{k}; \overline{k})$. In fact, because $Q_1(\overline{k}; \overline{k}) = 0$ for all $\overline{k}$, $0 = \frac{d}{d\overline{k}} Q_1(\overline{k}; \overline{k}) = Q_{11}(\overline{k}; \overline{k}) + Q_{12}(\overline{k}; \overline{k})$. Hence invoking the result in lemma 1, we find that $P(\overline{k}; \overline{k}) = -Q_{11}(\overline{k}; \overline{k}) \mathbb{E}^{k_0 = \overline{k}} \{\int_0^\infty e^{-rt} du_t\} \leq 0$ because Q is convex in k (note that $\mathbb{E}^{k_0 = \overline{k}} \{\int_0^\infty e^{-rt} du_t\} > 0$). When $\overline{k} < \lambda$, using $Q(\overline{k}, \overline{k}) > 0$ and $Q_1(\overline{k}, \overline{k}) = 0$, we find that Equation (A3) yields $Q_{11}(\overline{k}; \overline{k}) > 0$.

When $\overline{k} = \lambda$, Equation (A3) (with $f = 0$) is an ODE with an essential (irregular) singularity at λ. Our goal is to show that when $\overline{k} = \lambda$, $Q''(\lambda) = 0$. We first show that $Q''(\lambda-) = 0$, as the main concern is the explosion of Q'' near singularity. First, notice that $Q''(k)$ must be bounded in the vicinity of λ. Otherwise, because $Q'' \geq 0$, we can always find a point $\lambda - \varepsilon$ such that $Q'''(\lambda - \varepsilon) > 0$, $Q''(\lambda - \varepsilon) > B$ where ε is small enough and B is large enough. Then differentiating Equation (A3) at $\lambda - \varepsilon$, we observe that the term involving Q'' is greater than $(\beta(\lambda - \varepsilon) - \varepsilon \sigma^2) B > 0$ and $(r - \beta) Q'$ is bounded (note the fact that $Q(0) - Q(k^*) = 1$ and Q is convex implies $(r - \beta) Q'(\lambda - \varepsilon) < (r - \beta) Q'(k^*) < \frac{\beta - r}{k^*})$, and a contradiction follows. Now the mean-value-theorem argument similar to the proof of Proposition 2 shows that $Q''(\lambda-) = 0$. Finally, because it is easy to show $Q'(\lambda-) = 0$, $Q''(\lambda) = \lim_{h \to 0} \frac{Q'(\lambda) - Q'(\lambda - h)}{h} = \lim_{h \to 0} \frac{-Q'(\lambda - h)}{h} = \lim_{h \to 0} -\frac{r Q(\lambda - h) - \frac{h^2}{2} Q''(\lambda - h)}{h \beta(\lambda - h)} = -\frac{r Q'(\lambda)}{\beta \lambda} = 0$, where we use Equation (A3) with $f = 0$, and the fact that $Q(\lambda) = Q'(\lambda) = 0$. ∎

Now we can apply this lemma to show our claims. Note that $k c'(k)$ is positive for $k \in [0, k^*]$ and reaches 0 when $k = k^*$; moreover, it is decreasing for $k \in [k^*, \overline{k}]$. Hence $\partial k^* / \partial \gamma \propto d_2'(k^*) < 0$.

For $\partial k^*/\partial \mu \propto d_2'(k^*) - d_1'(k^*) > 0$, it is sufficient to see that $c(k) - kc'(k)$ in fact is increasing on $[0, \bar{k}]$. Finally, $\partial k^*/\partial l \propto -d_{rp}'(k^*) > 0$ follows immediately from (2) of the above lemma.

Appendix for Section 3.2.2

Based on the policy proposed in the main text, we construct the scaled value function with shirking $c^S(\cdot)$ as follows (see Figure 4). Starting from $(\frac{\phi}{r}, \frac{1}{r})$ above $c(\cdot)$, we extend $c^S(k)$ to the right according to $(r - \mu)c^S(k) = 1 + (\gamma - \mu)kc^{S'}(k) + \frac{1}{2}(\lambda - k)^2 \sigma^2 c^{S''}(k)$; to do this we just pick the appropriate value for $c_+^{S'}(\frac{\phi}{r})$, so that $c^S(\lambda)$ lands at $\frac{1}{r-\mu} - \lambda$. Comparing to $c(\cdot)$ one can show that $c_+^{S'}(\frac{\phi}{r}) < c'(\frac{\phi}{r})$. Similarly we extend $c^S(k)$ to the left of $\frac{\phi}{r}$. The next lemma states that we have $c_-^{S'}(\frac{\phi}{r}) > c'(\frac{\phi}{r})$ to meet the termination boundary condition.

Lemma 5. $c_-^{S'}(\frac{\phi}{r}) > c'(\frac{\phi}{r}) > c_+^{S'}(\frac{\phi}{r})$.

Proof. We consider the replacement case only (liquidation case is easier). Suppose $c_-^{S'}(\frac{\phi}{r}) \leq c'(\frac{\phi}{r})$. Because $\frac{1}{r} = c^S(\frac{\phi}{r}) > c(\frac{\phi}{r})$, according to Lemma 2 we know that $c^S(0) > c(0)$ for all $k \in [0, \frac{\phi}{r}]$. Moreover, we have $c^S(k^{*S}) - c^S(0) < c(k^*) - c(0)$. To see this, suppose that $k^{*S} \leq \frac{\phi}{r}$; then $c^S(k^{*S}) - c^S(0) < c(k^{*S}) - c(0) < c(k^*) - c(0)$. If $k^{*S} > \frac{\phi}{r}$, which implies that $k^* > \frac{\phi}{r}$ as well (because $c'(\frac{\phi}{r}) > c_+^{S'}(\frac{\phi}{r}) > 0$), we have $c^S(k^{*S}) - c^S(\frac{\phi}{r}) < c(k^{*S}) - c(\frac{\phi}{r})$ (using Lemma 2, part 3) and $c^S(\frac{\phi}{r}) - c^S(0) < c(\frac{\phi}{r}) - c(0)$. Therefore $c^S(k^{*S}) - c^S(0) < c(k^{*S}) - c(0) \leq c(k^*) - c(0)$. In conclusion, when $c_-^{S'}(\frac{\phi}{r}) \leq c'(\frac{\phi}{r})$, the resulting function $c^S(\cdot)$ fails to meet the termination condition. Because a large $c_-^{S'}(\frac{\phi}{r})$ could deliver an arbitrarily small $c^S(0)$, we conclude that $c_-^{S'}(\frac{\phi}{r}) > c'(\frac{\phi}{r}) > c_+^{S'}(\frac{\phi}{r})$. Also note that this proof shows that this result holds even if liquidation is optimal for c while replacement is optimal for c^S, because $c^S(k^{*S}) - c^S(0) < c(k^*) - c(0) \leq l$ given $c_-^{S'}(\frac{\phi}{r}) \leq c'(\frac{\phi}{r})$. ∎

Now we prove Proposition 4. Construct the auxiliary gain process G as in Equation (12) where $b(\delta, W) = \delta c^S(k)$. Note that the scaled value function $c^S(\cdot)$ with shirking has a kink at $\frac{\phi}{r}$, but it is still strictly concave over the whole domain $[0, \lambda]$. It follows that $\varphi^S(k) \equiv c^S(k) - (k - \frac{\phi}{r})c^{S'}(k)$ for $k \in [0, \lambda]$ is still quasiconvex in k, and the minimum takes place at $\varphi^S(\frac{\phi}{r}) = \frac{1}{r}$. For $k \in [0, \frac{\phi}{r}) \cup (\frac{\phi}{r}, \lambda]$, similar argument as in Theorem 1 shows that the above policy is optimal if we are trying to implement $a = \mu$. Furthermore, because we now have $\varphi^S(k) > \varphi^S(\frac{\phi}{r}) = \frac{1}{r}$ always, it is never optimal to induce shirking before k touches $\frac{\phi}{r}$.

When $k = \frac{\phi}{r}$, $c^S(\cdot)$ is kinked at $\frac{\phi}{r}$. We want to show that it is optimal to set $\sigma^W(\frac{\phi}{r}) = \frac{\phi\sigma}{r}$, and hence k_t stays at the constant $\frac{\phi}{r}$ (no diffusion). To show this, extend $c^S(\frac{\phi}{r})$ (from the right) to the left $\frac{\phi}{r} - \epsilon$ according to (4), and denote the curve as $\tilde{c}^S(k)$; it is strictly larger than $c^S(k)$ for $k < \frac{\phi}{r}$. Suppose that we want to implement working ($a = \mu$) instead of shirking ($a = 0$) at $k = \frac{\phi}{r}$, and k_t has a diffusion $(\lambda - \frac{\phi}{r})\sigma > 0$. If our scaled value function is $\tilde{c}^S(\cdot)$, then the drift for the gain process G is 0. But since our actual scaled value function $c^S(\cdot)$ has a concave kink at $\frac{\phi}{r}$, the drift under $c^S(\cdot)$ is negative (more formally, in the generalized Ito's Lemma, there is an additional negative local time term that aims to correct for the second-order impact on the kink—see Karatzas and Shreve (1991), p. 215). Hence, inducing working makes G a supermartingale at $k = \frac{\phi}{r}$, while shirking delivers a constant payoff $\frac{1}{r}$. Therefore it is optimal to implement shirking at $\frac{\phi}{r}$.

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