Inefficient Investment Waves*

Zhiguo He
University of Chicago and NBER

Péter Kondor
London School of Economics and CEPR

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Abstract

We show that firms’ individually optimal liquidity management results in socially inefficient boom-and-bust patterns. Financially constrained firms decide on the level of their liquid resources facing cash-flow shocks and time-varying investment opportunities. Firms’ liquidity management decisions generate simultaneous waves in aggregate cash holdings and investment, even if technology remains constant. These investment waves are not constrained efficient in general, because the social and private value of liquidity differs. The resulting pecuniary externality affects incentives differentially depending on the state of the economy, and often overinvestment occurs during booms and underinvestment occurs during recessions. In general, policies intended to mitigate underinvestment raise prices during recessions, making overinvestment during booms worse. However, a well-designed price-support policy will increase welfare in both booms and recessions.

Key Words: Pecuniary externality, overinvestment and underinvestment, market intervention

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*Email addresses: zhiguo.he@chicagobooth.edu, kondorp@ceu.hu. We are grateful to Ulf Axelson, Hans Gersbach, Arvind Krishnamurthy, Guido Lorenzoni, Semyon Malamud, John Moore, Tyler Muir, Martin Oehmke, Alp Simsek, Balazs Szentes, Jaume Ventura, Rob Vishny, and numerous seminar participants. We thank Miklós Farkas for excellent research assistance. Zhiguo He acknowledges the financial support from the Center for Research in Security Prices at the University of Chicago Booth School of Business. Péter Kondor acknowledges the financial support of the Paul Woolley Centre at the LSE and of the European Research Council (Starting Grant #336585).
1 Introduction

The history of modern economies is rich with boom-and-bust patterns. Boom periods during which vast resources are invested in new projects are followed by downturns during which long-run projects are liquidated early, liquid resources are hoarded in safe short-term assets, and there is little investment in new projects. While some of these patterns affect only certain industries, others affect the aggregate economy—e.g., the emerging market boom and bust at the end of 1990s, or the recent investment boom around the mid-2000s and the crisis afterwards. These investment cycles are in the forefront of the academic and policy debate.

In this paper, we show that firms’ individually optimal liquidity management results in socially inefficient boom-and-bust patterns. Financially constrained firms choose what level of liquid resources required to absorb cash-flow shocks and to take advantage of time-varying investment opportunities. Firms hold liquid resources both to avoid inefficient liquidation of productive capital in case of adverse cash-flow shocks and to be prepared for potentially cash-intensive future investment opportunities. Our focus is on the implications for the aggregate economy when cash-flow shocks are correlated across firms.

Our first observation is that firms’ liquidity management decisions generate simultaneous waves in firms’ aggregate holdings of liquid assets and investment and waves of the opposite phase in market value of liquidity, even if technology remains constant. We argue that the emerging picture partially rationalizes evidence on liquidity holdings of non-financial firms and the time variation in the market value of liquidity.

The main result of this paper is that we show that such investment waves are not constrained efficient when future investment opportunities are non-contractible. The social and private value of liquidity differs in general. In particular, the incentive to turn liquid resources into illiquid capital, which affects individual firms but not the planner, is stronger during booms (i.e. after a series of favorable cash-flow shocks so that the capital price is relatively high) than during recessions. We show that the externality is often two-sided depending on the aggregate state: there is overinvestment in capital during booms and underinvestment in capital during recessions. As a result, firm investment is too volatile.

The presence of a two-sided externality radically changes the outcome of policy interventions. In general, policies targeted on raising prices in recessions help mitigate underinvestment, but make overinvestment in booms worse. As an example, consider a transfer scheme that does not allow the price of capital to fall below a certain level during recessions. We show that setting the appropriate price level for such a policy is critical. If the set price for the recession is not sufficiently low, it may decrease welfare during both booms and recessions, as agents foresee the induced overinvestment in booms. We show how a specific price-floor policy can change incentives through all states of the economy in order to increase welfare during both booms and recessions.

1 For example, Hoberg and Phillips (2010) document a large number of examples of industry-specific boom-and-bust patterns beyond the well-known examples such as the boom and bust of the semi-conductor industry in the 1990s. (See also Rhodes-Kropf, Robinson and Viswanathan (2005) for related findings.)
For our analysis, we integrate a novel, analytically tractable, stochastic dynamic model of liquidity management into a macroeconomic context. Our model focuses on non-financial firms. We call their long-term risky asset capital, and their liquid asset holdings cash. Capital stands for certain fixed investment in long-term risky technology, which produces stochastic flows in cash. Cash can be stored safely, exchanged for consumption goods, or used to build new capital at a constant proportional cost. Capital can also be liquidated for a relatively smaller constant proportional benefit in terms of cash. Thus, aggregate cash holdings represent non-financial firms’ liquid financial claims on the rest of the economy. The risky cash flows generated by capital (which can also be interpreted as short-lived TFP shocks) represent aggregate shocks in our economy, and negative cash-flows imply that capital requires costly maintenance in terms of cash.

The economy is initially in the aggregate stage where identical firms facing the aggregate cash-flow shocks trade, build or liquidate capital, or consume. With Poisson intensity, firms move to the idiosyncratic stage. In that stage, some firms find a productive project, which uses the existing capital (capital firms), while others get a new idea for a project, which requires cash to be exploited (cash firms). Then, cash firms sell their capital to capital firms in a Walrasian market. After trading, cash firms invest all their cash into the new opportunity, whereas capital firms operate their capital holdings more productively. Finally, firms consume all their obtained wealth.

A crucial equilibrium implication of our setup is that the aggregate stage features simultaneous waves in investment, cash-holding of firms, and the price of capital in terms of cash, even with constant technology. Firms store the cash as a buffer in order to avoid inefficient liquidation of capital. As cash-flow shocks are perfectly correlated, a series of positive cash-flow shocks raise the aggregate level of cash holding. The larger buffer decreases the chance of a series of adverse shocks forcing firms to liquidate productive capital, and as a result raises the equilibrium price of capital. When the price of capital reaches the fixed cost of investment, firms decide to build new capital. Analogously, as a result of a series of negative cash-flow shocks the price of capital might drop to the level of the liquidation benefit, leading firms to liquidate capital. This process keeps the aggregate cash-to-capital ratio within the implied liquidation and investment thresholds. We think of the state when new capital is built as a boom period and the state when capital is liquidated as a recession.

We show that the equilibrium liquidation and investment thresholds do not coincide with a planner’s choice if the investment opportunities in the idiosyncratic stage are not contractible. In the planner’s solution, firms liquidate their productive capital only when the cash-to-capital ratio hits zero, and invest during booms when the cash-to-capital ratio hits a positive threshold, which is the socially optimal cash buffer in this economy. However, in the decentralized equilibrium, the investment and disinvestment thresholds are distorted. In particular, firms always liquidate capital at a strictly positive cash-to-capital ratio, implying that firms always underinvest in downturns. Interestingly, under some conditions firms invest in capital when the cash buffer is lower than the one the planner would choose. That is, they underinvest in capital (liquidate too much) in downturns and overinvest during booms. As a mirror image, they hoard too much cash during a
downturn, and hold too little cash during a boom.

Here is the economic intuition: As we noted, firms’ incentive to build liquidity buffers against cash-flow shocks generates procyclicality in aggregate liquidity holdings and countercyclicality in the value of liquidity, implying that the value of capital relative to cash, i.e. the capital price, has to be procyclical. Once investment opportunities arrive, cash firms can sell the capital they have, and capital firms buy the capital at the prevailing market price in terms of cash. Therefore, in booms, when the price of capital is higher, firms value their capital more than cash. That is, preparing for investment opportunities aggravates procyclicality in capital prices. However, this additional effect that influences private incentives is absent from social incentives, because one firm’s gain from trading capital to cash is the other firm’s loss. Therefore, there is a state-dependent wedge between the private and social valuation of capital (relative to cash), creating the possibility of overinvestment in booms and underinvestment in recessions.

This argument holds because we assume that certain markets are missing. For example, firms writing contracts ex ante on investment opportunities would insure each other against the gains and losses from ex post trading. Similarly, firms able to pledge the output of their investment opportunities, would exchange capital to cash at terms determined by the (fixed) output of these opportunities. These possibilities eliminate the wedge between the market price and the social value of capital, restoring the constrained efficiency for the decentralized economy.

As an extension of our model, we allow firms to pledge capital to obtain external credit by collateralized borrowing. This makes capital more valuable from both the private and social perspectives. We show that collateralized borrowing tends to push up the private benefit of capital more than it does on the capital’s social value. Therefore, collateralized borrowing could be excessive, in the sense that a sufficiently large borrowing capacity of capital brings a no-borrowing economy from “underinvestment always” to two-sided inefficiency, with overinvestment during the boom.

As an illustration of the potential of our mechanism to provide new explanations for existing problems in various contexts, we connect our results to the observed phenomenon of relative boom-and-bust patterns across industries, and to stylized facts that in less financially developed countries investment in productive technologies is more volatile and exhibits stronger procyclicality.

As a methodological contribution, we develop a novel dynamic model to analyze the effect of aggregate liquidity fluctuations on asset prices and real activity, with analytical tractability of the full joint distribution of states and equilibrium objects.

**Literature.** In our model, firms’ individually optimal liquidity management decisions generate aggregate waves in investment, market value of liquidity, and aggregate liquidity holdings. As a main contribution, we show that if future investment shocks are non-contractible, firms often have too much incentives to invest during booms and too little incentives to invest during recessions.

Ours is not the first paper to emphasize that firm-level constraints can generate inefficient investment waves. The literature with perhaps the largest influence on current policy discussions emphasizes the fire-sale feedback loops induced by a price-sensitive collateral constraint (e.g., Kiy-
In these models, firms fail to internalize that the more they borrow and invest during booms, the more they have to deleverage and disinvest during recessions, which depresses fire-sale prices and tightens the constraint faced by other firms as well. Compared to a social planner facing the same constraints, in these models firms’ incentives to borrow and/or invest are always too strong. Our research differs from this literature in two crucial dimensions. First, our mechanism is unrelated to any form of collateral-based or net-worth-based amplification mechanism. Second, and more important, the externality in our model changes sign with the state of the economy. As a result, policy measures limiting overexpansion in booms, which are unambiguously beneficial in an economy with collateral constraints, cause inefficient hoarding of liquidity in our economy and potentially decrease welfare everywhere.²

Like the literature on fire-sale feedback loops, our work also belongs to the literature analyzing the welfare effects of pecuniary externalities. This literature is based on the seminal papers of Stiglitz (1982), Greenwald and Stiglitz (1986), and Geanakoplos and Polemarchakis (1985), which, like the recent work of Farhi and Werning (2013), establish general conditions implying welfare-changing pecuniary externalities. Our application of this general principle is closest to the vein of research in which market incompleteness hinders the equalization of firms’ marginal utility of wealth across states or time (e.g., Shleifer and Vishny (1992), Allen and Gale (1994, 2004, 2005), Caballero and Krishnamurthy (2001, 2003), Lorenzoni (2008), Farhi, Golosov and Tsyvinski (2009) and Gale and Yorulmazer (2011)). Compared to a planner, this mechanism can imply that incentives to invest are either too strong or too weak, depending on the exact specification.³ Our main innovation is that we highlight the effect of interacting these types of pecuniary externalities with varying incentives to hold liquid assets over the cycle. This interaction leads to our main result that the sign of the distortion in investment incentives switches with the state of the economy.

A few recent papers cast in two-period settings investigate two-sided inefficiency and derive implications related to our work. Gersbach and Rochet (2012) study the moral hazard problem of incentivizing banks in a macroeconomic context, and show that banks extend too much credit in booms and too little in recessions. Their mechanism relies on the difference between the private and social solution of bank’s moral hazard problem. Additionally, in their two-period setting which models booms and recessions separately as two different states in period 1, the period 0 intervention can resolve the two-sided efficiency at once. In contrast, in our dynamic model booms and recessions

²This paper contributes to the discussion on the optimal mix of ex ante regulation and ex post intervention (e.g., Diamond and Rajan, 2011; Farhi and Tirole, 2012; Jeanne and Korinek, 2013), to the extent that we emphasize that a policy of intervention during a recession will also affect incentives during a boom. We characterize economies when, because of the two-sided externality, this fact has crucial consequences on the welfare effects of these policies.

³See Davila (2014) for a comparative analysis of different mechanisms connected to pecuniary externalities and the argument that collateral constraints always imply overinvestment ex ante. For uninsurable idiosyncratic liquidity shocks, see Holmstrom and Tirole (2011, chap.7.) of simplified versions and excellent discussion of Shleifer and Vishny (1992) and Caballero and Krishnamurthy (2003). Finally, a recent paper by Hart and Zingales (2011) studies the excessive supply of private money based on the idea of special pledgeability of certain assets. This friction always results in overinvestment in such assets, in contrast to our model.
occur in cycles, and the potentially inferior one-sided interventions emphasize the interconnected incentives between booms and recessions for forward-looking economic agents. Eisenbach (2013) studies banks financed with short-term debt in a general equilibrium setting, and show that in good (bad) times banks face too little (much) market discipline imposed by rolling over short-term debt. In contrast to our paper, in which idiosyncratic investment opportunities drive inefficiency, that paper emphasizes aggregate risk, and the fact that short-term debt lacks aggregate-state contingency.

In our model, firms hold liquid assets to avoid adverse effects of cash-flow shocks and to prepare for future investment opportunities. This is consistent with a large body of previous work on liquidity management (e.g., Almeida, Campello and Weisbach (2004), Bates, Kahle and Stulz (2009), Denis and Sibilkov (2010), Ivashina and Scharfstein (2010), Lins, Servaes and Tufano (2010), Eisfeldt and Muir (2013), Acharya, Almeida and Campello (2013)). This argument goes back to Keynes, who calls this the precautionary motive. However, instead of aiming for a detailed picture of firms’ individual saving and investment decisions, we focus on the consequences of such decisions to the aggregate economy.

The structure of our paper is as follows. In Section 2 we present the setup and the equilibrium of our model. In Section 3 we expose the inefficiencies of the market solution. Section 4 presents our findings on economic policy and other applications. We discuss the robustness of our mechanism in Section 5. We conclude in Section 6. All proofs are in Appendix, Online Appendix, or Additional Material available on the author’s website.

2 A Dynamic Model of Saving and Investment

2.1 Assets

We model an economy where firms facing cash-flow shocks and time-varying investment opportunities make saving and investment decisions. There is a single capital good representing risky and productive projects. The other asset in this economy is cash which serves both as a consumption good and as an input for building capital. We assume that there is a safe storage technology and that capital does not depreciate; thus both capital and cash are perfectly storable.

For each firm, there is a final date arriving at a stopping time \(\tau\) with Poisson intensity \(\xi\), where \(\xi\) is a positive constant. At this final date, firms receive potentially different investment opportunities (to be specified shortly), and any unused capital depreciates fully. For now, we think of the arrival of the final date as an aggregate shock (we offer an alternative interpretation in Section 2.4). Before the final date, each unit of capital generates random cash flows. This shock is common across capital units and driven by \(\sigma dZ_t\), where \(\sigma\) is a positive constant and \(Z \equiv \{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}\) is a standard Brownian-motion on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\). One can interpret the

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4 Others proposed the tax motive, the transaction motive, and the agency motive as alternative explanations (see Bates, Kahle and Stulz (2009) for detailed arguments and references).

aggregate cash-flow shocks $\sigma dZ_t$ as short-lived TFP shocks. When $\sigma dZ_t > 0$ the capital generates cash. When $\sigma dZ_t < 0$, the firm needs to spend $|\sigma dZ_t|$ amount of cash on this capital as maintenance cost; otherwise the capital turns unproductive.

Denote by $K_t$ the aggregate quantity of capital. Given the aggregate cash shock $\sigma dZ_t$ of each unit of capital, when firms do not invest or disinvest (to be introduced shortly), the aggregate level of cash accumulated in storage, $C_t$, would follow the evolution of

$$dC_t = K_t \sigma dZ_t.$$ (1)

### 2.2 Firms and frictions

The market is populated by a unit mass of risk-neutral firms who operate the capital. At each time instant, firms may decide to build new capital, trade capital for cash at the equilibrium price $p_t$, or liquidate the capital. Building new capital costs $h$ units of cash, while liquidating a unit of capital provides $l$ units of cash, where $h > l > 0$. Firms can also consume their cash at any moment of a constant marginal utility of 1. Because of linear technologies, in general it is optimal to have threshold strategies of (dis)investment. Thus, we can simply focus on thresholds in comparing different (dis)investment strategies.

The major friction in this economy is that firms can neither write contracts on the different investment opportunities they face, nor they can pledge the future return on these opportunities. Although firms are initially identical, they receive different investment opportunities. Specifically, in the random final date, each firm with probability half finds a project which uses the existing capital productively generating $R_K > 0$ unit of final consumption per each unit of used capital. The other group of firms find a new idea requiring liquid resources. Hence, this latter group have a superior use for liquid resources, and we assume that they receive $R_C > 1$ unit of final consumption per unit of cash invested. These shocks are independent across firms, and we refer to the earlier group as capital firms and the latter group as cash firms. $R_K$ and $R_C$ are positive constants. Our extreme assumption that neither group's project returns are pledgeable is a short-cut for agency and/or informational frictions.\(^6\) We partially relax this assumption in Section 5.2. Throughout we assume that

$$\frac{R_K}{R_C} > h; \quad (2)$$

which ensures that building capital is socially efficient when the economy has sufficient cash.\(^7\)

Firms learn which group they belong to only at the beginning of the final date. In the final date the conversion technology between capital and cash is no longer available, but firms have a last trading opportunity to trade capital for cash before final production. We refer to the potentially infinitely long interval before the final date $\tau$ as the aggregate stage of the economy, as at this stage

\(^6\)Appendix C of He and Kondor (2012), in the context of a simple two-period example, discusses the potential agency problems in detail.

\(^7\)Allowing for $h > \frac{R_K}{p_t} > l$ would leave the derivation and the characterization of the market equilibrium untouched. Although the comparison to the planner’s case remains similar, the derivation is more cumbersome. Hence, for easier readability, we discuss this case in Remark 1 in Section 3.2.
all shocks affect each agent the same way. By similar logic, we refer to the final date \( \tau \) (in which final trading occurs) as the idiosyncratic stage. We denote the price in the idiosyncratic stage by \( \hat{p}_\tau \) (recall that we denote by \( p_t \) the prices in the aggregate stage). Figure 1 summarizes the time line of events in our model. We expand on the interpretation of the two stages in Section 2.4.

### 2.3 Individual firm’s problem

Consider firm \( i \), which holds \( K^i_t \) units of capital and \( C^i_t \) amount of cash, with a wealth (in terms of cash) of \( w^i_t = p_t K^i_t + C^i_t \). Since the idiosyncratic stage arrives according to an exponential distribution with density \( e^{-\xi \tau} \), firm \( i \) is solving the following problem:

\[
\max_{\{\alpha^i_t \geq 0, K^i_t \geq 0, C^i_t \geq 0, dK^i_t\}} \mathbb{E} \left\{ \int_0^\infty \xi e^{-\xi \tau} \left( \int_0^\tau d\alpha^i_t + \left[ \frac{1}{2} \left( K^i_\tau + C^i_\tau \right) R_K + \frac{1}{2} \left( K^i_\tau \hat{p}_\tau + C^i_\tau R_C \right) \right] d\tau \right\}
\]

(3)

where \( \alpha^i_t \) is firm \( i \)'s cumulative consumption before the final date \( \tau \) (so it is non-decreasing with \( d\alpha^i_t \geq 0 \); later we see that it is zero in equilibrium), and \( dK^i_t \) is the amount of capital that it dismantles or builds. The term in the squared bracket is the consumption at the idiosyncratic stage. For instance, if the firm turns out to be cash-type, it will sell its capital holding \( K^i_\tau \) at the price of \( \hat{p}_\tau \) to receive \( K^i_\tau \hat{p}_\tau \), and then invest its cash together with \( C^i_\tau \) in exploiting new cash-intensive projects with return \( R_C \).

The problem in (3) is subject to the dynamics of individual wealth,

\[
dw^i_t = -d\alpha^i_t - \theta dK^i_t + K^i_t (dp_t + \sigma dZ_t),
\]

(4)

where \( \theta \) is the cost of changing the amount of capital, so that \( \theta = h \mathbf{1}_{\{dK^i_t \geq 0\}} + l \mathbf{1}_{\{dK^i_t < 0\}} \). Also, wealth cannot be negative at any point, i.e. \( w^i_t \geq 0 \) of all \( t \).

Recall \( K_t = \int_i K^i_t di \) is the aggregate capital. Combining the investment/disinvestment policy
\[ dK_t, \] (1) implies that the dynamics of aggregate cash level in the economy is\(^8\)

\[ dC_t = \sigma K_t dZ_t - \theta dK_t. \] (5)

The scale-invariance implied by the linear technology suggests that it is sufficient to keep track of the dynamics of the cash-to-capital ratio:

\[ c_t = \frac{C_t}{K_t}, \]

which evolves according to

\[ dc_t = \frac{dC_t}{K_t} - \frac{C_t}{K_t} \frac{dK_t}{K_t} = \sigma dZ_t - (\theta + c_t) \frac{dK_t}{K_t}. \] (6)

### 2.4 Interpretation of the aggregate and idiosyncratic stages

We stress that thinking of the arrival of the idiosyncratic stage as an aggregate shock and the resulting separation of the two stages is a didactic tool. It helps show how the incentives related to the idiosyncratic investment opportunities affect the incentives for saving and investing in the aggregate stage. In the real world, some firms might be in the idiosyncratic stage while others are still in the aggregate stage. Therefore, the final date does not correspond to an observable time point in the economy. Instead, we will think of recessions and booms and economic policies affecting saving and investment in these states within the aggregate stage of the economy. With this structure we can analyze the dynamic fluctuation of our economy without sacrificing analytical tractability.

Indeed, there is a formally equivalent economy where the arrival of the final date is idiosyncratic to individual firms. Under this interpretation, in each time interval \( dt \) a \( \xi dt \) fraction of firms randomly receive heterogeneous investment opportunities as above (i.e., \( \frac{\xi}{2} dt \) fraction are capital firms while the other \( \frac{\xi}{2} dt \) fraction cash firms), enter the idiosyncratic stage and trade cash for capital among themselves on a separate market, while the remaining firms continue to operate in the aggregate stage. Thus, under this interpretation the economy never terminates.

In this alternative economy, the individual firm’s problem (3) and the evolution of aggregate state (6) remain the same. Because at each instant there are equal fractions of cash and capital flowing out from the economy, the aggregate cash-to-capital ratio in the remaining economy is not affected; but the size of the remaining economy shrinks. The trading price also remains \( \hat{p}_r \) in the separate market, while all incumbent firms face a trading price of \( p_t \).

To further emphasize that this separation is a technical innovation, in Section 5.1 we present and analyze a version of our model where a fraction of firms learn about new investment opportunities in each time instant and trade cash and capital in a single market together with the firms who remain in the aggregate stage. That is, the aggregate stage and the idiosyncratic stage are not

\(^8\)To simplify notation we ignore the possibility that at any given point in time some firms create capital while some firms liquidate capital. It is easy to see that this never happens in equilibrium.
separated. While that version is not analytically tractable, we will illustrate by numerical analysis that our main result goes through.

2.5 Interpretation of cash and capital

Cash holding in the aggregate stage, \( C_i^t \), represents the financial slack of a firm—cash holdings; other short-term, liquid investments; or credit lines. It can be used either to cover any operating losses, or invest in any new opportunities (even outside the industry). As we illustrate in Figure 3, it is possible to map \( C_i^t \) to data by thinking of it as the liquid financial asset holdings of non-financial firms of the economy. Note that in reality these assets represent claims on the government, households, or foreigners: entities which we do not explicitly model.

\( K_i^t \) represents firms’ total gross property, plants, equipment, inventories, and intangible assets, which is much more specific to each industry and thus much less liquid. The process \( \sigma K_i^t dZ_t \) might represent cash flows from both operating and financing activities. In our abstract model without external financing, firms finance their investment from retained earnings only. However, in Section 5.2 we show that allowing for collateralized borrowing could make our main results more pronounced.

Importantly, \( R_C \) in the idiosyncratic stage should not be interpreted as the return from liquid investments. Instead, it is a reduced-form representation of the expected return from the cash-intensive development of a new idea. We follow a reduced-form treatment. In reality, the cash might have to be used to hire labor, or purchase specific capital for the new idea. \( R_K \) can be interpreted similarly, but for an idea that uses the same type of capital as the existing technology. Cash firms are the ones with comparative advantage in exploiting the former, whereas capital firms have comparative advantage in exploiting the latter.

2.6 Definition of equilibrium

**Definition 1** In the market equilibrium,

1. each firm chooses \( d\alpha_i^t, K_i^t, C_i^t, \text{ and } dK_i^t \) to solve (3), and
2. markets clear in every instant, during both the aggregate and the idiosyncratic stages.

As we will see, in our framework, the equilibrium only pins down the aggregate variables: prices, net trade, and net investment and disinvestment. Typically, any combination of individual actions consistent with the aggregate variables is an equilibrium. It is convenient to pick the particular market equilibrium where all firms follow the same action, which we refer to as the symmetric equilibrium.

Henceforth, we omit the time subscript \( t \) or \( \tau \) whenever it does not cause any confusion.
2.7 Market equilibrium

We solve for the market equilibrium in this section. As we show, in this economy consumption (of cash) before the idiosyncratic stage is strictly suboptimal, thus \( d\alpha_i = d\alpha_t = 0 \) always.

2.7.1 Equilibrium price in the idiosyncratic stage

Consider the idiosyncratic stage. The law of large numbers implies that there exists a half measure of capital (cash) firms. All capital firms use their cash holdings to buy capital holdings from cash firms, and the market clearing condition implies that

\[
\frac{1}{2} C = \frac{1}{2} K \hat{p} \Rightarrow \hat{p} = c.
\]

We still need to ensure that \( R_K \geq \hat{p} = c \). This is because capital firms have the option of consuming their cash holdings instead of purchasing capital, which puts an upper bound on \( \hat{p} \). Later we show that the full support of \( c \) is endogenous, because firms build (dismantle) capital whenever the aggregate cash is sufficiently high (low). For simplicity, we restrict the parameter space to ensure that the condition \( c \leq R_K \) holds always in equilibrium.

2.7.2 Equilibrium values, prices, and investment policies in the aggregate stage

Now we determine equilibrium objects in the aggregate stage. The next lemma states two useful features of our formalization: First, the only relevant aggregate state variable is the cash-to-capital ratio. Second, the value function of any individual firm is linear in its capital and cash holdings.

**Lemma 1** Let \( J(K^i, C^i, K, C) \) be the value function of firm \( i \) which holds capital \( K_i \) and cash \( C_i \) in an economy with aggregate capital \( K \) and aggregate cash \( C \). Then, for aggregate cash-to-capital ratio \( c = C/K \), there are functions \( v(c) \) and \( q(c) \) that,

\[
J(C, K, K^i, C^i) = K^i v(c) + C^i q(c).
\]

That is, regardless of the firm’s composition of asset holdings, the value of every unit of capital is \( v(c) \), and the value of every unit of cash is \( q(c) \). Both functions depend only on the aggregate cash-to-capital ratio. Because of linearity, the equilibrium price has to adjust in a way such that firms are indifferent to whether they hold capital or cash. That is, the equilibrium price of capital \( p(c) \) in the aggregate stage must satisfy that

\[
p(c) = \frac{v(c)}{q(c)}.
\]

Firms build capital whenever the capital price \( p \) reaches the cash cost \( h \), and they dismantle capital whenever the price falls to the liquidation benefit \( l \). Define \( c^*_h \) and \( c^*_l \) as the endogenous thresholds of the aggregate cash-to-capital ratio where firms start to build and dismantle capital,
respectively. These thresholds satisfy

\[ \frac{v(c_h^*)}{q(c_h^*)} = h, \quad \text{and} \quad \frac{v(c_l^*)}{q(c_l^*)} = l. \]  

(7)

Moreover, the linear technology implies that \( c_h^* \) and \( c_l^* \) are reflective boundaries of the process \( c \). Therefore, based on (6), the aggregate cash-to-capital ratio \( c \) must fluctuate in the interval \([c_l^*, c_h^*]\), with a dynamics of

\[ dc = \sigma dZ_t - dU_t + dB_t, \]  

(8)

where \( dU_t \equiv (h + c_h^*) \frac{dK_t}{K_t} \) reflects \( c \) at \( c_h^* \) from above, while \( dB_t \equiv (l + c_l^*) \frac{dK_t}{K_t} \) reflects \( c \) at \( c_l^* \) from below. The standard properties of reflective boundaries imply the following smooth-pasting conditions for our value functions (Dixit (1993)):

\[ v'(c_h^*) = q'(c_h^*) = q'(c_l^*) = v'(c_l^*) = 0. \]  

(9)

2.7.3 Characterizing the market equilibrium

Now we turn to characterizing the value functions \( v(c) \) and \( q(c) \) in the range \( c \in [c_l^*, c_h^*] \). Here we give a sketch; full details are available in the Online Appendix. Because of Lemma 1, firms are indifferent to the composition of their asset holdings, and we can consider the value function of a firm that holds only capital or only cash. The value function of a firm holding only cash gives an Ordinary Differential Equation (ODE) of \( q(c) \):

\[ 0 = \frac{\sigma^2}{2} q''(c) + \frac{\xi}{2} (R_C - q(c)) + \frac{\xi}{2} \left( \frac{R_K}{c} - q(c) \right), \]  

(10)

and the value function of a firm holding only capital, given \( q(c) \), yields the ODE for \( v(c) \):

\[ 0 = \frac{\sigma^2}{2} v''(c) + \frac{q'(c) \sigma^2}{2} + \frac{\xi}{2} (R_C c - v(c)) + \frac{\xi}{2} (R_K - v(c)). \]  

(11)

These ODEs are Hamilton-Jacobi-Bellman (HJB) equations for cash and capital given the dynamics of the state \( c \). We first explain the terms unrelated to \( \xi \) in each ODE. For (10), the Ito correction term \( \frac{\sigma^2}{2} q''(c) \) captures the impact of the evolution of the state variable \( c \); a similar term shows up in (11). In addition, we have \( q'(c) \sigma^2 \) in (11) because the capital generates random cash flows \( \sigma dZ_t \) which are perfectly correlated with the aggregate state \( c_t + dt = c_t + \sigma dZ_t \) (see (8)).

Multiplying by the intensity \( \xi \), the terms describe the change in expected utility once the idiosyncratic stage arrives. The first of these terms in (10) captures that, once a firm holding a unit of cash learns to be a cash firm, its value jumps to \( R_C \) from \( q(c) \). The second term says that it

\[ ^9\text{Heuristically, given } q(\cdot) \text{ as the marginal value of cash, the expected value of the cash flows } \sigma dZ_t \text{ standing at time } t = \mathbb{E}_t \left[ q(c + \sigma dZ_t) \sigma dZ_t \right] = \mathbb{E}_t \left[ q'(c) \sigma^2 (dZ_t)^2 \right] = q'(c) \sigma^2 dt. \]
uses the unit of cash to buy $1/\hat{p} = 1/c$ unit of capital, so its value jumps to $R_K/c$ from $q(c)$. The interpretation in (11) is analogous.

Define the constant $\gamma \equiv \sqrt{2\xi/\sigma}$. The ODE system in (10)-(11) has the closed-form solution:

$$q(c) = \frac{RC}{2} + e^{-c\gamma}A_1 + e^{c\gamma}A_2 + R_K \frac{\gamma e^{-c\gamma} \text{Ei}(-\gamma c) + e^{-c\gamma} \text{Ei}(c\gamma)}{2},$$

(12)

and

$$v(c) = R_K + \frac{RCC}{2} + e^{c\gamma}(A_3 - cA_2) - e^{-c\gamma}(A_4 + cA_1) + cR_K \frac{\gamma(e^{c\gamma} \text{Ei}(-\gamma c) - e^{-c\gamma} \text{Ei}(c\gamma))}{2},$$

(13)

where $\text{Ei}(x) \equiv \int_{-\infty}^{x} t^{-1}e^t dt$ is the exponential integral function, and the constants $A_1-A_4$ are determined from boundary conditions in (9).

Finally, we determine the endogenous investment/liquidation thresholds $c^*_l$ and $c^*_h$ using (7). The functions $v(c), q(c)$ and the thresholds constitute an equilibrium if the resulting price $p(c) = \frac{v(c)}{q(c)}$ falls in the range of $[l, h]$ when $c \in [c^*_l, c^*_h]$. The following proposition gives sufficient conditions for such a market equilibrium to exist and describes the basic properties of this equilibrium.\footnote{When $h - l$ is not sufficiently small, a variant of this equilibrium often prevails. Because this variant has very similar features, we relegate the discussion of it to Additional Material, available on the author’s website.}

**Proposition 1** If the difference between the benefit of liquidation, $l$, and the cost of building capital, $h$, is sufficiently small, then the market equilibrium exists with the following properties:

1. firms do not consume before the final date;

2. each firm in each state $c \in [c^*_l, c^*_h]$ is indifferent to the composition of its asset holdings and $0 < c^*_l < c^*_h < R_K$;

3. firms do not build or dismantle capital when $c \in (c^*_l, c^*_h)$ and, in aggregate, firms spend every positive cash shock to build capital if and only if $c = c^*_h$, and they cover negative cash shocks by liquidating a sufficient fraction of capital if and only if $c = c^*_l$;

4. the value of holding a unit of cash and the value of holding a unit of capital are described by $v(c)$ and $q(c)$, and the price in the aggregate stage is $p(c) = v(c)/q(c)$;

5. in the idiosyncratic stage, a capital firm sells all its capital to cash firms for the price $\hat{p}(c) = c$;

6. $q(c)$ is monotonically decreasing, $v(c)$ is monotonically increasing, and $p(c)$ is monotonically increasing. Furthermore, $q(c)$ has exactly one inflection point: there is a $c_q \in (c^*_l, c^*_h)$ such that $q''(c) < 0$ for $c \in (c^*_l, c_q)$ and $q''(c) > 0$ for $c \in (c_q, c^*_h)$.

**2.7.4 Investment waves**

The thick, solid curves on panels A-E of Figure 2 illustrate the properties of the market equilibrium. In panels A-C, the functions $p(c), v(c), q(c)$ describe the price of capital, the value of
cash, and the value of capital, respectively. Panels D-E depict the cash-to-capital ratio and the investment/disinvestment activity along one particular sample path.

The cash-to-capital ratio, $c$, represents the relative scarcity of liquid assets in the economy compared to illiquid capital. Thus, we refer to this ratio as “aggregate liquidity.” We also think of intervals with a large increase (drop) of capital as a boom (downturn). In our model, investment takes a simple threshold strategy, in such a way that investment (disinvestment) occurs only at $c_h$ ($c_l$). However, we believe the resulting clustered investment and disinvestment activities depicted in panel E captures the essence of boom-and-bust patterns observed in reality.

The economy fluctuates across states because the aggregate cash-flow shocks drive the level of aggregate liquidity. This is illustrated in panel D. This particular sample path starts with a series of positive shocks, which increase the capital value $v$ and decrease the cash value $q$. Thus, the price of capital increases along this path (not shown),\(^{11}\) because in these states the probability that the economy will slip into a downturn (and capital must be dismantled) is low. When the price hike reaches the cost of building capital, $h$, investment is triggered (as shown in Panel E). This keeps the cash-to-capital ratio below $c_h$. For symmetric reasons, as a series of negative shocks decrease aggregate liquidity, rising cash values and falling capital values lead to lower capital prices. When the price of capital drops to $l$, disinvestment in capital is triggered. This keeps the cash-to-capital ratio above $c_l$.

Figure 3 shows our first step in mapping our model to data. Based on FED Flow of Funds data, we construct a series of aggregate liquid financial assets for non-financial US firms, normalized by the nominal GDP, and showing NBER recessions as shaded areas. Based on the FRED database, we also plot the CD/T-bill spread as a proxy for the market value of liquidity; this spread is often used to measure the liquidity premium as CD is relatively less liquid compared to T-Bills. We also show the cyclical component of both series. These two series correspond to aggregate liquidity, $c_t$, and the value of a unit of liquidity, $q(c_t)$ in our model. In the data, the cyclical components of the two series are negatively correlated, with a coefficient of $-0.3$.

Note that in recessions, liquid financial assets tend to be low but the value of liquidity tends to be high. Indeed, the correlation between the cyclical component of liquid financial assets and the recession dummy is $-0.5$. These observations support our interpretation that recessions are associated with relatively low aggregate holdings of liquid assets and high valuations for liquidity.

As we will explain in the rest of the paper, the general pattern of investment waves, procyclical liquidity holdings, and countercyclical valuation for liquidity are a robust pattern in our economy. These features are present regardless of whether the economy is constrained efficient. It turns out that the efficiency properties of our economy are determined by whether the investment thresholds $c_l^*$ and $c_h^*$ are at their welfare-maximizing level. We examine this issue in the next section.

\(^{11}\) As a monotonically increasing function of $c$, the path of $p(c)$ looks qualitatively similar to the path of $c$, except that it fluctuates between $h$ and $l$ instead of $c_h^*$ and $c_l^*$. 

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Figure 2: Panels A-C depict the price of capital, the value of cash, and the value of capital. The solid vertical line on the right of each graph is at the investment threshold in the planner’s solution, \( c_{P} = 4.03 \); while the two dashed vertical lines are the disinvestment and investment thresholds in our baseline case, \( c_{l} = 1.13 \), \( c_{h} = 3.14 \). The horizontal lines on Panel A are at \( h \) and \( l \). Panels D-F depict a simulated sample path. Horizontal lines on panel D from top to bottom are \( c_{P} \), \( c_{h} \) and \( c_{l} \). Each panel shows objects of both the baseline model with competitive market (thick solid curves) and the planner’s solution (thin, dashed curves). Parameter values are \( K = 4.2 \), \( C = 2 \), \( \sigma = 0.6 \), \( \xi = 0.1 \), \( l = 1.8 \) and \( h = 2 \).
3 Welfare

To study pecuniary externalities, we first solve for constrained efficient allocation in this economy as a benchmark. We then show that our model features a two-sided inefficiency on investment waves: Firms underinvest in capital during downturns and often overinvest during booms.

3.1 Constrained efficient benchmark

We study the constrained efficient allocation with the technological constraint that the aggregate cash has to be kept non-negative by liquidating capital if necessary. Without this technological constraint, condition (2) implies that the planner should convert any amount of cash to capital.

We consider a social planner who can dictate investment policies but cannot know the realization of the idiosyncratic shock. Compared to the market equilibrium, the only difference is that in the market equilibrium investment and disinvestment are driven by the market price of capital. In contrast, the social planner ignores market prices and directly decides when to build or dismantle capital. The resulting outcome corresponds to the solution of the planner’s problem when he controls both investment in the aggregate stage and allocation in the idiosyncratic stage, given self-reporting (see He and Kondor (2012) for a detailed argument).

3.1.1 Social planner’s problem

Denote by \( J_P (K, C) \) the planner’s value function which decides when to build and dismantle capital. By the end of the idiosyncratic stage, at least as long as \( c_t \leq R_K \), due to linearity all cash ends up

Figure 3: Quarterly aggregate liquid financial assets for non-financial US-firms normalized by nominal GDP (calculated as the sum of items 1-14 in Flow of Funds Tables L.102, with NBER recessions as shaded areas); and CD/T-bill spread as a proxy for the market value of liquidity (CD6M/TB6M series in FRED database). Panel A plots the raw series, and Panel B plots the cyclical component applying the Baxter-King filter.
in the hands of cash firms and all capital ends up in the hands of capital firms.\footnote{This result relies on the linearity of technology and can be formally shown by the mechanism design approach (see Additional Material). Also, the conditions of Proposition 1 ensure that in the decentralized case \( \hat{p}_t \leq c_t^h < R_K \), therefore capital firms are willing to use all their cash to buy capital, instead of consuming their cash. However, in the planner’s solution, even for the same parameter values, it might be the case that the support of \( c_t \) is not a subset of \([0, R_K]\). Then, the planner who does not know idiosyncratic firm types cannot ensure that only cash firms are the end users of all cash. While our Propositions 2-6 are stated for the general case, we limit the discussion in the main text to the simpler case when \( c_t \in [0, R_K] \) in the planner’s solution. We show that the Propositions hold in the remaining cases in Additional Material by explicitly solving the planner’s problem based on the mechanism design approach when \( c_t > R_K \) has a positive support.} Therefore, the total value in the idiosyncratic stage is\footnote{Given \((K, C)\), the representative cash firm gets \( R_C (\hat{p}K + C) = R_C (cK + C) = 2CR_C \), while the representative capital firm gets \( R_k K + R_k C / (\hat{p}) = R_k K + \frac{R_k C}{c} = 2KR_K \). As both types are equally likely the expected total welfare is \( KR_K + CR_C \).} 

\[ KR_K + CR_C. \] (14)

Thus, given the aggregate state pair \((K, C)\), since the final date \( \tau \) arrives with exponential distribution with intensity \( \xi \), the social planner is solving

\[ J_P (K, C) = \max_{dK} \mathbb{E} \left[ \int_0^\infty \xi e^{-\xi \tau} (K \tau R_K + C \tau R_C) \, d\tau \mid K_0 = K, C_0 = C \right] \equiv K j_P \left( \frac{C}{K} \right) = K j_P (c) \]

subject to the constraint \( C_t \geq 0 \) and (5). In the second equality in (15), we have invoked the scale-invariance to define \( j_P (c) \) as the planner’s value per unit of capital.

Because of the linear technology, regulation with reflective barriers on \( c \) is optimal (Dixit (1993)). That is, there exists lower and upper thresholds \( c_l^P \geq 0 \) and \( c_h^P > c_l^P \), so that it is optimal to stay inactive whenever \( c \in (c_l^P, c_l^P) \), and dismantle (build) just enough capital to keep \( c = c_l^P \) (\( c = c_h^P \)).

Consider a given policy \( \{c_l, c_h\} \) in which \( c \) is regulated by reflecting barriers \( c_l < c_h \). Given initial state \( K_0 = K \) and \( C_0 = cK \), define the corresponding (scaled) social value as \( j_P (c; c_l, c_h) \), so that

\[ K \cdot j_P (c; c_l, c_h) = \mathbb{E} \left[ \int_0^\infty \xi e^{-\xi \tau} (K \tau R_K + C \tau R_C) \, d\tau \mid K_0 = K, C_0 = cK; c_l, c_h \right]. \] (16)

Using standard results in regulated Brownian motions, \( j_P (c) \) must satisfy

\[ 0 = \frac{\sigma^2}{2} j_P^\mu (c) + \xi (R_K + R_CC - j_P (c)), \quad \text{for} \; c \in (c_l, c_h), \] (17)

and at the reflective barriers \( c_l, c_h \) the smooth pasting conditions must hold:

\[ \frac{\partial [K j_P (c_l; c_l, c_h)]}{\partial K} = l \frac{\partial [K j_P (c_l; c_l, c_h)]}{\partial C}, \quad \text{and} \quad \frac{\partial [K j_P (c_h; c_l, c_h)]}{\partial K} = h \frac{\partial [K j_P (c_h; c_l, c_h)]}{\partial C}. \] (18)
(see Dixit (1993)). The ODE (17) has a closed-form solution

\[ j_P (c; c_l, c_h) = R_K + R_C c + D_1 e^{-\gamma c} + D_2 e^{\gamma c}. \] (19)

For any fixed \( \{c_l, c_h\} \), we solve for the constants \( D_1, D_2 \) based on (18).

Denote by \( \{c_l^P, c_h^P\} \) the social planner’s optimal barrier pair. With a slight abuse of notation, we denote the planner’s optimal value, \( j_P (c; c_l^P, c_h^P) \), simply by \( j_P (c) \):

\[ j_P (c) \equiv j_P (c; c_l^P, c_h^P) = \max_{c_l, c_h} j_P (c; c_l, c_h). \] (20)

Following Dumas (1991), we impose super-contact conditions to determine the optimal barrier pair. For the upper barrier \( c_h^P \), this is

\[ \frac{\partial^2 [K_j P (C/K; c_l^P, c_h^P)]}{\partial K \partial C} \bigg|_{C=Kc_h^P} = \frac{h}{(\partial C)^2} \frac{\partial^2 [K_j P (C/K; c_l^P, c_h^P)]}{\partial K \partial C} \bigg|_{C=Kc_h^P}. \] (21)

For the lower barrier \( c_l^P \), at the optimal choice the constraint \( C \geq 0 \) might bind. Thus, the super-contact condition is a complementarity slackness condition

\[ \frac{\partial^2 [K_j P (C/K; c_l^P, c_h^P)]}{\partial K \partial C} \bigg|_{C=Kc_l^P} \geq l \frac{\partial^2 [K_j P (C/K; c_l^P, c_h^P)]}{\partial C^2} \bigg|_{C=Kc_l^P}, \] with equality if \( c_l^P > 0 \) (22)

The next proposition shows that the optimal lower threshold is \( c_l^P = 0 \). However, the optimal upper threshold is characterized by the unique solution to an analytical equation. We explain the intuition in Section 3.1.3.

**Proposition 2** The planner dismantles capital whenever \( c \) reaches \( c_l^P = 0 \) and builds capital whenever \( c \) reaches a finite, strictly positive investment threshold \( c_h^P \). When the unique solution to the following equation

\[ \frac{R_K - h R_C}{R_K - l R_C} \left( e^{c_h^P \gamma (1 + l \gamma) - (1 - l \gamma) e^{-c_h^P \gamma}} \right) - 2 \gamma (c_h^P + h) = 0 \] (23)

lies in \([0, R_K]\), this solution is the socially optimal investment threshold. The optimal social value \( j_P (c) \) is concave over \([0, c_l^P]\).

While the market price in the aggregate stage is undefined in an economy where the social planner sets the investment and disinvestment thresholds, we can define the shadow price of capital, \( p_P (c) \), as the ratio of the planner’s marginal valuation of capital, \( \frac{\partial J_P(K,C)}{\partial K} \), over that of cash,

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14Heuristically, we can understand the super-contact condition as follows. Converting capital to cash at a cost of \( l \) brings a marginal gain of \( -J_K (c_l^P, K) + J_C (c_l^P, K) \), and the social planner is considering the marginal impact of reducing \( c_l^P \) on this marginal gain, i.e., \( J_K (c_l^P, K) - l J_C (c_l^P, K) \). At the optimal policy this marginal impact is zero. If the optimal policy is binding at \( c_l^P = 0 \), then this marginal benefit of reducing \( c_l^P \) remains strictly positive.
\[
\frac{\partial j_P(K,C)}{\partial C}, \text{ where}
\]
\[
\frac{\partial j_P(K,C)}{\partial C} = j_P(c) \quad \frac{\partial j_P(K,C)}{\partial K} = j_P(c) - cj_P'(c), \quad pp(c) = \frac{j_P(c) - cj_P'(c)}{j_P'(c)}.
\] (24)

We plot these objects in Figure 2 along with market equilibrium counterparts.

### 3.1.2 Investment thresholds, welfare, and expected investment volatility

As a preparation for our welfare analysis, we show that (scaled) social welfare, \( j_P(c; c_l, c_h) \), is monotonic in thresholds in the following sense: It is welfare improving to decrease the lower threshold (increase the upper threshold), whenever it is above (below) the choice of the social planner. This is a strong global result: First, it holds for any policy pair as long as \( c_l > 0 \) and \( c_h < c_P^l \). Second, the sign of welfare impact by changing investment thresholds is unambiguous everywhere.

**Proposition 3** For any \( c_h < c_P^l \) and \( c_l > 0 \), we have

\[
\frac{\partial j_P(c; c_l, c_h)}{\partial c_l} < 0, \quad \text{and} \quad \frac{\partial j_P(c; c_l, c_h)}{\partial c_h} > 0 \quad \text{for all } c \in [c_l, c_h].
\]

It is also useful to define a measure of the volatility of our investment waves. For this purpose, we define the expected total adjustment of capital, parameterized by the thresholds \( c_l, c_h \):

\[
T(c; c_l, c_h) = \mathbb{E} \left[ \int_0^\tau \frac{|dK_t|}{K_t} \right].
\] (25)

**Proposition 4** For any \( c_h \) and \( c_l \), we have

\[
\frac{\partial T(c; c_l, c_h)}{\partial c_l} > 0, \quad \text{and} \quad \frac{\partial T(c; c_l, c_h)}{\partial c_h} < 0.
\]

This proposition states that the expected investment volatility increases in the disinvestment threshold, \( c_l \), and decreases in the investment threshold, \( c_h \). Thus, if in the market equilibrium \( c_h^* < c_P^l \) and \( c_l^* > 0 \), then the economy exhibits more volatile investment compared with that in the constrained efficient benchmark.

### 3.1.3 Investment thresholds in market equilibrium and in the planner’s solution: intuition and comparative statics

As the welfare properties of our economy can be traced back to the investment thresholds, it is useful to understand the economic forces that determine them. As we have established in Propositions 1 and 2, the disinvestment threshold in the market equilibrium, \( c_l^* > 0 \), is strictly positive, whereas the planner disinvests only when it is unavoidable, \( c_l^P = 0 \). In the next proposition, we state further results and then proceed to the intuition.

**Proposition 5** The following results hold.
1. The solution of equation (23) determining the planner’s investment threshold, $c^P_h$

   (a) is converging to 0 as $\gamma \to \infty$, and decreasing in $\gamma$ given that $\gamma > \gamma^*$ for a given $\gamma^*$,

   (b) is decreasing in $l$ and $R_K$, and increasing in $h$ and $R_C$.

2. In contrast, in the market equilibrium determined in Proposition 1, we have

   (a) $c^*_h > h$ and $c^*_i < l$,

   (b) $c^*_h \to h$ and $c^*_i \to l$ as $\gamma \to \infty$.

The planner starts disinvesting only when he is forced to, i.e., $c^P_l = 0$. Intuitively, the planner does not want to dismantle capital as long as he has not run out of cash yet. A positive lower threshold would imply that a part of the cash buffer is never used for maintenance. Because capital is more productive than cash, that would be a waste. His choice of the investment threshold $c^P_h > 0$ is driven by a simple trade-off. While capital is more productive than cash, a cash buffer is useful to avoid the inefficient liquidation of capital in the case of a series of adverse cash-flow shocks.

Consider the role of the constant $\gamma = \sqrt{2\xi}/\sigma$. This parameter enters (23), which characterizes the constrained efficient solution, as well as the functions $q(c)$, $v(c)$ in (12) and (13), which characterize the market equilibrium. Figure 4 plots the planner’s investment threshold $c^P_h$ (dashed) as a function of $\gamma$.

Intuitively, $\gamma$ measures the relative importance of aggregate cash-flow shocks to idiosyncratic investment opportunities. When $\gamma$ is large, aggregate shocks are less important, either because their volatility is low, or because the idiosyncratic shock arrives with high intensity. Regardless of the particular reason, a larger $\gamma$ implies that the planner puts less weight on the possibility that a
sequence of negative cash-flow shocks force him to dismantle capital at the lower threshold. In fact, as Proposition 5 states, as $\gamma$ increases without bound, $c^P_h$ converges to zero as the planner decides not to store any cash (i.e., he will immediately convert any cash to capital) given that capital is relatively more productive $R_K > h R_C$. Figure 4 illustrates that the smaller the $\gamma$ (say, the larger the cash-flow volatility $\sigma$), the more weight the planner puts on the possibility of forced liquidation, and the larger cash buffer the planner wants to keep.

Figure 4 also plots the investment thresholds $c^*_h$ (solid) and the disinvestment threshold $c^*_l$ (dotted) for the market equilibrium. In the market solution, the same trade-off is present, which is behind the fact that $c^*_h$ is decreasing in $\gamma$ (just as $c^P_h$ does). However, there is an additional force: The firm in the market equilibrium knows that in the idiosyncratic stage the price of capital will be $\hat{p}_r = c_r$. A $c_t$ close to 0 implies that holding on to a bit of cash is a good idea because a small amount of cash can be exchanged for a large amount of capital in case the economy enters the idiosyncratic stage. (From the social perspective, the losses and gains from trade in the idiosyncratic stage are a wash.) Hence, the firm liquidates capital well before negative cash-flow shocks deplete all the capital stock, implying that $c^*_l$ is bounded away from 0. That is, the reason to liquidate capital before all cash is depleted is to turn this unit of capital to $l$ units of cash in the aggregate stage, instead of $\hat{p}_r = c_r$ units of cash in the idiosyncratic stage. Clearly, this logic makes sense only if $\hat{p}_r < l$, implying that $c^*_l$ must be smaller than $l$. Symmetric argument implies that $c^*_h$ must be above $h$. In fact, we show that in the limit $\gamma \to \infty$ so that aggregate shocks are unimportant, $c^*_l = l$ and $c^*_h = h$. As firms understand that the price of capital in the idiosyncratic stage is $\hat{p}_r = c_r$, when only that stage matters, they decide to (dis)invest exactly when that price reaches the cost of (dis)investing.

Turning to the other parameters, the higher $l$ and $R_K$, and the lower $h$ and $R_C$ (i.e., the lower the adjustment cost and the higher the relative benefit of capital to cash), the less the cash buffer that the planner is willing to build up. This reduces the upper threshold $c^P_h$, as stated in the second result in Proposition 5.

These results immediately imply that the disinvestment threshold is too high in the market equilibrium. Proposition 5 (or see Figure 4) suggests that the investment threshold $c^*_h$ in the market equilibrium can be either higher or lower than $c^P_h$ in the planner’s solution, depending on the parameters. That is, our economy might feature underinvestment always, or underinvestment during recessions but overinvestment during booms. In the next subsection, we identify the subset of parameters for the latter case, call it a two-sided inefficiency, and further explore the underlying mechanism.

### 3.2 Two-sided inefficiency

The following proposition states the main result of our paper.

**Proposition 6** Under the conditions of Proposition 1, the following statements hold:

1. Firms dismantle capital before the cash-to-capital ratio reaches zero, i.e., $c^*_l > 0$. Hence the
market equilibrium implies underinvestment in capital and over hoarding of cash in recessions.

2. If the difference between the productivity of capital and that of the new investment opportunity, $R_K/h - R_C$ is sufficiently small, then we have $c_h^* < c_h^P$. That is to say, the market equilibrium implies overinvestment in capital during booms.

Figure 2 illustrates a case of two-sided inefficiency. The thin, dashed curves on panels A-D of Figure 2 illustrate the properties of the solution of the planner’s problem. Panels B and C show the planner’s marginal valuation of cash and capital in the aggregate stage, while panel A shows the ratio of the two, which is the shadow price of capital as defined in (24). The dashed (solid) vertical lines show the thresholds of the market equilibrium (planner’s problem). As explained, in the market equilibrium firms dismantle capital when some cash is still around, $c_l^* > 0$. In this example, firms create new capital at a lower liquidity level than the social planner would, $c_h^* < c_h^P$. Panel D contrasts the resulting evolution of cash-to-capital ratio in the planner’s solution and in the market equilibrium under the same sample path of shocks $\{dZ_t\}$. Proposition 4 implies that, in the case of two-sided inefficiency, the resulting investment waves are too volatile in the market equilibrium (illustrated by Panels E and F) in the sense that the expected adjustment intensity of capital is too high.\(^{15}\) Finally, Proposition 3 implies that any policy that raises (decreases) the upper investment (lower disinvestment) threshold would unambiguously increase total welfare in this case.

The reason for the difference between the planner and the market is a wedge between the private and social valuation of capital relative to cash. To see this, consider firms’ marginal rate of substitution (MRS) between capital and cash in the idiosyncratic stage. As the value of capital is $R_K$ and $\hat{p}_r R_C$ and the value of cash is $R_K^* + R_C$ for capital firms and cash firms respectively, the MRS is

$$MRS \equiv \frac{1}{2} \left( \frac{\hat{p}_r R_C + R_K}{R_C + R_K^*} \right) = \hat{p}_r.$$ (26)

Intuitively, when capital price, $\hat{p}_r$, is higher, firms value more the capital they own, because cash firms can sell their capital and capital firms must buy the capital they lack at that higher price. In contrast, the relative social value of capital to cash is always $\frac{R_K}{R_C}$. From the social perspective, the main function of the idiosyncratic stage is that it allocates cash and capital to the highest-value user. The corresponding transfer across the two type of firms in the market equilibrium, pinned down by $\hat{p}_r$, is immaterial for the planner!

The wedge between the social and private valuation in the aggregate stage naturally follows from the wedge in the idiosyncratic stage. Because the price in the decentralized economy guides each individual firm’s investment decisions, it is this wedge that drives the inefficiency of the investment waves in the aggregate stage. What is more, in our model the valuation wedge fluctuates with the aggregate liquidity state. When $\gamma$ is finite, because firms are worried about cash-flow shocks, the

\(^{15}\)When comparing Panels E and F, recall Proposition 4 and the definition of $T(c; c_h, c_l)$. This excess volatility is in terms of the expected total adjustment intensity. Note that we cannot say whether, conditional on investment, the size of the adjustment is larger or smaller in the market equilibrium than in the planner’s solution.
aggregate liquidity holding of firms is procyclical. Our contracting frictions imply that \( \hat{\rho}_t \) positively depends on aggregate liquidity holdings. Therefore, from (26), the private incentive to hold capital instead of cash decreases during recessions with low prices and increases during booms with higher prices, compared with its social counterpart.

Figure 2 shows our mechanism in action. In Panel A, the solid line shows the market value of capital relative to cash in the aggregate stage, and the dashed line shows its social counterpart. The difference between them comes from our wedge.

Although underinvestment in recession is independent of the parameters, whether there is over- or underinvestment during booms depends on the parameter values. For example, consider again the case \( \gamma \rightarrow \infty \). Recall that \( R_K > hR_C \) implies \( c_h^P = 0 \) in this limit, while, as \( \hat{\rho}_t \) drives investment and disinvestment in the decentralized case, \( c_h^* = h \) and \( c_l^* = l \). That is, we have underinvestment both during booms and recessions. To generate overinvestment during booms, we need to make capital less attractive relative to cash for the planner.

As Proposition 5 describes, starting from a very large \( \gamma \), decreasing \( \gamma \) and/or increasing \( R_C \) does exactly that. In fact, as Proposition 6 shows, by decreasing \( R_K - hR_C \), we can raise \( c_h^P \) from zero to any positive level within \([0, R_K]\). While a smaller \( R_K - hR_C \) makes cash more attractive in both the market and planner’s solution, its effect on the market solution is much smaller because of the additional private incentive force we described above. Loosely speaking, this additional force keeps \( c_h^* \) and \( c_l^* \) close to \( h \) and \( l \) in the market equilibrium. Therefore, as Proposition 6 states, when \( R_K - hR_C \) is smaller than a given threshold, \( c_h^P > c_h^* \) i.e., we have overinvestment during booms.

We conclude this analysis with three remarks.

Remark 1 We can push the foregoing point further. So far, our analysis is performed under the parameter restriction of \( \frac{R_K}{R_C} > h \). What if \( l < \frac{R_K}{R_C} < h \), which says that capital is more attractive given the relatively small liquidation benefit and that cash is more attractive given the relatively large capital building cost? In Additional Material we show that, in this case, we always have a two-sided inefficiency. The intuition of the limiting case \( \gamma \rightarrow \infty \), which corresponds to a two-period static model, is rather simple. Given the relatively high adjustment cost, the planner would never want to convert capital to cash or vice versa, implying that \( c_l^P = 0 \) and \( c_h^P = \infty \). But in the market solution with \( c_l^* = l \), \( c_l^* = h \) remains the same as in our base setting, as firms still make their (dis)investment decisions in response to the market price \( \hat{\rho}_t = c_t \). It follows that \( 0 = c_l^P < c_l^* < c_h^* < c_h^P = \infty \), hence the two-sided inefficiency.

Remark 2 We emphasize that our market inefficiency result comes from non-contractible idiosyncratic investment opportunities. Without contracting frictions, say if \( R_K \) and \( R_C \) were pledgeable, \( \hat{\rho}_t = \frac{R_K}{R_C} \) would always hold and there would be no wedge between the private and social relative value of capital to cash. Just as in our baseline model, in the absence of contracting frictions firms still build up cash buffers against negative cash-flow shocks, because the precautionary motive to hold cash is still present. However, in this variant the investment and disinvestment thresholds are the same in both the planner’s solution and the market equilibrium (for a formal proof, see Additional
Material). That is, the precautionary motive alone does not create inefficiency. Nevertheless, in our model this precautionary motive to hold cash interacts with the externality.

Remark 3 Our mechanism is closely related to the main intuition behind the seminal papers on welfare affecting pecuniary externalities of Stiglitz (1982), Geanakoplos and Polemarchakis (1985), and Greenwald and Stiglitz (1986), which are followed by the more recent work of Shleifer and Vishny (1992), Allen and Gale (1994, 2004, 2005), Caballero and Krishnamurthy (2001, 2003), Lorenzoni (2008), Farhi, Golosov and Tsyvinski (2009), Farhi and Tirole (2012) and Gale and Yorulmazer (2011). The critical observation in these papers is that, because of frictions, agents’ marginal utility of wealth might not be equalized across time or states in the decentralized equilibrium. In this case, a price change can work as a transfer from low marginal utility states to high marginal utility states, creating ex ante welfare improvement. Indeed, there is a parallel argument in our model, as the marginal utility of wealth in the idiosyncratic stage is \( \frac{R_K}{p}\) for capital firms and \( R_C \) for cash firms. Whenever \( \hat{p}_r < \frac{R_K}{R_C} \), then the marginal value of wealth is higher for a capital firm. Therefore, if an intervention were to push down the threshold \( c_l^* \) to \( c_l^* - \varepsilon \) at that state, the delayed disinvestment would lower the capital price at the idiosyncratic stage \( \hat{p}_r \), which would be a transfer from the cash firms (sellers of capital) to capital firms (buyers of capital), increasing ex ante utility. However, note that, unlike in many other models in this literature, in our case it is not the transfer per se that is the source of the welfare effect, but the more efficient investment in the aggregate stage.\footnote{To see this, fix the aggregate stage investment but consider an unexpected intervention in the beginning of the idiosyncratic stage. This unexpected intervention transfers \( \varepsilon \) cash from cash firms to capital firms and then allows them to trade, produce, and consume. As cash firms would still exchange all their capital for cash and vice versa, the ex post allocation of the given \( (K, C) \) across firms would remain the same, hence this intervention would not affect ex ante welfare. A similar intervention in the bad state of the interim period of Lorenzoni (2008), i.e., transferring cash from consumers to firms, would change the amount of disinvestment, hence affecting welfare.}

4 Applications

In the first part of this section, as a main application, we discuss the role and limitations of economic policies in our context. In the second part we offer further applications connecting our findings to sectoral investment cycles and financial development.

4.1 Economic policies

Proposition 3 shows how social welfare in our economy changes as the investment and disinvestment thresholds change. However, in a market economy the policymaker cannot set these thresholds directly. Instead, the policymaker might be able to influence the investment/disinvestment threshold by changing the relative incentives of holding cash and capital, that is, by affecting the market price. In this section, we are interested in how various types of economic policies can serve this purpose. We first make the following definition:
Definition 2 A balanced (budget-neutral) policy is an intervention that changes the marginal value of capital only by the transfer scheme $\pi(c) \geq 0$, such that, given $c$, $c\pi(c)$ is the effective transfer for each unit of capital held, and $-\pi(c)$ is the effective transfer for each unit of cash held. An intervention equilibrium is a market equilibrium where the wealth dynamics in (4) is adjusted by transfers.

We refer to the equilibrium objects in an intervention equilibrium by the superscript $\pi$. In an intervention equilibrium, the policymaker influences the outcome only through the effect of $\pi(c)$ on the price in the aggregate stage.

The family of balanced policies is rich, because $\pi(c)$ might be defined and implemented in various ways. The simplest case is to impose a particular transfer between cash holders and capital holders. But the policymaker, to avoid inefficient liquidation, might also target a certain price path, $p^\pi(c)$, which will implicitly define $\pi(c)$. If $\pi(c)$ is positive in some range of $c$, the policymaker might implement $\pi(c)$ by buying a fraction of capital above market prices and selling it back to the market at some point. That is, in our abstract world, it is immaterial to the welfare effect whether policymakers choose to provide subsidies or bailouts to certain industries, to impose measures which affect the collateral value of assets, or to implement asset purchase programs, as long as the implied marginal transfers $\pi(c)$ in these programs are the same.

Note that by the argument derived in Section 3, the (scaled) value of the representative firm in an intervention equilibrium is still $j_P(c; c_l^\pi, c_h^\pi)$ as defined in (20), where $c_l^\pi, c_h^\pi$ are the implied investment/disinvestment thresholds. Therefore, Proposition 3 continues to hold: the welfare effect of a policy can be traced back to its effect on the thresholds.

In the rest of this section, we analyze interventions concentrated on certain stages of our investment waves. Since the 2008 financial crisis, there has been an ongoing debate on the potential adverse effects of interventions during recessions on incentives during booms and, relatedly, on the optimal mix of ex ante regulation and ex post intervention (e.g., Diamond and Rajan, 2011; Farhi and Tirole, 2012; Jeanne and Korinek, 2013). Our modelling approach emphasizes that a policy that, say, makes capital more attractive in a recession, affects the relative value of capital in every other state. What is more, the effect of that policy on the investment threshold in booms feeds back to agents’ welfare in recessions, too. As we demonstrate, when a two-sided externality is present, this interaction adds an interesting layer to this discussion.

We start our analysis with the following definition.

Definition 3 A balanced policy is concentrated on low (high) states if $\pi(c) = 0$ for any $c > c_0$ ($c < c_0$).

The next proposition specifies a simple criterion to decide how such a concentrated policy affects welfare.

Proposition 7 The following statements hold.
1. An intervention concentrated on low states to decrease the disinvestment threshold $c_l^\pi < c_l^\ast$, also reduces the investment threshold $c_l^\ast < c_l^\pi$, if $p^\ast (c_0) > p (c_0)$ and $q^\pi (c_0) \leq q (c_0)$. It increases the investment threshold $c_h^\ast > c_h^\pi$, if $p^\pi (c_0) < p (c_0)$ and $q^\pi (c_0) \geq q (c_0)$.

2. An intervention concentrated on high states to increase the investment threshold $c_h^\pi > c_h^\ast$, also increases the disinvestment threshold $c_l^\ast > c_l^\pi$, if $p^\ast (c_0) < p (c_0)$ and $q^\pi (c_0) \geq q (c_0)$. It decreases the disinvestment threshold $c_l^\pi < c_l^\ast$, if $p^\pi (c_0) > p (c_0)$ and $q^\pi (c_0) \leq q (c_0)$.

Proposition 7 states that, to understand a policy’s welfare consequences, it is sufficient to check the effect of the policy at the single state $c_0$, where the intervention stops. Together with Proposition 3 it also provides clear guidelines to the policymaker. For example, suppose that the economy features two-sided inefficiency. The policymaker might want to avoid inefficient liquidation by implementing a policy that increases the price of capital and decreases the value of cash in recessions. As long as the policy has the same effect at $c_0$, it unambiguously worsens the overinvestment problem during the boom. However, if the policymaker manages to find an alternative that partially avoids inefficient liquidation and decreases the price of capital (without decreasing the value of cash) at $c_0$ at the same time, then the intervention improves welfare everywhere.

To illustrate the usefulness of these guidelines, consider a particular family of policies. There, the policymaker chooses a price floor for capital $l + \delta$ with $\delta \geq 0$ together with a lower disinvestment threshold $c_l^\pi$ with $c_l^\pi < c_l^\ast$, and designs a policy that does not allow the price to fall below $l + \delta$ as long as $c \geq c_l^\pi$. With this intervention, the policymaker ensures that capital is liquidated only at $c_l^\pi$. As we show in Online Appendix B.4, the choice of $\delta$ and $c_l^\pi$ endogenously determines the corresponding transfer scheme $\pi (c)$ and the intervention threshold $c_0$. The policymaker can implement the transfer $\pi (c)$ as a direct subsidy to capital holders, or, for example, initiates a tax-financed program of buying assets at a markup $\psi$ above the market price $p^\pi (c)$ with some state-dependent intensity $\chi (c)$, where

$$\pi (c) = \chi (c) ((p^\pi (c) + \psi) q^\pi (c) - v^\pi (c)).$$

The dashed and dotted curves in Figure 5 show the main equilibrium objects of the intervention equilibrium for a $\delta = 0$ and a $\delta > 0$ price-floor policy with the same liquidation thresholds $c_l^\pi$. The intervention with $\delta = 0$ slightly decreases the price and increases the upper investment threshold $c_h^\pi$, thereby increasing the welfare everywhere. In contrast, the intervention with $\delta > 0$ has the opposite effect. The following proposition states the general result for the $\delta = 0$ case when $\gamma$ is large.

**Proposition 8** A price-floor policy with $\delta = 0$ and any $c_l^\pi < c_l^\ast$ improves welfare at every state $c \in [c_l^\ast, c_h^\ast]$ as long as $\gamma$ is sufficiently large.

---

17 The policymaker can resell the purchased assets to firms at the market price immediately, or after holding them for a given time interval. The latter case might be a reasonable description of the TARP program implemented by the US government in 2008.
The intuition behind the opposite effect of the two interventions is instructive. The high price floor close to the recession increases the value of capital during booms, encouraging more overinvestment. While there is less inefficient liquidation during the recession due to the support for capital holders, firms—even during the recession—foresee the resulting stronger overinvestment during booms. In the numerical example in Figure 5 with $\delta > 0$, the latter effect dominates the earlier and decreases welfare.

In contrast, when the price floor is sufficiently low, the policymaker prolongs the near-recession state of the economy by keeping the price very close to its minimal value through states of mild recovery. This can decrease the relative value of capital when the economy is booming. Thus, even if the policymaker manages to avoid some inefficient liquidation, the intervention can still decrease the value of capital, which makes the overinvestment problem less severe.

One take-away from this experiment is that setting the appropriate price is critical. If the set price for the recession is not sufficiently low, economic agents foresee the induced overinvestment in booms, thus decreasing welfare during both booms and recessions. If the set price is too low, then it does not stop inefficient liquidation. Hence, the policymaker should set a price floor that just discourages firms from selling the assets for lower-user value agents. This policy endogenously
keeps the price of capital low through mild states of recovery, which helps curb incentives for overinvestment during booms.

4.2 Sectoral and aggregate investment waves

In this section we flesh out two further applications that relate our model to sectoral investment waves and financial development. We keep the discussion here brief, and expand on these applications and on the related literature in He and Kondor (2012).

Sectoral investment waves It is well known that certain industries go through boom-and-bust patterns. Hoberg and Phillips (2010) argue that these patterns are widespread in the data, well beyond the handful of well-known episodes such as the 1990s tech-bubble and the 1980s biotech bubble. Interpreting our economy as one sector, our model implies that such cycles arise naturally, even if the technology does not change. The main implication of our mechanism is that in sectors with more non-contractible investment opportunities (e.g., sectors with a larger share of intangible input), other things being equal, these cycles are less efficient. That is, in these sectors too many resources would be spent on frequent adjustment of the capital level, reducing profitability.

Financial development and investment dynamics Our model also suggests a novel rationale for stylized facts on the connection of financial development and investment dynamics. Aghion et al. (2010) provide a useful starting point. The authors decompose aggregate investment to structural and other investment, arguing that structural investment is a proxy for investment in longer-term, riskier, but more productive projects. Then they show that in less financially developed countries structural investment is much more sensitive to productivity shocks, implying a more volatile and more procyclical pattern. They suggest that this difference in the dynamics of the composition of investment activities is an important way in which the lack of financial development hinders growth.

Our results are broadly consistent with the stylized facts in Aghion et al. (2010), if we take capital as a proxy for more productive and riskier projects, and the lack of contractibility on future investment opportunities as a proxy for a low level of financial development. Our two-sided inefficiency implies more volatile investment in capital (Proposition 4), a lower level of expected consumption (Proposition 3), and a lower growth rate of the economy in the long term for less financially developed countries.

5 Robustness

In this section, we present variants of our baseline model to show that the presence of a two-sided inefficiency is not linked to particular technical features of our model. First, we present a variant where the aggregate and idiosyncratic stages are not separated. We show that, due to a similar
intuition in the baseline model, two-sided externalities are present. Second, we show that allowing for collateralized borrowing could make the presence of two-sided externalities even more prevalent.

5.1 Contemporaneous aggregate and idiosyncratic shocks and a single market

We argue in Section 2.2 that the separation of the market into two segments, that in which firms in the aggregate stage trade and that in which firms in the idiosyncratic stage trade, is only a technical device. In this part, we build a variant in which we eliminate this segmentation of markets.

Just as in the alternative interpretation of our baseline case in Section 2.4, here we think of firms facing i.i.d. chances of being hit by idiosyncratic investment opportunities occurring with intensity $\xi$. Applying the law of large numbers, every instant there are a $\xi dt$ fraction of firms hit by the idiosyncratic skill shocks. These firms face the same investment opportunity sets as in our baseline model: with half probability each firm becomes either a capital or a cash firm. Again, as in our baseline, before final production firms can trade their holdings. However, in this variant, both firms with the investment opportunity and those without it trade at the same Walrasian capital price $p_t$. That is, there is no separate price $p^*_t$ for the idiosyncratic stage. Firms with investment opportunity, after exploiting it, exit and consume, but the economy goes on forever with the remaining firms who have not got any investment opportunity yet.

As the baseline model, incumbent firms face aggregate cash-flow shocks according to (1), and they can transfer cash to capital or vice versa at the same linear investment technology as in our base model. We further assume that in the aggregate stage there is another sector that combines capital and cash to produce perishable final consumption goods at the Cobb-Douglas technology $\phi K^\alpha C^{1-\alpha}$, where $\alpha \in (0,1)$ and $\phi > 0$ are positive constants. As we will discuss shortly, the Cobb-Douglas technology with Inada condition is only for ease of illustrating the welfare effect of small policy interventions.

We can solve the model by keeping track of the same state variable, cash-to-capital ratio $c_t = C_t/K_t$. There will be an upper (lower) boundary $c^*_h$ ($c^*_l$) so that firms start investing in (liquidating) capital when $c_t$ hits the boundary from below (above). Inside the inaction region $c_t \in (c^*_l, c^*_h)$, given the endogenous capital price $p(c_t)$, the cash-to-capital ratio follows

$$ dc_t = -\frac{\xi}{2} (p(c_t) + c_t) dt + \frac{\xi c_t}{2} \left(1 + \frac{c_t}{p(c_t)}\right) dt + \frac{\xi}{2} \left(-p(c_t) + \frac{c^2_t}{p(c_t)}\right) dt + \sigma dZ_t $$

Here, we have labeled the extra drift terms relative to (6). For example, $\frac{\xi}{2} dt$ fraction of capital firms causes an outflow of $\frac{\xi}{2} \left(1 + \frac{c_t}{p(c_t)}\right) dt$ on the scaled (by capital $K_t$) aggregate capital, which translates to a positive drift of $\frac{\xi c_t}{2} \left(1 + \frac{c_t}{p(c_t)}\right) dt$ for the cash-to-capital ratio.

Although the closed-form solution is no longer available once the drift of the state variable depends on the endogenous capital price $p(c)$, we can study the market equilibrium by numerically
solving a system of ODEs. For the marginal value of cash \( q(c) \), we have

\[
0 = q'(c) \left( -p(c) + \frac{c^2}{p(c)} \right) + \frac{\sigma^2}{2} q''(c) + \xi \left( \frac{1}{2} \left( \frac{R_C}{p(c)} + \frac{R_K}{p(c)} \right) - q(c) \right) + \phi (1 - \alpha) c^{-\alpha}. \tag{28}
\]

We highlight three terms in (28). The first term captures the drift of the state variable \( c_t \) as firms are exiting the economy. The second term gives the marginal value of cash when hit by idiosyncratic shocks: Each unit of cash either yields \( R_K \) if the firm becomes cash type, or \( \phi (1 - \alpha) c^{-\alpha} \) if capital type, each with half probabilities.

The third term, \( \phi (1 - \alpha) c^{-\alpha} = \partial [\phi K^\alpha C^{1-\alpha}] / \partial C \), which is new, gives the marginal value of cash at the aggregate stage before being hit by idiosyncratic investment opportunities. Due to Inada condition of the Cobb-Douglas technology, this marginal benefit \( \phi (1 - \alpha) c^{-\alpha} \) soars when \( c \) falls towards zero, guaranteeing that in equilibrium firms start liquidating capital for cash before \( c \) hits 0. As a result, the equilibrium disinvestment threshold \( c^* > 0 \) always takes an interior solution, which helps us illustrate the effect of a small distortionary tax scheme that we consider later. For details, see Additional Materials.

Similarly the marginal value of cash \( v(c) \) satisfies

\[
0 = q'(c) \sigma^2 + v'(c) \left( -p(c) + \frac{c^2}{p(c)} \right) + \frac{\sigma^2}{2} v''(c) + \xi \left( \frac{R_C p(c) + R_K}{2} - v(c) \right) + \phi \alpha e^{1-\alpha}. \tag{29}
\]

Finally, to pin down the market equilibrium, we have the same boundary conditions as in the base model

\[
v'(c^*_h) = q'(c^*_h) = q'(c^*_l) = v'(c^*_l) = 0, \tag{30}
\]

and (dis)investment optimality conditions are

\[
\frac{v(c^*_h)}{q(c^*_h)} = h, \quad \text{and} \quad \frac{v(c^*_l)}{q(c^*_l)} = l. \tag{31}
\]

For the planner’s solution, we consider a policy from the family of balanced tax/subsidy considered in Section 4.1:

\[
c \pi (c) = \begin{cases} 
-\pi_h & \text{if } c > (1 - \kappa_h) c^*_h \\
\pi_l & \text{if } c < (1 + \kappa_l) c^*_l \\
0 & \text{otherwise}
\end{cases} \tag{32}
\]

where \( \pi_h, \pi_l, \kappa_h \) and \( \kappa_l \) are positive constants. This policy taxes capital during booms (specifically, when the economy is close to investment boundary \( c^*_h \), or, a \( \kappa_h \) fraction below \( c^*_h \)) and/or subsidizes capital during recessions (specifically, when the economy is close to disinvestment boundary \( c^*_l \), or, a \( \kappa_l \) fraction above \( c^*_l \)). Budget-neutral policies imply that cash receives transfers of \( \pi (c) \). We obtain the new market equilibrium with intervention by numerically solving the new in-
vestment/disinvestment thresholds \((c_l^g, c_h^g)\), joint with new marginal capital and cash values \(v^g(\cdot)\) and \(q^g(\cdot)\), respectively.\(^{18}\) As before, the (scaled) social welfare in the economy is measured as \(j^g(c) = v^g(c) + cq^g(c)\).

### 5.1.1 Two-sided inefficiency

As an illustration, the solid curves in Panels A, B, and C on Figure 6 show the price of capital, the value of capital, and the value of cash, all as a function of \(c\) in the alternative setting. We observe that, in the alternative setting without segmented markets, the equilibrium objects are qualitatively similar to those in the base model. The dashed curves on the same panels show the corresponding objects under the policy intervention specified in (32). We observe that \(c_l^g < c^*_l\) and \(c_h^g > c^*_h\). The intuition is clear: As individual firms adjust their real decisions based on market prices, taxing (subsidizing) capital during booms (recessions) postpones the market investment (disinvestment) during that time. Because the policy intervention is small (we set \(\pi_h = \pi_l = 1.5\%\) and \(\kappa_h = \kappa_l = 7\%\) in this numerical example), the quantitative effect of this policy in Panels A, B, and C is only slightly visible.

Panel D shows the resulting improvement in social surplus for this policy (“two-sided intervention,” solid, U-shaped curve). To show the two-sided inefficiency more convincingly, we also calculate the change in social welfare under two different one-sided intervention policies, i.e., either only taxing capital during booms (“upper intervention” sets \(\pi_l = 0\), dotted, increasing curve) or only subsidizing capital during recessions (“lower intervention” sets \(\pi_h = 0\), dashed, decreasing curve). Each policy generates a strictly positive value improvement, and the curve of the two-sided intervention is the upper envelope of the one-sided interventions. These results imply the two-sided externality with underinvestment during recessions and overinvestment during booms in this economy.

For this example, we chose parameters satisfying \(l < \frac{R_K}{R_C} < h\) and picked a large \(\gamma\). Then, the intuition for two-sided externality follows from the argument in Remark 1 in Section 3.2. In particular, the parameters imply that the firm’s private relative valuation of capital to cash, i.e., its private \(MRS\) (which is close to \(p_t \in [l, h]\) according to the arguments in Section 3.2) will vary around its social counterpart (which is close to \(\frac{R_K}{R_C}\) according to the arguments in Section 3.2). Hence, in booms, when \(p_t\) is close to \(h\), the private valuation of capital is higher than its social counterpart. This implies that the investment threshold is lower (so the firm invests earlier) in the market equilibrium. The opposite argument holds during recessions, implying two-sided inefficiency.

### 5.2 Collateralized borrowing

We now go back to the base model and introduce collateralized borrowing. We present here only a sketch of this extension and the main results. A detailed analysis can be found in Additional Material, which is available on the the author’s website.

\(^{18}\)We have the same boundary conditions \(p(c_l^g) = l\) and \(p(c_h^g) = h\), but modify the ODEs slightly to reflect transfers: For capital, we need to add \(\epsilon \pi(c)\) in (28); for cash we need to subtract \(\pi(c)\) in (29).
Figure 6: Panels A-C depict the price of capital, the value of cash, and the value of capital in the alternative setting both in the market equilibrium (solid) and when the planner decreases the lower threshold and increases the upper threshold (dashed). Panel D depicts the percentage change in value due to the intervention. The vertical lines in each panel from left to right are the disinvestment threshold of the intervention and in the market equilibrium, $c^l = 0.6457$, $c^l = 0.7857$; and the investment threshold in the market equilibrium and of the intervention, $c^h = 3.5729$, $c^h = 3.7620$.

Parameter values are $R_K = 4.2$, $\sigma^2 = 1$, $\xi = 5$, $R_C = 2$, $l = 1.7$, $h = 2.6$, $\alpha = 0.5$, $\kappa_h = \kappa_l = 7\%$, and $\pi_h = \pi_l = 1.5\%$. We use MATLAB built-in ODE solver bvp4c to solve the model, with a convergence criterion of $10^{-7}$.

We assume that installing one unit of capital allows the firm to borrow a constant $b \in [0, l)$ units of cash from external creditors, during both the aggregate and the idiosyncratic stages, and there are no other borrowing technologies allowed. The welfare accounting remains the same, because external creditors obtain zero rent always. We can characterize both the market equilibrium and the planner’s solution in closed form for the economy with collateralized borrowing, using the fact that, in equilibrium, firms always maximize out their borrowing capacity.

Our main observation is that for a set of economies with small $h - l$, in the absence of borrowing firms underinvest even during booms, but, once allowing for collateralized borrowing, firms will start overinvesting during booms. This result indicates the potential social cost of excessive collateralized borrowing. Formally, define $\varepsilon \equiv h - l$, and imagine improved borrowing technologies modeled as an increasing sequence $b(k) > 0$ for $k \in (0, 1)$, so that $b(k) \equiv l - O(\varepsilon^k)$. The higher the $k$, the smaller the distance $O(\varepsilon^k)$, hence the higher the borrowing capacity $b(k)$ fixing $l$.

**Proposition 9** We have the following results:

For example, firms cannot borrow against the new investment opportunities at the idiosyncratic stage, potentially because of the non-contractible nature of new opportunities. Here, $b$ can be interpreted as the inefficient recovery that external creditors can obtain if the borrower reneges. This is in the tradition of Kiyotaki and Moore (1997), but to avoid complication we do not link the borrowing capacity to endogenous capital prices.
1. When $\varepsilon \to 0$, in the market equilibrium both $c_h^* \text{ and } c_l^*$ converges to $l$. However, in the social planner’s solution we have $c_l^P = 0 \text{ and } c_h^P \to 0$. Hence, there is underinvestment during both booms and recessions.

2. For sufficiently small $\varepsilon$, there is overinvestment during the boom $c_h^*(k) < c_h^P(k)$ for $k > 1/3$.

For the first part, from the planner’s view, cash-flow shocks pose little risk given little adjustment cost (as $h$ is close to $l$). Since capital is more productive ($R_K > hR_C$), the planner should hold almost no cash (i.e., $c_h^P \to 0$). However, the wedge between social and private incentives of holding capital remains. If $c_h^* \to h$, $c_l^* \to l$, and $h \to l$, then the price in the idiosyncratic stage is close to $l$, which suggests that in the competitive market firms liquidate capital as $c_l$ falls below $l$ and build as it rises above $l$.

The second result of Proposition 9 gives the main point of this subsection. A higher $k$ implies a higher $b(k)$, which increases the value of capital for both the planner and the market. We show that the positive effect on the market is stronger and, as $b(k)$ gets close to $l$, at some point $c_h^* < c_h^P$, i.e., overinvestment during booms. Intuitively, the planner’s solution is mainly determined by the adjustment cost $h - l = \varepsilon$. For the market, as $k$ increases, a higher $b(k) = l - O(\varepsilon^k)$ drives up $l$ and $h = l + \varepsilon$, which raises the capital price in the idiosyncratic stage and hence also strengthens the private incentives to hold capital in the aggregate stage. Therefore the increase of the borrowing capacity leads to a faster decrease in $c_h^*$ than in $c_h^P$.

6 Conclusion

We build an analytically tractable, dynamic stochastic model of investment and savings, in which investment cycles, i.e., boom periods with abundant investment and bust periods with low investment, arise naturally. In the presence of non-contractible idiosyncratic investment opportunities, a two-sided inefficiency can arise: there is too much investment in risky capital and cash buffers are too low during booms, and there is too little investment and too much cash hoarding during recessions. We show that in this case a one-sided policy targeting only the underinvestment during downturns might be ex ante Pareto inferior to the absence of intervention in all states (including downturns).

We acknowledge that there are standard ways to eliminate the inefficiency studied in this paper. In the working paper of this article (He and Kondor (2012)), we investigate this question in a simplified two-period setting. From an ex ante perspective, we show that the market can be completed by introducing Arrow-Debreu securities contingent on the realization of idiosyncratic opportunities, which restore investment incentives of individual firms in the competitive market. However, if idiosyncratic investment opportunities are not verifiable, so that the enforcement of Arrow-Debreu securities requires self-reporting, in general private investment incentives are still distorted away from the social ones. From an ex post perspective, if the final production output is fully pledgeable, then there will be no wedge between the social and private value of capital to cash,
even without Arrow-Debreu securities.\footnote{To understand this, note that at the idiosyncratic stage, a cash firm will attach a marginal value of $R_K$ to its capital, because such a cash firm, instead of selling its capital at the market, can hire another capital firm to operate its capital and extract all the output from production. Similarly, a capital firm values its cash by $R_C$, and the private value of capital to cash is always the social value ratio $R_K/R_C$.} Nevertheless, this result breaks down once we introduce imperfect delegation, which can be further microfounded by lack of commitment, hidden effort, or even information processing. In sum, just as in our two-period setting the social value ratio is independent of the state of cash-to-capital, our key result holds as long as market imperfections lead the price of capital to increase with the cash-to-capital ratio (in our model the idiosyncratic stage price $\hat{p}_r = c_r$ takes its extreme form due to the cash-in-the-market pricing).

Apart from analyzing two-sided inefficiencies, we also presented a novel way of modeling dynamic investment and savings. This method provides analytical tractability in a dynamic stochastic framework for the full joint distribution of states and equilibrium objects. Exploring the potential of the developed framework in various other contexts is a task for future research.

References


He, Zhiguo, and Arvind Krishnamurthy. 2012. “Intermediary Asset Pricing.” *American Eco-


A Appendix

In this Appendix we provide proofs for Propositions 2, 3, 4, the first part of Proposition 5 and Proposition 6. The proof of Proposition 9 is given in Additional Material available on the author’s website. Proofs for all the other results are provided in Online Appendix.
A.1 Proof of Proposition 2

Based on boundary conditions $R_K + D_1 + D_2 = l(R_C - \gamma D_1 + \gamma D_2)$ and $D_1 e^{-\gamma c_h^P} + D_2 e^{\gamma c_h^P} = 0$, the solutions for $D_1$ and $D_2$ are given by

$$
D_1 = -\frac{(R_K - lR_C) e^{2\gamma c_h^P}}{(1 + l\gamma) e^{2\gamma c_h^P} - (1 - l\gamma)}, \quad D_2 = \frac{R_K - lR_C}{(1 + l\gamma) e^{2\gamma c_h^P} - (1 - l\gamma)}. \tag{A.1}
$$

To verify that $c_h^P = 0$, we need to show that $j''_P(0) < 0$:

$$
\frac{1}{\gamma^2} j''_P(0) = D_1 + D_2 = -\frac{1}{(R_K - lR_C)} \frac{e^{2\gamma c_h^P} - 1}{e^{2\gamma c_h^P} + l\gamma (e^{2\gamma c_h^P} + 1)} < 0.
$$

The super-contact condition at the optimal upper investment threshold $c_h^P$ is

$$
0 = \frac{\partial^2 j_P (c; c_h^P, c_h^P)}{\partial c} \bigg|_{c=c_h^P} = \gamma^2 \left( D_1 e^{-\gamma c_h^P} + D_2 e^{\gamma c_h^P} \right). \tag{A.2}
$$

To show $c_h^P$ exists and unique, define a function $G(c)$ ($G(c)$ is proportional to (A.2) if we plug $D_1$ and $D_2$ in (A.1) into (A.2))

$$
G(c) = \frac{R_K - h R_C}{R_K - lR_C} (e^{c\gamma} (1 + l\gamma) - (1 - l\gamma) e^{-c\gamma}) - 2\gamma (c + h), \tag{A.3}
$$

with $G(0) = 2R_K \gamma \frac{l-h}{R_K - lR_C} < 0$ (recall $R_K - h R_C > R_K - lR_C > 0$) and $G(\infty) = \infty$. We have

$$
G'(c) = \gamma \left( \frac{(R_K - h R_C)}{(R_K - lR_C)} ((l\gamma + 1) e^{c\gamma} + e^{-c\gamma} (1 - l\gamma)) - 2 \right),
$$

$G'(0) = 2R_K \gamma \frac{l-h}{R_K - lR_C} < 0$, and $G'(c)$ changes sign only once. Consequently, there is a unique $\hat{c}$ that $G'(\hat{c}) = 0$, implying that $G(c)$ is decreasing for $c < \hat{c}$ and increasing for $c > \hat{c}$. As $G(0) < 0$ and $G(\infty) = \infty$, there must be a unique $c_h^P$ that $G(c_h^P) = 0$, verifying the equation (23).

The social planner’s value function $j_P(c)$ satisfies

$$
0 = \frac{\sigma^2}{2} j''_P(c) + \xi (R_K + R_C c - j_P(c)) \tag{A.4}
$$

with boundary conditions $j''_P(0) = j''_P(0)$, $j_P(c_h^P) = (h + c_h^P) j'_P(c_h^P)$, and $j''_P(c_h^P) = 0$. Note that the boundary conditions imply that $j_P(c_h^P) = R_K + R_C c_h^P$. For later reference, we show that $j_P(c)$ is concave and increasing over $[0, c_h^P]$, and $j_P(c) < R_K + R_C c$ for $c \in [0, c_h^P]$ First, from smooth pasting condition at $c_h^P$ we have

$$
R_C - j'_P(c_h^P) = R_C - \frac{j_P(c_h^P)}{h + c_h^P} = R_C - \frac{R_K + R_C c_h^P}{h + c_h^P} = \frac{R_K - R_C h}{h + c_h^P} < 0.
$$

Then, taking derivative again on (A.4) and evaluate at the optimal policy point $c_h^P$, we have

$$
j''_P(c_h^P) = -\frac{2\xi}{\sigma^2} (R_C - j'_P(c_h^P)) = -\frac{2\xi}{\sigma^2} \frac{R_K - R_C h}{h + c_h^P} > 0,
$$

and as a result $j''_P(c_h^P) < 0$. Suppose that $j_P$ fails to be globally concave over $[0, c_h^P]$. Then there exists
some point $j''_p > 0$, and pick the largest one $\hat{c}$ so that $j''_p$ is concave over $[\hat{c}, c^p]$, with $j''_p (\hat{c}) = 0$ and $j''_p (\hat{c}) < 0$. But since $j''_p$ is concave over $[\hat{c}, c^p]$, $j''_p (\hat{c}) > j''_p (c^p) > R_C$, therefore $\frac{\partial}{\partial c} j''_p (\hat{c}) = \xi (j''_p (\hat{c}) - R_C) > 0$, contradiction. Therefore $j_p$ is globally concave over $[0, c^p]$, which also implies that $j_p (c) < R_K + R_C c$ for $c \in [0, c^p]$ due to (A.4).

### A.2 Proof of Proposition 3

Suppose that we are given the policy pair $(c_l, c_h)$ with $0 < c_l < c_h < c^p$ where $c^p$ satisfies the super-contact condition $j''_p (c^p) = 0$. To avoid cumbersome notation we denote the social value $j_p (c_l, c_h)$ given the policy pair $(c_l, c_h)$ by $j (c_l, c_h)$, and denote the social value under the optimal policy $j_p (0, c^p)$ by $j_p (c)$. We need to show that

$$\frac{\partial j_p (c_l, c_h)}{\partial c_l} < 0 \text{ and } \frac{\partial j (c_l, c_h)}{\partial c_h} > 0.$$  

This result further implies that for $0 < c_l^1 < c_l < c_l^2 < c_l^3$, we have $j (c_l^1, c_l^2) < j (c_l^2, c_l^3)$. As preparation, we first show that $j'' (c_l, c_h) < 0$ and $j'' (c_l, c_h) < 0$. Because $(c_l, c_h)$ is sub-optimal, we must have $j (c_l, c_h) < j (c, c_h) < j (c, c_h)$ for all $(c_l, c_h)$, which implies that $j (c_l, c_h)$ is strictly concave at both ends. Second, for any policy pair $(c_l, c_h)$ (including the market solution or the planner’s solution), the smooth pasting condition (not optimality condition) at the regulated ends implies that

$$j (c_l, c_h) - (c_h + h) j' (c_l, c_h) = 0,$$

$$j (c_l, c_h) - (c_l + l) j' (c_l, c_h) = 0.$$  

Now we start proving the properties for the upper investment policy $c_h$. Define $F_h (c_l, c_h) = \frac{\partial}{\partial c_l} j (c_l, c_h)$, which is the marginal impact of changing the investment policy on the social value. Differentiating the ODE (A.7) by the policy $c_h$, we have $\frac{\partial^2}{\partial c_h^2} j'' (c_l, c_h) - \xi \frac{\partial}{\partial c_h} j (c_l, c_h) = 0$, or

$$\frac{\partial^2}{\partial c_h^2} F''_h (c_l, c_h) - \xi F_h (c_l, c_h) = 0.$$  

Moreover, take the total derivative with respect to $c_h$ on the equality (A.5), i.e., take derivative that affects both the policy $c_h$ and the state $c = c_h$, we have

$$\frac{\partial}{\partial c_h} j (c_l, c_h) + j' (c_l, c_h) + \frac{\partial}{\partial c_h} \left( j'' (c_h, c_l, c_h) \right) = 0$$

$$\Rightarrow \frac{\partial}{\partial c_h} j (c_l, c_h) = (c_l + l) \frac{\partial}{\partial c_h} j' (c_l, c_h) = (c_h + h) F''_h (c_l, c_h) - \xi F_h (c_l, c_h) < 0.$$  

which gives the boundary condition of $F_h (\cdot)$ at $c_h$. At $c_l$ we can take total derivative with respect to $c_h$ on the equality (A.6), we have the boundary condition of $F_h (\cdot)$ at $c_l$:  

$$\frac{\partial}{\partial c_h} j (c_l, c_h) = (c_l + l) \frac{\partial}{\partial c_h} j' (c_l, c_h) \Rightarrow F_h (c_l, c_h) - (c_l + l) F''_h (c_l, c_h) = 0.$$  

With the aid of these two boundary conditions, the next lemma shows that $F_h (\cdot)$ has to be positive always. Because of the definition of $F_h (c_l, c_h) = \frac{\partial}{\partial c_l} j (c_l, c_h)$, it implies that raising $c_h$ given any state $c$ and any lower policy $c_l$ improves the social value. The argument for the effect of $c_l$ is similar and thus omitted.

**Lemma A.1** We have $F_h (c) > 0$ for $c \in [c_l, c_h]$.

**Proof.** We show this result in two steps.
1. \( F_h(c) \) cannot change sign over \([c_l, c_h]\). Suppose that \( F_h(c_l) > 0 \); then from (A.9) we know that \( F'_h(c_l) > 0 \). Then simple argument based on ODE (A.7) implies that \( F_h(\cdot) \) is concave and always positive. Now suppose that \( F_h(c_l) < 0 \); then the similar argument implies that \( F_h \) is concave and negative always. Finally, suppose that \( F_h(c_l) = 0 \) but \( F_h \) changes sign at some point. Without loss of generality, there must exist some point \( \hat{c} \) so that \( F'_h(\hat{c}) = 0 \), \( F_h(\hat{c}) > 0 \) and \( F''_h(\hat{c}) < 0 \). But this contradicts with the ODE (A.7).

2. Define \( W_h(c) \equiv F_h(c) - (l + c) F'_h(c) \) so that

\[
W'_h(c) = -(l + c) F''_h(c) = -\frac{2\xi (l + c)}{\sigma^2} F_h(c).
\]

As a result, \( W'_h(c) \) cannot change sign. Because we have \( W_h(c_l) = 0 \), \( W_h(c) = 0 \) cannot change sign either.

3. Now suppose counterfactually that \( F_h(c) < 0 \) so that \( W'_h(c) > 0 \). Then (A.10) in Step 2 implies that \( W_h(c) > 0 \), and hence \( F'_h(c_h) = \frac{1}{(l + c_h)} (F_h(c_h) - W_h(c_h)) < 0 \). But we then have

\[
W_h(c_h) = F_h(c_h) - (l + c_h) F'_h(c_h) = F_h(c_h) - (h + c_h) F'_h(c_h) + (h - l) F'_h(c_h) < 0,
\]

where we have used (A.8). This contradicts with \( W_h(c) > 0 \). Thus we have shown that \( F_h(c) > 0 \).

\[\blacksquare\]

### A.3 Proof of Proposition 4

The expected total investment activity \( T(c) \) solves \( \frac{\sigma^2}{2} T''(c) = \xi T(c) \) with boundary conditions \( T'(c_l) = \frac{1}{l + c_l} \) and \( T'(c_h) = \frac{1}{h + c_h} \). For example, at \( c = c_h \), a positive shock hits with \( c = c_h + \epsilon \). To get back to the upper cash-to-capital ratio \( c_h \), the economy builds new capital of \( dK = \frac{K}{h + c_h} \); thus, we have

\[
T(c_h + \epsilon) = \frac{dK}{K} + T(c_h) = \frac{\epsilon}{h + c_h} + T(c_h) \Leftrightarrow T'(c_h) = \frac{1}{h + c_h}.
\]

Now we study the impact of policies \( c_l \) and \( c_l \) on \( T(\cdot;c_l,c_h) \). For illustration we analyze \( c_l \) only; a similar argument applies to \( c_h \). Define \( F(c) \equiv \frac{\partial}{\partial c} T(c;c_l,c_h) \); we have

\[
\frac{\sigma^2}{2} T''(c;c_l) = \xi T(c;c_l) \Rightarrow \frac{\sigma^2}{2} F''(c;c_l) = \xi F(c;c_l).
\]

To determine boundaries for \( F \), at \( c_h \) we have \( T'(c_h;c_l) = \frac{1}{l + c_l} \) which implies that

\[
F'(c = c_h) = \frac{\partial}{\partial c_l} T'(c = c_h;c_l) = 0.
\]

On the other end, \( T'(c_l;c_l) = \frac{1}{l + c_l} \) implies that \( F'(c = c_l) + T''(c = c_l;c_l) = -\frac{1}{(l + c_l)^2} \) or

\[
F'(c = c_l) = -\frac{1}{(l + c_l)^2} - T''(c = c_l;c_l) < 0.
\]

Here we used the fact that \( T''(c = c_l;c_l) > 0 \); this fact is implied by (A.11) together with \( T(c) > 0 \) by definition.
Now we show that $F(c) > 0$ so that the total investment activity goes up for a higher $c$. To see this, first note that $F(c)$ never changes sign. Otherwise, suppose that there exists some $c_1$ so that $F(c_1) = 0$. If $F'(c_1) > 0$ then it must be that $F$ is convex and positive for $c > c_1$, which contradicts with $F'(c_h) = 0$. Similarly we rule out $F'(c_1) < 0$. If $F'(c_1) = 0$, then combining with $F(c_1)$ we can solve for $F(c) = 0$ for all $c$, contradicting with $F'(c_1) < 0$.

Now since $F(c)$ never changes sign, it suffices to rule out $F(c) < 0$ always. If it were true, then $F$ is concave always due to (A.11). This contradicts with $F'(c_1) < 0 = F'(c_h)$. As a result, $F(c) > 0$.

A.4 Proof of Proposition 5

Recall $G(c)$ defined in (A.3). Since fixing $c$ we have

$$\lim_{\gamma \to \infty} \frac{R_K - h R_C}{R_K - l R_C} \left( e^{c \gamma} (1 + l \gamma) - (1 - l \gamma) e^{-c \gamma} \right) = \infty,$$

to ensure that $G(c_h^s) = 0$ as $\gamma \to \infty$ we must have $c_h^s \to 0$. This is the first part of the first statement.

For the second part of the first statement,

$$\frac{\partial G(c)}{\partial \gamma} = \frac{R_K - h R_C}{R_K - l R_C} (e^{c \gamma} (1 + l \gamma) + l e^{c \gamma} - c (l \gamma - 1) e^{-c \gamma} + le^{-c \gamma}) - 2 (c + h),$$

which is positive for sufficiently large $\gamma$. Finally, from the proof of Proposition 2 we know that $G'(c_h^s) > 0$. Hence, for sufficiently large $\gamma$, we have $\frac{\partial^2 G(c)}{\partial \gamma^2} = -\frac{\partial G(c_h^s)/\partial \gamma}{G'(c_h^s)} < 0$ which concludes the first part. The second part follows because $\frac{R_K - h R_C}{R_K - l R_C}$ is increasing in $R_K$ and decreasing in $R_C$, and

$$\frac{\partial G(c)}{\partial h} = -\frac{R_C}{R_K - l R_C} (e^{c \gamma} (1 + l \gamma) - (1 - l \gamma) e^{-c \gamma} - 2 \gamma < 0,$$

$$\frac{\partial G(c)}{\partial l} = (R_K - h R_C) \frac{R_C (1 - e^{-2c \gamma}) + R_K \gamma + R_K \gamma e^{2(-c \gamma)}}{e^{-c \gamma} (R_K - l R_C)^2} > 0.$$

Finally, fixing any $c$ we have $\lim_{R_K \to h R_C} G(c) = -2 \gamma (c + h) < 0$ always. This implies that for $\lim_{R_K \to h R_C} G(c_h^s) = 0$ to hold, it must be that $c_h^s \to \infty$ so that $\lim_{R_K \to h R_C} \frac{R_K - h R_C}{R_K - l R_C} (e^{c \gamma (1 + l \gamma)} - (1 - l \gamma) e^{-c \gamma}) \to 2 \gamma$. This concludes the first statement of the proposition. The second statement is in Online Appendix.

A.5 Proof of Proposition 6

The first statement comes from point 2 of Proposition 1. For the second statement, note that the proof of Proposition 1 goes through without any modification for the case when $R_K = R_C h$. That is, even in the limit $R_C h \to R_K$, the investment threshold in the market equilibrium $c_h^s$ is finite, and under the parameter restriction of Proposition 1 we have $c_h^s < R_K$. However, note that given any parameters, in the limit $R_C h \to R_K$ the solution to equation (23) diverges to $\infty$. Due to continuity, we can find $R_K - R_C h$ appropriately small so that $c_h^s$ sufficiently close to $R_K$ and hence $c_h^s < c_h^s$. And, even if the solution to (23) is above $R_K$, we can show that the resulting optimal investment threshold $c_h^s$ lies above $R_K$ and hence $c_h^s < R_K < c_h^s$. The details of this argument are in the Additional Material available on the author’s website. This completes the proof.
B Online Appendix for He and Kondor (2015): Proofs and Derivations

In this Online Appendix we provide proofs for Lemma 1 and Proposition 1, the second part of Proposition 5, and Propositions 7, 8 and 9.

B.1 Proof of Lemma 1 and Proposition 1

We construct the proof in steps. In particular, we separate Proposition 1 into the following four Lemmas. These four lemmas are sufficient to prove Proposition 1.

Lemma B.2 If the equation system (12)-(13), (7)-(9) has a solution where $c_h < R_K$, and both $v(c)$ and $q(c)$ are increasing in the range $c \in [c_l^*, c_h^*]$, then Proposition 1 holds.

Lemma B.3 The system (12)-(13), (7)-(9) always has at least one solution.

Lemma B.4 If $h - l$ is sufficiently small, then $c_h < R_K$.

Lemma B.5 $q(c)$ is decreasing in $c$. If $h - l$ is sufficiently small, then $v(c)$ is increasing for $c \in [c_l^*, c_h^*]$.

B.1.1 Step 1: Proof of Lemma 1 and Lemma B.2

Denote the dollar share of capital in the firm’s asset holdings by $\psi^i_t$, so that $\psi^i_t = K^i_t p_t / w^i_t$. According to our conjecture, the value function can be written as (recall the aggregate cash-to-capital ratio $c = C/K$)

$$J(K_t, C_t, K^i_t) = w^i_t \left( 1 - \psi^i_t q(c_t) + \frac{\psi^i_t}{p_t} v(c_t) \right)$$

is linear in $w_t$. This is equivalent to $J(C, K, K^i_t) = K^i_t v(c) + C^i_t q(c)$ stated in the Lemma. Also, we have the wealth dynamics, expressed in terms of capital share $\psi^i_t$, as

$$dw^i_t = -d\alpha^i_t - \theta dK^i_t + \psi^i_t w^i_t \frac{1}{p_t} (dp_t + \sigma dZ_t).$$

And, $q(c) \geq 1$ has to hold as firms can consume cash at the final date (and there is no discounting), which implies $d\alpha^i_t = 0$, i.e., firms do not consume in the aggregate stage.

As the firm is choosing capital share $\psi^i_t$, and the capital to build or dismantle $dK^i_t$, the Hamiltonian-Jacobi-Bellman (HJB) of problem (3) can be written as:

$$0 = \max_{d\psi^i_t, dK^i_t} d\alpha^i_t + J_C \mathbb{E}_t [dC_t] + \frac{1}{2} J_{CC} \mathbb{E}_t [dC_t^2] + J_w \mathbb{E}_t (dw_t) + J^K dK^i_t + J_{w,C} \mathbb{E}_t [dw_t dC_t].$$

The endogenous price dynamics (using Ito’s Lemma) is

$$dp_t = \frac{1}{2} \sigma^2 p''(c) dt + \sigma p'(c) dZ_t + dB^p_t - dU^p_t,$$

where $dB^p_t (dU^p_t)$ reflects $p$ at $p(c^*_l) = l$ ($p(c^*_h) = h$). This is because in any market equilibrium firms will create (dismantle) capital if $p_t = h$ ($p_t = l$), and keep doing it until the price adjusts. We derived the boundary conditions in the main text. Also, by risk neutrality and the initial homogeneity of firms, before the final date the price of the capital has to make firms indifferent whether to hold capital or cash. Otherwise markets could not clear. We also explained that $\tilde{p}_r = c_r$.  

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Thus, inside the reflection boundary \((c^*_l, c^*_h)\) the above HJB equation is (we drop \(i\) from now on)

\[
0 = \max_{\psi_i} \left\{ \frac{\sigma^2}{2} w_i q''(c_i) + q(c_i) \psi_i w_i \frac{\sigma^2 p''(c_i)}{p'_i} + q'(c_i) \left( \psi_i w_i \frac{\sigma + p'(c_i) \sigma}{p'_i} \right) \right\}.
\]

Since the problem is linear in \(\psi_i\), in equilibrium firms must be indifferent in their choice of \(\psi_i\). Thus, we can calculate the dynamics of the cash (capital) value by choosing \(\psi_i = 0\) \((\psi = 1)\). Setting \(\psi_i = 0\) directly implies (10). Choosing \(\psi_i = 1\) gives

\[
0 = \frac{\sigma^2}{2} q''(c) + q(c) \left( \frac{1}{p} (\sigma + p' \sigma) \sigma \right) + \frac{1}{p} \left( \frac{\xi}{2} (R_K + R_C) c - q(c) p \right).
\]

Since \(v(c) = p(c) q(c)\), \(v' = q'p + p'q\), and \(v'' = q''p + 2q'p' + p''q\), we can rewrite the above equation as (11).

Given that the ODEs for \(v(c)\) and \(q(c)\) were derived by substituting in \(\psi_i = 1\) and \(\psi_i = 0\), it is easy to see that these functions can be interpreted as the value of a capital and that of a unit of cash. This implies that

\[
J(C, K, w_i) = \left( w_i (1 - \psi_i) q(c) + \frac{\psi_i}{p_i} w_i v(c) \right) = q(c) w_t
\]

verifying both Lemma 1 and our conjecture on the form of \(J(C, K, w_i)\).

B.1.2 Step 2: Proof of Lemma B.3

First, note that for any arbitrary \(c_h\) and \(c_l\) from (9), we can express \(A_1\cdot A_4\) in (12)-(13) as functions of \(c_h\) and \(c_l\) only. Substituting back to (12)-(13) we get our functions parameterized by \(c_h\) and \(c_l\) which we denote as \(v(c; c_l, c_h)\) and \(q(c; c_l, c_h)\). Evaluating these functions at \(c = c_l\) and \(c = c_h\), we get the following expressions. Define

\[
\begin{align*}
f_l(c_l, c_h) &= \frac{e^{-\gamma c_h} (E_l[c_l \gamma] - E_l[c_l \gamma]) + e^{\gamma c_h} (E_l[-c_h \gamma] - E_l[-c_l \gamma])}{e^\gamma (c_h - c_l)} - e^{-\gamma (c_h - c_l)} , \\
g_l(c_l, c_h) &= \frac{e^{-\gamma c_h} (E_l[c_l \gamma] - E_l[c_l \gamma]) + e^{\gamma c_h} (E_l[-c_h \gamma] - E_l[-c_l \gamma])}{e^\gamma (c_h - c_l)} - e^{-\gamma (c_h - c_l)} , \\
f_h(c_l, c_h) &= \frac{e^{-\gamma c_l} (E_l[c_l \gamma] - E_l[c_l \gamma]) + e^{\gamma c_l} (E_l[-c_h \gamma] - E_l[-c_l \gamma])}{e^\gamma (c_h - c_l)} - e^{-\gamma (c_h - c_l)} , \\
g_h(c_l, c_h) &= \frac{e^{-\gamma c_l} (E_l[c_l \gamma] - E_l[c_l \gamma]) + e^{\gamma c_l} (E_l[-c_h \gamma] - E_l[-c_l \gamma])}{e^\gamma (c_h - c_l)} - e^{-\gamma (c_h - c_l)} ,
\end{align*}
\]

and

\[
m(c_l, c_h) = \frac{e^\gamma (c_h - c_l) - 1}{1 + e^\gamma (c_h - c_l)} \in (0, 1).
\]

Then the cash and capital values can be rewritten as

\[
q(c_l; c_l, c_h) = \frac{R_C}{2} + \frac{R_K \gamma}{2} f_l(c_l, c_h) ,
q(c_h; c_l, c_h) = \frac{R_C}{2} + \frac{R_K \gamma}{2} f_h(c_l, c_h) ,
\]

\[
v(c_l; c_l, c_h) = R_K + \frac{c_l R_C}{2} + \frac{R_C}{2} m(c_l, c_h) + \frac{R_K \gamma}{2} \left( g_l(c_l, c_h) - c_l f_l(c_l, c_h) \right) ,
\]

and

\[
v(c_h; c_l, c_h) = R_K + \frac{c_h R_C}{2} - \frac{R_C}{2} m(c_l, c_h) + \frac{R_K \gamma}{2} \left( g_h(c_l, c_h) - c_h f_h(c_l, c_h) \right).
\]

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For any \( c_h \), define the function \( H(c_h) \) implicitly as the corresponding lower threshold \( c_l \) so that at \( c = c_h \) the market price is just \( h \), i.e.,

\[
p(c_h; c_l = H(c_h), c_h) = \frac{v(c_h; c_l = H(c_h), c_h)}{q(c_h; c_l = H(c_h), c_h)} = h. \]

Similarly, define \( L(c_h) \) is defined implicitly by

\[
p(c_l; c_l = L(c_h), c_h) = \frac{v(c_l; c_l = L(c_h), c_h)}{q(c_l; c_l = L(c_h), c_h)} = l, \]

which makes the market price to be \( l \) at \( c = c_l \). Obviously, once we find such \( c_h \) that \( H(c_h) = L(c_h) \), then this particular \( c_h \) and the corresponding \( c_l = H(c_h) = L(c_h) \) is a solution of (7)-(9), (12)-(13). To show that this solution exists, we first establish properties of \( L(c_h) \) then we proceed to the properties of \( H(c_h) \).

**Properties of \( L(c_h) \)** It is useful to observe that

\[
\frac{\partial f_l}{\partial c_l} = \frac{e^{2\gamma c_h}}{e^{2\gamma c_h} - e^{2\gamma c_l}} \left( \gamma f_l - \frac{1}{c_l} \right), \quad \frac{\partial^2 f_l}{\partial^2 c_l} = \frac{1}{c_l} - \gamma f_l,
\]

\[
\frac{\partial g_l}{\partial c_l} = \frac{e^{2\gamma c_h} + e^{2\gamma c_l}}{e^{2\gamma c_h} - e^{2\gamma c_l}} \frac{\gamma g_l}{\partial c_l} = -\frac{2\gamma g_l}{e^{\gamma (c_h - c_l) - e^{\gamma (c_l - c_l)}}},
\]

\[
\lim_{c_l \to c_h} f_l = \frac{1}{\gamma c_h}, \quad \lim_{c_l \to c_h} g_l = 0, \quad \lim_{c_l \to c_h} m = 0.
\]

1. We show that \( f_l(c_h, c_l) \) is monotonically decreasing in \( c_l \). Its slope in \( c_l \) is

\[
\frac{\partial f_l}{\partial c_l} = \frac{e^{2\gamma c_h} + e^{2\gamma c_l}}{e^{2\gamma c_h} - e^{2\gamma c_l}} \left( \gamma f_l(c_h, c_l) - \frac{1}{c_l} \right), \quad (B.12)
\]

and the second derivative is

\[
\frac{\partial^2 f_l}{\partial^2 c_l} = -\left( 4\gamma e^{2\gamma c_h} \frac{e^{2\gamma c_l}}{(e^{2\gamma c_h} - e^{2\gamma c_l})^2} - \frac{e^{2\gamma c_h} + e^{2\gamma c_l}}{(e^{2\gamma c_h} - e^{2\gamma c_l})^2} \gamma \right) \left( \frac{1}{c_l} - \gamma f_l(c_h, c_l) - \frac{-\gamma}{c_l} \right) \frac{e^{2\gamma c_h} + e^{2\gamma c_l}}{(e^{2\gamma c_h} - e^{2\gamma c_l})} = \gamma \left( \frac{1}{c_l} - \gamma f_l(c_h, c_l) \right) + \frac{e^{2\gamma c_h} + e^{2\gamma c_l}}{(e^{2\gamma c_h} - e^{2\gamma c_l})} \frac{1}{c_l^2}
\]

Note that if the first derivative is zero, then the second derivative is positive implying that \( f_l(c_h, c_l) \) can have only local minima, but no local maxima in \( c_l \). At the limit one can check that

\[
\lim_{c_l \to c_h} \frac{\partial f_l}{\partial c_l} = \lim_{c_l \to c_h} \left( \frac{1}{c_l} \frac{e^{2\gamma c_h} + e^{2\gamma c_l}}{(e^{2\gamma c_h} - e^{2\gamma c_l})} \gamma c_l f_l(c_h, c_l) - 1 \right) = \frac{1}{c_h} \left( -\frac{1}{2\gamma c_h} \right) < 0.
\]

Thus, \( f_l(c_h, c_l) \) is decreasing at \( c_h = c_l \). Suppose that it is not monotonic over the range of \( c_l < c_h \) in \( c_l \). Then the largest \( c_l \) where the first derivative is 0, would be a local maximum. But we have just ruled out the existence of a local maximum. Thus \( f_l(c_h, c_l) \) monotonically decreasing over the whole range of \( c_l < c_h \) in \( c_l \). This statement is equivalent to \( \gamma f_l(c_h, c_l) - \frac{1}{c_l} < 0 \) for \( c_l < c_h \), for any fixed \( c_h \).
2. We show that \( X(c_l) \equiv f_1(c_h, c_l) - \frac{1}{\gamma c_l} \) is increasing in \( c_l \). We would like to show that

\[
X'(c_l) = \gamma \left( \frac{e^{2\gamma c_h} + e^{2\gamma c_l}}{e^{2\gamma c_h} - e^{2\gamma c_l}} \right) X(c_l) + \frac{1}{\gamma c_l^2} > 0. \tag{B.13}
\]

Clearly, we have

\[
X(c_l = c_h) = 0, \quad X'(c_l = c_h) = f_1'(c_h, c_h) + \frac{1}{\gamma c_h^2} = \frac{1}{2\gamma c_h^2} > 0.
\]

We know that when \( c_l \to 0 \), \( f(c_h, c_l) \) has the order of \( \text{Ei}(\gamma c_l) \) which is \( O(\ln c_l) \); this implies that \( X(c_l) \to -\infty \) when \( c_l \to 0 \). Then, if \( X(c_l) \) is not monotone, we must have two points \( x_1 < x_2 \) closest to (but below) \( c_h \) so that

\[
0 > X(x_1) > X(x_2), \quad X'(x_1) = X'(x_2) = 0.
\]

Setting (B.13) to be zero, we have (because \( 0 < x_1 < x_2 \))

\[
X(x_1) = -\left( \frac{e^{2\gamma c_h} - e^{2\gamma x_1}}{\gamma x_1^2 (e^{2\gamma c_h} + e^{2\gamma x_1})} \right) < -\left( \frac{e^{2\gamma c_h} - e^{2\gamma x_2}}{\gamma x_2^2 (e^{2\gamma c_h} + e^{2\gamma x_2})} \right) = X(x_2),
\]

in contradiction with \( X(x_1) > X(x_2) \). Thus (B.13) holds always.

3. We show that the function \( \frac{g_l(c_h, c_l)}{\gamma} - c_l f_1(c_h, c_l) \) is monotonically increasing in \( c_l \). Its first derivative is (all the derivatives in this part are with respect to \( c_l \))

\[
\left( \frac{g_l}{\gamma} - c_l f_1 \right)' = \frac{1}{\gamma c_l} + \left( \frac{e^{2\gamma c_h} + e^{2\gamma c_l}}{e^{2\gamma c_h} - e^{2\gamma c_l}} \right) g_l(c_h, c_l) - \left( \frac{e^{2\gamma c_h} + e^{2\gamma c_l}}{e^{2\gamma c_h} - e^{2\gamma c_l}} \right) (c_l \gamma f_1(c_l, c_h) - 1) + f_1(c_l, c_h)
\]

\[
= \frac{1}{\gamma c_l} + \left( \frac{e^{2\gamma c_h} + e^{2\gamma c_l}}{e^{2\gamma c_h} - e^{2\gamma c_l}} \right) \left( \frac{g_l}{\gamma} - c_l f_1 \right) + \left( \frac{e^{2\gamma c_h} + e^{2\gamma c_l}}{e^{2\gamma c_h} - e^{2\gamma c_l}} \right) - f_1
\]

Whenever the first derivative is zero, at that point we have

\[
\frac{g_l}{\gamma} - c_l f_1 = \frac{f_1 - \frac{1}{\gamma c_l}}{\frac{e^{2\gamma c_h} + e^{2\gamma c_l}}{e^{2\gamma c_h} - e^{2\gamma c_l}}} - \frac{1}{\gamma}.
\tag{B.14}
\]

We also know that

\[
\lim_{c_l \to c_h} \left( \frac{g_l}{\gamma} - c_l f_1 \right)' = 0, \quad \text{and} \quad \lim_{c_l \to c_h} \left( \frac{g_l}{\gamma} - c_l f_1 \right)'' = -\frac{1}{3\gamma c_h^2} < 0;
\]

so for any fixed \( c_h \), \( c_l = c_h \) is a local maximum. Thus to show that \( \frac{g_l}{\gamma} - c_l f_1 \) is monotone, it suffices to rule out the case of a local minimum \( \hat{c}_l < c_h \) so that \( \left( \frac{g_l}{\gamma} - c_l f_1 \right)' = 0 \) and \( \left( \frac{g_l}{\gamma} - c_l f_1 \right)'' > 0 \). In general

\[
\left( \frac{g_l}{\gamma} - c_l f_1 \right)'' = -\frac{1}{\gamma c_l^2} + \gamma \left( \frac{e^{2\gamma c_h} + e^{2\gamma c_l}}{e^{2\gamma c_h} - e^{2\gamma c_l}} \right) \left( \frac{g_l}{\gamma} - c_l f_1 \right) - f_1' + \frac{4e^{2\gamma c_h} e^{2\gamma c_l}}{\left( e^{2\gamma c_h} - e^{2\gamma c_l} \right)^2} \gamma^2 \left( \frac{g_l}{\gamma} - c_l f_1 \right) + \frac{1}{\gamma}
\]

Thus, if there were a \( \hat{c}_l \) that \( \left( \frac{g_l}{\gamma} - c_l f_1 \right)' = 0 \), using (B.12) and (B.14) we have \( \left( \frac{g_l}{\gamma} - c_l f_1 \right)'' \) to be
equal to
\[ -\frac{1}{\gamma c_l^2} - f'_l + \frac{4\gamma^2 e^{2\gamma c_h} e^{2\gamma c_l}}{(e^{2\gamma c_h} - e^{2\gamma c_l})^2} \left( \frac{f_l - \frac{1}{\gamma c_l}}{e^{2\gamma c_h} + e^{2\gamma c_l}} - \frac{1}{\gamma} + \frac{1}{\gamma} \right) = -\frac{1}{\gamma c_l^2} - \gamma \frac{e^{2\gamma c_h} - e^{2\gamma c_l}}{e^{2\gamma c_h} + e^{2\gamma c_l}} (f_l - \frac{1}{\gamma c_l}). \]

But from (B.13) we know the above term is strictly negative, which proves the contradiction.

4. We show that \( q(c_l; c_l, c_h) \) is also decreasing in \( c_l \) for any \( c_l < c_h \). Given that \( \left( \frac{g_l}{c_l} - c_l f_l \right)' > 0 \) and
\[
\partial \left( \frac{R c e^\gamma}{2} + \frac{R c e^{-\gamma (c_h - c_l)}}{2(1 - e^{-\gamma (c_h - c_l)})} \right) / \partial c_l = \frac{1}{2} R c e^{-\gamma (c_h + c_l + 1)} > 0, \]
\( v(c_l; c_l, c_h) \) is increasing in \( c_l \). Thus, \( p(c_l; c_l, c_h) \) is increasing in \( c_l \) for any \( c_l < c_h \). Also one can show that \( \lim_{c_l \to 0} = p(c_l; c_l, c_h) = \frac{\gamma}{\tanh(\gamma c_h)} < 0 \), and

\[
\lim_{c_l \to c_h} p(c_l; c_l, c_h) = \frac{R_K + c_h R_G}{2} + \frac{R_K \gamma}{2 \gamma c_h} \frac{1}{c_h} = \frac{R_K + c_h R_G - R_K}{2 c_h},
\]

which is larger than \( l \) as long as \( c_h > l \). Thus, as long as \( c_h > l \), \( \lim_{c_l \to c_h} p(c_l; c_l, c_h) \geq l \) and there is a unique solution \( c_l \) for any \( c_h \) of \( p(c_l; c_l, c_h) = l \). Therefore \( L(c_h) \) exist. From the monotonicity in \( c_l \), and continuity of \( p(c_l; c_l, c_h) \) we also know that \( L(c_h) \) is continuous.

Properties of \( H(c_h) \) First, we show that for any \( c_h \in [l, R_K] \), \( H(c_h) \) is a continuous function and \( H(c_h) \in [0, c_h] \). Again, the notation \( ' \) means we are taking the derivative with respect to \( c_l \). We use the following facts:

\[
\frac{\partial f_h}{\partial c_l} = 2 \frac{\gamma f_l (c_l, c_l) - \frac{1}{c_l}}{e^{\gamma (c_h - c_l)} - e^{-\gamma (c_h - c_l)}}, \quad \frac{\partial g_h}{\partial c_l} = \frac{2 \gamma g_l (c_l, c_l)}{e^{\gamma (c_h - c_l)} - e^{-\gamma (c_h - c_l)}} \]
\[
\frac{\partial f_h}{\partial c_h} = \frac{e^{2\gamma c_h} + e^{2\gamma c_l}}{e^{2\gamma c_h} - e^{2\gamma c_l}} \left( \frac{1}{c_h} - \gamma f_h (c_l, c_l) \right), \quad \frac{\partial g_h}{\partial c_h} = \frac{e^{2\gamma c_h} + e^{2\gamma c_l}}{e^{2\gamma c_h} - e^{2\gamma c_l}} \gamma g_h (c_l, c_h) \]
\[
\lim_{c_l \to c_h} f_h = \frac{1}{\gamma c_h}, \quad \lim_{c_l \to c_h} g_h = 0.
\]

1. The result of \( \frac{\partial f_h}{\partial c_l} = \frac{2 \gamma f_l (c_l, c_l) - \frac{2}{c_l}}{e^{\gamma (c_h - c_l)} - e^{-\gamma (c_h - c_l)}} < 0 \) follows from the step 1 in the previous subsection.

2. We show \( \left( \frac{g_l}{c_l} - f_h c_h \right)' > 0 \) for \( c_l < c_h \). We have \( \left( \frac{g_l}{c_l} - f_h c_h \right)' = 2 \frac{g_l - c_h \gamma f_l + c_h}{e^{\gamma (c_h - c_l)} - e^{-\gamma (c_h - c_l)}} \) and
\[
\frac{\partial^2 \left( \frac{g_l}{c_l} - f_h c_h \right)}{\partial^2 c_l} = 2g_l' - c_h 2\gamma f_l' - \frac{2 c_h}{c_l} + e^{-\gamma (c_h - c_l)} e^{2\gamma (c_h - c_l)} - 1 \left( 2g_l - c_h 2\gamma f_l + \frac{2 c_h}{c_l} \right).
\]
If the first derivative is zero at a point \( c_h > c_l \), then the second derivative is
\[
\frac{2 c_h - \frac{2 c_h}{c_l}}{e^{\gamma (c_h - c_l)} - e^{-\gamma (c_h - c_l)}} < 0.
\]
Intercept of \( H (c_h) \) and \( L (c_h) \)

1. We show that \( H (h) > L (h) \). We know that \( H (h) = h \) because

\[
\lim_{c_l \to h} v(c_h; c_l, c_h) = \frac{v(c_h; c_l, c_h)}{q(c_h; c_l, c_h)} = \frac{R_K c_h + c_h^2 \frac{R_C}{2} - \frac{R_C}{2} c_h}{c_h^2 + \frac{R_C}{2}} = \frac{c_h^2 R_C + R_K c_h}{R_C c_h + R_K} = c_h.
\]

Because \( \lim_{c_l \to h} p(c_h; c_l, c_h) = -c_h \), hence we know that for any \( c_h > h \) there is a unique \( c_l \in [0, c_h] \) which solves \( p(c_h; c_l, c_h) = h \). From the monotonicity of \( p(c_h; c_l, c_h) \) in \( c_l \) and the continuity in \( c_h \), the resulting function \( H (c_h) \) is continuous in \( c_h \).

2. Now we show that \( \lim_{c_h \to \infty} H (c_h) = 0 < \lim_{c_h \to \infty} L (c_h) \). It is easy to check that

\[
\lim_{c_h \to \infty} f_l = -Ei(-c_h \gamma) / e^{\gamma(-c_h)}, \quad \lim_{c_h \to \infty} g_l = Ei(-c_h \gamma) / e^{\gamma(-c_h)}, \quad \lim_{c_h \to \infty} f_h = 0, \quad \lim_{c_h \to \infty} g_h = 0
\]

Thus, \( \lim_{c_h \to \infty} \frac{v(c_h; c_l, c_h)}{q(c_h; c_l, c_h)} \) takes the value of

\[
\lim_{c_h \to \infty} \frac{R_K + \frac{c_h R_C}{2} + \frac{R_C m(c_l, c_h)}{2 \gamma} + \frac{R_C \gamma}{2} \left( q_l(c_h, c_l) / \gamma - c_l f_l (c_l, c_h) \right)}{R_C + \frac{R_C \gamma}{2} f_l (c_l, c_h)} = \frac{R_K + \frac{c_h R_C}{2} + \frac{R_C m(c_l, c_h)}{2 \gamma} + \frac{R_C \gamma}{2} \left( \frac{Ei(-c_h \gamma)}{\gamma e^{\gamma(-c_h)}} - c_l - \frac{Ei(-c_h \gamma)}{e^{\gamma(-c_h)}} \right)}{R_C + \frac{R_C \gamma}{2} \left( \frac{Ei(-c_h \gamma)}{\gamma e^{\gamma(-c_h)}} - c_l - \frac{Ei(-c_h \gamma)}{e^{\gamma(-c_h)}} \right)} = l.
\]
In contrast, \( \lim_{c_h \to -\infty} \frac{v(c_h; c_l, c_{b})}{q(c_h; c_l, c_{b})} \) takes the value of

\[
\lim_{c_h \to -\infty} \frac{R_K + \frac{c_h R_C}{2} - \frac{R_C}{\gamma} m(c_l, c_h) + \frac{R_K}{2} \left( \frac{g_h(c_l, c_h)}{\gamma} - c_h f_h(c_l, c_h) \right)}{R_C + \frac{R_K}{2} f_h(c_l, c_h)} = \lim_{c_h \to -\infty} \frac{R_K}{R_C + \frac{R_K}{2} f_h(c_l, c_h)} = \lim_{c_h \to -\infty} \frac{R_K}{R_C + \frac{R_K}{2} f_h(c_l, c_h)} = \infty,
\]

Hence, \( \frac{v(c_h; c_l, c_{b})}{q(c_h; c_l, c_{b})} \) grows without bound for any fixed \( c_l \) and \( c_{b} \). This, and the continuity of \( H(\cdot) \) and \( L(\cdot) \) in \( l \), implies that at the limit \( l \to h \), there is a solution of the system (7)-(9), (12)-(13) that \( c^*_{b} - c^*_{b} \to 0 \) and \( c^*_h, c^*_l \to h \). Then, the statement comes from \( h < hR_C < R_K \) (as \( R_C > 1 \)).

### B.1.3 Step 3: Proof of Lemma B.4

We have shown that \( H(h) = h \). Note also that if \( c_h = c_l \) then \( \frac{c_h}{q_h} = \frac{c_l}{q_l} \). This, and the continuity of \( H(\cdot) \) and \( L(\cdot) \) in \( l \), implies that at the limit \( l \to h \), there is a solution of the system (7)-(9), (12)-(13) that \( c^*_{b} - c^*_{b} \to 0 \) and \( c^*_h, c^*_l \to h \). Then, the statement comes from \( h < hR_C < R_K \) (as \( R_C > 1 \)).

### B.1.4 Step 4: Proof of Lemma B.5

First we show that \( q(c) \) is always deceasing, and there exists a critical value \( \tilde{c} \in (c_l, c_h) \) so that \( q''(c) < 0 \) for \( c \in (c_l, \tilde{c}) \) and \( q''(c) > 0 \) for \( c \in (\tilde{c}, c_h) \). Moreover, for \( c \in (c_l, \tilde{c}) \) where \( q''(c) < 0 \), we have that \( q'''(c) > 0 \).

1. To show that \( q' < 0 \), we differentiate the ODE \( 0 = \frac{\sigma^2}{2} q''' + \frac{\xi}{2} (R_C + \frac{R_K}{c^2}) - \xi q \) again to reach

\[
0 = \frac{\sigma^2}{2} q''' - \frac{\xi}{2} \frac{R_K}{c^2} - \xi q'.
\]

Due to boundary conditions, we have at both ends \( c^*_l \) and \( c^*_h \), the function \( q'(c) \) equals zero and its second derivative \( \frac{\sigma^2}{2} q'''' = \frac{\xi}{2} \frac{R_K}{c^2} > 0 \). Suppose to the contrary that \( q'(\tilde{c}) > 0 \) for some point \( \tilde{c} \in (c_l, c_h) \); then we can pick \( \tilde{c} \) so that \( q''(\tilde{c}) < 0 \) and \( q'''(\tilde{c}) = 0 \) (otherwise the function \( q'(c) \) is zero at one end, is convex globally, and thus never comes back to zero at the other end). But because \( \frac{\sigma^2}{2} q''''(\tilde{c}) = \frac{\xi}{2} \frac{R_K}{c^2} + \xi q'(\tilde{c}) > 0 \), contradiction. This proves that \( q' < 0 \).

2. We know that \( q''(c_l) < 0 \) and \( q''(c_h) > 0 \), and therefore there exists \( \tilde{c} \) so that \( q''(\tilde{c}) = 0 \). We show this point is unique. Because \( 0 = \frac{\sigma^2}{2} q''' + \frac{\xi}{2} (R_C + \frac{R_K}{c^2}) - \xi q \), we have \( 0 = \frac{\sigma^2}{2} q''' - \frac{\xi}{2} \frac{R_K}{c^2} - \xi q' \), and

\[
0 = \frac{\sigma^2}{2} q''' + \frac{\xi R_K}{c^3} - \xi q''.
\]

Suppose we have multiple solutions for \( q''(\tilde{c}) = 0 \). Clearly, it is impossible to have \( q''(\tilde{c}) < 0 \) and \( q''(\tilde{c}^+) > 0 \); otherwise \( q'''(\tilde{c}) > 0 \) which contradicts with (B.16). Then there must exist two points \( c_1 > \tilde{c} \) and \( c_2 > c_1 > \tilde{c} \) that \( q''(c_1) = 0 \), \( q''(c_2) < 0 \) and \( q'''(c_2) > 0 \), but \( q''(c) < 0 \) for \( c \in (c_1, c_2) \). This implies that \( \frac{\sigma^2}{2} q'''(c_1) = -\frac{\xi R_K}{c_1^2} + \xi q''(c_1) < 0 \). As a result, there exists another point \( c_3 \in (c_1, c_2) \) so that \( q'''(c_3) = 0 \) with \( q''(c_3) < 0 \). But this contradicts with (B.16).

3. Now we show that for \( c \in (c_l, \tilde{c}) \) with \( q''(c) < 0 \), we have \( q''(c) > 0 \), i.e., \( q''(c) \) is increasing. Suppose not. Since \( q'''(c_4) > 0 \) so that \( q''(c) \) is increasing at the beginning, there must exist some reflecting point \( c_4 \) for the function \( q'' \) so that \( q'''(c_4) = 0 \). But because \( q''(c_4) < 0 \), it contradicts with (B.16).
Second, we show that \( v(c) \) is increasing if \( h - l \) is sufficiently small.

1. We show that if \( v''(c_1) > 0 \), then \( v(c) \) is increasing in \( c \). Let \( F(c) \equiv v'(c) \), so that

\[
0 = q'' \sigma^2 + \frac{\sigma^2}{2} F'' + \frac{\xi}{2} R_C - \xi F
\]

with boundary conditions that \( F(c_1) = F(c_h) = 0 \). The assumption \( v''(c_1) > 0 \) implies that \( F'(c_1) > 0 \). Thus, if there are some points with \( F(c) < 0 \) in the range of \((c_1, c_h)\) then we can find two points \( c_1 \) and \( c_2 \) (a maximum and a minimum) so that \( c_1 < c_2 \) but \( F''(c_1) < 0, F''(c_2) > 0, F'(c_1) = F'(c_2) = 0 \) and \( F(c_1) > 0 > F(c_2) \). We can apply the ODE to these two points:

\[
\begin{align*}
0 &= q''(c_1) \sigma^2 + \frac{\sigma^2}{2} F''(c_1) + \frac{\xi}{2} R_C - \xi F(c_1), \\
0 &= q''(c_2) \sigma^2 + \frac{\sigma^2}{2} F''(c_2) + \frac{\xi}{2} R_C - \xi F(c_2).
\end{align*}
\]

The second equation implies that \( q''(c_2) < 0 \), which implies that \( c_1 < c_2 < \hat{c} \). However, the above two equations also imply that

\[
q''(c_1) \sigma^2 > \frac{\xi}{2} R_C > q''(c_2) \sigma^2
\]

contradiction with the previous lemma which shows that \( q'' \) is increasing over \([c_1, \hat{c}]\).

2. Now we show that if \( h - l \) is sufficiently small, then \( v''(c_1) > 0 \); with the first result we obtain our claim. From our ODE,

\[
v''(c_1) = -\frac{\xi}{\sigma^2} \left( \frac{(R_C c_1 + R_K)}{2} - v(c_1) \right) = \frac{\xi}{\sigma^2} \left( \frac{R_K}{2} + \frac{R_K}{2} h(c_1, c_h) + \frac{R_K \xi}{\gamma \sigma^2} \left( \frac{q_1(c_1, c_h)}{\gamma} - c_1 f_1(c_1, c_h) \right) \right).
\]

We know that as \( h - l \to 0 \), \( c_h - c_l \to 0 \). We will prove the statement by showing that (1) \( \lim_{c_1 \to c_h} \left( \frac{(R_C c_1 + R_K)}{2} - v(c_1) \right) = 0 \), because \( \lim_{c_1 \to c_h} \left( \frac{(R_C c_1 + R_K)}{2} - v(c_1) \right) \) equals

\[
\lim_{c_1 \to c_h} \left( \frac{R_K}{2} + \frac{R_K}{2} h(c_1, c_h) + \frac{R_K \gamma}{2} \left( \frac{q_1(c_1, c_h)}{\gamma} - c_1 f_1(c_1, c_h) \right) \right) = \frac{R_K}{2} + 0 + \frac{R_K \xi}{\gamma \sigma^2} \left( 0 - \frac{1}{\gamma} \right) = 0
\]

and (2) \( \lim_{c_1 \to c_h} \frac{\partial (\frac{(R_C c_1 + R_K)}{2} - v(c_1))}{\partial c_1} = \lim_{c_1 \to c_h} \frac{\partial \left( \frac{R_K}{2} h(c_1, c_h) + \frac{R_K \gamma}{2} \left( \frac{q_1(c_1, c_h)}{\gamma} - c_1 f_1(c_1, c_h) \right) \right)}{\partial c_1} < 0 \), because it equals

\[
\lim_{c_1 \to c_h} \left( - \frac{R_C e^{\gamma(c_h - c_l)}}{(e^{\gamma (c_h - c_l)} + 1)^2} + \frac{R_K \gamma}{2} \left( \frac{1}{\gamma c_l} + \frac{e^{2 \gamma c_l} + e^{2 \gamma c_l}}{(e^{2 \gamma c_l} - e^{2 \gamma c_l}) q_1 - \left( e^{2 \gamma c_l} + e^{2 \gamma c_l} \right) (c_l f_1 - 1)} \right) \right) = - \frac{R_C}{(1 + 1)^2} + \frac{R_K \gamma}{2} \left( \frac{1}{\gamma c_h} - \frac{1}{2 \gamma c_h} - \frac{1}{2 \gamma c_h} \right) = - \frac{R_C}{4} < 0.
\]

These two statements imply that if \( c_h - c_l \) is small enough then \( v''(c_1) > \lim_{c_1 \to c_h} v''(c_1) = 0 \).

### B.2 Proof of the Second Part of Proposition 5

The result \( c_h^* > h \) is a consequence of the fact that we defined \( c_h^* \) as the unique \( c_l \) solving \( v_h^*(c_l, c_h) = h \) when \( c_h > h \). (see part 4 in section B.1.2.)

For the result \( c_l^* \leq h \), consider the possibility that \( c_l^* > l \). The following lemma states that in this case \( p''(c_l^*) < 0 \). This implies that this is not an equilibrium. To see this, we have \( q'(c_l^*) = 0 \) by the boundary
conditions \( v'(c_i^*) = q'(c_i^*) = 0 \). Thus \( p''(c_i^*) < 0 \), combined with \( p(c_i^*) = l \) and \( p'(c_i^*) = 0 \), would imply that \( p(c) < l \) for \( c \) sufficiently close to \( c_i^* \).

**Lemma B.6** The sign of \( p''(c_i^*) \) is the same as that of \( l - c_i^* \).

**Proof.** Simple algebra implies that

\[
p''(c_i^*) = \left( \frac{v'q - qv}{q^2} \right)' = \frac{(v''q + v'q' - (q''v + v'q'))}{q^2} - 2q^{-3}(v'q - qv)
\]

\[
= \frac{v''q - qv''}{q^2} = \left( -\frac{\xi}{2} (RCc_i^* + R_K) + \xi lq(c_i^*) \right) \frac{2}{q^2} q - \left( -\frac{\xi}{2} (RCc_i^* + R_K) + \xi c_i^* q(c_i^*) \right) \frac{2}{q^2} q
\]

\[
= \frac{\left( -\frac{\xi}{2} (RCc_i^* + R_K) + \xi c_i^* q(c_i^*) \right) \frac{2}{q^2} q - \left( l - c_i^* \right) \xi q(c_i^*) \frac{2}{q^2} q}{q^2}
\]

\[
= (l - c_i^*) \frac{1}{c_i^*} \left( \frac{\xi}{2} (RCc_i^* + R_K) - \xi c_i^* q(c_i^*) \right) \frac{2}{q^2} q + \xi q(c_i^*) \frac{2}{q^2} q
\]

which gives the lemma by noticing that \( q \) is decreasing in \( c \) and the boundary \( q'(c_i^*) = 0 \) implies that

\[
-\frac{\xi}{2} (RCc_i^* + R_K) + \xi c_i^* q(c_i^*) \propto q''(c_i^*) < 0.
\]

The third statement is a consequence of the following Lemma.

**Lemma B.7** We have the following limiting results:

\[
\lim_{\gamma \to \infty} \gamma f_i = \frac{1}{c_i}, \quad \lim_{\gamma \to \infty} \gamma f_h = \frac{1}{c_h}, \quad \lim_{\gamma \to \infty} g_h = 0, \quad \lim_{\gamma \to \infty} g_i = 0;
\]

and \( \lim_{\gamma \to \infty} c_h^* = h \), \( \lim_{\gamma \to \infty} c_i^* = l \).

**Proof.** The first four results are based on L’Hopital rule. Take the first result for illustration:

\[
\lim_{\gamma \to \infty} \gamma f_i = \lim_{\gamma \to \infty} \frac{\gamma (Ei[-c_i\gamma] - Ei[-c_i\gamma])}{e^{\gamma(-c_i)}} = \lim_{\gamma \to \infty} \frac{Ei[-c_i\gamma] - Ei[-c_i\gamma]}{\frac{\gamma}{2} e^{\gamma(-c_i)}}
\]

\[
= \lim_{\gamma \to \infty} \frac{e^{-c_i\gamma} - e^{-c_i\gamma}}{\frac{\gamma}{2} e^{\gamma(-c_i)} + \frac{(c_i\gamma)}{2} e^{\gamma(-c_i)}} = \lim_{\gamma \to \infty} \frac{-e^{-c_i\gamma}/\gamma}{\frac{(c_i\gamma)}{2} e^{\gamma(-c_i)}} = \frac{1}{c_i}.
\]

These four results imply that

\[
\lim_{\gamma \to \infty} \frac{v}{q_h} = \frac{R_K + \frac{c_h Rc}{2} - \frac{Rc}{2} m(c_i, c_h) + R_K \frac{2}{2} \left( \frac{g_h(c_i, c_h)}{\gamma} - c_h f_h(c_i, c_h) \right)}{\frac{Rc}{2} + R_K \frac{2}{2} f_h(c_i, c_h)} = \frac{R_K + \frac{c_h Rc}{2} - \frac{Rc}{2} \frac{1}{2}}{\frac{Rc}{2} + R_K \frac{1}{2}}
\]

Thus, in the limit the solution of \( \frac{v}{q_h} = h \) is the solution for the equation of

\[
\frac{Rc}{2} + R_K \frac{1}{2} = h.
\]
which gives \( \lim_{\gamma \to \infty} c_h^* = h \). Similarly, the following calculation implies that \( \lim_{\gamma \to \infty} c_l^* = l \):

\[
\lim_{\gamma \to \infty} v_l = \lim_{\gamma \to \infty} \frac{R_K + \frac{c \gamma c}{2} + \frac{R_c}{2\gamma} m(c_l, c_h) + R_K \frac{\gamma}{2} \left( \frac{q_l(c_l, c_h)}{\gamma} - c_l f_l(c_l, c_h) \right)}{R_c + R_K \frac{\gamma}{2} f_l(c_l, c_h)} = \frac{R_K + \frac{c \gamma c}{2} + R_K \frac{1}{2}}{R_c + R_K \frac{1}{2\gamma}}.
\]

\[\blacksquare\]

### B.3 Proof of Proposition 7

The proofs of the two statements follow the same logic. Thus, we prove the first statement in detail and explain the necessary modifications for the second statement at the end of the proof.

Consider the functions \( \tilde{q}(c; q_0, v_0, c_h) \) and \( \tilde{v}(c; q_0, v_0, c_h) \) of \( c \) parameterized by \( q_0, v_0, \) and \( c_h \):

\[
0 = \frac{\sigma^2}{2} \tilde{q}''(c) + \frac{\xi}{2} (R_c - \tilde{q}(c)) + \frac{\xi}{2} \left( \frac{R_K}{c} - \tilde{q}(c) \right) \tag{B.17}
\]

\[
0 = \tilde{q}'(c) \sigma^2 + \frac{\sigma^2}{2} \tilde{v}''(c) + \frac{\xi}{2} (R_c c - \tilde{v}(c)) + \frac{\xi}{2} (R_K - \tilde{v}(c)). \tag{B.18}
\]

and the boundary conditions

\[
\tilde{v}'(c_h) = \tilde{q}'(c_h) = 0, \tag{B.19}
\]

\[
\tilde{q}(c_0) = q_0, \tilde{v}(c_0) = v_0. \tag{B.20}
\]

The general solution is

\[
\tilde{q}(c) = \frac{R_c}{2} + e^{-c_1} A_1 + e^{c_1} A_2 + \frac{R_K \gamma e^{-c_1} Ei(-\gamma c) + e^{-c_1} Ei(c_1)}{2} \tag{B.21}
\]

\[
\tilde{v}(c) = R_K + \frac{c R_c}{2} + e^{c_1} (A_3 - c A_2) - e^{-c_1} (A_4 + c A_1) + \frac{c R_K \gamma e^{c_1} Ei(-\gamma c) - e^{-c_1} Ei(c_1)}{2} \tag{B.22}
\]

\[
= R_K + R_c c + e^{c_1} A_3 - e^{-c_1} A_4 - \tilde{q}(c). \tag{B.23}
\]

where \( A_1 - A_4 \) (may differ from those in (12) and (13)) are pinned down by (B.19)-(B.20). We have

\[
\tilde{q}'(c) = -\gamma e^{-c_1} A_1 + \gamma e^{c_1} A_2 - \frac{R_K \gamma^2 (e^{-c_1} Ei[c_1] + e^{c_1} Ei[-c_1])}{2},
\]

\[
\tilde{v}'(c) = \frac{R_c}{2} + \frac{R_K \gamma (-e^{-c_1} Ei[c_1] + e^{c_1} Ei[-c_1])}{2} + \frac{R_K c \gamma^2 (e^{-c_1} Ei[c_1] + e^{c_1} Ei[-c_1])}{2}
\]

\[
+ e^{c_1} ((-\gamma c - 1) A_2 + \gamma A_3) + e^{-c_1} ((\gamma c - 1) A_1 + \gamma A_4).
\]

Define the function \( c_h(q_0, v_0) \) implicitly by \( \tilde{v}(c_h; q_0, v_0, c_h) = h \tilde{q}(c_h; q_0, v_0, c_h) \), and we are interested in the derivatives

\[
\frac{\partial c_h}{\partial q_0} = \frac{\tilde{v}'_{q_0} - h \tilde{q}'_{q_0}}{\tilde{v}'_{c_h} - h \tilde{q}'_{c_h}} \quad \frac{\partial c_h}{\partial v_0} = -\frac{\tilde{v}'_{v_0} - h \tilde{q}'_{v_0}}{-\tilde{v}'_{c_h} + h \tilde{q}'_{c_h}}.
\]

We proceed as follows. First we show that \( \frac{\partial c_h}{\partial q_0} > 0 \) and \( \frac{\partial c_h}{\partial v_0} < 0 \). This proves that if \( q_\pi(c_0) \leq q(c_0) \) and \( v_\pi(c_0) \geq v(c_0) \) then \( c_h^* < c_i^* \), that is, such policies make the overinvestment problem worse. Then we show that this is true even if \( q_\pi(c_0) \leq q(c_0) \) and \( v_\pi(c_0) \leq v(c_0) \), as long as the policy increases the price at \( c_0 \), i.e., \( \frac{v_\pi(c_0)}{q_\pi(c_0)} > \frac{v(c_0)}{q(c_0)} \).

We start with the following lemmas.
Lemma B.8 We have
\[ \frac{\partial \tilde{q}(c_h; q_0, v_0, c_h)}{\partial q_0} = \frac{2}{e^{c_h \gamma} e^{-c_0 \gamma} + e^{-c_h \gamma} e^{c_0 \gamma}} > 0, \quad \text{(B.24)} \]
\[ \frac{\partial \tilde{v}(c_h; q_0, v_0, c_h)}{\partial v_0} = \frac{2}{e^{-\gamma(c_h - c_0)} + e^{-\gamma(c_h - c_0)}} > 0, \quad \frac{\partial \tilde{q}(c_h; q_0, v_0, c_h)}{\partial v_0} = 0. \quad \text{(B.25)} \]

**Proof.** We show (B.24) first. We know that \( \tilde{q}(c_0) = q_0 \), which based on (B.21) can be written as \( e^{-c_0 \gamma} A_1 + e^{c_0 \gamma} A_2 + l_q = q_0 \) (where \( l_q \) is independent of \( q_0 \)) which implies
\[ A_1 = \frac{-l_q - e^{c_0 \gamma} A_2 + q_0}{e^{-c_0 \gamma}}. \quad \text{(B.26)} \]
and \( \tilde{q}'(c_h) = 0 \) which can be rewritten as \( -e^{-c_h \gamma} A_1 + e^{c_h \gamma} A_2 + s_q = 0 \) (where \( s_q \) is independent of \( q_0 \)) which implies
\[ A_2 = \frac{e^{-c_h \gamma} A_1 - s_q}{e^{c_h \gamma}} = \frac{e^{-c_h \gamma} A_1 - l_q - e^{c_0 \gamma} A_2 + q_0 - s_q}{e^{c_h \gamma}} \Rightarrow A_2 = \frac{e^{-c_h \gamma} A_1 - l_q + q_0 - s_q}{(1 + e^{-2c_h \gamma} e^{2c_0 \gamma}) e^{c_h \gamma}}. \quad \text{(B.27)} \]
Thus, (B.27) and (B.26) imply that
\[ \frac{\partial A_2}{\partial q_0} = \frac{e^{-c_h \gamma}}{e^{c_h \gamma} e^{-c_0 \gamma} + e^{-c_h \gamma} e^{c_0 \gamma}} \quad \text{(B.28)} \]
\[ \frac{\partial A_1}{\partial q_0} = \frac{1}{e^{-c_0 \gamma} - e^{c_0 \gamma}} - \frac{e^{-c_h \gamma}}{e^{c_h \gamma} e^{-c_0 \gamma} + e^{-c_h \gamma} e^{c_0 \gamma}} = \frac{e^{c_h \gamma}}{e^{c_h \gamma} e^{-c_0 \gamma} + e^{-c_h \gamma} e^{c_0 \gamma}}. \quad \text{(B.29)} \]
Using (B.21) we obtain our result.

The first result in (B.25) follows similarly. The second result \( \frac{\partial \tilde{q}(c_h; q_0, v_0, c_h)}{\partial v_0} = 0 \) comes from the fact that (B.17) and the boundary conditions \( \tilde{q}'(c_h) = 0 \) and \( \tilde{q}(c_0) = q_0 \) are independent of \( v_0 \). \[ \blacksquare \]

Lemma B.9 We have
\[ \frac{\partial \tilde{v}(c_h; q_0, v_0, c_h)}{\partial q_0} = \frac{2 e^{\gamma(c_h - c_0)} - e^{-\gamma(c_h - c_0)} - \gamma (c_h - c_0) (e^{-\gamma(c_h - c_0)} + e^{\gamma(c_h - c_0)})}{\gamma (e^{\gamma(c_h - c_0)} e^{-c_h \gamma} + e^{-\gamma(c_h - c_0)} e^{c_h \gamma})^2} < 0, \]
\[ \frac{\partial \tilde{v}(c_h; q_0, v_0, c_h)}{\partial q_0} - h \frac{\partial \tilde{q}(c_h; q_0, v_0, c_h)}{\partial q_0} = \frac{2 e^{\gamma(c_h - c_0)} - e^{-\gamma(c_h - c_0)} - \gamma (c_h + h - c_0) (e^{-\gamma(c_h - c_0)} + e^{\gamma(c_h - c_0)})}{\gamma (e^{\gamma(c_h - c_0)} e^{-c_h \gamma} + e^{-\gamma(c_h - c_0)} e^{c_h \gamma})^2} < 0 \]

**Proof.** We show the first result. We rewrite \( \tilde{v}(c_0) \) and \( \tilde{v}'(c_h) \) as (as before here \( l_{vq} \) and \( s_{vq} \) are independent of \( q_0 \))
\[ \tilde{v}(c_0) = e^{c_0 \gamma} (A_3 - c_0 A_2) - e^{-c_0 \gamma} (A_4 + c_0 A_1) + l_{vq}, \]
\[ \tilde{v}'(c_h) = s_{vq} + e^{c_h \gamma} ((-c_h - 1) A_2 + \gamma A_3) + e^{-c_h \gamma} ((\gamma c_h - 1) A_1 + \gamma A_3) \]
Thus, the boundary conditions \( \tilde{v}(c_0) = v_0 \) and \( \tilde{v}'(c_h) = 0 \) imply that
\[ A_3 = c_0 A_2 + e^{-2c_0 \gamma} v_0 + e^{-2c_0 \gamma} (A_4 + c_0 A_1), \]
\[ (-e^{c_h \gamma} (c_h - c_0 + 1)) A_2 + (e^{-c_h \gamma} (c_h - 1) + c_0 e^{-2c_0 \gamma} e^{c_h \gamma}) A_1 + \gamma A_3 \]
\[ = - \frac{(-e^{c_h \gamma} + c_0 e^{-2c_0 \gamma} e^{c_h \gamma}) v_0 + (s_{vq} - \gamma e^{-c_0 \gamma} e^{c_h \gamma} l_{vq})}{\gamma e^{-c_h \gamma} + e^{-2c_0 \gamma} e^{c_h \gamma}} \]
Thus, using the result in (B.28) and (B.29) one can derive that
\[
\frac{\partial A_1}{\partial q_0} = e^{\gamma c_h} \frac{2e^{\gamma c_0}e^{-\gamma c_h} - \gamma c_0 \left( e^{\gamma c_0}e^{-\gamma c_h} + e^{-\gamma c_0}e^{\gamma c_h} \right)}{\gamma \left( e^{\gamma c_0}e^{-\gamma c_h} + e^{-\gamma c_0}e^{\gamma c_h} \right)^2}.
\]
Similarly it implies that
\[
\frac{\partial A_3}{q_0} = \frac{\partial A_1}{q_0} e^{-2c_0\gamma} + \frac{\partial A_2}{q_0} c_0 + \frac{\partial A_4}{q_0} e^{-2c_0\gamma} = \frac{2e^{-\gamma c_0} + \gamma c_0 \left( e^{\gamma c_0}e^{-2\gamma c_h} + e^{-\gamma c_0} \right)}{\gamma \left( e^{\gamma c_0}e^{-\gamma c_h} + e^{-\gamma c_0}e^{\gamma c_h} \right)^2}.
\]
Consequently, using (B.23), we have (where we have used (B.24))
\[
\frac{\partial (c_h)}{\partial q_0} = \frac{e^{\gamma c_h} \frac{\partial A_3}{q_0} - e^{-\gamma c_h} \frac{\partial A_1}{q_0} - c_h \frac{\partial \tilde{q}(c_h)}{\partial q_0}}{\gamma} = \frac{2 \left( e^{\gamma c_h} - e^{-\gamma (c_h-c_0)} - \gamma (c_h-c_0) \left( e^{-\gamma (c_h-c_0)} + e^{\gamma (c_h-c_0)} \right) \right)}{\gamma \left( e^{\gamma c_0}e^{-\gamma c_h} + e^{-\gamma c_0}e^{\gamma c_h} \right)^2} < 0.
\]
The last inequality comes from the fact that the function $e^x - e^{-x} - x (e^{-x} + e^x)$ is negative and monotonically decreasing for all $x > 0$. The second statement comes directly from the expression for $\frac{\partial \tilde{q}(c_h)}{\partial q_0}$. 

**Lemma B.10** If $\frac{c_0}{q_0} < h$, then $\tilde{v}(y; q_0, v_0, y) - h\tilde{q}(y; q_0, v_0, y) > 0$.

**Proof.** We parameterize $c_h$ by $y$. The idea is that if the function $\tilde{v}(y; q_0, v_0, y) - h\tilde{q}(y; q_0, v_0, y)$ is negative at $y = c_0$ and positive as $y \to \infty$, then there is a $y = c_h$ so that this function is zero (satisfying the definition of $c_h$) and where the slope of this function is positive, which is the claim of our lemma.

The function $\tilde{v}(y; q_0, v_0, y) - h\tilde{q}(y; q_0, v_0, y)$ can be solved by imposing the boundary conditions
\[
\tilde{v}'(y) = \tilde{q}'(y) = 0, \quad \tilde{q}(c_0) = q_0, \quad \tilde{v}(c_0) = v_0.
\]
for all $y \geq c_0$. Thus, by setting $y = c_0$, we must have
\[
\tilde{v}(c_0; q_0, v_0, c_0) - h\tilde{q}(c_0; q_0, v_0, c_0) = v_0 - hq_0 < 0,
\]
by the condition of the proposition.

Now we show that $\lim_{y \to \infty} \tilde{v}(y; q_0, v_0, y) = \lim_{y \to \infty} \tilde{q}(y; q_0, v_0, y) = \infty$ as $y \to \infty$. We first show calculate $\lim_{y \to \infty} \tilde{q}'(y; q_0, v_0, y) = \infty$ in (B.21). For this, we solve for $e^{-\gamma y}A_1$ and $e^{\gamma y}A_2$ from (B.21)-(B.22) and (B.30):
\[
e^{-\gamma y}A_1 = \frac{q_0 - R\gamma}{e^{(\gamma c_0 - \gamma) + e^{\gamma (c_0 - y)}}} - \frac{R\gamma}{e^{\gamma (c_0 - y)}} M(c_0), \quad e^{\gamma y}A_2 = \frac{q_0 - R\gamma}{e^{(\gamma c_0 - \gamma) + e^{\gamma (c_0 - y)}}} - \frac{R\gamma}{e^{\gamma (c_0 - y)}} M(c_0).
\]
where $M(y) = e^{-\gamma y}E(y) + e^{-\gamma y}E(y')$. Using $\lim_{y \to \infty} M'(y) = 0$, it is easy to show that $\lim_{y \to \infty} e^{\gamma y}A_2 = \lim_{y \to \infty} e^{-\gamma y}A_1 = 0$, which implies that $\lim_{y \to \infty} \tilde{q}(y; q_0, v_0, y) = \frac{R\gamma}{e^{\gamma (c_0 - y)}}$ in (B.21). A similar argument implies that $\lim_{y \to \infty} \tilde{v}(c_0; q_0, v_0, c_0) - h\tilde{q}(y; q_0, v_0, c_0) = \infty$. Thus, $\tilde{v}(c_0; q_0, v_0, c_0) - h\tilde{q}(c_0; q_0, v_0, c_0) = \infty$. This prove the statement. 

Putting together the above three lemmas, we have
\[
\frac{\partial c_h}{\partial q_0} = \frac{\tilde{v}' - h\tilde{q}'}{\tilde{v}' - h\tilde{q}'} > 0, \quad \text{and} \quad \frac{\partial c_h}{\partial v_0} = \frac{\tilde{v}' - h\tilde{q}'}{\tilde{v}' - h\tilde{q}'} < 0
\]
This implies that $c_h^\pi < c_h^\pi$ whenever $q_\pi(c_0) \leq q(c_0)$ and $v_\pi(c_0) \geq v(c_0)$.

For the last step, as $\frac{\partial c_h}{\partial v_0} = \frac{\tilde{v}' - h\tilde{q}'}{\tilde{v}' - h\tilde{q}'} < 0$, it suffices to show that this result holds for the worst $v_0$ drop to maintain $p_0$, i.e., $v_0$ and $q_0$ decrease proportionally so $v_0/q_0$ remains at constant.
To this end, we consider decreasing \( q_0 \) to \( \tilde{q}_0 = q_0 - \varepsilon \) where \( \varepsilon \) is very small. To make sure that \( \frac{\tilde{q}_0}{q_0} = \frac{q_0 - \varepsilon}{q_0} \), we need that \( v_0 = v_0 - a \varepsilon \) where \( a = \frac{\tilde{q}_0}{q_0} \). Let us refer to all the objects after the change with the bar. Our goal is to show that \( \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} / \frac{v(c_h)}{q(c_h)} \) would increase; then \( \tilde{v}'_c - h\tilde{q}'_c > 0 \) implies that \( c_h^q < c_h^g \). Using the first two Lemmas above, we have (denoting \( x \equiv (c_h - c_0) \gamma \))

\[
\tilde{q}(c_h) = \frac{\tilde{q}(c_h) - \varepsilon}{e^{x} + e^{-x}} \\
\tilde{v}(c_h) = \frac{\tilde{v}(c_h) - 2\varepsilon x - e^{-x} - x (e^{-x} + e^x)}{\gamma (e^x + e^{-x})^2} \frac{v_0}{q_0} \frac{2\varepsilon}{e^x + e^{-x}}
\]

Hence for sufficiently small \( \varepsilon \) we have (up to the first order)

\[
\frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} = \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} - \frac{2\varepsilon}{\tilde{q}(c_h)} \left( \frac{e^x - e^{-x} - x (e^{-x} + e^x)}{\gamma (e^x + e^{-x})^2} + \frac{v_0}{q_0} \frac{1}{e^x + e^{-x}} \right) + \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} \frac{2\varepsilon}{e^x + e^{-x}}
\]

\[\frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} = \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} - \frac{2\varepsilon}{\tilde{q}(c_h)} \left( \frac{e^x - e^{-x} - x (e^{-x} + e^x)}{\gamma (e^x + e^{-x})^2} + \left( \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} \right) \right) \frac{\varepsilon}{\tilde{q}(c_h)}
\]

Here, the third inequality in (B.31) is because the term \( e^x - e^{-x} - x (e^{-x} + e^x) < 0 \) for all \( x > 0 \) and \( \tilde{q}_0 - \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} \) is strictly negative because \( \frac{\tilde{q}_0}{q_0} < \frac{\tilde{v}(c_h)}{\tilde{q}(c_h)} = h \); hence the first order impact of decreasing \( q_0 \) is an increase in \( \tilde{v}(c_h) / \tilde{q}(c_h) \). Because the above argument holds for any \( v_0 \) and \( q_0 \), tracing out the first-order effect implies that any intervention which lowers cash value but keeps capital price unchanged will lower \( \tilde{v}(c_h) / \tilde{q}(c_h) \). Compared to that change, an increase in \( v_0 \) just decreases \( c_h^q \) further. That concludes our proof.

The second statement follows the same steps with the following modifications. Each \( c_h \) has to be changed to \( c_l \) and each \( h \) has to be changed to \( l \) at every point of the proof. Then the first lemma remains the same, the first statement in the second lemma changes to \( \frac{\partial v(c_l, q_0, v_0, c_l)}{\partial q_0} \) > 0, while the second statement does not change. Also, in the proof of the first statement we use that \( e^x - e^{-x} - x (e^{-x} + e^x) > 0 \) for all \( x < 0 \), and the proof of the second statement we use that \( e^x - e^{-x} - (x + y) (e^{-x} + e^x) < 0 \) for all \( x < 0 \) and \( y > 0 \). In the last part we follow the same steps, but the inequality (B.31) in the modified version is switched. This gives that \( c_l^q > c_l^g \) under the conditions of the statement.

### B.4 Solution for Price Floor Policy and Proof of Proposition 8

#### B.4.1 Characterizing the equilibrium with price floor policy

We first derive the solutions for price floor policy. A price floor policy \( \pi(c) \) is defined as

\[
0 = q'(c) \sigma^2 + \frac{\sigma^2}{2} v''(c) - v(c) + \frac{\xi}{2} (RC_c + RK_c) + c\pi(c),
\]

\[
0 = \frac{\sigma^2}{2} q'' - q(c) + \frac{\xi}{2} \left( RC + \frac{RK_c}{c} \right) - \pi(c).
\]

so that 1) for \( c \in (c_0, c_0^q] \), \( \pi(c) = 0 \), and at the upper investment threshold \( p(c_0^q) = h \); and 2) for \( c \in [c_0^q, c_0] \), \( v(c) = (l + \delta) q(c) \) always. Here, \( v(c) \), \( q(c) \), \( \pi(c) \), \( c_0 \) and \( c_0^q \) are endogenous. We have the following lemma.

**Lemma B.11** Given the lower disinvestment threshold \( c_0^q \), the solution to the price floor policy can be calculated as follows.

1. Given the upper investment threshold \( c_0^q \), first calculate the welfare function \( j_0(c) = RK + RKC + \)
\[ D_1 e^{-\gamma c} + D_2 e^{\gamma c}, \]
where the constants \( D_1, D_2 \) are given by the boundary conditions

\[
\begin{align*}
    j_g (c_g^0; c_h^0) &= (c_g^0 + l) j_g' (c_g^0), \quad \text{and} \quad j (c_g^0; c_h^0) = (c_h^0 + h) j_g' (c_h^0).
\end{align*}
\]

2. For \( c \in (c_0, c_h^0) \), the capital price and cash price is given by

\[
\begin{align*}
    v (c) &= R_K + \frac{R_C c}{2} + e^{\gamma t} (A_3 - c A_2) - e^{-\gamma t} (A_4 + c A_1) + c R_K \gamma \left( e^{\gamma t} Ei(-\gamma c) - e^{-\gamma t} Ei(\gamma c) \right), \\
    q (c) &= \frac{R_C c}{2} + e^{-\gamma t} A_1 + e^{\gamma t} A_2 + R_K \gamma \left( e^{\gamma t} Ei(-\gamma c) + e^{-\gamma t} Ei(\gamma c) \right).
\end{align*}
\]

Here, \( A_4 = -D_1 \) and \( A_3 = D_2 \). The other four constants, i.e., \( A_1 - A_2, c_0 \) and \( c_h^0 \), are determined by the following four boundary conditions

\[
\begin{align*}
    v' (c_h^0) &= 0, q' (c_h^0) = 0, v (c_0) = (l + \delta) q (c_0), v' (c_0) = (l + \delta) q' (c_0)
\end{align*}
\]

3. For \( c \in [c_l^0, c_0] \), we have

\[
q (c) = \frac{j_g (c)}{l + c} \quad \text{and} \quad v (c) = \frac{l + \delta}{l + c} j_g (c)
\]

and the taxation is given by

\[
\pi (c) = \frac{\sigma^2}{2} q'' - \xi q (c) + \frac{\xi}{2} \left( R_C + \frac{R_K}{c} \right) > 0
\]

**Proof.** The total welfare function \( j (c) = v (c) + cq (c) \) given in the step 1 of Lemma B.11 only depends on the investment/disinvestment policies \( c_l^0 \) and \( c_h^0 \) (see explanations around equation (18) and (19)). For \( c \in (c_0, c_h^0) \), there is not taxation and the derivation is the same as before, except that at the endogenous intervention point \( c_0 \) we are value-matching and smooth-pasting so that the price is the implemented floor price \( l + \delta \). Note that by construction we have \( v (c_h^0) = h q (c_h^0) \) (due to \( j (c_h^0) = (c_h^0 + h) j (c_h^0) \)). For \( c \in [c_l^0, c_0] \), notice that \( v (c) = (l + \delta) q (c) \) always; (B.34) follows because of \( j_g (c) = v (c) + cq (c) = (l + c) q (c) \). The endogenous taxation \( \pi (c) \) follows from (B.33). \( \blacksquare \)

### B.4.2 Proof of Proposition 8

Now we set \( \delta = 0 \) and prove Proposition 8. There are three steps.

**Step 1. Rewrite the problem** Clearly, for \( c \in (c_0, c_l^0] \) the same structure solution applies without policy, with the only difference at the lower end \( c_0 \) so that \( v' (c_0) = l q' (c_0) \) might not be zero. This allows us to draw connection between the equilibrium with policy and the one without. We first show that for \( c_l^0 < c_l^0 \), the resulting slope at \( c_0 \) has to be negative, i.e.

\[
v' (c_0) = l q' (c_0) < 0.
\]

To show this, focus on \( c \in [c_l^0, c_0] \). By \( v (c) = \frac{l}{l + c} j_g (c) \) and boundary condition of \( j_g (c) \), we have

\[
v' (c_l^0) = \frac{l \left[ j_g' (c_l^0) (l + c_l^0) - j_g (c_l^0) \right]}{(l + c_l^0)^2} = 0.
\]
Moreover, since \( j''_g (c) < 0 \) (see Proposition 2 and its proof), we have \([j'_g (c) (l + c) - j_g (c)]' = j''_g (c) (l + c) < 0\). As a result, since \( c_0 > c^*_h \), we have

\[
\text{sign} \left[ q' (c_0) \right] = \text{sign} \left[ j'_g (c_0) (l + c_0) - j_g (c_0) \right] < 0.
\]

This proves (B.35).

This suggests us to introduce \( \{ \nu (\cdot), q (\cdot), c_0, c^*_h; x \} \) indexed by \( x \) as the solution to the ODE system (10) and (11), with modified boundary conditions

\[
\begin{align*}
v' (c^*_h) &= q' (c^*_h) = 0, v (c^*_h) = hq (c^*_h) , \\
v' (c_0) &= -xl, q' (c_0) = -x, v (c_0) = lq (c_0).
\end{align*}
\]

Here, the parameter \( x > 0 \) captures the negative slope of \( \nu' (c_0) = lq' (c_0) < 0 \). As shown shortly, our key result does not depend on the exact value of \( x \), which will be determined by pre-determined lower disinvestment threshold \( c^*_h \).

It is easy to show that if \( c^*_0 = c^*_h \), i.e., the policy sets the lower disinvestment threshold as the one in the market solution, then \( x = 0 \) and we have \( c_0 = c^*_h = c^*_h \) and \( c^*_0 = c^*_h \). Given this result, the claim in Proposition 8 is equivalent to show that

\[
\lim_{\gamma \to \infty} \frac{\partial c_h}{\partial x} > 0.
\]

**Step 2. Solve the new ODE system** For simplicity, we denote \( c^*_0 \) by \( c_h \). Given \( c_0 \) and \( c_h \), the boundary conditions \( \nu' (c_h) = q' (c_h) = 0 \) and \( \nu' (c_0) = -xl, q' (c_0) = -x \) imply that

\[
\begin{align*}
q (c_0; c_0, x, c_h) &= q (c_1; c_1, c_h) |_{c_1 = c_0} + \frac{x (e^{2\gamma c_0} + e^{2\gamma c_h})}{\gamma (e^{2\gamma c_h} - e^{2\gamma c_0})}, \\
q (c_h; c_0, x, c_h) &= q (c_h; c_1, c_h) |_{c_1 = c_0} + \frac{2xe^{\gamma (c_0 + c_h)} - e^{2\gamma c_0}}{\gamma (e^{2\gamma c_h} - e^{2\gamma c_0})}, \\
v (c_0; c_0, x, c_h) &= v (c_1; c_1, c_h) |_{c_1 = c_0} + \frac{x (e^{2\gamma c_0} (\gamma l + 1) + e^{2\gamma c_h} (\gamma l - 1))}{\gamma^2 (e^{2\gamma c_h} - e^{2\gamma c_0})}, \\
v (c_h; c_0, x, c_h) &= v (c_h; c_1, c_h) |_{c_1 = c_0} + \frac{2xe^{\gamma (c_0 + c_h)} (c_0 - c_h + l)}{\gamma (e^{2\gamma c_h} - e^{2\gamma c_0})}.
\end{align*}
\]

where \( q (c_1; c_1, c_h), q (c_h; c_1, c_h), v (c_1; c_1, c_h), v (c_h; c_1, c_h) \) has been defined above. Then, \( c_0 \) and \( c_h \) solve \( F_h (c_0, x, c_h) = F_l (c_0, x, c_h) = 0 \) where we define

\[
\begin{align*}
F_h (c_0, x, c_h) &= v (c_h; c_0, x, c_h) - hq (c_h; c_0, x, c_h), \\
&= R_K \left( \frac{(c_h - h) R_C}{2} - \frac{R_C}{2\gamma} m (c_0, c_h) + \frac{R_K \gamma}{2} \left( \frac{g_h (c_0, c_h)}{\gamma} - (c_h + h) f_h (c_0, c_h) \right) \right) \\
&\quad + \frac{2xe^{\gamma (c_0 + c_h)} (c_0 - c_h + l)}{\gamma (e^{2\gamma c_h} - e^{2\gamma c_0})} - \frac{2xe^{\gamma (c_0 + c_h)}}{\gamma (e^{2\gamma c_h} - e^{2\gamma c_0})}, \quad \text{and} \\
F_l (c_0, x, c_h) &= v (c_0; c_0, x, c_h) - lq (c_0; c_0, x, c_h), \\
&= R_K \left( \frac{(c_0 - l) R_C}{2} - \frac{R_C}{2\gamma} m (c_0, c_h) + \frac{R_K \gamma}{2} \left( \frac{g_l (c_0, c_h)}{\gamma} - (c_0 + l) f_l (c_0, c_h) \right) \right) \\
&\quad + \frac{x (e^{2\gamma c_0} (\gamma l + 1) + e^{2\gamma c_h} (\gamma l - 1))}{\gamma^2 (e^{2\gamma c_h} - e^{2\gamma c_0})} - \frac{x (e^{2\gamma c_0} + e^{2\gamma c_h})}{\gamma (e^{2\gamma c_h} - e^{2\gamma c_0})}.
\end{align*}
\]
Simple derivation reveals

\[
\frac{\partial F_i}{\partial c_0} = \frac{R_C}{2} - \frac{R_C}{2} \frac{2e^{(c_0 + c_h)\gamma}}{(e^{c_0\gamma} + e^{c_h\gamma})} + \frac{R_K \gamma}{2} \left( \frac{1}{\gamma} + \frac{e^{2c_h\gamma} + e^{2c_0\gamma}}{e^{2c_h\gamma} - e^{2c_0\gamma}} \right) \left( g_i - (c_0 + l) \frac{e^{2c_h\gamma} + e^{2c_0\gamma}}{e^{2c_h\gamma} - e^{2c_0\gamma}} \right) \left( \gamma f_i - \frac{1}{c_0} \right)
\]

\[
\frac{\partial F_i}{\partial c_h} = \frac{R_C}{2} \frac{2e^{(c_0 + c_h)\gamma}}{(e^{c_0\gamma} + e^{c_h\gamma})^2} + \frac{R_K \gamma}{2} \left( \frac{2gh}{e^{(c_h - c_0)\gamma} - e^{-\gamma(c_h - c_0)}} - (c_h + h) \right) \frac{\gamma f_i - \frac{1}{c_0}}{e^{(c_h - c_0)\gamma} - e^{-\gamma(c_h - c_0)}}
\]

\[
\frac{\partial F_i}{\partial c_h} = \frac{R_C}{2} \frac{2e^{(c_0 + c_h)\gamma}}{(e^{c_0\gamma} + e^{c_h\gamma})^2} + \frac{R_K \gamma}{2} \left( \frac{2gh}{e^{(c_h - c_0)\gamma} - e^{-\gamma(c_h - c_0)}} - (c_h + h) \right) \frac{\gamma f_i - \frac{1}{c_0}}{e^{(c_h - c_0)\gamma} - e^{-\gamma(c_h - c_0)}}
\]

\[
\frac{\partial F_i}{\partial c_h} = \frac{R_C}{2} \frac{2e^{(c_0 + c_h)\gamma}}{(e^{c_0\gamma} + e^{c_h\gamma})^2} + \frac{R_K \gamma}{2} \left( \frac{2gh}{e^{(c_h - c_0)\gamma} - e^{-\gamma(c_h - c_0)}} - (c_h + h) \right) \frac{\gamma f_i - \frac{1}{c_0}}{e^{(c_h - c_0)\gamma} - e^{-\gamma(c_h - c_0)}}
\]

**Step 3. Prove the claim**  Now we are ready to show our desired result \( \lim_{\gamma \to \infty} \frac{\partial F_i}{\partial c_0} > 0 \). First of all, it is easy to show that when \( \gamma \to \infty, c_h \to h \) and \( c_0 \to l \) are bounded. The Cramer’s rule (or implicit function theorem) implies

\[
\lim_{\gamma \to \infty} \frac{\partial F_i}{\partial c_0} = \lim_{\gamma \to \infty} \left( \frac{\partial F_i}{\partial c_0} \frac{\partial F_i}{\partial c_h} \right) = \lim_{\gamma \to \infty} \left( \frac{\partial F_i}{\partial c_0} \frac{\partial F_i}{\partial c_h} \right) = \lim_{\gamma \to \infty} \frac{\partial F_i}{\partial c_0} \frac{\partial F_i}{\partial c_h} = 0,
\]

Focus on the denominator first. It is easy to show that

\[
\lim_{\gamma \to \infty} \frac{\partial F_i}{\partial c_0} = \frac{R_C}{2} + \frac{R_K l}{c_0}, \quad \lim_{\gamma \to \infty} \frac{\partial F_i}{\partial c_h} = \frac{R_C}{2} + \frac{R_K h}{2c_h}, \quad \text{and} \quad \lim_{\gamma \to \infty} \frac{\partial F_i}{\partial c_0} = \lim_{\gamma \to \infty} \frac{\partial F_i}{\partial c_h} = \frac{\partial F_i}{\partial c_0} = 0,
\]

implying

\[
\lim_{\gamma \to \infty} \frac{\partial F_i}{\partial c_0} = \frac{R_C}{2} + \frac{R_K h}{2c_h} = \frac{R_C}{2} + \frac{R_K l}{2c_0}
\]

(B.36)

For the numerator, since \( \frac{\partial F_i}{\partial x} = -\frac{1}{\gamma} \) and \( \frac{\partial F_i}{\partial x} = -\frac{2e^{(c_0 + c_h)\gamma}(h - l + c_h - c_0)}{(e^{2c_h\gamma} - e^{2c_0\gamma})\gamma} \), we can show the following two limiting results:

\[
\lim_{\gamma \to \infty} \left( e^{(c_h - c_0)} - e^{-\gamma(c_h - c_0)} \right) \frac{\partial F_i}{\partial c_0} = -2 \left( h - l + (c_h - c_0) \right) \left( \frac{R_C}{2} + \frac{R_K l}{2c_0} \right)
\]

(B.37)
and

\[
\lim_{\gamma \to \infty} \gamma \left( e^{\gamma (c_h - c_0)} - e^{-\gamma (c_h - c_0)} \right) \frac{\partial F_h}{\partial c_0} \frac{\partial F_i}{\partial x} = \\
\lim_{\gamma \to \infty} \frac{1}{\gamma} \left( \frac{R_c}{2} \frac{2 (e^{\gamma c_h} - e^{\gamma c_0})}{(e^{\gamma h} + e^{\gamma c_h})} + \frac{R_k}{2} \frac{2 y (c_h, c_0) - (c_h + h) 2 \left( \gamma f_1 - \frac{1}{c_0} \right)}{(e^{2 \gamma c_h} - e^{2 \gamma c_0})} \right) = 0. \tag{B.38}
\]

Hence, applying (B.37) and (B.38) to (B.36), we have

\[
\lim_{\gamma \to \infty} \gamma \left( e^{\gamma (c_h - c_0)} - e^{-\gamma (c_h - c_0)} \right) \frac{\partial c_h}{\partial x} = \frac{2 \left( h - l + (c_h - c_0) \right)}{\left( \frac{R_c}{2} + \frac{R_k h}{2 c_h} \right) \left( \frac{R_c}{2} + \frac{R_k}{2 c_0} \right)} > 0.
\]

QED.
C Additional Material for He and Kondor (2015)

C.1 What if $c^p_h > R_K$?

Throughout the main body of the paper we have restricted our attention to the case where the equilibrium range of cash-to-capital ratio, which is $[c^l, c^h]$, is below $R_K$. This ensures that in the idiosyncratic stage the price $\bar{p}_r = c_r \leq R_K$ clears the market in a way that cash (capital) firms get all the cash (capital). Otherwise, suppose that $c^h > R_K$. Then, along the equilibrium path it is possible that $c_r > R_K$, and the price of capital at the idiosyncratic stage will be capped at the capital’s final output $R_K$. As a result, cash firms sell all their capital $K_2$ to capital firms at a price of $R_K$, ending up with a total amount of cash of $\frac{C + R_K K_2}{2}$; while the capital firms will have $K_r$ units of capital but with $\frac{C - R_K K_2}{2}$ units of cash in their hands. This allocation is inefficient as cash firms are holding cash.

This concern is also relevant in the planner’s constrained efficient allocation. Recall that overinvestment requires the planner’s upper investment threshold $c^P_h > c^h$, hence it is quite likely that in a wide range of parameters $c^P_h > R_K$. Because the planner is facing the same information constraint, i.e., the planner cannot tell a cash firm from a capital firm, and the constrained outcome at the idiosyncratic stage is likely to be inefficient.

To illustrate that our main results hold in this case, in this Appendix we relax our parameter restriction to $R_K > h R_C$, and $R_K < h$. (C.39)

Note that we are replacing $R_K < h R_C$ by $R_K < h$; it just says that on the margin capital is better than cash if we just consume cash (for a utility of 1).

We will fully characterize the planner’s solution when $c^P_h > R_K$, which is relatively simpler than the market solution (solving for the market solution fully is much more involved). It turns out that when

$$l < \frac{R_K}{R_C} < h$$

holds, then $c^P_h > R_K$ always holds. More importantly, this Appendix shows that when $c^P_h > R_K$, then our key Proposition 6 in the main text remains valid. It is because in Proposition 6 we show the overinvestment result by establishing in the limit that $c^*_r \rightarrow h < R_K$. Then, since $c^P_h > R_K$, we know that $c^*_h < c^P_h$ in this case automatically, and the key overinvestment result holds.

C.1.1 Mechanism design approach: constrained efficient allocation at idiosyncratic stage

We first show that the mechanism design approach yields the same result as if the planner opens the trading market at the idiosyncratic stage: as discussed above, cash firms ends up with $\frac{C + R_K K_2}{2}$ amount of cash, while capital firms have $K_r$ units of capital but with $\frac{C - R_K K_2}{2}$ amount of cash.

Recall that cash firms cannot operate capital, but capital firms can consume cash at its reservation value of 1. Denote the allocations by $\{C_C, C_K, K_C, K_K\} \in \mathbb{R}_+^4$ where the subscript indicates the reported type. Given resource pair $(C, K)$, the planner is maximizing

$$\max_{\{C_C, C_K, K_C, K_K\} \in \mathbb{R}_+^4} \frac{1}{2} (R_C C_C) + \frac{1}{2} (R_K K_K + C_K)$$

s.t.

$$C_C + C_K \leq C,$$  \hspace{1cm} (C.40)
$$K_C + K_K \leq K,$$  \hspace{1cm} (C.41)
$$C_C \geq C_K,$$  \hspace{1cm} (C.42)
$$R_K K_K + C_K \geq R_K C_C + C_K.$$  \hspace{1cm} (C.43)
Proposition C.1 The solution to the above problem is
\[ K_K = K, \ K_C = 0, \ C_K = \frac{C - R_K K}{2}, \ \text{and} \ C_C = \frac{C + R_K K}{2}, \]
which is identical to the market solution, where the capital price is capped at \( \bar{p}_r = R_K \).

**Proof.** We have several key observations. First, reducing \( K_C \) and increasing \( K_K \) until (C.41) binds can improve objective, relax (C.43), but still satisfies (C.41). Hence \( K_C = 0 \) and \( K_K = K \). Second, (C.40) holds with equality. Otherwise, let \( \varepsilon = (C - C_C - C_K)/2 > 0 \) and raise \( C_C \) and \( C_K \) by \( \varepsilon \), which improves objective and satisfies (C.42) and (C.43). Then we guess (C.43) binds before (C.42). Solve the problem with binding (C.42), we have \( C_K = \frac{C - R_K K}{2} \) and \( C_C = \frac{C + R_K K}{2} \). Since (C.42) holds with strict inequality under this solution, our claim follows. \( \blacksquare \)

C.1.2 Property of the planner’s value function \( j_P(c) \)
The previous subsection shows that when \( c > R_K \), then the aggregate surplus at the idiosyncratic stage is
\[ R_K + R_C \left( \frac{c + R_K}{2} \right) + \frac{c - R_K}{2} \]
where the second (third) term captures the cash in the cash (capital) firms. For \( c \leq R_K \), the surplus is still \( R_K + c R_C \). As a result, we can write the HJB equation as
\[ 0 = \frac{\sigma^2}{2} j''_P(c) + \xi \left( R_K + \frac{R_C + 1}{2} c + \frac{R_C - 1}{2} \min(c, R_K) - j_P(c) \right) \quad \text{for} \ c \in (0, c_h^P) \]
where the flow payoff \( f(c) = R_K + \frac{R_C + 1}{2} c + \frac{R_C - 1}{2} \min(c, R_K) \) is piecewise linear, increasing, and concave in \( c \).

The general solution can be written as
\[ j_P(c) = \begin{cases} 
R_K + R_C c + D_1^{\text{below}} e^{-\gamma c} + D_2^{\text{below}} e^{\gamma c} & \text{for} \ c \in (0, R_K) \\
R_K + R_C c + D_1^{\text{upper}} e^{-\gamma c} + D_2^{\text{upper}} e^{\gamma c} & \text{for} \ c \in (R_K, c_h^P) 
\end{cases} \]
where \( D_1^{\text{below}}, D_2^{\text{below}}, D_1^{\text{upper}}, \) and \( D_2^{\text{upper}} \) are coefficients to be determined. Now we list all the boundary conditions. At \( c_h^P = 0 \) we have the smooth pasting condition:
\[ j_P(0) = l_j P(0). \quad \text{(C.44)} \]
At \( c = R_K \) we have value matching and smooth pasting conditions on both sides
\[ j_P(R_K-) = j_P(R_K+) \quad \text{and} \quad j''_P(R_K-) = j''_P(R_K+) \quad \text{(C.45)} \]
At \( c = c_h^P \) we have smooth pasting condition
\[ j_P(c_h^P) = (h + c_h^P) j'_P(c_h^P) \quad \text{(C.46)} \]
and super contact condition
\[ j''_P(c_h^P) = 0 \quad \text{(C.47)} \]
These five conditions (C.44)-(C.47) pin down five unknowns, in which four of them are coefficients for \( j_P(c) \) in two intervals, and one of them is the optimal upper threshold \( c_h^P \).

Now we show that our main result hold in this case of \( c_h^P > R_K \). First, we the counterpart of Proposition
2.

**Proposition C.2** The planner’s value function under optimal policy $j_P(c)$ is strictly concave, and $j_P(c) < f(c) = R_K + \frac{R_K+1}{2} c + \frac{R_K-1}{2} \min(c, R_K)$.

**Proof.** Denote the flow payoff by

$$f(c) = R_K + \frac{R_K+1}{2} c + \frac{R_K-1}{2} \min(c, R_K).$$

The value function $j_P(c)$ satisfies

$$0 = \frac{\sigma^2}{2} j''_P(c) + \xi (f(c) - j_P(c)) \tag{C.48}$$

with boundary conditions $j_P(0) = l'_P(0)$, $j_P(c^*_h) = (h + c^*_h) j'_P(c^*_h)$, $j''_P(c^*_h) = 0$, and two conditions at $c = R_K$. Note that the boundary conditions imply that $j_P(c^*_h) = f(c^*_h) = \frac{R_K+1}{2} R_K + \frac{R_K+1}{2} c^*_h$ given $c^*_h > R_K$.

We show that $j_P(c)$ is concave over $[0, c^*_h]$, which implies $j_P(c) < f(c)$. First, from smooth pasting condition at $c^*_h$ we have (recall the parameter restriction of $R_K > h$)

$$\frac{R_K+1}{2} - j'_P(c^*_h) = \frac{R_K+1}{2} - \frac{j_P(c^*_h)}{h + c^*_h} = \frac{R_K+1}{2} - \frac{\frac{R_K+1}{2} R_K + \frac{R_K+1}{2} c^*_h}{h + c^*_h} = \frac{R_K+1}{2} \frac{R_K - h}{h + c^*_h} < 0.$$

Then, taking derivative again on (C.48) and evaluate at the optimal policy point $c^*_h$, we have

$$j'''_P(c^*_h) = -\frac{2\xi}{\sigma^2} \left( \frac{R_K+1}{2} - j'_P(c^*_h) \right) = \frac{2\xi}{\sigma^2} \frac{R_K+1}{2} \frac{R_K - h}{h + c^*_h} > 0 \tag{C.49}$$

and as a result $j''_P(c^*_h) < 0$. Suppose that $j_P$ fails to be globally concave over $[0, c^*_h]$. Then there exists some point $j''_P > 0$, and pick the largest one $\tilde{c}$ so that $j''_P$ is concave over $[\tilde{c}, c^*_h]$ with $j''_P(\tilde{c}) = 0$ and $j''_P(\tilde{c}) < 0$. At $\tilde{c}$ we have $j_P(\tilde{c}) = f(\tilde{c})$, $j_P(\tilde{c}) < f'(c)$ (because $j_P(c)$ cross $f(c)$ from above), and $j_P(c)$ is strictly convex around the vicinity of $c < \tilde{c}$. Using standard argument one can show that $j_P(c)$ is strictly convex over $[0, \tilde{c}]$ (if not, pick the largest point point $\tilde{c}$ so that $j''_P(\tilde{c}) = 0$. But due to convexity of $f(c)$ and linearity of $f(c)$ in $[\tilde{c}, \tilde{c}]$ we have $f(c) < j_P(c)$ strictly, contradicting with $j''_P(\tilde{c}) = 0$). Thus $j_P(c)$ is strictly convex over $[0, \tilde{c}]$. Since $f(c)$ is concave, we have $f'(c) < f'(0) = R_C$ and $j_P(0) > f(0) = R_K$. It contradicts with the boundary condition at $c = 0$, because $j_P(0) = l'_P(0) < lR_C < R_K$.

To sum up, $j_P(c)$ is globally concave over $[0, c^*_h]$, which also implies that $j_P(c) < f(c)$ due to (C.48).

We now show that when $R_K < hR_C$ holds, we have $c^*_h > R_K$.

**Corollary 1** Under parameter restriction C.29, and suppose that $R_K < hR_C$. Then $c^*_h > R_K$.

**Proof.** Suppose that $c^*_h < R_K$; then we should have the same characterization in Proposition 2. However,

$$\frac{R_K - hR_C}{R_K - lR_C} \left( e^{c^*_h \gamma} (1 + l\gamma) - (1 - l\gamma) e^{-c^*_h \gamma} \right) - 2\gamma (c^*_h + h) = 0$$

admits no solution: if $R_K - hR_C < 0$ then even the first term is negative.\(^{21}\) Hence $c^*_h > R_K$. And, when $c \to \infty$ the marginal value of cash is 1 and the marginal value of capital is $R_K$. Hence when $R_K < h$, holding cash is strictly dominated by holding capital when $c \to \infty$, implying $c^*_h < \infty$.\(^{22}\)

\(^{21}\) Note that $e^{c^*_h \gamma} (1 + l\gamma) - (1 - l\gamma) e^{-c^*_h \gamma} > e^{c^*_h \gamma} - e^{-c^*_h \gamma} > 0$.

\(^{22}\) This result is also consistent with condition (C.49) which says that postponing liquidating for $c > c^*_h$ gives $j''_P(c^*_h) > 0$. 

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Intuitively, when accumulated cash (relative to the capital stock) is below \( R_K \), then the marginal value of cash is \( R_K \). If \( R_K < h R_C \), then the benefit of capital is below the cost of building capital, it is never optimal to build the capital for \( c < R_K \). When \( c > R_K \), the marginal value of cash is just its consumption value 1. As \( R_K > h \) says the benefit of capital exceeds the cost, then the planner starts building capital when \( c = c^P_h > R_K \) for sufficiently high \( c^P_h \).

### C.1.3 Welfare results

Now we have the counterpart of Proposition 3, which gives the key result about investment inefficiency. The argument is almost identical to Proposition 3 which only relies on the concavity of \( j_P \) for all policy \( c \in [c_l, c_h] \) if \( c_l < c^P_h \) and \( c_l > 0 \).

**Proposition C.3** For any \( c_l < c^P_h \) and \( c_l > 0 \), we have

\[
\frac{\partial j_P (c; c_l, c_h)}{\partial c_l} < 0, \quad \text{and} \quad \frac{\partial j_P (c; c_l, c_h)}{\partial c_h} > 0 \quad \text{of all} \quad c \in [c_l, c_h].
\]

**Proof.** Suppose that we are given the policy pair \( (c_l, c_h) \) with \( 0 < c_l < c_h < c^P_h \), where \( c^P_h \) satisfies the super-contact condition \( j^\prime P (c^P_h; 0, 0) = 0 \). To avoid cumbersome notation we denote the social value \( j_P (c; c_l, c_h) \) given the pair \((c_l, c_h)\) by \( j(c; c_l, c_h) \), and denote the social value under the optimal policy \( j_P (c; 0, c^P_h) \) by \( j_P (c) \). We need to show that

\[
\frac{\partial j (c; c_l, c_h)}{\partial c_l} < 0 \quad \text{and} \quad \frac{\partial j (c; c_l, c_h)}{\partial c_h} > 0.
\]

This result further implies that for \( 0 < c^2 < c^2_l < c^2_h < c^2_P \), we have \( j(c; c^2_l, c^2_h) < j(c; c^2, c^2_h) \).

As preparation, we first show that \( j'''' (c_l; c_l, c_h) < 0 \) and \( j'''' (c_l, c_l, c_h) < 0 \). Because \((c_l, c_l)\) is suboptimal, we must have \( j(c; c_l, c_h) < j_P (c) \leq f (c) \) (recall the above proposition). Then \( 0 = \frac{\sigma^2}{2} j'''' (c) + \xi (f (c) - j (c)) \) implies that \( j (c) \) is strictly concave at both ends. Second, for any policy pair \((c_l, c_h)\) (including the market solution or the planner’s solution), the smooth pasting condition (not optimality condition!) at the regulated ends implies that

\[
j (c_l; c_l, c_h) - (c_l + h) j' (c_l; c_l, c_h) = 0, \tag{C.50}
\]

\[
j (c_l, c_l, c_h) - (c_l + h) j' (c_l; c_l, c_h) = 0. \tag{C.51}
\]

Now we start proving the properties for the top policy \( c_h \). Define \( F_h (c; c_l, c_h) \equiv \frac{\partial}{\partial c_h} j (c; c_l, c_h) \), which is the marginal impact of changing the top investment policy on the social value. Differentiating the basic ODE by the policy \( c_h \), we have \( \frac{\sigma^2}{2} \frac{\partial}{\partial c_h} j''' (c; c_l, c_h) - \xi \frac{\partial}{\partial c_h} j (c; c_l, c_h) = 0 \), or

\[
\frac{\sigma^2}{2} F_h''' (c; c_l, c_h) - \xi F_h (c; c_l, c_h) = 0. \tag{C.52}
\]

Moreover, take the total derivative with respect to \( c_h \) on the equality \( (C.50) \), i.e., take derivative that affects both the policy \( c_h \) and the state \( c = c_h \), we have

\[
\frac{\partial}{\partial c_h} j (c_l; c_l, c_h) + j' (c_l; c_l, c_h) = j' (c_l; c_l, c_h) + (c_l + h) \left( \frac{\partial}{\partial c_h} j' (c_l; c_l, c_h) + j''' (c_l; c_l, c_h) \right)
\]

\[
\Rightarrow \frac{\partial}{\partial c_h} j (c_l; c_l, c_h) - (c_l + h) \frac{\partial}{\partial c_h} j' (c_l; c_l, c_h) = (c_l + h) j''' (c_l; c_l, c_h) < 0
\]

\[
\Rightarrow F_h (c_l; c_l, c_h) - (c_l + h) F_h' (c_l; c_l, c_h) < 0. \tag{C.53}
\]

which gives the boundary condition of \( F_h (\cdot) \) at \( c_h \). At \( c_l \) we can take total derivative with respect to \( c_h \) on
the equality (C.51), we have the boundary condition of \( F_h (\cdot) \) at \( c_l \):

\[
\frac{\partial}{\partial c_l} j (c_l; c_l, c_h) = (c_l + l) \frac{\partial}{\partial c_l} j' (c_l; c_l, c_h) \Rightarrow F_h (c_l; c_l, c_h) - (c_l + l) F'_h (c_l; c_l, c_h) = 0.
\]

(C.54)

With the aid of these two boundary conditions, the next lemma shows that \( F_h (\cdot) \) has to be positive always. Because of the definition of \( F_h (c; c_l, c_h) \equiv \frac{\partial}{\partial c_h} j (c; c_l, c_h) \), it implies that raising \( c_h \) given any state \( c \) and any lower policy \( c_l \) improves the social value. The argument for the effect of \( c_l \) is similar and thus omitted.

**Lemma B.1** We have \( F_h (c) > 0 \) for \( c \in [c_l, c_h] \).

**Proof.** We show this result in three steps.

First, \( F_h (c) \) cannot change sign over \([c_l, c_h]\). Suppose that \( F_h (c_l) > 0 \); then from (C.54) we know that \( F'_h (c_l) > 0 \). Then simple argument based on ODE (C.52) implies that \( F_h (\cdot) \) is convex and always positive. Now suppose that \( F_h (c_l) = 0 \); then the similar argument implies that \( F_h \) is concave and negative always.

Finally, suppose that \( F_h (c_l) = 0 \) but \( F_h \) changes sign at some point. Without loss of generality, there must exist some point \( \bar{c} \) so that \( F'_h (\bar{c}) = 0 \), \( F_h (\bar{c}) > 0 \) and \( F''_h (\bar{c}) < 0 \). But this contradicts with the ODE (C.52).

Second, define \( W_h (c) \equiv F_h (c) - (l + c) F'_h (c) \) so that \( W'_h (c) = -(l + c) F''_h (c) = -2l(l+c) F_h (c) \). As a result, \( W'_h (c) \) cannot change sign. Because we have \( W_h (c_l) = 0 \), \( W_h (c) \) cannot change sign either.

Third, suppose counterfactually that \( F_h (c) < 0 \) so that \( W'_h (c) > 0 \). Step 2 implies that \( W_h (c) > 0 \), and \( F'_h (c_l) = \frac{h-l}{l+c} (F_h - W_h) < 0 \). But we then have

\[
W_h (c_l) = F_h (c_l) - (l + c) F'_h (c_l) = F_h (c_l) - (h + c) F'_h (c_l) + (h - l) F''_h (c_l) < 0,
\]

where we have used (C.53), contradiction. Thus we have shown that \( F_h (c) > 0 \). 

The next proposition naturally follows.

**Proposition C.4** When \( \gamma \to \infty \) then in the market solution \( c^*_h \to h < R_K \). For planner’s solution, we have \( c^*_h > R_K \) when either \( R_K - hR_C > 0 \) is sufficiently small or when \( R_K - hR_C < 0 \). It implies that firms overinvest in capital in booms in the market solution.

### C.1.4 A Model without Idiosyncratic Investment Opportunities

Suppose that at the final date, every firm with holdings \((K,C)\) can produce \( R_K K + R_C C \) units of final consumption goods. This formulation is also equivalent to the base model but with complete market, i.e., introducing some Arrow-Debreu securities contingent on firms’ idiosyncratic type realization—either \( K \) or \( C \)—which obviously complete the market. Ex ante, each firm will fully hedge using these Arrow-Debreu contracts, so that each unit of capital pays off \( R_K \) units of consumption goods while each unit of cash pays off \( R_C \) units of consumption goods. This is exactly identical to the hypothetical precautionary-saving-motive model without idiosyncratic investment opportunities.

We show that the precautionary-saving-motive model is constraint efficient, a reminiscent of the first welfare theorem. To prove this result formally, denote the value functions in the complete market equilibrium by \( v_{cm} (c) \) and \( q_{cm} (c) \). The HJB equation for value functions become

\[
0 = \frac{\sigma^2}{2} v''_{cm} (c) + \xi (R_C - q_{cm} (c)) \quad \text{volatility of } dc_l \quad \text{final date realization}
\]

(C.55)

\[
0 = \frac{\sigma^2}{2} q''_{cm} (c) + q'_{cm} (c) \sigma^2 + \xi (R_K - v_{cm} (c)) \quad \text{volatility of } dc_l \quad \text{expected value of dividends} \quad \text{final date realization}
\]

(C.56)
In contrast, in our model the valuation equations for \( q \) and \( v \) given idiosyncratic investment opportunities are

\[
0 = \frac{\sigma^2}{2} q''_{cm} (c) + \frac{\xi}{2} (R_C - q_{cm} (c)) + \frac{\xi}{2} \left( \frac{R_K}{c} - q_{cm} (c) \right) \tag{C.57}
\]

\[
0 = \frac{\sigma^2}{2} v''_{cm} (c) + \frac{\xi}{2} q'_{cm} (c) \sigma^2 + \frac{\xi}{2} (R_C c - v_{cm} (c)) + \frac{\xi}{2} (R_K - v_{cm} (c)) \tag{C.58}
\]

Take \( q \) equation as example. In \((C.55)\), the term "final date realization" captures that with intensity \( \xi \), the firm can use its cash holding to obtain \( R_C \) units of consumption goods. While in \((C.57)\), this term has two components: if becoming a cash firm with intensity \( \frac{\xi}{2} \), then it obtains \( R_C \); while if becoming a capital firm with intensity \( \frac{\xi}{2} \), it obtains \( \frac{R_K}{c} \) by purchasing capital (which generates \( R_K \)) at the price of \( \tilde{p} = c \).

Comparing \((C.55)-(C.56)\) to the planner’s solution in Section 3.1.1, we see that

\[
J_p (c) = v_{cm} (c) + q_{cm} (c).
\]

Denote the new endogenous (dis)investment boundaries by \( c^e_{h,m} \) and \( c^f_{l,m} \). Because we knew that the constrained efficient solution features \( c^f_{l,m} = 0 \), we need to be careful in the associated boundary conditions at the lower bound boundary. At upper boundary, we have

\[
\frac{v (c^e_{hm})}{q (c^e_{hm})} = h, \quad v' (c^e_{hm}) = 0, \quad \text{and} \quad q' (c^e_{hm}) = 0. \tag{C.59}
\]

At the lower boundary, taking into account of possibility that \( c^f_{l} = 0 \) binds at zero, we have the complementary-slack condition:

\[
\frac{v (c^f_{l,m})}{q (c^f_{l,m})} = l, \quad v' (c^f_{l,m}) = 0, \quad \text{and} \quad q' (c^f_{l,m}) \leq 0 \quad \text{with strict inequality if} \quad c^f_{l,m} = 0. \tag{C.60}
\]

The first condition \( \frac{v (c^f_{l,m})}{q (c^f_{l,m})} = l \) have to hold because in equilibrium only a fraction of firms are liquidating their capital at \( c^f_{l,m} \), who must be indifferent between selling or liquidating their capital. The intuition behind \( v' (c^f_{l,m}) = 0 \) and \( q' (c^f_{l,m}) < 0 \) is as follows. Note that \( c \) in the functions \( v (\cdot) \) and \( q (\cdot) \) capture the aggregate cash liquidity. Loosely speaking, if the aggregate liquidity is strictly negative say \( -\varepsilon \), then the value of cash is higher (relative to \( q (c = 0) \)) because cash can be used to reduce the amount of capital that needs to be liquidated. In contrast, given \( c = -\varepsilon \) or 0, the optimal policy with regard to existing capital is unchanged (think about those capitals that end up not to be liquidated), which explains \( v' (c^f_{l,m} = 0) = 0 \).

The following proposition formally shows that \( c^e_{l,m} = 0 \) and \( c^f_{h,m} = c^f_{l,m} \), which coincide with the planner’s solution.

**Proposition C.5** In the complete market economy, there is an equilibrium for any set of parameters where

1. firms do not consume before the aggregate stage,
2. each firm in each state \( c \in [0, c^e_{h,m}] \) is indifferent in the composition of her portfolio,
3. each firm holding capital use every positive cash shock to build capital if and only if \( c = c^e_{h,m} \) and finance the negative cash shocks by liquidating the capital if and only if \( c = 0 \),
4. the value of holding a unit of cash and the value of holding a unit of capital are described by

\[
q_{cm} (c) = R_C + e^{-\gamma_1} L_1 + e^{-\gamma_2} L_2, \tag{C.61}
\]

\[
v_{cm} (c) = R_K + e^{-\gamma_1} (D^p_1 - c L_1) + e^{-\gamma_2} (D^p_2 - c L_2). \tag{C.62}
\]

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where constants $L_1, L_2, D_1^P, D_2^P$ and $c_h^{cm}$ are determined by boundary conditions from (C.59) and (C.60):

\[
\begin{align*}
v_{cm}(c_h^{cm}) &= h, \quad v_{cm}(0) = 1, \quad v'_{cm}(c_h^{cm}) = q_{cm}(c_h^{cm}) = v'_{cm}(0) = 0. \\
\end{align*}
\]  

(C.63)

In this equilibrium, $v_{cm}(c)$ is increasing in $c$, $q_{cm}(c)$ is decreasing in $c$. Hence $p_{cm}(c) \equiv \frac{v_{cm}(c)}{q_{cm}(c)}$ is increasing in $c$, implying the optimality of $c_h^{cm} = 0$.

5. Finally, we have $c_h^{cm} = c_h^P$, $D_1^P = D_1$ and $D_2^P = D_2$ given in the planner’s solution, so that $j_P(c) = v_{cm}(c) + q_{cm}(c)$ for all $c \in [0, c_h^P]$.

**Proof.** Given (C.61)-(C.62), the boundary conditions in (C.63) are

\[
\begin{align*}
R_K + R_Cc_h^{cm} + e^{c_h^{cm}\gamma}D_2^P + e^{-c_h^{cm}\gamma}D_1^P &= h + c_h^{cm}; \\
R_C + e^{-c_h^{cm}\gamma}L_1 + e^{c_h^{cm}\gamma}L_2 &= L; \\
\gamma e^{c_h^{cm}\gamma}(D_2^P - c_h^{cm}L_2) - L_2 e^{c_h^{cm}\gamma} - \gamma e^{-c_h^{cm}\gamma}(D_1^P - c_h^{cm}L_1) - L_1 e^{-c_h^{cm}\gamma} &= 0; \\
-\gamma e^{-c_h^{cm}\gamma}L_1 + \gamma e^{c_h^{cm}\gamma}L_2 &= 0; \\
\gamma D_2^P - L_2 - \gamma D_1^P - L_1 &= 0.
\end{align*}
\]  

(C.64)  

(C.65)  

(C.66)  

(C.67)  

(C.68)

Adding $c_h^{cm}$ times (C.67) to (C.66) gives

\[
e^{c_h^{cm}\gamma}(\gamma D_2^P - L_2) - e^{-c_h^{cm}\gamma}(\gamma D_1^P + L_1) = 0.
\]

Together with (C.68), this implies

\[
\gamma D_2^P = L_2, \quad \text{and} \quad -L_1 = \gamma D_1^P. \\
\]  

(C.69)

Substituting this into (C.67) gives

\[
e^{-c_h^{cm}\gamma}D_1^P + e^{c_h^{cm}\gamma}D_2^P = 0. \\
\]  

(C.70)

Also, as $L_1 + L_2 = \gamma L_3 - \gamma L_4$, (C.65) implies that

\[
R_K + D_2^P + D_1^P = l(R_C + \gamma D_2^P - \gamma D_1^P) \\
\]  

(C.71)

and by (C.69), (C.64) is equivalent to

\[
R_K + R_Cc_h^{cm} + e^{c_h^{cm}\gamma}D_2^P + e^{-c_h^{cm}\gamma}D_1^P = (h + c_h^{cm}) \left( R_C - \gamma D_1^P e^{-c_h^{cm}\gamma} + \gamma D_2^P e^{c_h^{cm}\gamma} \right). \\
\]  

(C.72)

Then, we observe that the system (C.70)-(C.72) is equivalent with the following system for the planner’s problem with $D_2^P = D_2$, $D_1^P = D_1$ and $c_h^{cm} = c_h^P$:

\[
R_K + D_1 + D_2 = l(R_C - \gamma D_1 + \gamma D_2), \\
R_K + R_Cc_h^P + D_1 e^{-c_h^P} + D_2 e^{c_h^P} = (h + c_h^P) \left( R_C - \gamma D_1 e^{-c_h^P} + \gamma D_2 e^{c_h^P} \right), \\
D_1 e^{-c_h^P} + D_2 e^{c_h^P} = 0.
\]

with $D_1 = \frac{(R_K - lR_C)e^{c_h^P}}{(1+l)e^{c_h^P} - (1-l)e^{-c_h^P}}$, $D_2 = \frac{R_K - lR_C}{(1+l)e^{c_h^P} - (1-l)e^{-c_h^P}}$.

Finally, to show that price is monotonically increasing in this economy we show that $v'_{cm}(c) > 0$ and
\( q'_{cm}(c) < 0 \) for every \( c \in (0, e^{\text{cm}}_h) \). It is easy to check that

\[
q'_{cm}(c) = -\gamma e^{-\gamma t} L_1 + \gamma e^{-\gamma t} L_2 = \gamma^2 e^{-\gamma t} D_1 + \gamma^2 e^{-\gamma t} D_2 = \\
\gamma^2 (R_K - lR_C) e^{-\gamma t} \frac{1 - e^{2\gamma t c^0}}{e^{2\gamma t c^0} + l\gamma (e^{2\gamma t c^0} - 1)} < 0.
\]

This also verifies the complementarity-slackness condition in (C.60). And,

\[
v'_{cm}(c) = \gamma e^{-\gamma t} (D_2 - cL_2) - L_2 e^{-\gamma t} - \gamma e^{-\gamma t} (D_1 - cL_1) - L_1 e^{-\gamma t} = \\
-\gamma^2 D_2 e^{-\gamma t} - \gamma^2 D_1 e^{-\gamma t} = \gamma \gamma^2 (R_K - lR_C) e^{-\gamma t} \frac{1 - e^{2\gamma t c^0}}{e^{2\gamma t c^0} + l\gamma (e^{2\gamma t c^0} - 1)} \geq 0.
\]

\[Q.E.D.\]

C.2 The role of Cobb-Douglas technology in Section 5.1

In Section 5.1 we introduce a Cobb-Douglas technology which combines cash and capital to produce some final goods; and recall in the base model this technology is not needed. This note explains why we introducing this technology.

It is a tradition in the pecuniary externality literature to think about distortionary tax scheme as small transfer which results in some first-order incentive/welfare implications. But, because the linear production technology used in the main model may well push individual firms to take some cornered solution (as we find in our numerical solution), firms are insensitive to marginal tax transfers. In other words, the linear technology implies a cornered solution on the lower side, and as a result distortionary tax schemes which only affect incentives slightly have a hard time to induce some real effect.

We introduce the Cobb-Douglas technology \( \phi K^\alpha C^{1-\alpha} \) in the aggregate stage in order to break the corner solution for the disinvestment threshold. Cobb-Douglas technology naturally implies a higher marginal value of cash when the aggregate cash level is lower. This can be seen here:

\[
0 = q'(c) \sigma + v'(c) \frac{\xi}{2} \left( -p(c) + \frac{c^2}{p(c)} \right) + \frac{\sigma^2}{2} v''(c) + \frac{\xi}{2} \left( \frac{R_K - lR_C p(c)}{2} - v(c) \right) + \phi \alpha c^{1-\alpha} \quad \text{(C.73)}
\]

\[
0 = q'(c) \frac{\xi}{2} \left( -p(c) + \frac{c^2}{p(c)} \right) + \frac{\sigma^2}{2} v''(c) + \frac{\xi}{2} \left( \frac{R_K - lR_C}{p(c)} - q(c) \right) + \phi (1-\alpha) c^{-\alpha} \quad \text{(C.74)}
\]

relative to (12), there is an extra flow \( \phi (1-\alpha) c^{-\alpha} \) in the cash value equation which increases when \( c \) drops. In fact, because it features the Inada condition so that the marginal value of cash goes to infinity when \( C_t = 0 \), it automatically guarantees that the equilibrium disinvestment threshold take an interior solution with a zero-order first condition (rather than be cornered at \( c = 0 \) with non-zero first-order condition).

As a result, with Cobb-Douglas technology, in the market solution firms are taking interior solutions on both the upper investment and lower disinvestment thresholds, both with zero first-order conditions. Because policy interventions are in the form of small distortionary tax schemes, this helps us illustrate the pecuniary externality with small policy interventions, an exercise that we are performing in Section 5.

C.3 The model with collateralized borrowing and proof of Proposition 9

As a preparation we first analyze the model with collateralized borrowing; we then give the proof of Proposition 9 in Section 5.2.

We highlight three parameters with superscript \( b \) which have special roles in characterizing the equilibrium. In the economy with collateralized borrowing, we use \( R^b_K \) to denote the productivity of capital; and,
per unit of capital firms get \( l^b \) units of cash when liquidating while need to invest \( h^b \) units of cash when investing.

We conjecture that firms always max out their borrowing capacity (and verify later this holds in equilibrium). In the idiosyncratic stage after the heterogenous technology shocks hit, there are \( K_r/2 \) units of capital to be sold. On the cash side, in addition to \( C_r/2 \) units of the cash, collateralized borrowing implies that there are \( bK_r \) units of extra cash in aggregate from external creditors. Hence the equilibrium capital price is \( \tilde{p}_r = \frac{bK_r + C_r/2}{K_r/2} = 2b + c_r \). For capital firms being willing to purchase capital, we require

\[
R^b_K > 2b + c_r, \tag{C.75}
\]

which, as in the base model, can hold in equilibrium because \( c_r \) will be bounded endogenously.

For assets that can be used as collateral, it is useful to note that \( \tilde{p}_r - b \) is the “effective” capital price. This is because each unit of capital can be used to borrow \( b \) units of cash, and firms with their own cash of \( \tilde{p}_r - b = b + c_r \) can buy a unit of capital. When \( c_r = -b \) which is lower bound of the aggregate net cash holding, the effective price of capital drops to zero.

For capital firms with capital \( K_r \) and cash \( C_r \), they will use their cash holding \( C_r \), together with credit \( K_r b \), to purchase capital from the market at the effective capital price \( b + c_r \). The final consumption goods, net of borrowing payment, is

\[
\frac{K_r (R^b_K - b)}{b + c_r} + \frac{C_r + K_r b}{b + c_r} (R^b_K - b) = K_r \frac{(2b + c_r) (R^b_K - b)}{b + c_r} + C_r \frac{R^b_K - b}{b + c_r} \tag{C.76}
\]

Note, under condition (C.75), it is optimal to exhaust the borrowing capacity (at a marginal cost of 1) to purchase the capital from the market (at a marginal benefit of \( \frac{R^b_K - b}{b + c_r} \)). For cash firms, their payoff is

\[
\frac{C_r R_C}{b + c_r}. \tag{C.77}
\]

Hence, the marginal payoff for capital is \( K_r (2b + c_r) \), while for cash it is \( R_C \).

Now we move on to aggregate stage, and denote the value of capital and cash by \( v^b (c) \) and \( q^b (c) \), respectively. The similar structure for the market equilibrium as in the base model prevails, i.e. firms build (dismantle) capital when the aggregate cash-to-capital ratio \( c_t \) reaches an endogenous upper (lower) threshold \( c_{h}^b \) \( (c_{l}^b) \). In the inaction region \( c^b \in (c_{l}^b, c_{h}^b) \), we have the same evolution of state variable \( dc_t = \sigma dZ_t \), and the values of capital and cash satisfy

\[
0 = q^{b^b} (c_t^b) = v^{b^b} (c_t^b) = 0, \quad v^{b^b} (c_h^b) = p^b q^{b^b} (c_h^b), \quad \sigma^2 = \frac{2}{2} q^{b^b} (c_t^b) - \xi q^b (c) + \frac{\xi}{2} \left[ \frac{(2b + c) (R^b_K - b)}{b + c} + R_C (c + 2b) \right], \tag{C.78}
\]

Here, we have used the marginal payoffs of capital and cash for either capital or cash firms given in (C.76) and (C.77). The boundary conditions are the same as the base model:

\[
q^{b^b} (c_l^b) = 0, \quad v^{b^b} (c_l^b) = p^b q^b (c_l^b), \quad q^{b^b} (c_h^b) = 0, \quad v^{b^b} (c_h^b) = h^b q^b (c_h^b). \tag{C.79}
\]

We now show that there is a simple relationship between the economy with and without borrowing both
in the planner’s case and in the decentralized case, define three “translated” parameters as

\[ R_K = R^b_K - b, \quad h = h^b - b, \quad \text{and} \quad l = l^b - b, \]

and consider the no-borrowing economy with the above three parameters. For the market equilibrium, denote the resulting capital and cash value functions \( v(c) \) and \( q(c) \) respectively, with equilibrium thresholds \( (c^*_l, c^*_h) \). Analogously, denote \( j(\cdot) \) and \( c^*_l \) as the social planner’s solution. We have the following proposition.

**Proposition C.6** Consider the economy with borrowing. For the social planner, we have \( j^b(c) = j(c + b) \), \( c^b_l = -b \) and \( c^b_h = c^b - b \). For the market equilibrium the capital and cash value functions are given by

\[ v^b(c) = v(c + b) + b q(c + b), \quad \text{and} \quad q^b(c) = q(c + b). \]  

Hence the capital price is \( p^b(c) = p(c + b) + b \), and the investment and disinvestment thresholds are given by \( c^b_l = c^l - b \) and \( c^b_h = c^h - b \).

**Proof.** Recall that in the base model without borrowing, if \( R = R^b - b \), our value functions satisfy

\[ 0 = \frac{\sigma^2}{2} q''(c) + \frac{\xi}{2} (R_C - q(c)) + \frac{\xi}{2} \left( \frac{R^b_K - b}{c} - q(c) \right), \]  

\[ 0 = q'(c) \sigma^2 + \frac{\sigma^2}{2} v''(c) + \frac{\xi}{2} (R_C - v(c)) + \frac{\xi}{2} \left( \frac{R^b - b - v(c)}{c} \right). \]

We only show \( q^b \). If \( q^b(c) = q(c + b) \), we have

\[ 0 = \frac{\sigma^2}{2} q''(c + b) + \frac{\xi}{2} (R_C - q^b(c + b)) + \frac{\xi}{2} \left( \frac{R^b_K - b}{c + b} - q^b(c + b) \right) \]

\[ \Leftrightarrow 0 = \frac{\sigma^2}{2} q''(c + b) + \frac{\xi}{2} (R_C - q(c + b)) + \frac{\xi}{2} \left( \frac{R^b_K - b}{c + b} - q(c + b) \right) \]

which holds always as we can view \( c + b \) as \( c \) in (C.82). Similarly we can show the result for \( v^b(c) \). The investment and disinvestment thresholds and the social planner’s solution are obvious given this result. ■

### C.3.1 Proof of Proposition 9

For simplicity we set \( \gamma = 1 \). Our results rely on two lemmas. The first lemma gives the market solution.

**Lemma B.2** For any \( a > 1 \), \( c_l = x, c_h = ax \) is an equilibrium in the limit \( x \to 0 \), if

\[ \frac{3(a-1) - (a+1) \ln a}{\ln a} x = l, \]

\[ \frac{3(a-1) - (a+1) \ln a}{\ln a} x + o(x) = h. \]

**Proof.** One can show \( \lim_{x \to 0} x f_1(x, ax) = \frac{\ln(a)}{4(a-1)} \), \( \lim_{x \to 0} g_l(x, ax) = 1 - \frac{a \ln(a)}{a-1} \). Thus, we have

\[ \lim_{x \to 0} \frac{p_l(x; x, ax)}{x} = \lim_{x \to 0} \frac{R_K + x R_C + R_K - m(x, ax) + R_K - R_K (q_l(x, ax) - x f_1(x, ax))}{x R_C + R_K x f_1(x, ax)} = \lim_{x \to 0} \frac{R_K + R_K \left( 1 - \frac{a \ln(a)}{a-1} - \frac{\ln(a^*)}{4(a-1)} \right)}{x R_C + R_K \frac{\ln(a^*)}{4(a-1)}} = \frac{3(a-1) - (a+1) \ln a}{\ln a}. \]
Hence if \( l = \frac{3(a-1)-(a+1)\ln a}{\ln a} x \) then \( x = L(ax) \) in the limit. Similarly

\[
\lim_{x \to 0} ax f_h(x, ax) = \frac{a \ln(a)}{\gamma (a-1)}, \quad \lim_{x \to 0} g_h(x, ax) = 1 - \frac{\ln(a)}{a-1}.
\]

Thus, for

\[
\lim_{x \to 0} \frac{p_h(x; x, ax)}{x} = \lim_{x \to 0} \frac{R_K + \frac{axR_C}{2} + \frac{R_C}{2} m(x, ax) + \frac{R_K}{2} (g_h(x, ax) - ax f_h(x, ax))}{x} = \frac{R_K + \frac{axR_C}{2} + \frac{R_C}{2} m(x, ax) + \frac{R_K}{2} \left(1 - \frac{\ln(x)}{a-1} - \frac{a \ln(a)}{a-1}\right)}{x} = \frac{1 + \frac{1}{2} (g_h(x, ax) - ax f_h(x, ax))}{\frac{1}{2} x f_h(x, ax)} = \frac{3(a-1)-(a+1)\ln(a)}{\ln(a)}
\]

Hence if

\[
h = \frac{3(a-1)-(a+1)\ln(a)}{\ln(a)} x + o(x)
\]

Then \( x = H(ax) \) in the limit. QED. ■

Because \( l(k) = O(\varepsilon^k) \), \( h(k) = O(\varepsilon^k) + \varepsilon \) and \( k < 1 \), \( \varepsilon = o(O(\varepsilon^k)) \), which implies the above lemma applies. Hence the market solution has

\[
c_l^* = O(\varepsilon^k), \quad c_h^* = aO(\varepsilon^k)
\]

where \( a > 1 \) is the solution to \( 3(a-1) = a \ln a \). The next lemma gives the planner’s solution.

**Lemma B.3** For \( x \) being sufficiently small, suppose that

\[
l = x, \quad h = x + O\left(x^{3(1+\alpha)}\right)
\]

where the constant \( \alpha \) can be either positive or negative. Then the social planner’s solution satisfies

\[
c_h^p \propto \begin{cases} 
  x^{1+\alpha} & \text{if } \alpha < 0 \\
  x & \text{if } \alpha = 0 \\
  x^{1+1.5\alpha} & \text{if } \alpha > 0
\end{cases}
\]

**Proof.** Without loss of generality we fix \( \gamma = 1 \). The planner’s solution \( c_h^p \) satisfies

\[
\left(R_K - \left(x + O\left(x^{3(1+\alpha)}\right)\right)R_C\right) [e^{c_h^p}(1+x) - (1-x)e^{-c_h^p}] = 2\left(R_K - xR_C\right) \left(c_h^p + x + O\left(x^{3(1+\alpha)}\right)\right)
\]
and (C.84) therefore is

\[ x \] is sufficiently small. We have the following three cases to consider.

Here, term (1) on LHS is dominated by (a) on RHS, term (2) on LHS is dominated by (c) on RHS, and the term (3) on LHS is dominated by (b) on RHS. As a result, it must be that \( c_h^p \) is determined by LHS = 0 when \( x \) is sufficiently small. We have the following three cases to consider.

1. If \( \alpha > 0 \), we conjecture that term (b) is at a higher order so it is negligible. Thus \( c_h^p \) is determined by

\[ x (c_h^p)^2 R_K - 2R_K O \left(x^{3(1+\alpha)}\right) = 0, \] or

\[ c_h^p = \sqrt{2} O \left( x^{3(1+\alpha)} / x \right)^{\alpha} = \sqrt{2} O \left( x^{1+1.5\alpha} \right). \]

This also implies that \( (c_h^p)^3 = O \left( x^{3+4.5\alpha} \right) \) which is indeed at a higher order than term (c).

2. If \( \alpha = 0 \), then we LHS = 0 implies that

\[ 2R_K O (c^3) - \frac{R_K}{3} (c_h^p)^3 = R_K x (c_h^p)^2 \]
which implies that \( c_h^P = O(x) \).

3. If \( \alpha < 0 \), we conjecture that term \( (a) \) is at a higher order so it is negligible. Then \( c_h^P \) is determined by

\[
\frac{R_K}{3} \left( (c_h^P)^3 \right) - 2R_K O \left( x^{3(1+\alpha)} \right) = 0
\]

which implies that \( c_h^P = \sqrt[3]{O(x)^{1+\alpha}} \).

Recall that \( l(k) = O(\varepsilon^k) \), \( h(k) = O(\varepsilon^k) + \varepsilon \); applying the above lemma, we know that

\[
c_h^P(k) \propto \begin{cases} 
O(\varepsilon^{1/3}) & \text{if } k < 1/3 \\
O(\varepsilon^k) & \text{if } k = 1/3 \\
O(\varepsilon^{(1-k)/2}) & \text{if } k > 1/3
\end{cases}
\]

Because \( c_i^P(k) = O(\varepsilon^k) \) and \( c_h^P(k) = aO(\varepsilon^k) \), for \( k > 1/3 \) we have \( c_h^P(k) > c_h^*(k) \), i.e. overinvestment in booms.

### C.4 An alternative equilibrium

In the main text we showed that an equilibrium exist when \( h - l \) is sufficiently small; it is possible that the type of equilibrium presented in the main text does not exist. In this subsection we provide some insights on the type of equilibrium that arises instead.

While the system (12)-(13), (7)-(9) always have a solution, for some parameters this solution implies that for a \( c \) sufficiently close to \( c_i^* \), the price is below the threshold \( l \). This obviously cannot be an equilibrium—firms would dismantle whenever the price drops below the liquidation benefit \( l \). For that set of parameters we can construct the equilibrium as follows. There exists a \( c_x \in (c_i^*, c_h^*) \), so that for every \( c \in [c_i^*, c_x] \) we have \( p(c) = \frac{v(c)}{q(c)} = l \), and an endogenous fraction of capital are dismantled at every instant. That is, in this range the price is constant in \( c \) and firms dismantle an increasing fraction of their capital as \( c \) drops further from \( c_x \). The following proposition describes this equilibrium.

**Proposition C.7** Suppose that there is a \( c_h^* < R_K \), \( c_x \in (l, c_h^*) \), \( q_0, A_1, A_2, A_3, A_4 \) solving (12)-(13), (31)

\[
\frac{\xi}{2\sigma^2} \left( R_C + \frac{R_K}{c_x} \right) (l - c_x) = q'(c_x) \\
\frac{l - \xi}{2\sigma^2} \left( R_C + \frac{R_K}{c_x} \right) (l - c_x) = v'(c_x) \\
\frac{v(c_x)}{q(c_x)} = l, \quad \frac{v(c_h^*)}{q(c_h^*)} = h, \quad v'(c_h^*) = q'(c_h^*) = 0.
\]

Then there exists a market equilibrium with partial liquidation where

1. firms do not consume before the final date,

2. each firm in each state \( c \in [l, c_h^*] \) is indifferent in the composition of its asset holdings,

3. firms do not build or dismantle capital when \( c \in (c_x, c_h^*) \) and, in aggregate, firms spend every positive cash shock to build capital iff \( c = c_h^* \) and cover the negative cash shocks by liquidating a fraction of capital iff \( c \in [l, c_x] \). When \( c = l \), firms finance every negative cash shock by liquidating capital.
4. the value of cash and the value of capital are given by \( q(c) \) and \( v(c) \) are the same as in the base model for \( c \in (c_x, c_x^0) \); for \( c \in (l, c_x) \) they are

\[
q(c) = q_0 + \frac{\xi}{2\sigma^2} \left[ (RCl - RK)(c - l) - \frac{RC}{2} (c^2 - l^2) + lRK (\ln c - \ln l) \right],
\]

and the price in the aggregate stage is \( p = l \) when \( c \in [l, c_x] \).

5. In the idiosyncratic stage, each cash firm sells all its capital to the cash firms who are not hit by the shock of the price \( \tilde{p}_t = c \).

**Proof.** Under the conditions of the Proposition, firms start to disinvest whenever \( p(c) = l \). Given the liquidation rate \( y(c) = dK/K \), then its impact on the aggregate cash-to-capital ratio \( c \) is

\[
x(c) \, dt = \frac{dC}{K} - \frac{C}{K} \, dt - \frac{lK}{K} \, dt - \frac{C}{K} \, dK = (l + c) \, y(c) \, dt,
\]

so \( c \) evolves as \( dc = x(c) \, dt + \sigma dZ_t \). We must have \( v(c) = lq(c) \) as firms are always indifferent in liquidating the capital, and \( v \) and \( q \) satisfies:

\[
\begin{align*}
0 &= x(c) q'(c) + \frac{\sigma^2}{2} q''(c) + \frac{\xi}{2} \left( R_C + \frac{R_K}{c} \right) - \xi q(c) \\
0 &= x(c) v'(c) + q'(c) \sigma^2 + \frac{\xi}{2} (RC c + R_K) - \xi v(c)
\end{align*}
\]

Using \( v(c) = lq(c) \), we obtain

\[
\begin{align*}
0 &= x(c) lq'(c) + \frac{\sigma^2}{2} lq''(c) + \frac{\xi l}{2} \left( R_C + \frac{R_K}{c} \right) - \xi l q(c) \\
0 &= x(c) lq'(c) + q'(c) \sigma^2 + \frac{\xi}{2} (RC c + R_K) - \xi l q(c)
\end{align*}
\]

Eliminating identical terms, we get

\[
q'(c) = \frac{\xi}{2\sigma^2} \left( R_C + \frac{R_K}{c} \right)(l - c) = 0.
\]

As \( q'(c_l) = 0 \) has to hold, \( c_l = l \). The closed-form solution is

\[
q(c) = q_0 + \frac{\xi}{2\sigma^2} \left[ (RCl - RK)(c - l) - \frac{RC}{2} (c^2 - l^2) + lRK (\ln c - \ln l) \right]
\]

And, we have \( q''(c) = -\frac{\xi}{2\sigma^2} (RC + \frac{R_K}{c}) < 0 \). We know that of \( c \in [l, c_x] \) we have \( v(c) = lq(c) \) which allows us to back out the endogenous drift of \( c \):

\[
x(c) = \frac{-\sigma^2 q''(c) - \frac{\xi}{2} (RC + \frac{R_K}{c}) + \xi q(c)}{q'(c)},
\]

and thus the endogenous liquidation rate \( y(c) = \frac{x(c)}{1 + \sigma} \). For \( c > c_x \) we have the ODE as usual. We then search of the \( c_x, c_h \) pair that satisfies the conditions of the proposition.

Plotting \( v, q \) and \( p \) give very similar graphs to Figure 2 with the main difference that at the range \( c \in [l, c_x] \) the price is flat at the level \( l \). In the same range \( q(c) \) is decreasing implying that \( v(c) = lq(c) \) is also decreasing.