Optimal Long-Term Contracting with Learning

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Motivation

- Many long-term contractual relationships feature uncertainty and learning: e.g., unknown project’s quality or agent’s ability
- Dynamic contracting with uncertainty & learning: empirically relevant but theoretically challenging
- Moral hazard interacts with learning — persistent belief manipulation effect
  - Agent’s shirking lowers today’s output
    - Principal mistakenly thinks the project is worse, and belief distortion persists
    - Agent keeps getting more compensations (information rent) in the future
- Incentive provisions are intertemporally linked
  - higher future incentives $\implies$ more information rents $\implies$ more shirking today
  - not in typical dynamic agency literature with repeated moral hazard
This Paper

- We introduce uncertainty into an infinite-horizon Holmstrom-Milgrom setting
  - both transitory “hidden action” and persistent “hidden information”

- We derive the optimal contract, which best trades off working incentive provision against information rent extraction

- Our model is tractable so that the optimal contract is fully characterized by an ordinary differential equation (ODE) w.r.t. information rent only
Main Results

> The optimal effort policy is fully stochastic, surprising in a CARA-Normal framework
  > a result of both long-term contracting and learning
    > w/o learning, Holmstrom-Milgrom (1987): constant optimal effort
    > w. learning but short-term contracts, Holmstrom (1999): deterministic effort

> Under the optimal contract, Agent works harder in earlier periods
  > otherwise stationary framework
  > but later incentive provisions lead to greater information rents

> Option-like feature: incentives increase following good performance
  > mitigating Agent’s belief-manipulation incentives
  > new insight on the popularity of option-based compensation
Literature Review

- Dynamic agency models with hidden information and learning
  - DeMarzo and Sannikov (2008) and Prat and Jovanovic (2010)
  - Adrian and Westerfield (2009)

- Short-term contracting with adverse selection and moral hazard

- Long-term contracting with adverse selection and moral hazard
The Model

- An infinite-horizon continuous-time principal-agent model
- Principal hires Agent to manage a project with output

\[
\begin{align*}
dY_t &= \left( \begin{array}{c} \mu_t \\ \theta \end{array} \right) dt + \sigma dB_t \\
&= \left( \begin{array}{c} \text{unobservable} \\
\text{effort} \\
\text{unknown} \\
\text{quality/ability} \end{array} \right) dt + \sigma dB_t
\end{align*}
\]

- \( \theta \) is *unknown* quality (or managerial ability), \( B_\theta \perp B \):

\[
d\theta_t = \sigma \phi dB_{\theta,t}
\]

- common prior \( \theta \sim \mathcal{N}(m_0, \Sigma_0) \), posterior

\[
m^\mu_t = \mathbb{E}^\mu[\theta | y^t, \mu^t]
\]

- stationary Bayesian learning: assume \( \Sigma_0 = \sigma^2 \phi \)

\[
\Sigma_t = \text{Var}[\theta | y^t, \mu^t] = \Sigma_0 \quad \text{for all } t
\]
The Model (Cont’d)

- Principal is risk-neutral & offers a contract \( \{ c_t, \mu_t \}_{t=0}^{\infty} \)
- Risk averse Agent with exponential utility & reservation value \( v_0 \) chooses actual effort \( \hat{\mu}_t \) and actual consumption \( \hat{c}_t \)

\[
\mathbb{E}^{\hat{\mu}} \left[ \int_0^\infty e^{-rt} u (\hat{c}_t, \hat{\mu}_t) \, dt \right]
\]

where \( u (\hat{c}_t, \hat{\mu}_t) = -\frac{1}{a} \exp \left[ -a (\hat{c}_t - g (\hat{\mu}_t)) \right] \)

- moral hazard: effort is unobservable and costly

\[
g (\hat{\mu}_t) = \frac{1}{2} \hat{\mu}_t^2
\]

- private savings account with balance \( S_t \):

\[
dS_t = rS_t \, dt + c_t \, dt - \hat{c}_t \, dt \text{ with } S_0 = 0
\]

- private saving in CARA framework helps tractability
The Principal-Agent Problem

- Agent. Given contract with wages and recommended effort \( \{ c_t, \mu_t \} \)

\[
\max_{\{\hat{c}, \hat{\mu}\}} \mathbb{E}^{\hat{\mu}} \left[ \int_0^\infty e^{-rt} u(\hat{c}_t, \hat{\mu}_t) \, dt \right]
\]

s.t. \[
dY_t = \left( \hat{\mu}_t + \hat{\mu}_t^\mu \right) dt + \sigma dB^\hat{\mu}_t
\]

\[
dS_t = rS_t dt + c_t dt - \hat{c}_t dt \text{ with } S_0 = 0
\]

- w.l.o.g, focus on incentive-compatible & no-savings contracts

- Principal.

\[
\max_{\{c, \mu\}} \mathbb{E}_0^{\mu} \left[ \int_0^\infty e^{-rt} (dY_t - c_t dt) \right]
\]

s.t. \[
dY_t = \left( \mu_t + \mu_t^\mu \right) dt + \sigma dB^{\mu}_t
\]

\[
\mathbb{E}_0^{\mu} \left[ \int_0^\infty e^{-rt} u(c_t, \mu_t) \, dt \right] = v_0
\]

\( \{c_t, \mu_t\} \) satisfy incentive-compatible & no-savings constraints
Belief Manipulation

- If Agent follows \( \{\mu_t\} \) recommended by Principal (on-equilibrium), both share same posterior \( m_t^\mu = \mathbb{E}^\mu [\theta | y^t, \mu^t] \):

\[
dm_t^\mu = \phi \left[ dY_t - (\mu_t + m_t^\mu) \right] dt
\]
Belief Manipulation

- If Agent follows \( \{\mu_t\} \) recommended by Principal (on-equilibrium), both share same posterior \( m^\mu_t = \mathbb{E}^\mu [\theta | y^t, \mu^t] \):

\[
dm^\mu_t = \phi \left[ dY_t - \left( \mu_t + m^\mu_t \right) dt \right]
\]

- Otherwise, if Agent shirked \( \hat{\mu}_t = \mu_t - \epsilon \) only at \( [t, t + dt] \) (off-equilibrium), then Principal’s belief would be distorted

Agent knows the truth: \( dm^\mu_t = \phi \left[ dY_t - \left( \hat{\mu}_t + m^\mu_t \right) dt \right] \)

- Persistent belief-manipulation effect for \( s \geq t \) — hidden information

\[
m^\mu_s - \hat{m}^\mu_s = -\phi e^{-\phi(s-t)} \epsilon \cdot dt < 0 \text{ for any } s \geq t
\]
Belief Manipulation (Cont’d)

One time shirking at \([t, t + dt]\) leads to persistent belief distortion

\[
m^\mu_s - \hat{m}^\mu_s = -\phi e^{-\phi(s-t)} \epsilon \cdot dt < 0 \text{ for any } s \geq t
\]
The Agent’s Problem: Continuation Value

- Given the contract \( \{c_t, \mu_t\} \), continuation value, no-savings

\[
v_t \equiv \mathbb{E}_t^\mu \left[ \int_t^\infty e^{-r(s-t)} u(c_s, \mu_s) \, ds \right]
\]
The Agent’s Problem: Continuation Value

- Given the contract \( \{c_t, \mu_t\} \), continuation value, no-savings

\[
v_t \equiv \mathbb{E}^\mu_t \left[ \int_t^\infty e^{-r(s-t)} u(c_s, \mu_s) \, ds \right]
\]

- Martingale Representation Theorem (Sannikov, 2008):

\[
dv_t = rv_t \, dt - u(c_t, \mu_t) \, dt + \beta_t \left( -arv_t \right) \left( dY_t - \left( \mu_t + m^\mu_t \right) \, dt \right)
\]

\$\text{incentive}\ \text{Marginal } u, \ \$\rightarrow \text{utils} \quad \text{performance } \sigma dB^\mu_t

- Private savings + CARA imply

\[
rv_t = u(c_t, \mu_t) \quad \text{and} \quad u_c = -arv_t
\]

  - CARA utility \( \Rightarrow u_c \) is proportional to \( u \)

- \( v_t \) is an exponential Martingale, with loading being incentives \( \{\beta_u\} \)

\[
v_s = v_t \exp \left( - \int_t^s ar\beta_u \sigma dB^\mu_u - \frac{1}{2} \int_t^s a^2 r^2 \beta_u^2 \sigma^2 \, du \right), \quad \text{for } s > t
\]
The Agent’s Problem: Incentive-Compatibility Condition

- Shirking \( \hat{\mu}_t = \mu_t - \epsilon \) at \([t, t + dt]\) affects the agent’s total payoff

\[
u (c_t, \mu_t - \epsilon) dt + v_t + E_t^\mu \left[ \int_t^\infty e^{-r(s-t)} dv_s \right]
\]

\[
= u (c_t, \mu_t - \epsilon) dt + E_t^\mu \left[ \beta_t (-arv_t) \left( dY_t (\mu_t - \epsilon) - \mu_t dt - m_t^\mu dt \right) \right]
\]

Saving effort cost (+)  
Hurting performance today (-)  

\[
+ E_t^\mu \left[ \int_{t+dt}^\infty e^{-r(s-t)} \beta_s (-arv_s) \left( dY_s (\mu_s) - \mu_s dt - m_s^\mu dt \right) \right] + v_t
\]

Creating future belief divergence (+)
The Agent’s Problem: Incentive-Compatibility Condition

- Shirking $\hat{\mu}_t = \mu_t - \epsilon$ at $[t, t + dt]$ affects the agent’s total payoff

$$u (c_t, \mu_t - \epsilon) dt + v_t + \mathbb{E}^\mu_t \left[ \int_t^\infty e^{-r(s-t)} dv_s \right]$$

$$= u (c_t, \mu_t - \epsilon) dt + \mathbb{E}^\mu_t \left[ \beta_t (-arv_t) \left( dY_t (\mu_t - \epsilon) - \mu_t dt - m^\mu_t dt \right) \right]$$

- Saving effort cost (+)
- Hurting performance today (-)

$$+ \mathbb{E}^\mu_t \left[ \int_{t+dt}^\infty e^{-r(s-t)} \beta_s (-arv_s) \left( dY_s (\mu_s) - \mu_s dt - m^\mu_s dt \right) \right] + v_t$$

- Creating future belief divergence (+)

- The incentive-compatibility (IC) condition

$$u_\mu + (-arv_t) \beta_t - \mathbb{E}^\mu_t \left[ \int_t^\infty \phi e^{-(\phi+r)(s-t)} \beta_s (-arv_s) ds \right] = 0$$

- future incentives $\beta_s \uparrow$, shirking to manipulate belief $\uparrow$, offsetting today’s incentive $\beta_t$
The Incentive-Compatibility Condition — Information Rent

- Recall $u_\mu = -u_c \mu_t = (arv_t) \mu_t$
  
  $v_s = v_t \exp \left( -\int_t^s ar \beta_u \sigma dB_u^\mu - \frac{1}{2} \int_t^s a^2 r^2 \beta_u^2 \sigma^2 du \right)$

- Cancel $v_t$ on both sides, IC condition becomes

$$\mu_t = \beta_t - \mathbb{E}_t^\mu \left[ \int_t^\infty \phi e^{-(\phi + r)(s-t)} \beta_s e^{-\int_t^s ar \beta_u \sigma dB_u^\mu - \frac{1}{2} \int_t^s a^2 r^2 \beta_u^2 \sigma^2 du} ds \right]$$

- $\mu_t = \beta_t - \rho_t$, only $\{\beta\}$ enters in the IC condition!

- Intuitively, CARA implies level invariance, the level of today’s $v$ does not matter

- We use Pontryagin’s Maximum Principle to rigorously derive it and prove its sufficiency
The Principal’s Problem

The Principal’s Problem:

\[ \max_{\{\beta_t\}} \mathbb{E} \left[ \int_0^{\infty} e^{-rt} (dY_t - c_t dt) \right] \]

s.t.

\[ dY_t = (\mu_t + m_t)dt + \sigma dB_t \quad \text{and} \quad dm_t = \phi \sigma dB_t \]

\[ c_t = \frac{1}{2} \mu_t^2 - \frac{\ln(-ar v_t)}{a} \]

\[ dv_t = \beta_t (-ar v_t) \sigma dB_t \quad \text{given} \quad v_0 \]

\[ \mu_t = \beta_t - pt (\{\beta_s\}) \]

Value function is separable w.r.t. two state variables: \((v_t, p_t)\)

\[ J(v, p) = V(p) - \frac{\ln(-ar v)}{ar} \]

- Project Value
- Certainty Equivalent to Agent

CARA setting, \(v_t\) only matters to the extent of certainty equivalent
The Principal’s Problem: Dynamic Programming

- Hamilton-Jacobi-Bellman equation for $V(p)$, with only state variable $p_t$

$$
V(p) \equiv \max_{\{\beta_t, \sigma_t^P\}} \mathbb{E} \left[ \int_0^\infty e^{-rt} \left( \begin{array}{c}
(\beta_t - p_t) - \frac{1}{2} (\beta_t - p_t)^2 \\
-\frac{1}{2} ar\sigma^2 \beta_t^2
\end{array} \right) dt \right]
$$

s.t. $dp_t = [ (\phi + r) p_t + \beta_t \left( ar\sigma^P \sigma_t - \phi \right) ] dt + \sigma_t^P dB_t$, $p_0 = p$

- Optimal control is on $\beta_t$ (drift of $dp_t$) and $\sigma_t^P$ (volatility of $dp_t$)
- Mapping to consumption/portfolio problem: $\beta_t$ is like consumption affecting the drift of wealth, $\sigma_t^P$ is like portfolio choice affecting the volatility of wealth
Benchmark Case: Deterministic Contracts

- Under the restriction $\sigma^p = 0$,
  - $\{\beta_t\}$ is deterministic and
    
    $$(IC) \quad \mu_t = \beta_t - \int_t^\infty \phi \beta_s e^{-(\phi + r)(s-t)} ds$$

- The optimal deterministic contract is solved in closed-form with quadratic value function
  
  $$V^d(p) = -\frac{1}{2}A^d p^2 + B^d p$$

- Result: $p^d$, $\beta^d$, and $\mu^d$ declines exponentially over time toward zero
  - more costly to provide incentives later, because later incentives give agent information rents early on
  - remains to hold in the stochastic optimal contract
The Optimal Contract: Characterization

- Value function $V(p)$ satisfies ODE: for $p \in [0, \bar{p}]$

\[
rx \frac{1}{2} \frac{(1 + p - \phi V_p)^2}{1 + ar\sigma^2 + a^2r^2\sigma^2 \frac{V^2_p}{V_{pp}}} - p - \frac{1}{2}p^2 + V_p (\phi + r) p
\]

where $p_t$ is bounded between an endogenous entry-no-exit upper boundary $\{\bar{p}\}$ and an absorbing lower boundary $\{0\}$

- Under the optimal contract, effort $\mu_t = \beta_t - p_t$ is stochastic!
Optimal Contracting: Option-Like Feature

- Besides time-decreasing effort, the optimal contract exhibits an **option-like** feature
  - incentives increase after good performance, i.e., diffusion of $\beta$ is positive
- Interestingly, it is due to an asset pricing intuition
- To reduce the agent’s information rent

$$p_t = \frac{1}{u_c(c_t)} \mathbb{E}_t \left[ \int_t^\infty \phi e^{-(\phi+r)(s-t)} \beta_s u_c(c_s) \, ds \right].$$

- Intuition: (i) following good performance, $u_c(c_s, \mu_s) \downarrow$
  (ii) increasing incentives, $\beta_s \uparrow$, induces negative covariance
  $$\text{Cov}_t[\beta_s, u_c(c_s, \mu_s)] < 0$$
  \Rightarrow as a result, information rent $p_t$ is lower
- Option: give belief manipulation rewards (high incentives) in states where Agent does not care much (low marginal utility)
Concluding Remarks

- We study optimal dynamic contracting in a context of uncertainty and learning
- Our continuous time agency model features both hidden action and hidden information
  - the optimal effort policy is stochastic and front-loaded
- The optimal contract has an option-like feature, which is used to mitigate the agent’s belief-manipulation motives
  - it is typically hard to have option in optimal contracting standard models (Dittmann and Maug, 2007)
- Empirical predictions/evidence
  - Core and Guay (1999) find that the annual grant of options and stocks to CEO is increasing in past stock returns
  - the industry/firm with higher uncertainty should have more option-based contracts for managerial compensation
  - Ittner, Lambert, and Larcker (2003) and Murphy (2003) who document new-economy firms (e.g., computer) grant more stock options to managers than old-economy firms