Financial Frictions in Production Networks

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November 9, 2015

Abstract

We study how an economy’s production architecture determines the qualitative and quantitative effects of financial shocks. In our framework, production is organized through an input-output production network where financial constraints induce a sub-efficient use of inputs. We decompose the effects of financial shocks into a total factor productivity effect and a labor market distortion. This decomposition explains how financial constraints propagate depending the network’s architecture. We calibrate this model to the US and report various exercises. We compare this economy with an economy with a representative firm.

Keywords: Production Networks, Financial Frictions, Business Cycles

JEL classification: C67, G10, E42

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*We thank Daron Acemoglu, Bob Hall, Vasco Carvalho, Gabriel Chodorow-Reich, Michael Gofman, Mark Gertler, James Hamilton, Ricardo Lagos, Alireza Tahbaz-Salehi and Randall Wright for valuable feedback and suggestions. We improved the paper thanks to discussions by Chad Jones, V.V. Chari, Marco Ortiz, and Alberto Trejos. We benefitted from seminar comments at the Chicago Booth Junior Finance Symposium, the Central Bank of Peru 30th Research Meeting of Economists, 2013 SED meetings, the Richmond FED, the 2014 AEA meetings, the 2015 LAEF conference, Paris School of Economics, Ente Einaudi, Cambridge University, and the 7th Annual Conference on Macroeconomics Across Time and Space hosted by the Federal Reserve Bank of Philadelphia and NBER.
1 Introduction

Since the onset of the Great Recession, financial frictions have been reinstated in the forefront of business cycle research. A production-side view suggests that the tightening of credit to firms was a main contributor of the Great Recession, the most significant and prolonged contraction since the Depression. Yet, the precise mechanics of financial shocks are not fully understood. This paper is a qualitative and quantitative study on how financial shocks on firms are transmitted when production is organized through an input-output production network.

Close to a major metropolitan area, anyone looking down from an airplane’s window can appreciate the outstanding complexity involved in economic production. One can identify warehouses, silos, loading levers, etc., all laid out in the neighborhood of manufacturing and energy plants. A grid of roads and railroads allows materials to flow from one site to another and add value to a final product that is, eventually, sold in retail posts inside the city. Although we are trained economists, we are still perplexed by how this spontaneous, harmonious, yet complex organization is achieved. We are trained to think that general equilibrium forces do the magic of organizing production. At the same time, it makes us wonder how resilient is this organization? Most of modern macroeconomics abstracts away from this complexity. Our most standard models present firms that don’t buy inputs from other firms. However, given the vast amount of interfirm trade that we observe in the real world, could financial shocks that disrupt a small number of sectors also disrupt an entire economy? By how much are financial shocks amplified when we consider input-output linkages? Which sectors are the most vulnerable and which create the most vulnerabilities? How should we detect the presence of financial shocks in aggregate data? This paper uses the discipline of general equilibrium analysis to provide qualitative and quantitative answers.

The answers to these questions are important to understand the nature of business cycles. A first generation of macro-finance models has been seriously challenged on quantitative grounds. Those criticisms may lead to the conclusion that financial constraints on firms cannot be a major source of strong recessions.\footnote{Typically, quantitative models that have been successful in delivering large output fluctuations from financial shocks employ strong amplification mechanisms; nominal rigidities for example. This doesn’t resolve the issue because those amplification mechanisms can also amplify a variety of non-financial shocks. These puts financial shocks at the same level of play as any other shock.} This study investigates if input-output linkages can resolve those challenges. There are reasons that make us think that a simple model with input-output linkages is a good starting point. When firms are interconnected, they need funds to purchase intermediate inputs. This implies that firms may be constrained even if, as has been observed by some studies, firms have enough internal funds to purchase capital and labor expenditures. Furthermore, with interfirm trade, the effect of financial constraints on only a small portion of firms may have disproportional effects through the supply-chain. General equilibrium theory teaches us that if an industry generates less output, the price of the goods produced by that industry should increase, and the price of its inputs fall. At the micro level, these induced changes in relative prices and quantities will make
other firms perceive that their main problems are demand or cost related, even when they are also hit by higher funding costs. Whereas at the micro-level, firms perceive demand problems, at the macro-level, the economy suffers only because small financial shocks get compounded. Though it is thought that there was a large credit-crunch during the Great Recession, survey data revealed that only a small portion of firms found funding costs to be among their major concerns. Our analysis is an attempt to reconcile those challenges.

Our approach to these questions consists on modifying a classic static general equilibrium model with production by introducing financial constraints. Our frictionless structure is a modest generalization of earlier work of Acemoglu et al. (2012) which was focused on studying the amplification of productivity shocks. In this framework, firms within a production sector operate using a Cobb-Douglass technology. That technology requires inputs from a subset of other sectors. The cross-sector input requirements defines an input-output production network. At the aggregate level, the only exogenous input is the labor supplied by households. We introduce frictions modelled as unexplained wedges between the marginal revenue of a firm and its marginal costs —this wedges can be derived from interest rates on working capital or enforcement constraints. Although the approach is simple, it is a valuable benchmark because we can use it to distill the effects of financial shocks. We found that in this economy, aggregate output can be summarized by an aggregate production function that uses labor as the only input. The effects of financial shocks can be decomposed into two effects whose strengths depend on the location of shocks and the overall network architecture. The first effect shows up as a multiplicative Total Factor Productivity (TFP) shock —i.e., it induces an efficiency wedge. The second effect operates by distorting labor supply because it induces an effect equivalent to a labor income tax —i.e., it induces a labor wedge.

We first solve the most general class of networks by developing a method that can be used in a more general class of models. We employ this method to provide qualitative answers by solving several example economies. These examples teach us lessons about how the network architecture determines the effects of financial shocks. We construct these examples so that without financial frictions these economies are allocationally equivalent.

Example 1. In our first example, production is organized through a vertical production chain: only the first sector uses labor and sells its output to the next sector —this structure corresponds to the vertical chain economies in Kiyotaki and Moore (1997a) and Kalemli-Ozcan et al. (2014). The second sector uses the output of the previous sector, and so on, until the final sector produces a final good. In this example, financial frictions do not induce an efficiency wedge. The reason is that an equilibrium amount of labor cannot be misallocated —there is only one way to route labor inputs. However, financial frictions do induce a labor wedge because, given an amount of labor, real wages must fall to compensate the decline in the firms’ funds.

Example 2. In our second example, firms produce and operate in isolation. This is the typical structure of standard macroeconomic models. In that economy, sectors use labor as their only input. Their outputs are consolidated through a consumption basket. A common financial shocks operates
as in the chain example because it does not affect the relative use of hours across sectors. However, just as in the production chain, a labor wedge appears to adjust for the financial frictions. Things are different once we alter that symmetry. When shocks enter asymmetrically both efficiency and labor-supply decisions are affected.

Example 3. Our third example is a blend of the previous two and is the closest to the actual architecture of the US economy. In that example, intermediate inputs are routed as in the vertical chain, but labor inputs are used throughout the production chain. The qualitative properties of that economy are different. Common and individual shocks move both, the efficiency and labor wedges. That example also reveals that sectoral shocks have greater aggregate effects as they hit sectors closer to final goods. The converse is also true: aggregate shocks have greater effects on the sectoral output of the most primary sectors.

After we develop these examples, we use the US input-output matrix to calibrate the model. This exercise provides a list of quantitative answers: First, we find that a shock that raises working capital costs by 1% in every sector, leads to an aggregate output fall of 2.5%. If we decompose effects, we obtain that 20% of this effect emerges from a movement in efficiency and 80% from the response in labor supply because of the decline in real wages. In contrast, in the representative firm economy, output would fall only by 0.7%. Thus, we obtain a strong multiplier solely from input-output linkages. We find that the most vulnerable sectors are the sectors that supply materials to manufacturing: e.g., metal products, chemical products, fabricated metal products, hydrocarbons, etc. The most influential sectors are those closest to final goods production and that are integrated to a manufacturing clusters: motors and construction. We report a sensitivity analysis.

The simplicity of the environment yields a transparent analysis. At the same time, it points to additional modeling features that seem important for follow-up work. Before placing the model in the context of the literature, it is important to acknowledge what our analysis is missing. One restriction of our framework is restricted to Cobb-Douglas technologies. These technologies allow a high degree of substitution across inputs. This is a clear restriction because certain inputs cannot be substituted away in the short run: e.g., energy or water. Although our solution method can be adapted to allow for more general CES technologies, we chose to work with Cobb-Douglas because obtain analytic solutions and we are not aware of a calibration method that allows to identify the additional parameters introduced when bringing in a CES technology. Another feature that dampens effects of financial shocks is the assumption that labor can be immediately be reallocated across sectors. On the flip side, one reason why our model may overstate the effects of financial shocks is that we assume that financial shocks hit entire sectors, so there’s no way firms can substitute the output of others. In the practice, if firms don’t face capacity constraints, there can be a lot of substitution across companies within the same industry. Finally, we remain agnostic about the sources of financial shocks and any dynamic aspects.

Related Literature
It is useful to place this paper in the context of the literature to understand the value of the results we find. Our paper falls in the boundary of two literatures, the literature on financial frictions and the literature on production networks. The pioneers of modern macro finance models are Bernanke and Gertler (1989) and Kiyotaki and Moore (1997b). Both models, and much of the literature that followed, have in common that financial frictions distort investment decisions only. Presumably, those papers were influenced corporate finance models that study a firm’s biggest challenge, the funding of large infrequent investment projects. However, when it comes to explaining business cycle analysis, those models were unable to generate strong output responses —see for example, Cordoba and Ripoll (2004), Kocherlakota (2000) and Bigio (2015). The reason is that in standard business cycle theory, investment is too small relative to the capital stock. Typically, with other frictions such as nominal rigidities —e.g., Bernanke et al. (1999)—these class of models can generate strong responses to financial shocks, but at the same time, those additional frictions will amplify a broad class of shocks: monetary policy, government spending, etc. This means that financial shocks are not particularly powerful. At the same time, this class of models has been criticized because it conflicts with micro-data evidence. Chari et al. (2008), Kahle and Stulz (2013) and Shourideh and Zetlin-Jones (2012) show that for the Great Recession, at least at the aggregate, non-financial corporations can fund their capital expenditures entirely through retained earnings, challenging the idea that firms were constrained in the crisis. Finally, models that distort investment decisions cannot be reconciled with the business cycle decomposition in Chari et al. (2007) who found little evidence of counter-cyclical investment wedges.

A second generation of quantitative financial frictions models introduce frictions in a way that do generate stronger effects on output. There are other two ways common approaches. One is the working capital approach in Jermann and Quadrini (2011) and Bigio (2015). Although this approach can deliver strong responses to output shocks and is consistent with the decomposition in Chari et al. (2007), it has its own drawbacks. Bigio (2015) shows that the same reasons that lead to strong output responses, lead to very high implied financial costs. These implied costs are either counterfactual interest rates or shadow costs that eventually lead to high profitability margins. The lessons out of this papers, though, is that to get strong responses, financial shocks must affect the use of intermediate inputs, but in a way that makes not seem that large. The second approach is to depart from a representative firm environment by introducing heterogeneous producers. The idea of this approach is that financial shocks affect the scale of the most efficient firms, so inputs must be reallocated to more inefficient firms, lowering output. Buera and Moll (2012) multiple variants of those frictions. The problem with that approach is that in order to create strong output responses, one needs (a) a substantial amount of heterogeneity, and (b), large flows of capital and labor during the business cycle, something that seems counterfactual.

In contrast to all of this previous work, ours is the first paper to highlight the role of the organization of production structures —networks— in amplifying financial shocks. The idea is that small financial shocks can have large effects through input-output linkages. This approach builds
on the second literature this paper relates to. In particular, our paper shares the spirit of early multi-sector real business cycles models. A first such model was Long and Plosser (1983) multi-sectoral model of real business cycles. That literature was famously criticized by Robert Lucas, who argued that sectoral shocks could not explain the fact that during recessions, all all sectors shrink together. Following Long and Plosser (1983), a debate ensued between Horvath (1998, 2000) and Dupor (1999) over whether sectoral shocks could lead to strong observable aggregate TFP shocks.

[Incomplete. Missing connection to Networks and Finance literature. Finish the literature review.]

2 The General Production Network Model

We first introduce the general input-output structure of an economy with multiple sectors. The economy is populated by a representative household and $N$ production sectors indexed by $i = 1, \ldots, N$. Each sector consists of a continuum of identical and competitive firms, each producing an identical good. However, while goods are identical across all firms within a sector, goods are thus differentiated across sectors. We index goods by $j = 1, \ldots, N$, with the understanding that there is a one-to-one mapping between sectors and goods. We will thus use the indices $i$ and $j$ interchangeably, to denote either the sector $i$ or the good $j$ produced by sector $j = i$.

Production. Firms are perfectly competitive. We assume that the technology and the financial constraint is the same across all firms within a sector. This implies the existence of a representative firm per sector. We thus simply refer to the sector as the individual production unit, with the understanding that there is a representative firm per sector.

Sector $i$ produces output according to the following Cobb-Douglas technology

$$y_i = (z_i x_i)^{\eta_i}.$$  \hspace{1cm} (1)

where $y_i$ is the output of sector $i$, $z_i$ is input-augmenting sector-specific productivity, and $x_i$ is a composite of inputs used by sector $i$. The parameter $\eta_i < 1$ implies that there is decreasing returns to scale in sector $i$.\footnote{We assume that in the short run there is a fixed-factor, say, capital or land.}

The composite $x_i$ of sectoral inputs is given by the Cobb-Douglas composite

$$x_i \equiv \ell_i^{\alpha_i} \left( \prod_{j=1}^{N} x_{ij}^{w_{ij}} \right)^{1-\alpha_i}$$

where $\ell_i$ is sector $i$’s labor input and $x_{ij}$ is the amount of commodity $j$ used by sector $i$. The \footnote{There does not necessarily have to be a one-to-one mapping between sectors and goods. In fact the input-output matrix does not necessarily assume this (see data appendix). For the purposes of this paper, however, we abstract from this issue and assume that all firms within a sector produce only one good, the good that corresponds to that sector.}
exponent $w_{ij} \in [0, 1]$ denotes the share of good $j$ in the total intermediate input use of sector $i$. Note that if sector $i$ does not use a good $j$ as an input to production, then $w_{ij} = 0$. Finally, without loss of generality, we assume that $\sum_{j=1}^{N} w_{ij} = 1, \forall i \in \{1, ..., N\}$.\(^5\)

Next, we impose financial constraints on the sector’s purchase of intermediate inputs. Firm $i$ takes prices as given and maximizes profits subject to the financial constraint:

$$\ell_i + \sum_{j=1}^{N} p_j x_{ij} \leq \chi_i p_i y_i$$

where the wage is normalized to 1 and $p_j$ denotes the price of good $j$. The expenditure of firm $i$ on all inputs is constrained to be less than $\chi_i$ of its earnings. This representation can be adapted to adjust for interest rate costs, limited enforcement, moral-hazard, and funding costs for trade-credit. We show a possible microfoundation for this constraint stemming from a limited enforcement problem in Appendix D.

The objective of the representative firm of each sector is to choose inputs and output so as to maximize profits subject to its financial constraint (2), taking the wage and prices as given.

The Household. The preferences of the representative household are given by

$$U(c, \ell) \equiv \frac{c^{1-\gamma}}{1-\gamma} - \frac{\ell^{1+\epsilon}}{1+\epsilon},$$

where $c$ is the household’s final consumption basket and $\ell$ is the household’s labor supply. Here, $\gamma \geq 0$ parametrizes the income elasticity of labor supply,\(^6\) and $\epsilon > 0$ corresponds to the inverse Frish elasticity of labor supply. The final consumption basket of the household is a composite of the differentiated goods in the economy.

$$c = \prod_{j=1}^{N} c_j^{v_j}$$

where $c_j$ is the household’s consumption of good $j$. The parameter $v_j \in [0, 1]$ is the household’s expenditure share on good $j$. If the household does not consume good $j$, then $v_j = 0$. Again without loss of generality we set $\sum_{j=1}^{N} v_j = 1$.\(^7\)

The budget constraint of the household is given by

$$\sum_{j=1}^{N} p_j c_j \leq \ell + \sum_{i=1}^{N} \pi_i$$

\(^4\)In general, $w_{ii}$ need not be equal to 0, as sectors may use their own output as an input to production.

\(^5\)This is without loss of generality because we may always rescale the value of $\eta_i$.

\(^6\)Note that risk aversion and intertemporal substitution play no role in our setting because all idiosyncratic risk is insurable and the model is static. Therefore, $\gamma$ controls only the sensitivity of labor supply to income for a given wage. Note that when $\gamma = 0$, there is no income effect (as in GHH preferences).

\(^7\)We may always rescale the value of $\gamma$. 

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\(^1\)
where $\pi_i$ are the profits of sector $i$. Thus, total expenditure must be weakly less than the wage income of the household plus dividends from owning the firms. The household’s objective is to choose consumption and labor so as to maximize utility subject to its budget constraint, taking the wage and the vector of prices as given.

*Market Clearing.* The output of any given sector may be either consumed by the household or used by other sectors as an input to production. Commodity market clearing for each good $j$ is thus given by

$$y_j = c_j + \sum_{i=1}^{N} x_{ij}, \forall j \in \{1, \ldots, N\}.$$  (3)

Similarly, labor market clearing satisfies

$$\ell = \sum_{i=1}^{N} \ell_i.$$  

Finally, we set household labor to be the numeraire, thereby normalizing the wage to 1.

*Input-Output Matrix.* We let $W$ denote the $N \times N$ input-output matrix of the economy with entries $w_{ij}

$$W = \begin{bmatrix}
    w_{11} & w_{12} & \cdots & w_{1N} \\
    w_{21} & w_{22} & \cdots & \vdots \\
    \vdots & \vdots & \ddots & \vdots \\
    w_{N1} & w_{N2} & \cdots & w_{NN}
\end{bmatrix}.$$  (4)

We let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)'$ and $\eta = (\eta_1, \ldots, \eta_N)'$, denote the vectors of the labor shares and the vector of the decreasing returns to scale parameters for each firm, respectively. Furthermore, the vector of household expenditure shares is given by $v = (v_1, v_2, \ldots, v_N)'$. Note that the rows of the matrix $W$ sum to one due to our normalization that $\sum_{j=1}^{N} w_{ij} = 1, \forall i$. However, the columns of the matrix $W$ need not sum to one; the column sum is what is known as the weighted outdegree. This corresponds to the share of sector $i$’s output in the input supply of the entire economy.\(^8\)

For future reference we let $e_n$ denote a column vector of 1’s of length $n$ and we let $I_n$ denote the identity matrix of size $n \times n$. Finally, we let $1 - \alpha = e_n - \alpha$.

*Equilibrium Definition.* We define an equilibrium for this economy as follows.

**Definition 1** A competitive equilibrium consists of a vector of commodity prices $p = (p_1, p_2, \ldots, p_N)$, a consumption bundle \(\{c_j\}_{j=1}^{N}\), and output, input, and labor allocations \(\{y_i, \{x_{ij}\}_{j=1}^{N}, \ell_i\}_{i=1}^{N}\) such that

(a) the household and firms are at their respective optima

\(^8\)Finally, input-output relationships between different sectors can equivalently represented by a directed weighted graph on $n$ vertices, each corresponding to a particular sector of the economy.
(b) prices clear commodity markets, and
(c) the wage, normalized to 1, clears the labor market.

Remarks. First, in terms of notation for the firm’s production, labor is treated differently as an input to production because later we will do an auxiliary step in which labor is exogenous.

Next, we want to think of the underlying production unit as sectors, in which we have a representative firm in each sector. The productivity and financial shocks will operate at the sectoral level. In previous network papers (see e.g. Acemoglu et al) the question of wh

There are multiple important aspects of production economies that our paper isn’t capturing. First, we have Cobb-Douglass production functions—we are not using CES production function. This means that we have an elasticity of substitution of 1 across all products. While this makes the analysis simpler, it does mean that ... See Atalay (), for work on this issue.

Similarly, our model that labor is perfectly substitutable across sectors. We aren’t capturing this at all. Missing Labor mobility.

Due to these low elasticities of goods substitution and labor substitution, we believe our estimates on the effects of shocks are biased downward—with elastic substitution of goods and labor, disruptions in the production network. Thus we believe our estimates are a lower bound on the actual effects of shocks.

3 Equilibrium Characterization of the General Model

We now turn to characterizing the general equilibrium in this production network economy. To do so, we proceed in three steps. In the following subsection we first characterize the optimality conditions for the representative (sector-level) firms and the representative household. The second step in our method is to use these optimality conditions along with the market clearing conditions in order to solve for equilibrium output and consumption for a given labor supply—this is done in subsection 3.2. That is, for a moment, we take the household’s labor supply as given and we solve for aggregate consumption given the exogenously fixed amount of labor. The solution to this intermediate step essentially traces out the aggregate production function of this economy. This allows us to highlight the effects of financial frictions on aggregate TFP, also known as the efficiency wedge; we show how movements in individual firm financial constraints results in upward and downward shifts of the economy’s aggregate production function.

The final step in our solution method is to incorporate the household’s endogenous labor supply decision. In subsection 3.3, we combine the household’s optimality condition between consumption and labor and with the aggregate production function found in the previous step. Together these conditions allow us to finally characterize equilibrium output (consumption) and labor in this economy. We furthermore use this representation to examine how the sectoral financial frictions manifest themselves also as a labor wedge between the marginal substitution between consumption and labor and the marginal rate of transformation.
To conclude, in subsection 3.4 we illustrate the effects of financial frictions on the economy’s efficiency and labor wedges. All proofs for this section are provided in Appendix A.

3.1 Firm and Household Optimality

Firms. We first consider the representative (sector-level) firm’s problem, which may be written explicitly as follows.

**Problem 1** Given the real wage and the vector of prices, the representative firm of sector \( i \) chooses inputs \( \{x_{ij}\}_{j=0}^{N} \) and output \( y_{i} \) in order to maximize profits

\[
\max p_i y_i - \ell_i - \sum_{j=1}^{N} p_j x_{ij}
\]

subject to the firm’s production function

\[
y_i = \left[ z_i^\ell_i \alpha_i \left( \prod_{j=1}^{N} x_{ij}^{w_{ij}} \right)^{1-\alpha_i} \right]^{\eta_i},
\]

and financial constraint

\[
\ell_i + \sum_{j=1}^{N} p_j x_{ij} \leq \chi_i p_i y_i.
\]

We solve for the firm’s optimal choice of inputs, labor, and output in two steps. The first is an expenditure minimization problem: given an arbitrary amount of total output, we solve for the firm’s cost-minimizing choice of inputs and labor that would achieve that output. The next step is to then solve the firm’s optimal level of output. The solution to the firm’s dual problem is presented in the following lemma.

**Lemma 1** Let \( u_i = \ell_i + \sum_{j=1}^{N} p_j x_{ij} \) denote the total expenditure of sector \( i \) on inputs (including its expenditure on labor) and let \( g_i = p_i y_i \) denote the total revenue of sector \( i \). Given a vector of prices and the real wage, the firm’s optimal choice of inputs and output satisfies the following conditions:

(i) expenditure on any particular good \( j \) is proportional to the firm’s total expenditure

\[
p_j x_{ij} = (1 - \alpha_i) w_{ij} u_i \tag{7}
\]

\[
\ell_i = \alpha_i u_i \tag{8}
\]

(ii) total expenditure and total revenue satisfy

\[
u_i = \phi_i \eta_i g_i \quad \text{where} \quad \phi_i = \min \left\{ \frac{\chi_i}{\eta_i}, 1 \right\} \tag{9}
\]
The first part of this lemma is a direct result of the Cobb-Douglas intermediate input structure. It states that expenditure on any individual input is proportional to the firm’s total expenditure on inputs, where \( \alpha_{ij} \) is the constant of proportionality. This proportionality holds no matter the firm’s total expenditure of inputs. This is a Classic result of Cobb-Douglas which is constant expenditure shares.

The second part of this lemma states that firm expenditure is proportional to its revenue. However, this constant of proportionality now depends on the financial friction, parameterized by \( \phi \). In particular, if the firm is unconstrained, then expenditure is equal to total revenue times the returns to scale parameter; that is, \( u_i = \eta_i g_i \). This is the profit maximizing choice of scale for the firm in the absence of financial constraints. However, if the firm is financially constrained, it cannot reach its profit maximizing scale. Instead, the financial constraint in (6) holds with equality. This constraint thus binds when \( \chi_i < \eta_i \).

The Household. We next consider the representative household’s problem, written explicitly as follows.

**Problem 2** Given the real wage, the vector of prices, and the vector of firm profits, the household chooses consumption and labor in order to maximize utility

\[
\max c^{1-\gamma} - \ell^{1+\epsilon} \\
\text{subject to} \\
c = N \prod_{j=1}^{N} c_{j}^{w_{j}} \\
\sum_{j=1}^{N} p_{j} c_{j} \leq \ell + \sum_{i=1}^{\eta} \pi_{i}
\]

In a fashion similar to the firm’s problem, we solve for the household’s optimal choice of consumption over goods and labor supply in two steps. We first solve the household’s expenditure minimization problem: given a total amount of final consumption, we solve for the household’s expenditure-minimizing choice over commodities. The expenditure minimizing cost of these goods results in the ideal price of the household’s consumption basket. Given this price (and therefore the real wage), we may then solve for the household’s optimal choice of total consumption and labor supply. The solution to the household’s dual problem is presented in the following lemma.

Trivial. Nothing new here, all classical results.

**Lemma 2** Let \( u_0 = \sum_{j=1}^{N} p_{j} x_{0j} \) denote the household’s total expenditure on the differentiated goods. Given a vector of prices and the real wage, the household’s optimal choices of consumption and leisure are characterized as follows.
(i) expenditure on any particular good $j$ is proportional to the household’s total expenditure

$$p_j c_j = v_j \bar{p}c$$

where $\bar{p}$ is the ideal price index for the household’s consumption basket, given by

$$\bar{p} = \prod_{j=1}^{N} \left( \frac{p_j}{v_j} \right)^{v_j}$$

(ii) labor and total consumption must jointly satisfy

$$\frac{\ell^c}{c^{-\gamma}} = \frac{1}{\bar{p}} \quad \text{and} \quad u_0 = \ell + \sum_{i=1}^{N} \pi_i$$

where $u_0 = \bar{p}c$.

Again due to the Cobb-Douglas structure of preferences, household expenditure on any individual good is proportional to the household’s total expenditure, where $v_j$ is the constant of proportionality. Total expenditure on consumption is denoted by $\bar{p}c$, where $\bar{p}$ is the ideal price of the final consumption basket, given by equation (11).

Given the price of consumption, and hence the real wage, the second part of this lemma provides the household’s optimality condition for consumption and labor. Equation (12) equates the marginal rate of substitution between consumption and labor with the marginal rate of transformation, that is, the real wage $1/\bar{p}$. Finally, any allocation the household chooses must satisfy its budget constraint.

**Tax (Wedge) Representation.** There is a simple isomorphism between our economy and the taxation literature. The solution to the firm’s problem is also characterized by the same first order condition to the problem of firm that faces a sales tax, or equivalently to a firms facing input taxes. It is convenient then to characterize the firm’s problem in terms of a firm facing sales taxes. To be more precise, a firm faces a sales tax maximizes profits:

$$\max_{x_i} (1 - \tau_i) p_i (z_i x_i)^{\eta_i} - \ell_i - \sum_{j=1}^{N} p_j x_{ij}.$$ 

The first order conditions to this problem will yield $(1 - \tau_i) \eta_i g_i = u_i$. Thus, the firm’s problem is equivalent to a the problem of a firm facing a sales tax that satisfies $(1 - \tau_i) = \phi_i$. This representation shows how our environment is similar to the input-output model with distortions of Jones (2011a).
3.2 Aggregate Production Function and the Efficiency Wedge

The second part of our solution method is to combine market clearing conditions with the firms’ and household’s optimality conditions in Lemmas 1 and 2 in order to characterize aggregate output and consumption for an exogenously fixed amount of labor. That is, for a moment we take the household’s labor supply as given and proceed through a series of intermediate steps: we first solve for equilibrium sales, then equilibrium prices, and finally equilibrium allocations. As a result, we characterize aggregate consumption given an exogenous level of labor, thereby obtaining the aggregate production function of this network economy.

Equilibrium Sales. In order to obtain the aggregate production function, our first step is to solve for the equilibrium sales and expenditure of each sector given a fixed labor supply. We let $g \equiv (g_1, \ldots, g_N)'$ denote the vector of sectoral sales and let $u \equiv (u_1, \ldots, u_N)'$ denote the vector of sectoral expenditures. Furthermore, we let $\phi \equiv (\phi_1, \ldots, \phi_N)'$ denote the vector of frictions across firms where $\phi_i$ is defined in (9).

The Cobb-Douglas structure of preferences and technologies implies constant expenditure shares of firms and households over the commodities, as seen in Lemmas 1 and 2. Substituting these constant expenditure shares into the economy’s market clearing conditions and combining this with the household’s budget constraint admits a fixed point in sectoral sales and expenditures. The solution to this fixed point is given in the following proposition.

**Proposition 1** Take the household’s supply of labor $\ell$ as given. For given $\ell$, any equilibrium will be characterized by a vector of sectoral sales $g$ and a vector of firm expenditures $u$ such that

$$g(\phi) = a(\phi) \ell \quad \text{and} \quad u(\phi) = (\phi \circ \eta \circ a(\phi)) \ell \quad (13)$$

where

$$a(\phi) \equiv \left[ I_N - \left( (1 - \alpha) e_N' \circ W \right)' \circ (e_N (\phi \circ \eta)') - v (e_N - (\phi \circ \eta))' \right]^{-1} v \quad (14)$$

and $\circ$ denotes the Hadamard (entrywise) product.

By solving this fixed point, we obtain the equilibrium sales and expenditure of each sector for a given level of aggregate labor. Proposition 1 shows that for any fixed amount of labor, the equilibrium revenue and expenditure of each sector is linear in the aggregate labor endowment. Furthermore, equilibrium sales and expenditures depend on $\phi$, the vector of financial frictions.

Equilibrium Prices. The next step is to calculate equilibrium prices. Using the equilibrium sales vector as well as the sector-specific production functions, we may back out the vector of equilibrium sectoral prices which is given in the following proposition.

**Proposition 2** Given the vector of equilibrium sales, $g$, the vector of sectoral prices is given by

$$\log p(\phi) = -B [\eta \circ (\log z + \log \phi + \log \eta + \kappa) - (e_N - \eta) \circ \text{log}(g(\phi))] \quad (15)$$
where \( \kappa = (\kappa_1, \ldots, \kappa_N)' \) is an \( N \times 1 \) vector of constants and \( B \) is an \( N \times N \) matrix defined by

\[
B \equiv \left[ I_N - \left( (\eta \circ (1 - \alpha)) e_N' \right) \circ W \right]^{-1}
\] (16)

The aggregate price level (the ideal price for the household’s consumption basket) is then given by the scalar

\[
\log \bar{p} = v' \log p(\phi) - v' \log v
\] (17)

From Proposition 2 one can see that there are two effects of financial frictions \( \phi \) on equilibrium prices. First, there is the log-linear effect of \( \phi \) on prices which is similar to that of productivity \( z \). A decrease in \( \phi \), i.e. a tightening of financial constraints leads to an increase in prices. This is because tightening a firm’s financial constraint forces it to produce at a level lower than it would otherwise. This decrease in supply increases the price for that sector. This increase in price, however, also means that the sector’s customers’ input costs go up, thereby leading to an increase in the customer’s price. This affects their customers’ customers’ price, and so on. The cumulative effect on prices is captured in the \( B \) matrix.

In addition to this aforementioned network effect, there is also an attenuating effect in the term \( (e_N - \eta) \circ \log g(\phi) \) stemming from the sectors’ decreasing returns to scale. We will discuss this effect shortly, after characterizing the equilibrium allocation.

Equilibrium Allocations. Finally, given equilibrium sales, expenditures, and prices, we may then back out equilibrium allocations and hence obtain aggregate GDP. Note that sectoral gross output is given by the vector \( \log y(\phi) = \log g(\phi) - \log p(\phi) \). However, we are interested in sectoral value added, which we will now denote by \( \mu \). Given sales, expenditures, and prices, sector \( i \)'s value added is given by

\[
\mu_i = \bar{p}^{-1} (g_i - (u_i - \ell_i))
\] (18)

That is, value added is firm sales minus firm expenditure on inputs (not including labor costs). We denote sectoral value added by the vector \( \mu = (\mu_1, \ldots, \mu_N)' \).

Real GDP is simply equal to the consumption of the household given by

\[
GDP = c = \bar{p}^{-1} u_0
\] (19)

where \( u_0 = \ell + e_N' (g - u) \) denotes household wealth. It is easy to check that sectoral value added (18) summed across all sectors is equal to aggregate value added, (19).

Next, we substitute our solution for \( g \) and \( u \) from (13) into equation (19). As a result, we may express real GDP as follows

\[
GDP = \bar{p}^{-1} (1 + \psi(\phi)) \ell
\] (20)
where the scalar $\psi(\phi)$ is given by

$$\psi(\phi) \equiv e'_N ((e_N - (\phi \circ \eta)) \circ a(\phi))$$  \hfill (21)

This $\psi(\phi)$ term arises due to the fact that the household receives profits of the firm. In the special case in which there are constant returns to scale and no financial frictions, i.e. $\phi_i = 1$ and $\eta_i = 1$ for all $i$, the term $\psi(\phi)$ reduces to 0 as firms make zero profit. In this special case, $GDP = \bar{p}^{-1} \ell$.

Equation (20) thus provides a simple expression for the aggregate GDP which depends on the inverse of the price level and mechanically on the household’s profits. However, the endogeneity of the price level means that we must combine equation (20) with our characterization of equilibrium prices in Proposition 2 in order to obtain an expression for equilibrium GDP in terms of financial frictions and productivity shocks alone. This characterization is given in the following theorem.

**Theorem 1** The log of GDP may be expressed as the following function of productivity shocks, financial frictions, and aggregate labor,

$$\log GDP = q \log z + q \log \phi + \log \ell$$

$$+ \log (1 + \psi(\phi)) - d \log g(\phi) + K$$

where $\log g(\phi) = \log a(\phi) + \log \ell$ is the sectoral sales vector, $q$ is a $1 \times N$ row vector given by

$$q \equiv (v' B) \circ \eta' = (v' \left[ I_N - \left( \eta \circ (1 - \alpha) e'_N \right) \circ W \right]^{-1}) \circ \eta'$$

$d$ is a $1 \times N$ row vector given by

$$d \equiv v' (e_N (1 - \eta') \circ B)$$

and $K$ is a scalar constant.

To understand this result, let us first consider a special case without financial frictions.

**Special Case: no Financial Frictions and CRS Technology.** We first consider the special case in which there are no financial frictions: $\phi_i = 1$ for all $i$, and in which all sectors operate constant returns to scale technologies: $\eta_i = 1$ for all $i$. The constant returns to scale assumption implies that $d = (0, \ldots, 0)$. Furthermore, the combination of both CRS and no financial frictions implies that $\psi(\phi) = 0$. As a result, GDP in this case may be expressed as

$$\log GDP = q \log z + \log \ell + \text{const}$$

where in this special case

$$q = v' \left[ I_N - \left( (1 - \alpha) e'_N \circ W \right) \right]^{-1}.$$
To make things simple, from these examples we abstract from the constant $K$ the above expression, which is a simply a function of exogenous parameters.

Therefore, in this special case aggregate output is linear in labor. Aggregate GDP furthermore depends on the individual productivity shocks of the sectors, represented in the vector $\log z$. The effect of each sectoral productivity shock on aggregate GDP is represented by the vector $q$—what is often called the “influence vector” of productivity. Each component in the influence vector gives the effect on aggregate GDP of a productivity shock to sector $i$.

To understand this equation, note that the influence vector is simply an expansion given by

$$B = \left[ I_N - \left( (1 - \alpha) e_N' \circ W \right) \right]^{-1} = I_N + \left( (1 - \alpha) e_N' \circ W \right) + \left( (1 - \alpha) e_N' \circ W \right)^2 + \cdots$$

where the first term gives the direct effect of the productivity shock on output in each sector, the second term gives the effect of the shock on the sectors’ customers, the third term gives the effect of the shock on the sectors’ customers’ customers, and so on. Thus, the matrix $B$ catches all terms.

**General Case with both Financial Frictions and Decreasing Returns to Scale Technology.**

Next, we relax our restriction on $\phi$ and allow for binding financial constraints, i.e., $\phi_i \in (0, 1)$. With financial constraints, we must also impose decreasing returns to scale in firm technology to ensure that a firm optima exists—that is, $\eta_i \in (0, 1)$. In this case, we obtain equation (22) in Theorem 1 for the aggregate production function.

To understand this, consider the first line in equation (22). Aggregate GDP now depends on the individual financial constraints of the $N$ sectors, represented in the vector $\log \phi$. As with the productivity vector, the vector of financial frictions is pre-multiplied by the influence vector $q$. Note that the influence vector in (23), compared to (26), now reflects the decreasing returns to scale parameter $\eta_i$. That is, the effects of financial frictions and productivity are attenuated by the sector’s decreasing returns to scale parameter. However, the intuition for the influence vector remains the same.

In addition to these log linear terms, there are two other terms that emerge in the second line of equation (22). First, there is a term given by $\psi(\phi)$. Because of binding financial constraints, firms make profits which are then sent to the households. Households then spend these resources on final good consumption; this allocation of spending would be different had these resources stayed within the firms and been spent on inputs within the production network. Note that if instead the profits

---

9To make expression (26) even simpler, suppose that the labor share is constant across all sectors, that is, assume $\alpha_i = \bar{\alpha}$ for all $i$. Furthermore assume that also the household’s consumption shares are constant across all commodities, i.e. $v_i = 1/N$. In this special case, the influence vector reduces to

$$q = \frac{1}{N} e_N' [I_N - (1 - \bar{\alpha}) W]^{-1}$$

This is the influence vector presented in Acemoglu et al (2012) nested as a special case of our model. Thus, one can think of equation (26) as a generalization of the influence vector in Acemoglu et al (2012) to the case of heterogeneous labor shares across industries and heterogeneous consumption shares across goods.
of the firm were not given to the household and just thrown out, or if they were taxed completely by the government (and not used for any consumption), the $\psi(\phi)$ term would disappear from equation (??).

Finally, there is one last term in (22), given by $-d \log g(\phi)$ where $d$ is defined in (24) and $\log g(\phi)$ is the vector of sectoral sales. This term originates in a price effect stemming from firms’ scale of production. This indirect effect results from the fact that as you tighten the constraint on firms, they operate at a lower scale (given their productivity). Operating at a lower scale implies greater marginal returns to output from any additional input. Thus, lower sales of each firm leads to a decrease in prices–as we saw above. Moreover a lower price index leads to greater output. Thus, the tightening of financial constraints (a decrease in $\phi$) leads to an increase in output due to this price effect. This is a subtle but attenuating effect which is only a result of the decreasing returns to scale. In the limit in which technology approaches CRS ($\eta_i \to 1$), all entries of vector $d$ approach zero: $d \to (0, \ldots, 0)$, and this term disappears.

The Efficiency Wedge. Theorem 1 provides the aggregate production function for this network economy. As a corollary, we may rewrite the aggregate production function in the following way, which highlights the effect of financial frictions on the efficiency wedge of our economy.

**Lemma 3** The aggregate production function in this economy is given by

$$GDP = \bar{z}(z) \zeta(\phi) \ell^0$$

where

$$\bar{z}(z) \equiv \exp\{q \log z\}$$

$$\zeta(\phi) \equiv \exp\{q \log \phi - d \log a(\phi)\} (1 + \psi(\phi))$$

$$\bar{\eta} \equiv 1 - de_N$$

Equation (27) follows directly from exponentiating equation (22). Here $\bar{z}(z)$ represents the effect of sectoral productivity shocks on the aggregate production function, while $\zeta(\phi)$ represents the effect of sectoral financial frictions. Following Chari Kehoe McGrattan (2007), we call movements in $\zeta(\phi)$ the efficiency wedge. Therefore, we find that financial frictions affect the efficiency wedge: movements in individual firm financial constraints results in upward and downward shifts of the economy’s aggregate production at any particular level of labor.

**Comment on Hulten’s Theorem.** Finally, we note that our model breaks Hulten’s (1978) theorem which states that the influence of any sector on aggregate output is simply equal to its equilibrium share of sales. That is, Hulten finds an equivalence between the influence vector and the equilibrium sales vector. As also emphasized in Gabaix (2011), one can think of the share of sales
vector as a “sufficient statistic” of the influence of any firm or sector, and that the information of the entire network structure is unnecessary for answering particular questions. This has also been shown in Gabaix (2011), and Acemoglu et al. (2012). Here we show that this result is in fact just an artifact of having perfect competition; with sectoral wedges, this equivalence breaks down.

First, consider the share of sales vector \( a(\phi) \) in our economy, given in equation (14). To make the analysis simpler, again consider the CRS limit, in which \( \eta_i \to 1 \) for all \( i \). In this case, the equilibrium sales vector reduces to

\[
a(\phi) = \left[ I_N - \left( (1 - \alpha) e_N' \circ W \right)' \circ (e_N \phi') - v (e_N - \phi)' \right]^{-1} v. \tag{28}
\]

Second, consider the influence vector \( q \), defined as the effect of individual sectoral productivity shocks on aggregate output.\(^{10} \)

We may first take the transpose of this vector (to make it a column vector similar to \( a \)). In the limit of case of CRS technology the transposed influence vector \( q' \) reduces to

\[
q' = \left[ I_N - \left( (1 - \alpha) e_N' \circ W \right)' \right]^{-1} v \tag{29}
\]

It is clear that the vectors in (28) and (29) are not identical due the presence of financial frictions. This is because financial frictions divert some of the sales of firms to households in the form of profits. If instead there were no financial frictions, these two vectors would then be identical:

\[
a = \left[ I_N - \left( (1 - \alpha) e_N' \circ W \right)' \right]^{-1} v = q',
\]

thereby confirming the result found in Hulten (1978), Gabaix (2011), and Acemoglu et al. (2012). This leads us to the following proposition.

**Proposition 3** In the special case without financial frictions and with constant returns to scale, the influence vector is identical to the vector of sectoral sales.

\[ q' = a \]

However, with wedges, even in the limit case of constant returns to scale, the influence vector diverges from the vector of sectoral sales, \( q' \neq a(\phi) \).

### 3.3 General Equilibrium and the Labor Wedge

The last step in our solution method is to combine our characterization of the aggregate production function with the optimality of consumption and labor for the household. That is, we locate the

\(^{10}\)If one were to consider the effect of financial frictions on aggregate GDP, it would be clearly non-log-linear, due to the fact that equation (1) is not log-linear in the \( \phi \)'s. Thus, it is almost immediate that the effect of financial frictions is the not the same as the aggregate sales vector.
point on the aggregate production function that also satisfies the household’s optimal choice between consumption and labor. From Lemma 2, we have that the optimal consumption-labor choice of the household satisfies
\[ \frac{c^{-\gamma}}{\ell^\epsilon} = \bar{p} \]  
(30)

We thus combine this optimality condition with the results of the previous subsection in order to solve for equilibrium labor supply and equilibrium aggregate output. This yields the following result.

**Theorem 2** Let \( \beta \equiv \frac{1-\gamma}{\epsilon+\gamma} \). Equilibrium output and equilibrium labor are given by the following functions of productivity and financial friction shocks:

\[
\log GDP(\phi) = \Gamma_q [q \log z + q \log \phi - d \log a(\phi) + K] + \Gamma_\psi \log (1 + \psi(\phi)) 
\]  
(31)

and

\[
\log \ell(\phi) = \Lambda_q [q \log z + q \log \phi - d \log a(\phi) + K] - \Lambda_\psi \log (1 + \psi(\phi)) ,
\]  
(32)

where \( \Gamma_q, \Gamma_\psi, \Lambda_q, \) and \( \Lambda_\psi \) are scalars given by

\[
\Gamma_q = \frac{1 + \beta}{1 + \beta \text{de}_N} \quad \text{and} \quad \Gamma_\psi = 1 - \frac{\gamma \beta (1 - \text{de}_N)}{1 - \gamma (1 + \beta \text{de}_N)}
\]  
(33)

\[
\Lambda_q = \frac{\beta}{1 + \beta \text{de}_N} \quad \text{and} \quad \Lambda_\psi = \frac{\gamma \beta}{1 - \gamma (1 + \beta \text{de}_N)}
\]  
(34)

Theorem 2 characterizes both equilibrium labor and equilibrium GDP in this economy. Comparing our results here for GDP to that in Theorem 1, we see that when labor is endogenous, the effect of productivity and financial shocks on aggregate output are the same, modulo a scalar multiple—this scalar multiple is given by \( \Gamma_q \). Therefore, it has a similar interpretation to that given for Theorem 1 except with a scalar multiplier on these effects.

**Limit case of CRS Technology.** To gain intuition, we first consider the limit case as sectoral production functions approach constant returns to scale. In the limit in which \( \eta_i \to 1 \) for all \( i \), the entries of \( d \) approach zero \( d \to (0, \ldots, 0) \). This implies that \( \Gamma_q \to \frac{\epsilon + 1}{\epsilon + \gamma} > 0 \) and \( \Gamma_\psi \to \frac{\epsilon}{\epsilon + \gamma} > 0 \). Therefore, in this limit case we may rewrite the equations (31) and (32) as follows:

\[
\log GDP(\phi) = \frac{\epsilon + 1}{\epsilon + \gamma} (q \log z + q \log \phi) + \frac{\epsilon}{\epsilon + \gamma} \log (1 + \psi(\phi)) + \text{const},
\]  
(35)

and

\[
\log \ell(\phi) = \frac{1 - \gamma}{\epsilon + \gamma} (q \log z + q \log \phi) - \frac{\gamma}{\epsilon + \gamma} \log (1 + \psi(\phi)) + \text{const},
\]  
(36)

where we have again abstracted from the constant term.

Recall that in the aggregate production function, the aggregate effects of the sectoral-level productivity shocks and financial frictions are given by the influence vector, \( q \) (abstracting from the
profits effect and that of decreasing returns to scale.) With endogenous labor supply this log-linear form remains intact, except that the influence vectors are now multiplied by a positive coefficient \((\varepsilon + 1) / (\varepsilon + \gamma)\), as can be seen in equation (35). Thus, output is increasing in both positive productivity shocks and the loosening of financial constraints. Whether this scalar multiple is greater than one or less than one depends on the magnitude of \(\gamma\).

Next, from equation (36) we see that the effect of productivity and financial frictions on labor is similar. However, whether labor is increasing or decreasing in \(\log z\) and \(\log \phi\), depends on the magnitude of \(\frac{1 - \gamma}{\varepsilon + \gamma}\). In particular, one may think of the term \(\frac{1}{\varepsilon + \gamma}\) as controlling substitution effect and the term \(-\frac{2}{\varepsilon + \gamma}\) as controlling the income effect of the real wage. One can see the income effect quite clearly from how labor depends on \(\psi(\phi)\). This term clearly identifies the income effect: household dividends is just an increase in the wealth of the household. Greater wealth has a negative impact on labor supply; the term \(\frac{2}{\varepsilon + \gamma}\) thus parameterizes this negative wealth effect. Therefore the higher the \(\gamma\), the greater the income effect.

In particular, what matters for the effect on labor is whether \(\gamma\) is greater or less than 1. If \(\gamma < 1\), then the income effect is quite weak and the substitution effect outweighs the income effect. As a result, labor is increasing in productivity and the loosening of financial constraints. On the other hand, if \(\gamma > 1\) then there is a large income effect of the real wage; the income effect then outweighs the substitution effect and in this case, labor is decreasing in productivity and the loosening of financial frictions. Finally, if we are in the knife-edge case in which \(\gamma = 1\) (i.e. log utility), the income effect exactly cancels out with the substitution effect, and labor does not respond at all to these shocks (except through the profits term).

As for aggregate output, it is always increasing in productivity and the loosening of financial frictions. However, the scalar multiplying the influence vector is greater than 1 when \(\gamma < 1\), and less than 1 if \(\gamma > 1\). Thus, as long as the income effect is weak, the introduction of endogenous household labor supply amplifies the effect of both productivity and financial friction shocks.

**General Case.** Moving away from the CRS limit, when there are decreasing returns to scale \(d \neq (0, \ldots, 0)\) and the overall picture is not as clear. For this we find sufficient conditions for the sign of these multipliers to be positive.

**Lemma 4** \(\gamma < 1\) is a sufficient condition for \(\beta > 0\). In this case \(\Gamma_q > 0\) and \(\Lambda_q > 0\).

The restriction that the CRRA parameter \(\gamma\) be less than 1 may not standard. However, note that \(\gamma\) in our environment only represents the income elasticity of labor, this seems a reasonable assumption.\(^{11}\) Thus, similar to Greenwood-Hercowitz-Huffman (GHH) preferences, we are assuming that the income effect on labor is low. In this case, the substitution effect always outweighs the income effect, and hence labor rises with an increase in the real wage.

\(^{11}\)The model is static, so \(\gamma\) does not control the usual elasticity of intertemporal substitution. Similarly, there is no uncertainty, so \(\gamma\) does not control risk aversion.
For the rest of this paper, we thus continue under the assumption that $\gamma < 1$ and move forward with the understanding that $\Gamma_q$ and $\Lambda_q$ are both positive multipliers.

**Labor Wedge.** Finally, we may characterize the aggregate labor wedge in this economy. Following Chari, Kehoe, and McGrattan (2007), we explore the movement in the aggregate labor wedge which captures distortions at the aggregate level. A large literature has documented large labor wedges as well as the countercyclicality of this wedge (see e.g., Hall, 1997; Rotemberg and Woodford, 1999; Chari, Kehoe, and McGrattan, 2007; Shimer, 2009).\footnote{A recent paper by Karabarbounis (2013) shows that most of the time variation in the labor wedge is on the household rather than on the firm side. However, this is not true during the current recession.} Following this literature, we define the aggregate labor wedge $(1 - \tau_\ell (\phi))$ implicitly by

$$
(1 - \tau_\ell (\phi)) \left( \frac{c_\phi (\phi)}{\ell_\phi (\phi)} \right) = \frac{\ell (\phi)^\ell}{c (\phi)^{1 - \gamma}}
$$

That is, the aggregate labor wedge is simply the wedge between the aggregate marginal rate of transformation of aggregate labor, computed from the production function found in (27), and the marginal rate of substitution between consumption and labor. The equilibrium labor wedge that arises in this economy is characterized in the following proposition.

**Proposition 4** *In equilibrium, an aggregate labor wedge emerges which satisfies*

$$
\log (1 - \tau_\ell (\phi)) = - \log (1 + \psi (\phi)) - \log \bar{\eta}
$$

We therefore see that the labor wedge in this economy is simply driven by the profits distortion $\psi (\phi)$. In fact, the labor wedge is inversely proportional to the amount of profits that accrue to the household.

$$
1 - \tau_\ell (\phi) \propto \frac{1}{1 + \psi (\phi)}
$$

Thus, financial frictions create a labor distortion by sending profits to households. Thus, the labor wedge that ends up looking similar to the distortion caused by an aggregate labor tax or an aggregate mark-up.

### 3.4 An Econ 101 Description

To conclude this section, we discuss a graph that summarizes the effects of financial shocks. The figures are built using the parameters from the benchmark calibration that we show later in Section 5. Equilibrium output and hours and the effects of financial shocks can be summarized with two curves. Figure 1a equilibrium outcomes when frictions are not present. The x-axis corresponds to an amount of hours and the y-axis to output. The red —dashed— curve plot the equilibrium production function. The equation that determines this curve is presented Theorem 1. That curve
summarizes the aggregate technology of this economy. The blue —solid— curve is a household’s indifference curve, leveled at an equilibrium point, indicated by the red dot. The thin –black—line, is the the tangency curve at the meeting point of both curves. Without frictions, equilibrium output and hours is determined by that tangency point, in the same way taught in elementary economics class.

Figure 1b summarizes the effects of financial shocks. Depending on the networks’ architecture, and which sectors are affected, two things happen relative to Figure 1a. A first effect is that the shock reduces the efficiency of the economy, which is why the red curve shifts inwards in the figure. A second effects is a distortion in the labor market. As firms become constrained, equilibrium cannot be at a tangency point because this would mean firms are unconstrained. The effect of the labor market distortion depends on the household’s preferences. The overall effect on out depends on the combination of both effects as noted in Theorem 1.

4 Exploration of Simple Network Economies

In this section we explore the effects of financial frictions in three simple economies. Each economy consists of three firms (sectors), but the production is arranged differently in each setting. All three economies are special cases of the general network model we have outlined in the previous section. Given their simple nature, we hope that they give some intuition for the underlying forces at work and how different network structures alter the effect of shocks.

The first economy we study is a vertical economy in which production is arranged in a chain: each firm buys its intermediate good from the firm below it. In this economy, only the first firm in the chain uses labor as input—all other firms only use an intermediate good as input. The second economy we study is a horizontal economy. In this economy all three firms operate in isolation from one another and each use labor their sole input. Finally, we consider a hybrid economy. In this economy firms are again arranged in a vertical chain. However, unlike the vertical economy, all firms use labor as input, as in the horizontal economy.

All proofs for this section are provided in Appendix B.

**Vertical Economy.** In this economy firms use only one commodity as an input to production. Firm 1 uses labor as its sole input. Firm 2 uses the good produced by firm 1 as its sole input. Similarly, firm 3 uses the good produced by firm 2 as it’s sole input. Thus, the firms in this economy are arranged in a vertical chain of production. We write the firms’ production functions
(a) Aggregate Production Function and Consumption Labor Isocurve

(b) Effects of Financial Shocks
as follows

\[
\begin{align*}
y_1 &= z_1 \ell_1 \\
y_2 &= z_2 x_{21} \\
y_3 &= z_3 x_{32}
\end{align*}
\]

where for simplicity we have assumed constant returns to scale for all technologies. Market clearing in commodities 1 and 2 are given by \( x_{21} = y_1 \) and \( x_{32} = y_2 \), while the household consumes only the last good, so that market clearing in this good is given by \( c = y_3 \). Finally, in terms of labor market clearing, \( \ell = \ell_1 \) so that labor is supplied only to firm 1.

In this economy, the vector of labor shares is \( \alpha = (1, 0, 0)' \), the vector of household consumption shares is \( v = (0, 0, 1)' \), and the input-output matrix is given by

\[
W = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\] (39)

Note that the fact that the entries in the first row do not add up to one doesn’t matter because firm 1’s labor share is 1 (and hence its input share is zero).

In this economy, we find that the influence vector is given by

\[
q = (1, 1, 1)
\]

Thus, in terms of the influence vector, all shocks are equally important in this economy. In this vertical production network, there is only one route in which labor can be transformed into the final good. Thus, a friction affecting any part of the chain of production will have an effect on output, and moreover no part of the chain is more important than any other part.

We may furthermore solve for the profit distortion; we find that it is given by

\[
1 + \psi (\phi) = \frac{1}{\phi_1 \phi_2 \phi_3}
\]

Therefore, the profit distortion is decreasing in all of the \( \phi \) wedges symmetrically.

We may use these results to characterize GDP as well the efficiency and labor wedges in this economy. Assuming that productivities are constant, i.e. there are no shocks to the \( z \)'s, these objects are given in the following proposition.

**Proposition 5** In the vertical chain production network, equilibrium log GDP is given by

\[
\log GDP (\phi) = \frac{1}{\epsilon + \gamma} \left( \log \phi_1 + \log \phi_2 + \log \phi_3 \right) + \text{const}
\]
and the efficiency and labor wedges are given by

\[ \zeta(\phi) = 1 \quad \text{and} \quad 1 - \tau_\ell(\phi) = \phi_1 \phi_2 \phi_3 \]

Therefore, there is no efficiency wedge in this economy, there is only a labor wedge. This is because in this vertical production network, there is only one route in which labor can be transformed into the final good. Thus, the aggregate production function is unchanged by any movement in financial constraints. Instead, the financial constraints affect the final price and hence the labor wedge. In particular, the frictions affect the labor wedge symmetrically.

**Horizontal Economy.** We next consider a completely horizontal economy. In this economy, there are three firms that each use labor as input to produce a good. These firms operate in isolation—they do not buy any intermediate goods from one another as inputs. The production functions of these firms are given by the following,

\[
\begin{align*}
y_1 &= z_1 \ell_1 \\
y_2 &= z_2 \ell_2 \\
y_3 &= z_3 \ell_3
\end{align*}
\]

where again we have assumed constant returns to scale for all technologies. Finally the household consumes a basket of the three goods, given by

\[
c = \prod_{j=1}^{3} c_j^{v_j}
\]

where for simplicity let us assume that the consumption shares of the household are equal across all three goods, that is, \( v_j = 1/3 \ \forall j \). Market clearing for the commodities is given by \( c_j = y_j \) for each \( j \), and the market clearing condition in the labor market is given by \( \ell_1 + \ell_2 + \ell_3 = \ell \).

In this economy, the vector of labor shares is \( \alpha = (1, 1, 1)' \), the vector of household consumption shares is \( \mathbf{v} = (1/3, 1/3, 1/3)' \), and the input-output matrix is given by

\[
W = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\] (40)

All entries are zero because no commodity is bought by any firm as input. (Again, note the fact that for all rows, the entries do not add up to one does not matter because all firms have a labor share of 1.)
In this economy, we find that the influence vector is given by

$$q = (1/3, 1/3, 1/3)$$

Thus, again, all shocks are equally important in this economy. This would be expected because all firms produce in isolation and are identical to each other in terms of technology and the household’s consumption basket.

We furthermore find that the profit distortion is given by

$$1 + \psi (\phi) = \frac{1}{\frac{1}{3} (\phi_3 + \phi_2 + \phi_1)}$$

which is decreasing in each of the $\phi$’s. Again, each $\phi$ enters equally into this equation. The following proposition characterizes GDP as well the efficiency and labor wedges in this horizontal economy (again abstracting from productivity shocks).

**Proposition 6** In the horizontal production network, equilibrium GDP is given by

$$\log GDP (\phi) = \frac{\epsilon + 1}{\epsilon + \gamma} \left( \log \phi_1 + \log \phi_2 + \log \phi_3 \right) - \frac{\epsilon}{\epsilon + \gamma} \log (\phi_3 + \phi_2 + \phi_1) + \text{const}$$

and the efficiency and labor wedges are given by

$$\zeta (\phi) = \frac{\left( \phi_1 \phi_2 \phi_3 \right)^{1/3}}{\frac{1}{3} (\phi_3 + \phi_2 + \phi_1)} \quad \text{and} \quad 1 - \tau_\ell (\phi) = \frac{\phi_3 + \phi_2 + \phi_1}{3}$$

The results for this economy are what one would expect. Each $\phi$ has an equal effect on GDP. The same is true for both the efficiency and the labor wedge. Also note that if each wedge is hit equally, there is no efficiency wedge, only a labor wedge.

**Hybrid economy.** Finally, we consider an economy that is a hybrid between both the vertical and the horizontal economies examined above. In this economy, firms are again... firms use as inputs a commodity produced by the downstream firm as well as labor

$$y_1 = z_1 \ell_1$$

$$y_2 = z_2 \ell_2 x_{21}^{1-\alpha_2}$$

$$y_3 = z_3 \ell_3 x_{32}^{1-\alpha_3}$$

Again, market clearing in commodities 1 and 2 are given by $x_{21} = y_1$ and $x_{32} = y_2$, while the household consumes only the last good: $c = y_3$. Finally, the market clearing condition in the labor market is given by $\ell_1 + \ell_2 + \ell_3 = \ell$.

In this economy, the vector of labor shares is $\alpha = (1, \alpha_2, \alpha_3)$, the vector of household consumption
shares is \( v = (0, 0, 1) \), and the input-output matrix is the same as in the vertical economy \((39)\). In this economy, we find that the influence vector is given by

\[
q = ((1 - \alpha_2)(1 - \alpha_3), 1 - \alpha_3, 1)'
\]

This nests the one found for the vertical economy when \( \alpha_3 = \alpha_2 = 0 \). Relative to the vertical economy, the effect of firm 1 and firm 2’s financial constraints are attenuated because labor can now be used as a substitutable input in firm 3.

We furthermore find that the profit distortion is given by

\[
1 + \psi(\phi) = \frac{1}{\alpha_3 \phi_3 + \alpha_2 (1 - \alpha_3) \phi_3 \phi_2 + (1 - \alpha_2) (1 - \alpha_3) \phi_3 \phi_2 \phi_1}
\]

Again, this nests the one found for the vertical economy when \( \alpha_3 = \alpha_2 = 0 \). Therefore, the profit distortion is decreasing in all \( \phi \)'s, with \( \phi_3 \) having the greatest impact, then \( \phi_2 \), then \( \phi_1 \).

The following proposition characterizes GDP as well the efficiency and labor wedges in this hybrid economy (again abstracting from productivity shocks).

\[
\log GDP(\phi) = \frac{\epsilon + 1}{\epsilon + \gamma} (q \log z + q \log \phi) + \frac{\epsilon}{\epsilon + \gamma} \log (1 + \psi(\phi)) + \text{const}
\]

**Proposition 7** In the simple hybrid production network, equilibrium GDP is given by

\[
\log GDP(\phi) = \frac{\epsilon + 1}{\epsilon + \gamma} ((1 - \alpha_2)(1 - \alpha_3) \log \phi_1 + (1 - \alpha_3) \log \phi_2 + \log \phi_3) - \frac{\epsilon}{\epsilon + \gamma} \log [\alpha_3 \phi_3 + \alpha_2 (1 - \alpha_3) \phi_3 \phi_2 + (1 - \alpha_2) (1 - \alpha_3) \phi_3 \phi_2 \phi_1] + \text{const}
\]

and the efficiency and labor wedges are given by

\[
\zeta(\phi) = \frac{\phi_1^{(1-\alpha_2)(1-\alpha_3)} \phi_2^{(1-\alpha_3)} \phi_3}{\alpha_3 \phi_3 + \alpha_2 (1 - \alpha_3) \phi_3 \phi_2 + (1 - \alpha_2) (1 - \alpha_3) \phi_3 \phi_2 \phi_1}
\]

\[
1 - \tau_\ell(\phi) = \alpha_3 \phi_3 + \alpha_2 (1 - \alpha_3) \phi_3 \phi_2 + (1 - \alpha_2) (1 - \alpha_3) \phi_3 \phi_2 \phi_1
\]

## 5 Quantitative Assessment

This section provides a quantitative assessment of the model. The goal is to get a sense of how much input-output features can amplify financial frictions, which sectors can generate the largest effects and which are the most vulnerable. Since the Great Recession is the epitome for a systematic increase in financial constraints, we also test our model’s behavior during the years of that recession.
We conclude this section describing the dimensions of success and failure and provide avenues for new research.

5.1 A Birds-Eye View of the US I-O Architecture

To frame ideas, it is convenient to begin our quantitative assessment with a description of how the production network of the US economy is structured in practice. At a five digit level, the US matrix contains 137,641 entries. Even at 2 digits, there are already 196 entries. Fortunately, the Bureau of Economic Analysis classifies organizes information in a way that allows us to uncover some visual patterns that we discuss next. We will refer back to these patterns when we describe the quantitative results. The visual patterns we discuss here are based on the reconstruction of the US I-O matrix that we explain in the next section.

Figure 2 presents a Sankey plot that shows the intermediate input flows and added value flows across sectors. The figure contains a number of curves that join vertical bars. Each vertical bar at the left corresponds to a 2-digit sector. The length of each bar is scaled by the dollar-value of the intermediate goods production of each sector. The shaded curves emerging from the bars at the left represent the scaled flows of commodities from the sectors represented at the left bars to the sectors in the middle bars. On the flip side, the bars at the middle also indicate the use of commodities in the middle sectors. Another set of curves emerges from the bars in the middle and connects to the set of bars at the right. The bars at the right of the figure represent scaled flows of commodities as final uses —consumption, investment and government expenditures. The curves at the right represent flows of commodities as final uses. The length of the bars at the middle is the max of the intermediate inflows and final goods outflows: a sector that receives a large inflow from the left, but generates a small outflow to the right, produces mostly intermediate inputs. Sectors that have a small inflow from the left, but thicker bars to the right, generate a large amount of added value without employing many resources from other sectors —using capital and labor only.

Several things can be understood from this Sankey plot. There are three broad categories of sectors. A first group —the primary sectors— almost exclusively produces production inputs. These sectors are composed by agriculture, mining, and utilities. The second group sectors is composed of sectors that either use a large amount of intermediate goods, or supply a large amount of intermediate inputs in relation to their added value. These sectors include manufacturing, construction, and service sectors that support production. We can observe that manufacturing is both, the biggest intermediate producer and user of intermediate inputs. At the same time, manufacturing also produces a large proportion of aggregate consumption and investment goods. Construction plays a small role as an intermediate goods producer since it products investment goods, that reenter production as capital, but not as intermediate inputs. Sectors that support production include professional services —lawyers, accountants, management firms—, reallocation services —transportation, warehousing—, information services —IT and advertisement—, and wholesale. A
third set of sectors that are not connected to a major supply chain. Those sectors do not use a large amount of intermediate goods nor supply intermediate goods to other sectors. These are sectors primarily employ capital and labor, but don’t pull resources from other sectors. Among these we find: consumer and other services, education, health, government and retail —note that when a commodity is sold to retail, it’s not counted as an input because the product is not transformed.
Figure 2: Flows of Intermediate Inputs and Added Value.

Note: The width of gray curves are proportional to cross-sectoral flows. Total sectoral flows.
Figure 3 presents the 5-digit input-output matrix of the US. There are 371 five digit sectors. Although it contains 137,641 entries, we can learn something from the heatmap of this matrix. In the figure, each row represents a sector supplying inputs to the sectors in the column. The entries of the matrix correspond to the share of uses of each sector in a given column. Lighter colors reflect more intensive input uses. We also added labels to that correspond to the 2-digit family of five digit sectors.

Several additional patterns emerge from this figure. First, observe that the diagonal features light colors. This reflects that many transactions occur within the same sectors. We would expect this diagonal term to vanish as we reach higher levels of disaggregation—in fact, we study the robustness of our results by setting diagonal elements to zero. We also find a concentration of light colors among certain rows which indicates that some sectors supply inputs to many other sectors. For example, the rows corresponding to utilities sectors are light for almost all column entries—which reflects that all sectors need energy.

A similar patterns is found found for service sectors that support production—a reflection that there’s a substantial amount of outsourcing of tasks outside the firm. We also find that manufacturing sectors are special in that they belong to a cluster of input-outputs. We see light colors from and towards manufacturing sectors. We can also observe that sectors that belong to construction, supply few inputs to other sectors, but demand inputs from many other sectors. Again, this is because construction produces investment goods that are not treated as commodities.
Figure 3: US 5-Digit Input-Output Matrix

*Note:* Rows in the matrix correspond to sectors supplying intermediate goods to the sectors located in the column. The colors indicate the logarithm of sectoral flows.
5.2 Data and Calibration

**Data.** The Bureau of Economic Analysis (BEA) provides data on the gross output —gross sales— for all sectors. It also provides information on the distribution of gross sectoral outputs as intermediate inputs of other sectors and final uses. These transactions are recorded in the US input-output tables where sectors are classified according to the North American Industry Classification System (NAICS). The BEA reports annual data for two- and three-digit industry levels from 1997 to 2012 and quinquennial data at the five-digit industry level. There are 15, 65, and 384 sectors at the two-, three-, and five-digit classification levels.

In Appendix D we describe construct an input-output matrix that is consistent with our model from the actual US input-out matrices, but here we outline what we do. First, we make adjustments so that there are the same number of input as output sectors. For example, the scrapping of steel is considered an intermediate input, but scrapping is not supplied by any particular sector so we eliminate the row corresponding to scrapping. Second, we extract the rows and columns corresponding to the Finance, Insurance, and Real-Estate (FIRE) sectors. We don’t want to treat the income of FIRE sectors as a production input, but rather, as part of capital gains. At the same time, we use the expenses of all other sectors in FIRE services as one method to obtain measurements for $\phi_i$. Third, our model of production is closed economy, but the actual input-output matrices include imports and exports. In the data, several industries show a negative trade balance, which in principle means that certain intermediate inputs could be partially supplied by an external sector. It turns out that for all industries, except for oil, domestic absorption exceeds any negative trade balance. Thus, we can treat imports and exports as if they are part of final uses —assuming that all foreign trade occurs between households and the external sector, but doesn’t affect internal production. Along those lines, we assume that all production inputs are supplied domestically, and imports are used only for final uses. Consequently, when we calibrate expenditure shares, we will use the series on net final uses, calculated as absorption minus net exports. For the oil industry, we can’t take that approach because oil imports exceed domestic absorption —this means that at some amount of oil is used as an intermediate input. For the oil industry only, we scale up the sector’s production by the proportion of net imports minus domestic absorption. Fourth, we correct for inventories. In the data, part of investment uses includes inventories. Yet, inventories are stored to be used as final uses at a later date or to be distributed for final uses. Since our model is static, we subtract inventories from total investment, and redistribute the dollar value of inventories supplied by a given sector by adding a share of total inventories to the uses by other sectors. The share of inventories we add to uses is proportional to the uses of each sector over total uses —i.e., if cotton mills sell a given amount to the textile industry, we increase the sales of cotton mills to the textile industry, by adding the stock of cotton inventories times the proportion of sales cotton to the to the textile industry to the total sales cotton mills to the textile industry.

**Preference Parameters.** The household’s preference parameters control the labor supply in the model. As explained earlier, $\gamma$ governs the wealth effect of labor supply decisions and $\epsilon$ governs
the substitution effect. We set $\gamma = 0$ and $\epsilon = 1/2$ to isolate any wealth effect, but we report results for a activating the wealth effect. The Frisch elasticity of labor supply implied by $\epsilon$ is higher than the average value of micro studies, but is standard in the macroeconomics literature. Since we report the responses of output and efficiency, we are automatically reporting the effects for both, inelastic and elastic labor supply.\footnote{13Labor supply elasticity is one of the most contentious parameters in the macroeconomics literature. It is understood that macroeconomic models with frictionless labor markets need a highly elastic labor supply to generate large output responses. These high elasticities can be obtained through indivisible labor. A useful discussion is found in Chetty et al. (2011) and Ljungqvist and Sargent (2011). Here, we follow other macroeconomic papers, for example Hall (2009), and study the behavior of our model for a Frisch elasticity of 2 hoping that this elasticity labor market frictions left out of our model. Of course, we report results for lower labor supply elasticities, but we want to warn our readers that our benchmark calibration will assume the macro elasticity, as opposed to the lower micro elasticity.}

**Expenditure Shares.** Let $\hat{N}$ be the number of sectors of a given input output matrix at a given level of disaggregation. Under Cobb-Douglas utility, the household’s expenditure share in commodity $i$ is constant. The BEA does not report prices and quantities separately, but it does report the sales of each sector as final uses. We treat final uses as the analogue of the household’s expenditure shares in the model. Following the corrections we discussed in Appendix D, we construct final uses as domestic absorption minus net imports. Let $\hat{u}^c_{i,t}$ be the final uses in the data for year $t$ industry $i$. We set $\nu_{i,t}$ to:

$$\nu_{i,t} = \frac{\hat{u}^c_{i,t}}{\sum_{j=1}^{\hat{N}} \hat{u}^c_{j,t}}.$$ 

Figure 4 reports $\log \nu_{i,t}$ for each industry at the 3-digit classification level from 2006-2010. All expenditure shares at this level are stable.

**Technology Parameters.** For each $i$ we must calibrate the parameters that affect the intermediate composite, $\alpha_i$ and $\{w_{ij}\}_{j=1}^{\hat{N}}$. Calibrating $\alpha_i$ and $w_{ij}$ is straightforward: Each entry of a Uses table presents the purchases of intermediate inputs of a sector corresponding to a given column, of commodities supplied by a sector at a given row. That matrix will include an extra row for labor expenses. Since our firms have a Cobb-Douglas production technology, we proceed in the same way as we did for households: Let $u_{ij,t}$ be the total expenses of sector $i$ in commodities produced by sector $j$ in year $t$. We calibrate the entries of the input-output matrix in our model according to:

$$w_{ij,t} = \frac{\hat{u}_{ij,t}}{\sum_{j=1}^{\hat{N}} \hat{u}_{ij,t}}.$$ 

The values of this calibration are used in the construction of Figure 3 that we discussed earlier. To calibrate the labor shares $\alpha_i$, we divide the labor expenses of sector $i$, name it $\hat{l}_i$, by the total expenses of the sector: $\hat{l}_i + \sum_{j=1}^{\hat{N}} \hat{u}^c_{ij,t}$.

There are two additional parameters that affect the technology. The main challenge is to separately identify the coefficients of decreasing returns to scale parameter $\eta_i$ from the financial shocks $\phi_i$. We do not have observables that we can use to obtain these parameters directly, as we have
for the components of the input-output matrix. The problem is that input shares cannot be independently identified from the wedges \( \phi \), a problem faced by all of the literature on misallocation, including the well known work of Hsieh and Klenow (2009) and Jones (2011b). Concerning this issue, Jones (2011b) writes that “there is a fundamental identification problem: we see data on observed intermediate goods shares, and we do not know how to decompose that data into distortions and differences in technologies.” In terms of the model, the firm \( i \)'s optimality conditions imply that, absent frictions, total expenses relative to sales should equal \( \eta_i \). With frictions, this ratio equals the product \( \eta_i \phi_i \), so we cannot use this ratio to separately identify \( \eta_i \phi_i \).

In the exercises that follow, there are two important things. One is to calibrate \( \eta_i \) and a benchmark value for \( \phi_i \). The other is to obtain measurement series for the shocks \( \phi_i \) that we can use to evaluate the model’s response. We will use a single method to calibrate the levels of \( \eta_i \) and \( \phi_i \), and then try three approaches to calibrate the changes in \( \phi_{i,t} \) that we can use to make comparisons.

**Values of \( \eta_i \) and Benchmark Levels for \( \eta_i \).** We use the assumptions of the model to obtain benchmark values for \( \phi_{i,t} \) and \( \eta_i \). Recall that equilibrium in our model requires:

\[
\frac{u_{i,t}}{p_{i,t}y_{i,t}} = \phi_{i,t} \eta_i. \tag{42}
\]

Thus, our model attributes all changes in the costs-to-sales ratio, \( \frac{u_{i,t}}{p_{i,t}y_{i,t}} \), in each sector to movements in \( \phi_i \) because technology is assumed to be fixed in the short run. In the data, the cost-to-sales ratio
of each industry moves every year so we exploit this time variation to obtain bounds for $\eta_i$. A natural upper bound is one because, as we explained earlier, our model does not admit increasing returns to scale. A lower bound is obtained by the assumption that $\eta_i$ is constant. If $\eta_i$ is constant, the lowest value possible for $\eta_i$ is the highest costs-to-sales at a given sample of years. This is because at best, $\phi_{i,t} = 1$ during the year at which the cost to sales ratio was the highest. Thus, a lower bound $\eta_i$ must be the highest cost-to-sales ratio in the data. Thus, we have bounds for the possible values of the returns to scale parameter, $[\eta_i, 1)$. Notice that if we had adopted a Bayesian approach with uniform prior, the posterior would have been a uniform over $[\eta_i, 1)$. When we studied the experiments that we report below, we found that our model gives the largest responses as $\eta_i$ approaches 1 and as $\phi_i$ becomes the lowest. To understate our results, whenever we need to set a particular benchmark value for $\eta_i$ we set it to $\eta_i$. When we report results, we report results at both limits so that our reader gets a sense of the range of multipliers delivered by the network effect.

**Measured $\phi_i$.** Once we set a value for $\eta_i$ —from our bounds— we can obtain a time series for measured $\phi_{i,t}$ directly from the relation in (42) and the data. Figure 5 shows the values of $\phi_{i,t}$ for the sub-sample of years around the Great Recession, 2006-2010 when we set the benchmark $\eta_i = \eta_i$. It is clear that the year 2009, the year that the Great Recession unfolded, features lower imputed values for $\phi_{i,t}$. We denote this first benchmark measurement by $\phi_{i,t}^1$.

We construct two alternative measurements for $\phi_{i,t}^1$. The second method uses information from FIRE expenses and the third method uses a direct measure of financing costs. For the FIRE method, we choose a benchmark year, 2007 —the year of the onset of the Great Recession. We maintain the same benchmark values of $\eta_i$, but we compute a new measurement for the financial frictions, $\phi_{i,t}^{FIRE}$.
For each sector \( i \), we sum the expenses in all FIRE sectors—which we excluded from the input output matrix. Let \( \hat{u}_{i,\text{FIRE},t} \) denote the data analogue of those expenses. We obtain an alternative sequence of financial shocks through the formula:

\[
\phi^\text{FIRE}_{i,t} = \left[ \sum_{j=1}^{\hat{N}} \hat{u}_{ij,t} \hat{u}_{i,\text{FIRE},t} - \sum_{j=1}^{\hat{N}} \hat{u}_{ij,2007} \hat{u}_{i,\text{FIRE},2007} \right] + \hat{\phi}_{i,2007}.
\]

Let’s explain the meaning of this formula. In Appendix C, we described how interest on working capital would enter the firm’s first-order condition in the same way as \( \phi_{i} \) does. FIRE expenses can be interpreted as net interest rate payments on working capital loans, so \( \hat{u}_{i,\text{FIRE},t} / \sum_{j=1}^{\hat{N}} \hat{u}_{ij,t} \) can be the analogue of an interest rate. The inverse of this ratio is an analogue of \( \phi_{i} \). In the formula above, the difference in the brackets gives us a change in our financial shocks that can be attributed to changes in FIRE expenses only. We add the term \( \hat{\phi}_{i,2007} \) so that \( \phi^\text{FIRE}_{i,t} \) and \( \hat{\phi}_{i,t} \) yield the same value during 2007. That way, we can compare the effects of financial shocks under each measurement.

One important observation is that uses of FIRE don’t fully capture a sector’s financial expenses, it simply captures the expenses attributed to the services of the financial industry by the BEA. Yet, part of the financial expenses in the data will be captured by operating revenues, which don’t include financial expenses directly. In that sense, we can think of \( \phi^\text{FIRE}_{i,t} \) as a lower bound.

The third method is based on the excess-bond premium (EBP) measurements of Gilchrist and Zakrajšek (2012).\(^{14}\) The EBP is computed in several steps. First, the authors compare the price of a synthetic bond reconstructed from coupon payments on actual bonds versus the yields of Treasury Bills. From that price difference, the authors factor a component that predicts corporate defaults and a component that is orthogonal to credit risk. That component yields a time series for excess-bond premia, which the authors show is correlated with multiple measures of economic activity and financial distress. The authors also built sectoral indices for EBP based on the 12 industry of Fama and French which is widely used in the finance literature. We map that classification to the classification in NAICS and obtain a time series, \( EBP^GZ_{i,t} \), for the EBP for each of our industries. We then construct a new measurement labeled \( \phi^GZ_{i,t} \) according to the following formula:

\[
\phi^GZ_{i,t} = \left\{ \frac{1}{1 + EBP^GZ_{i,2007}} \right\} \hat{\phi}_{i,2007}.
\]

As with the interpretation of our \( \phi^\text{FIRE}_{i,t} \) measurement, the interpretation of this formula is that the EBP should relate to a higher financing cost in our model. The formula for \( \phi^GZ_{i,t} \) factors in an increase in \( EBP^GZ_{i,t} \) in the same way an interest-rate surcharge would map into our model, but also makes sure that \( \phi^GZ_{i,t} \) takes the same value as \( \hat{\phi}_{i,2007} \). One thing to note is that the measurement of \( \phi^GZ_{i,t} \) doesn’t fully capture increments in financial costs because it factors out solvency risk. Another difficulty with this measurement is that the industry classifications in the BEA data are more broad—multiple NAICS sectors will belong to the same Fama and French sector.

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\(^{14}\)We would like to Simon Gilchrist and Egon Zakrajšek for kindly sharing their data with us.
5.3 Experiments

5.3.1 Studying our Measurements

Thus far, we have explained how we obtained three measurements for $\phi_{i,t}$ that coincide with the same value in 2007. Before we analyze the model, we investigate our measurements. First, we report the correlations between $\hat{\phi}_{i,t}$, $\phi_{FIRE}^{i,t}$ and $\phi_{GZ}^{i,t}$ to see if their variation is potentially capturing the same events. Second, we investigate the correlation of our measurements with sectoral outputs. Finally, we investigate sector effects are present in the data. Once we have a sense the behavior of our measurements, we use the model to perform some experiments.

**Measurements and Correlations.** We first test if $\hat{\phi}_{i,t}$, $\phi_{FIRE}^{i,t}$ and $\phi_{GZ}^{i,t}$ relate in any way. Under all measures, 2009, the year where the Great Recession reached its peak, is the year with the tightest financial constraints. The largest average drop is obtained the benchmark measure $\hat{\phi}_{i,t}$ which on average falls by -0.0230 —or equivalently, as a 2.3% increase in interest rates. The average drop using the $\phi_{FIRE}^{i,t}$ method is one order of magnitude lower -0.0015, but again, this measure is missing many forms of external finance. The average drop using the GZ method is about one third than the benchmark method -0.0082. However, since according to Gilchrist and Zakrajiˇsek (2012), the EBP captures only about one half of the firm’s financing cost, we think that our benchmark method and the GZ method are of the same order of magnitude. Standard deviations also differ substantially. These are 0.060, 0.014 and 0.003 respectively for the three measures. Thus, in terms of their cross-sectional variation, our benchmark method yields the greatest variation. We compute the correlation of the changes of $\hat{\phi}_{i,2009} - \hat{\phi}_{i,2007}$ and $\phi_{FIRE}^{i,2009} - \phi_{FIRE}^{i,2007}$ feature a statistical correlation of 0.29 which is significant at the 5% probability. The correlation between $\hat{\phi}_{i,2009} - \hat{\phi}_{i,2007}$ and $\phi_{GZ}^{i,2009} - \phi_{GZ}^{i,2007}$ is almost close to zero and non-significant, perhaps due to the more granular classification in NAICS.

The next test asks if our measurements are correlated with sectoral outputs. Our model has two obvious predictions: sectoral outputs should be positively correlated with (a) $\phi_{i,t}$ and (b) $\nu_{i,t}$. In addition, there is the prediction are network spillovers, because sector $i$’s output can be affected by the shock $\phi_{j,t}$ to sector $j$. Which sector affect what sector should depends on the calibration of the input-output matrix. We test for these predictions running several regressions —one for for each of our measurements of $\phi_{i,t}$. Observations correspond to the pairs $\{i, t\}$ with $i$ being a sector and $t$ all years from 2006 to 2010. The dependent variable is the change in gross sales of a given sector at a given year, relative to gross sales at the benchmark year 2007. The first regressor are expenditure shares, $\nu_{i,t}$, to capture any effects coming from relative demand. The second regressor is the change in our measurement of $\phi_{i,t}$, that is $(\phi_{i,t} - \phi_{i,2007})$, for each of the three methods described above. To study the presence of network effects, for each sector $i$, we calculate ranking the sectors that generate the biggest drop in sector $i$, after a 1% drop in $\phi_{j}$ of any other sector —we denote by $n_{m}^{i}$ the identity of the sector that has the m-th ranking effect on sector $i$, period at $t$. Thus, $\hat{\phi}_{n_{m}^{i}, t}$ us the value of $\phi_{j}$ for the sector that has the m-th largest effect on $i$. We include the $\phi'$s of the first five
neighbors of each sector $i$ as regressors. The results of these regressions are summarized by Table ??\). The first column reports results for our benchmark measure $\hat{\phi}_{i,t}$. Results suggest a positive and significant relationship between sectoral output growth and $\hat{\phi}_{i,t}$. Furthermore, the regression also detects the presence of network effects up to the first neighbor. The second regression in the table shows that $\phi_{FIRE}^{i,t}$ is also significantly correlated with sectoral growth, but the neighborhood effects are no longer significant under this measure. The third regressions, uses the EBP method, but shows no significance —possibly due to the aggregation issues we discussed earlier.

Figure 9 in the Appendix shows a scatter plot of the data points, together with the hyperplane on the $(\hat{\phi}_{i,t} - \hat{\phi}_{i,2007})$ and $(\nu_{i,t} - \nu_{i,2007})$ axis corresponding to the regression results of the first column of Table ??\). Years are indicated by a different symbol. We can observe that for 2009, the underpredicts the declines in gross sales for almost all sectors, but overall, the measurement of $\hat{\phi}_{i,t}$ is positively correlated with sectoral output.

### 5.3.2 Robustness and Sensitivity

Before we report our results, we want to warn our reader of the things he should and shouldn’t be concerned about. We performed several sensitivity tests to all of the results we provide. First, we studied results for all of the levels of disaggregation when possible. In addition, we studied results using the IO tables before and after redefinitions —see data appendix for more details. We also tested if the exclusion of the diagonal term of the IO matrix alters results. For none of these variations do we find changes in results.

By contrast, what does matter for results is how we calibrate the average level of $\phi_i$ and $\eta_i$. There are two things to note. First, the closest we are to constant returns to scale, the greater the responses, and in particular, the greater the difference between the calibrated economy and one with a representative firm. As we explained in the theoretical analysis, this is because the curvature

<table>
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<th>Benchmark</th>
<th>FIRE</th>
<th>GZ</th>
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<tbody>
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<td>$\Delta \phi_{i,t}$</td>
<td>30.7*</td>
<td>311***</td>
<td>-16.5</td>
</tr>
<tr>
<td>$\Delta \nu_{i,t}$</td>
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<td>2362.4***</td>
<td>2434***</td>
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<td>$\Delta \phi_{n_1,t}$</td>
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<td>$\Delta \phi_{n_2,t}$</td>
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<td>$\Delta \phi_{n_3,t}$</td>
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<td>55.9</td>
<td>-107.6</td>
</tr>
<tr>
<td>$\Delta \phi_{n_4,t}$</td>
<td>2.7</td>
<td>33.9</td>
<td>-59.3</td>
</tr>
<tr>
<td>$\Delta \phi_{n_5,t}$</td>
<td>-10.6</td>
<td>29.6</td>
<td>31.8</td>
</tr>
<tr>
<td>Observations</td>
<td>315</td>
<td>315</td>
<td>315</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.2348</td>
<td>0.184</td>
<td>0.11</td>
</tr>
</tbody>
</table>

*p < 0.05, **p < 0.01, ***p < 0.001

Table 1: Measurement of $\phi$ and Sectoral Growth
of the production partially offsets the misallocation of output. Second, changes in $\phi_i$ have greater effects, the further is $\phi_i$ from 1. This is because the optimal labor allocation is a convex problem. so the closest we set $\phi_i$ to one, the closest we are setting the model to the global optimum. The the model is to the global optimum, the lower will responses. We report responses at both extremes of $[\bar{\eta}_i, 1]$.

In our benchmark methodology, we use the model’s properties to reverse engineer $\phi_i$ and $\eta_i$. When $\eta_i = \bar{\eta}_i$, the value of $\hat{\phi}_i$ is the lowest value given the assumption that all the variation in the costs sales follows from financial shocks. However, the reader should still be aware that even for this calibration, $\hat{\phi}_i, 2007$ is 0.9. This means that in our baseline calibrations calibrations, firms are already operating at the interior of their first order conditions. A generous interpretation is that the model is missing any form of market power, capacity constraints, price rigidities, etc., that would add to average frictions. When study the model’s response, we increase $\phi_i$ in small amounts, amounts that translated into interest rates, are not abnormal. However, we warn our reader that we are departing from active constraints. Results are dampened as we set $\hat{\phi}_i, 2007$ closer to 1. Finally, the parameters that govern labor supply are also important, so we report effects for multiple values.

5.3.3 Great Recession Fit

Fit to Macroeconomic Aggregates. In a first experiment we insert our measured values of $\phi_i$ in the model. We compare several macroeconomic aggregates between the years 2007 and 2009, from the year prior to the Great Recession to the year when it ended. The results are reported in Table 2. We report results calibrating the input output matrix to the 3-digit after redefinitions (3AR) table, for both limits of our calibration. We insert the measured series for $\hat{\phi}_i$ for 2007 and 2009, and compute changes in macroeconomic aggregates. For the lower bound, the model predicts an output drop of almost 4%. This response corresponds to a decline in hours if 5% which is aggravated by an almost 1% TFP. This latter effect is entirely responding to a misallocation inputs. For comparison, we present the responses when we alter the baseline calibration by setting the entries of the input-output matrix to $w_{ij} = 0$, to be consistent with a horizontal economy. All other parameters are kept the same. For this alternative calibration, we find that the response is almost halved, for both output and hours. We find that the labor wedge responds in almost the same amount because, real wages absorb part of the financial shock. What is different in this economy is that efficiency is not distorted because all sectors are hit in the same proportion, and there is no change in relative prices among commodities —although there is a change in relative prices relative to labor. If we call the ratio of the response in the response of the calibrated economy to the response of the horizontal economy, the liquidity multiplier, we get multiplier of almost 2%. At the decreasing returns to scale limit, the responses of both economies are much larger, but the liquidity multiplier reaches almost 7. Notice that we are not imputing any output trend into the model.

Sectoral Fit. We can compare the responses in the cross-section of sectors in our model and in the data after we input the measurements $\hat{\phi}_i$ into the model. Again, we compare the sectoral
returns-to-scale: $\eta_i = \eta_i \rightarrow 1$

<table>
<thead>
<tr>
<th>Variable</th>
<th>3AR</th>
<th>Horizontal</th>
<th>3AR</th>
<th>Horizontal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>-3.8%</td>
<td>-2.1%</td>
<td>-28.3%</td>
<td>-4.5%</td>
</tr>
<tr>
<td>Hours Worked</td>
<td>-4.9%</td>
<td>-2.7%</td>
<td>-21.79%</td>
<td>-4.3%</td>
</tr>
<tr>
<td>Efficiency</td>
<td>-0.9%</td>
<td>0.0%</td>
<td>-8.4%</td>
<td>0.13%</td>
</tr>
</tbody>
</table>

Table 2: Model Fit - Great Recession

Figure 6: Measured $\phi_i$

responses in the cross section for the year 2009, vis-à-vis the responses for 2008. We just plot the results at the lower bound. The cross-section of responses is plotted in Figure 6 which shows gross sales growth for every sector, both in the model and in the data. An immediate observation is that sectoral growth rates are much more volatile than in the model. This probably follows from a much more rigid production structure in certain sectors in the actual economy. For example, if the supply of oil is fixed in the short run, the price of oil will respond much more than in our model, to any aggregate shock. Despite this differences, the correlation between growth rates in the model and the data is high, 0.44.
Returns-to-Scale:
\[ \eta_i = \eta_j \rightarrow 1 \]

<table>
<thead>
<tr>
<th>Δ Efficiency</th>
<th>Hor.</th>
<th>3AR</th>
<th>Symm.</th>
<th>Hor.</th>
<th>3AR</th>
<th>Symm.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Δ GDP for ( \gamma = 0, \nu = 0.5 )</td>
<td>0%</td>
<td>-0.1%</td>
<td>-0.1%</td>
<td>0%</td>
<td>-0.8%</td>
<td>-0.8%</td>
</tr>
<tr>
<td>Δ GDP for ( \gamma = 0, \nu = 1 )</td>
<td>-0.6%</td>
<td>-0.9%</td>
<td>-0.9%</td>
<td>-1.0%</td>
<td>-2.3%</td>
<td>-2.3%</td>
</tr>
<tr>
<td>Δ GDP for ( \gamma = 0, \nu = 2 )</td>
<td>-0.3%</td>
<td>-0.5%</td>
<td>-0.5%</td>
<td>-0.5%</td>
<td>-1.1%</td>
<td>-1.2%</td>
</tr>
<tr>
<td>Δ GDP for ( \gamma = 0.5, \nu = 0.5 )</td>
<td>-0.3%</td>
<td>-0.5%</td>
<td>-0.5%</td>
<td>-1.0%</td>
<td>-2.7%</td>
<td>-2.8%</td>
</tr>
</tbody>
</table>

Table 3: Model Responses to 1% drop in all \( \phi_i \)

5.3.4 Aggregate and Cross-Sectional Impact of Financial Frictions

Network-Liquidity Multiplier. When we feed the model with the \( \phi_{it} \) we measured from the great recession, many sectors are shocked at the same time, and in a different magnitude, so the the shock is not symmetric. We also study the effects of symmetric increase in \( \phi \) of 1% in every sector, to get a pure liquidity multiplier. We ask (a) what is the fall in aggregate output that we obtain from our calibration? (b) how does this compare with a horizontal economy and other common structures. Since results depend on preferences and the calibration of technology, we report values for multiple calibrations. The results are summarized in Table 3.

Each column corresponds to a different calibration—or model—for the input output tables, leaving all other parameters fixed. Rows report changes in TFP and output for different values of labor supply parameters. The first column corresponds to a horizontal economy that ignores all intermediate inputs so \( w_{ij} = 0 \). The second column presents the results for \( W \) calibrated to the 3AR Uses table. The third column is a calibration where \( \alpha_i \) equals the share used in the 3AR model, but sets equal expenditure shares across all inputs so \( w_{ij} = 1/N \). We will see that this symmetric economy model, is a good approximation to the model calibrated to the 3-digit I-O table. That last 3 columns repeat the exercises for the constant returns to scale limit.

Let’s begin discussing the results for benchmark calibration. The first row presents results for TFP. Recall that this is the also the response for GDP under inelastic labor supply. The first thing to note is that, just as in the simple example of the horizontal economy, productivity does not not change when all sectors face the same shock. Things are different when we introduce a non-zero input-output matrix. Efficiency in the symmetric economy responds in the magnitude as when we calibrate the model to US data. As we explained in the characterization, all of the effects on efficiency stem from the misallocation of labor across sectors. Here, symmetry is key: When we study a common shock and all sectors have the same degree of decreasing returns to scale and labor shares, the overall effect almost entirely vanishes. There is a slight effect present, when expenditure shares differ across sectors. The effect is gone when we make expenditure shares constant. In summary, the cross-sectoral heterogeneity in decreasing returns to scale, and labor shares, induce a strong misallocation of labor out of financial shocks.
We then compare the response of aggregate GDP including the endogenous response of hours. We want to highlight two directions of effects: first, the stronger the labor supply elasticity, the strongest the is the response of hours to the induced labor wedge. Second, $\gamma$ tends to dampen the overall response of hours. Working with the standard parameter choice in macroeconomics, $\nu$ and reducing the effect, projects the effect in efficiency into a response that is 1.5 times higher. The level of the response and the liquidity multiplier is here smaller than for the Great Recession fit, because the shock is smaller, and there’s no dispersion, which gives an important additional kick to the overall response. In the constant returns to scale limit, all responses are amplified, but the liquidity multiplier increases to 2.5.

**Vulnerable Sectors.** We also use the model to explain which sectors are the most affected after an increase in $\phi$ of 1%, in all sectors. Figures 7a and 7b present the per cent responses of gross commodity output for 2- and 5-digit level sectors. Figures 7a includes all 2-digit sectors and 7b includes the most and least sensitive to the aggregate shock. Since at two digits, there are only 14 sectors, 7a presents results for all sectors. Figure 7b presents results for the most and least sensitive sectors. Figure 7a can also be used as a two-digit color code.\(^{15}\)

We find that the sectors most affected from an aggregate liquidity shock belong to a cluster of sectors that supply inputs to the manufacturing industry. These sectors include the processing of metal products, chemical products, textile products, etc. This result is consistent with the analysis of the simple networks of the vertical-mix economy we studied above and the location of these sectors in figures 2 and 3. These sectors are located upstream in the production chain and belong to a manufacturing cluster. On the flip side, sectors that belong to miscellaneous professional services and government, that interact little with the rest of the industry, are less sensitive. This is an interesting result in it of itself. Typically, we think of manufacturing industries as producing durable products and we attribute a greater business cycle volatility of these sectors, to household inventory management. Our models shows a different story, where manufacturing sectors are more volatile because manufacturing requires many steps, and many steps make more upstream firms more vulnerable to aggregate shocks. This result is in sharp contrast with the responses to TFP presented in Acemoglu *et al.* (2012) or here, because an aggregate response to a common TFP shock would have the same effect on all sectors—at the constant-returns-to-scale limit.

**Influential Sectors.** Our last experiment studies where is an idiosyncratic financial shock be more damaging for GDP. We present two responses. The first is the ranking of a 1% drop in $\phi$, affecting one sector at a time. The second is the same experiment, but scale shocks in proportion to a sector’s size. The interpretation of the second set of results is the response to a fixed real amount of funding.

\(^{15}\)The colors of the bars in Figure 7b and the ones that follow correspond to the same colors of the two digit level sectors that a five digit sector belongs to.
(a) Sectoral Sensitivity to Aggregate Shock - 2 Digits

(b) Sectoral Sensitivity to Aggregate Shock - 5 Digits
When the shock is scaled by a sector’s sales, Figure 8a reports that the sectors that lead to the largest drop in aggregate output are those who are largest in terms of added value: different branches of government, wholesale, hospitals, etc. There’s also a set of sectors that provide services that support industries. This is the analogue effect of what we know for TFP shocks: Acemoglu et al. (2012) already showed that the ranking of sales leads to the same ranking of the impact of effects.\textsuperscript{16} Once we normalize the effects by the sales, in Figure 8b we find that sectors in the manufacturing cluster of the automobile industry yields the largest multipliers. For example, our ranking suggests that providing a one dollar to vehicle body industries, generate 14 dollars of aggregate GDP. Although we know that the volatility of the automobile industry is large in comparison to other industries, it is typically attributed to the durability of cars (Ramey and Vine, 2006), and not the complexity of production, as this result highlights.

\textsuperscript{16} Although the correlation between ranking size and influence is not perfect here.
(a) Aggregate Response to Sector Specific Shock - Levels

(b) Aggregate Response to Sector Specific Shock - Weighted by Sales
6 Conclusion

This paper studies that the network of production links in an economy can matter substantially for the transmission of financial shocks. To make this point, we formulated an economy in the spirit of static general equilibrium models with production. We introduced financial frictions as an exogenous wedge on the purchase of intermediate inputs and show how the classic aggregation result in Cobb-Douglas economies can be adapted to incorporate those frictions. The aggregation result shows that financial frictions will show up as an efficiency and labor wedge. The exact mapping from one to the other depends on the network’s architecture and the location of shocks.

We provided several analytic examples of liquidity shocks to analyze their propagation in particular network structures. We then used data from the US I-O tables and calibrated the model. We asked what what is the multiplier that we get from the network effects. Our experiment showed a multipliers that go from 1.5 to 7 depending on the exercise we perform.

Our theory is, of course, incomplete. However, we hope that the lessons with collect in this paper will prompt new avenues to understand the transmission of financial factors during business cycles.
References


A Proofs of Propositions in Section 3

Proof of Lemma 1. We may split the firm’s problem into a dual problem. We let $q_i x_i$ denote the firm’s total expenditure on inputs, where $q_i$ is a composite cost of inputs.

$$q_i x_i \equiv \min \ell_i + \sum_{j=0}^N p_j x_{ij}, \text{ subject to } x_i = \ell_i^\alpha \left( \prod_{j=1}^N x_{ij}^{w_{ij}} \right)^{1-\alpha_i}$$

The firm maximizes profits.

$$\max p_i y_i - q_i x_i$$

subject to

$$y_i = (z_i x_i)^{\eta_i}$$

$$q_i x_i \leq \chi_i p_i y_i$$

Clearly if the financial constraint isn’t binding, then firm optimality implies that $q_i x_i = \eta_i p_i y_i$. On the other hand, if the financial constraint is binding, we have that $q_i x_i = \chi_i p_i y_i$. Therefore,

$$q_i x_i = \min \{ \chi_i, \eta_i \} p_i y_i = \min \left\{ \frac{\chi_i}{\eta_i}, 1 \right\} \eta_i p_i y_i$$

We may thus write

$$q_i x_i = \phi_i \eta_i p_i y_i \quad \text{where} \quad \phi_i = \min \left\{ \frac{\chi_i}{\eta_i}, 1 \right\}$$

The variable $\phi_i$ represents the firm’s specific wedge stemming from the financial constraint, which appears between the marginal cost of input composite and the marginal revenue. This proves part (ii) of Lemma 1.

Next, given some amount of output, the firm can solve its cost minimization problem over the individual inputs. This is given by

$$q_i x_i \equiv \min \ell_i + \sum_{j=0}^N p_j x_{ij}$$

subject to

$$x_i = \ell_i^\alpha \left( \prod_{j=1}^N x_{ij}^{w_{ij}} \right)^{1-\alpha_i}$$
The first order conditions with respect to $x_{ij}$ and $\ell_i$ are given by

\[ p_j - \lambda_i (1 - \alpha_i) w_{ij} \frac{x_i}{x_{ij}} = 0 \]
\[ 1 - \lambda_i \alpha_i \frac{x_i}{\ell_i} = 0 \]

where $\lambda_i$ is the multiplier on the constraint. Thus $p_j x_{ij} = \lambda_i (1 - \alpha_i) w_{ij} x_i$ and $\ell_i = \lambda_i \alpha_i x_i$. Summing over the expenditure on goods,

\[ q_i x_i = \ell_i + \sum_{j=1}^{N} p_j x_{ij} = \lambda_i \alpha_i x_i + \sum_{j=1}^{N} \lambda_i (1 - \alpha_i) w_{ij} x_i = \lambda_i x_i \]

We thus have that $\lambda_i = q_i$, which implies that $p_j x_{ij} = (1 - \alpha_i) w_{ij} u_i$ and $\ell_i = \alpha_i u_i$ as in part (i) of Lemma 1. QED.

**Proof of Lemma 2.** We may split the household’s problem into a dual problem. Let $u_0 = q_0 c$ be firm’s total expenditure on inputs, where $q_0$ is the cost-minimizing composite cost of inputs. The firm maximizes profits.

\[ \max \frac{c^{1-\gamma}}{1-\gamma} - \frac{\ell^{1+\epsilon}}{1+\epsilon} \]

subject to the household’s budget constraint

\[ \bar{p} c \leq \ell + \sum_{i=1}^{N} \pi_i \]

The first order conditions of this problem yields the following optimality condition

\[ \frac{c^{-\gamma}}{\ell^\epsilon} = \bar{p} \]

where $q_0$ is the effective price of consumption relative to labor. This proves part (ii) of Lemma 2.

Next we solve the household’s expenditure minimization problem.

\[ \bar{p} c = \min \sum_{j=1}^{N} p_j c_j \]
\[ \text{s.t. } c = \prod_{j=1}^{N} \frac{v_j}{c_j} \]

The household’s first order conditions are given by

\[ p_j - \lambda_0 v_j \frac{c}{c_j} = 0 \]
where $\lambda_0$ is the multiplier on the household’s constraint. Thus $p_j c_j = \lambda_0 v_j c$. Summing over the expenditure on goods,
\[
\bar{p} c = \sum_{j=1}^{N} p_j c_j = \sum_{j=1}^{N} \lambda_0 v_j c = \lambda_0 c
\]
We thus have that $\lambda_0 = \bar{p}$, which implies $p_j c_j = v_j \bar{p} c = v_j u_0$, as in part (i) of Lemma 2.

Finally, to get the ideal price index, we have that
\[
c = \prod_{j=1}^{N} \left( \frac{v_j}{p_j} \right)^{v_j} = \prod_{j=1}^{N} \left( \frac{v_j}{p_j} \right)^{v_j} \bar{p} c
\]
Therefore,
\[
\bar{p} = \prod_{j=1}^{N} \left( \frac{p_j}{v_j} \right)^{v_j}
\]
Thus, this is the ideal price index for the household. QED.

**Proof of Proposition 1.** From market clearing, we have that total revenue of a sector $g_i = p_i y_i$ must satisfy
\[
g_i = p_i c_i + \sum_{j=1}^{N} p_i x_{ji}
\]
From firm and household optimality conditions for expenditure on each good, (7) and (10), we obtain the following equation
\[
g_i = v_i u_0 + \sum_{j=1}^{N} (1 - \alpha_i) w_{ji} u_j
\]
Stacking equation (43) for each sector atop one another, we have
\[
\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} u_0 + \begin{bmatrix} 1 - \alpha_1 \\ 1 - \alpha_2 \\ \vdots \\ 1 - \alpha_N \end{bmatrix} \begin{bmatrix} w_{11} & w_{21} & \cdots & w_{N1} \\ w_{12} & w_{22} & \cdots & \vdots \\ \vdots & \vdots & \cdots & \ddots \\ w_{1N} & w_{2N} & \cdots & w_{NN} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}
\]
where $\circ$ denotes the Hadamard product (entrywise product). Next, let $(1 - \alpha)$ denote the following vector
\[
1 - \alpha = e_N - \alpha = \begin{bmatrix} 1 - \alpha_1 \\ 1 - \alpha_2 \\ \vdots \\ 1 - \alpha_N \end{bmatrix}
\]
We thus write the system of equations in (44) as
\[
g = ((1 - \alpha) e_N' \circ W') u + v u_0
\]
Thus, using market clearing and the optimality of expenditure on each good, we can relate total revenue of each firm, to the total expenditures of each sector and the expenditure of the household.

Next, we use the household’s budget constraint. Note that $\pi_i = g_i - u_i$. Household expenditures must thus satisfy the household budget constraint given by

$$u_0 = \ell + \sum_{i=1}^{N} (g_i - u_i) = \ell + e'_N (g - u)$$

Substituting this expression for $u_0$ into our market clearing equation (45), we have that

$$g = ((1 - \alpha) e'_N \circ W') u + v (\ell + e'_N (g - u))$$

Next, the from firm optimality condition (9) we have that firm expenditure satisfies $u_i = \phi_i \eta_i g_i$. Stacking this equation for each sector atop one another, we have

$$u = \begin{bmatrix} \phi_1 \eta_1 g_1 \\ \phi_2 \eta_2 g_2 \\ \vdots \\ \phi_N \eta_N g_N \end{bmatrix} = \phi \circ \eta \circ g \quad (47)$$

Substituting (47) for $u$ into (46), yields

$$g = ((1 - \alpha) e'_N \circ W') (\phi \circ \eta \circ g) + ve'_N (g - (\phi \circ \eta \circ g)) + v \ell$$

With a little bit of algebraic manipulation, we may re-write this as

$$g = \left[ (1 - \alpha) e'_N \circ W' \right] (e_N (\phi \circ \eta)' + ve_N (g - (\phi \circ \eta))) + v \ell$$

Total revenue is thus given by

$$g = \left[ I_N - ((1 - \alpha) e'_N \circ W') \circ (e_N (\phi \circ \eta)') - v (e_N - (\phi \circ \eta))' \right]^{-1} v \ell$$

where $I_N$ is the the identity matrix of size $N \times N$. This, along with (47) gives us (13) with the vector $a(\phi)$ given by

$$a(\phi) \equiv \left[ I_N - ((1 - \alpha) e'_N \circ W') \circ (e_N (\phi \circ \eta)') - v (e_N - (\phi \circ \eta))' \right]^{-1} v$$

QED.

**Proof of Proposition 2.** Substituting (7) into (5), we have that total revenue must satisfy
\[ g_i = p_i \left[ u_i z_i (\alpha_i)^{\alpha_i} \left( (1 - \alpha_i) \prod_{j=1}^{N} \left( w_{ij} \frac{p_j}{p_i} \right)^{1 - \alpha_i} \right) ^{\eta_i} \right] \]

Taking logs of both sides results in the following expression

\[ \log g_i = \log p_i + \eta_i \log z_i + \eta_i \log u_i + \eta_i \left[ \alpha_i \log \alpha_i + (1 - \alpha_i) \log (1 - \alpha_i) + (1 - \alpha_i) \sum_{j=1}^{N} w_{ij} \log (w_{ij} - \log p_j) \right] \]

Therefore

\[ \log g_i = \log p_i + \eta_i \log z_i + \eta_i \log u_i + \eta_i \left[ \kappa_i - (1 - \alpha_i) \sum_{j=1}^{N} w_{ij} \log p_j \right] \quad (48) \]

where

\[ \kappa_i \equiv \alpha_i \log \alpha_i + (1 - \alpha_i) \log (1 - \alpha_i) + (1 - \alpha_i) \sum_{j=1}^{N} w_{ij} \log w_{ij} \]

is a constant for each sector \( i \). Stacking equation (48) for each sector atop one another, we get

\[
\begin{bmatrix}
\log g_1 \\
\log g_2 \\
\vdots \\
\log g_N
\end{bmatrix} =
\begin{bmatrix}
\log p_1 \\
\log p_2 \\
\vdots \\
\log p_N
\end{bmatrix} + 
\begin{bmatrix}
\eta_1 \\
\eta_2 \\
\vdots \\
\eta_N
\end{bmatrix} \circ 
\begin{bmatrix}
\log z_1 \\
\log z_2 \\
\vdots \\
\log z_N
\end{bmatrix} + 
\begin{bmatrix}
\log u_1 \\
\log u_2 \\
\vdots \\
\log u_N
\end{bmatrix} + 
\begin{bmatrix}
\kappa_1 \\
\kappa_2 \\
\vdots \\
\kappa_N
\end{bmatrix} - 
\begin{bmatrix}
\eta_1 \\
\eta_2 \\
\vdots \\
\eta_N
\end{bmatrix} \circ 
\begin{bmatrix}
1 - \alpha_1 \\
1 - \alpha_2 \\
\vdots \\
1 - \alpha_N
\end{bmatrix} \circ 
\begin{bmatrix}
(w_{11}, w_{12}, \ldots, w_{1N}) \\
(w_{21}, w_{22}, \ldots, w_{2N}) \\
\vdots \\
(w_{N1}, w_{N2}, \ldots, w_{NN})
\end{bmatrix}
\]

We may re-write this in vector form as follows

\[ \log g = \log p + \eta \circ (\log z + \log u + \kappa) - \eta \circ (1 - \alpha) \circ (W \log p) \]

where \( \kappa \equiv (\kappa_1, \ldots, \kappa_N)^\prime \). We may now use this expression to solve for the log vector of prices \( p \),

\[ \log p - \eta \circ (1 - \alpha) \circ (W \log p) = \log g - \eta \circ (\log z + \log u + \kappa) \]

Taking logs of equation (47), we have that \( \log u = \log \phi + \log \eta + \log g \). Substituting this expression for \( \log u \) into the above equation yields

\[ \log p - \eta \circ (1 - \alpha) \circ (W \log p) = \log g - \eta \circ (\log z + \log \phi + \log \eta + \log g + \kappa) \]
With some algebraic manipulation, this becomes

$$\left[ I_N - \left( \eta \circ (1 - \alpha) e_N' \right) \circ W \right] \log p = (e_N - \eta) \circ \log g - \eta \circ (\log z + \log \phi + \log \eta + \kappa)$$

Therefore,

$$\log p = \left[ I_N - \left( \eta \circ (1 - \alpha) e_N' \right) \circ W \right]^{-1} \left[ (e_N - \eta) \circ \log g - \eta \circ (\log z + \log \phi + \log \eta + \kappa) \right]$$

We thus obtain equation (15) in Proposition 2 with the properly defined matrix $B$.

Next, we obtain the aggregate price index as follows. Taking the log of equation (11) we have that

$$\log \bar{p} = \sum_{j=1}^{N} v_j (\log p_j - \log v_j)$$

Re-writing this in vector form, gives us (17). Finally, we may solve for log gross output per sector. Given that $g_i = p_i y_i$, taking logs we have that

$$\log g_i = \log p_i + \log y_i.$$ 

It follows that the vector of sectoral output is given by $\log y = \log g - \log p$. QED.

**Proof of Equations (19) and (20).** Real GDP in this economy is simply household consumption. Household consumption satisfies the following budget constraint

$$\bar{p} c = \ell + \sum_{i=1}^{N} \pi_i$$

Therefore, real GDP is given by

$$c = \bar{p}^{-1} (\ell + e_N' (g - u))$$

which corresponds with equation (19). Value added for each sector is sectoral sales minus input costs, aside from labor, divided by the aggregate price level:

$$\mu_i = \bar{p}^{-1} (g_i - (u_i - \ell_i))$$

It’s clear that aggregating over value added of all sectors results in aggregate consumption (i.e. GDP).

$$\sum_{i=1}^{N} \mu_i = \frac{1}{\bar{p}} \left[ \sum_{i=1}^{N} (g_i - u_i) + \sum_{i=1}^{N} \ell_i \right] = c$$
Next, substituting in from (13) for \( g \) and \( u \) into equation (19) for real GDP, we have that

\[
c = \bar{p}^{-1} \left( \ell + \mathbf{e}'_N \left( \mathbf{a}(\phi) \ell - \left( \phi \circ \eta \circ \mathbf{a}(\phi) \right) \ell \right) \right)
\]

With a little bit of algebraic manipulation we may re-write this as

\[
c = \bar{p}^{-1} \left( 1 + \mathbf{e}'_N \left( \left( \mathbf{e}_N - \left( \phi \circ \eta \right) \circ \mathbf{a}(\phi) \right) \right) \ell \right)
\]

Therefore, real GDP satisfies (20) where the scalar \( \psi(\phi) \) is as defined in (21). QED.

**Proof of Theorem 1.** We first obtain an expression for the ideal price index in terms of sectoral sales. From (11) we have that the aggregate price index satisfies

\[
\log \bar{p} = \mathbf{v}' \log \mathbf{p} - \mathbf{v}' \log \mathbf{v} 
\]

(49)

where the vector of sectoral prices is given by

\[
\log \mathbf{p} = \mathbf{B} \left[ (\mathbf{e}_N - \eta) \circ \log \mathbf{g} - \eta \circ (\log \mathbf{z} + \log \phi + \log \eta + \kappa) \right]
\]

(50)

With a little bit of algebraic manipulation, the vector of sectoral prices may be re-written as

\[
\log \mathbf{p} = \left( (\mathbf{e}_N (\mathbf{e}_N - \eta)) \circ \mathbf{B} \right) \log \mathbf{g} - \mathbf{B} \left( \eta \circ (\log \mathbf{z} + \log \phi + \log \eta + \kappa) \right)
\]

(51)

Next, taking logs of equation (20) and substituting in our expression for the log price index from (49), we have that the log of GDP must satisfy

\[
\log GDP = -\mathbf{v}' \log \mathbf{p} + \mathbf{v}' \log \mathbf{v} + \log \left( 1 + \psi(\phi) \right) + \log \ell.
\]

Next, using (51) to replace the vector of sectoral prices in the above equation, gives us the following

\[
\log GDP = -\mathbf{v}' \left( (\mathbf{e}_N (\mathbf{e}_N - \eta)) \circ \mathbf{B} \right) \log \mathbf{g} + \mathbf{v}' \mathbf{B} \left( \eta \circ (\log \mathbf{z} + \log \phi + \log \eta + \kappa) \right)
\]

\[
+ \log \ell + \log \left( 1 + \psi(\phi) \right) + \mathbf{v}' \log \mathbf{v}
\]

Finally, from Proposition 1 we have that \( \log \mathbf{g} = \log \mathbf{a}(\phi) + \mathbf{e}_N \log \ell \). Substituting this in for \( \log \mathbf{g} \) in the above expression yields

\[
\log GDP = -\mathbf{v}' \left( (\mathbf{e}_N (\mathbf{e}_N - \eta)) \circ \mathbf{B} \right) \left( \log \mathbf{a}(\phi) + \mathbf{e}_N \log \ell \right)
\]

\[
+ \mathbf{v}' \mathbf{B} \left( \eta \circ (\log \mathbf{z} + \log \phi + \log \eta + \kappa) \right) + \log \ell
\]

\[
+ \log \psi(\phi) + \mathbf{v}' \log \mathbf{v}
\]
Collecting terms,

\[
\log GDP = -v' \left( (e_N (e_N - \eta)' \circ B) \log a (\phi) + [1 - v' \left( (e_N (e_N - \eta)' \circ B) e_N \right) \log \ell \right. \\
+ v' \left( B (\eta \circ (\log z + \log \phi + \log \eta + \kappa)) + \log (1 + \psi (\phi)) + v' \log v \right)
\]

We may thus reduce the above equation as follows

\[
\log GDP = -d \log a (\phi) + (1 - de_N) \log \ell \\
+ ((v'B) \circ \eta') (\log z + \log \phi) \\
+ \log (1 + \psi (\phi)) + K
\]

where

\[
d = v' \left( (e_N (e_N - \eta)' \circ B) \right) \\
K = v' B (\eta \circ (\log \eta + \kappa)) + v' \log v
\]

QED.

**Proof of Theorem 2.** First, in the household’s optimality condition (30) between consumption and labor, we substitute in our expression for the household’s consumption from equation (20). This gives us

\[
\frac{(\bar{p}^{-1} (1 + \psi (\phi)) \ell^{-\gamma}}{\ell^\epsilon} = \bar{p}
\]

Rearranging, we have that equilibrium labor must satisfy

\[
\ell = \bar{p}^{\frac{1}{\epsilon+\gamma}} (1 + \psi (\phi))^{-\frac{\gamma}{\epsilon+\gamma}}
\]

Taking logs of both sides of this equation yields

\[
\log \ell = - \left( \frac{1 - \gamma}{\epsilon + \gamma} \right) \log \bar{p} - \frac{\gamma}{\epsilon + \gamma} \log (1 + \psi (\phi))
\]

Letting \( \beta = \frac{1 - \gamma}{\epsilon + \gamma} \), we may re-write this as

\[
\log \ell = -\beta \log \bar{p} - \frac{\gamma}{\epsilon + 1} (1 + \beta) \log (1 + \psi (\phi)). \tag{52}
\]

Using our expression (49) for the price index, we may re-express (52) as

\[
\log \ell = -\beta (v' \log p - v' \log v) - \frac{\gamma}{\epsilon + 1} (1 + \beta) \log (1 + \psi (\phi))
\]
Furthermore combining this with equation (51) for the sectoral price vector, we get
\[ \log \ell = -\beta \mathbf{v}' \left( (\mathbf{e}_N (\mathbf{e}_N - \mathbf{\eta})') \circ \mathbf{B} \right) \log \mathbf{g} - \mathbf{B} \log \mathbf{z} - \mathbf{B} \left( \mathbf{\eta} \circ (\log \phi + \log \mathbf{\eta} + \mathbf{\kappa}) \right) \]
\[ + \beta \mathbf{v}' \log \mathbf{v} - \frac{\gamma}{\epsilon + 1} (1 + \beta) \log (1 + \psi (\phi)) \]

Finally, substituting \( \log \mathbf{g} = \log a(\phi) + \mathbf{e}_N \log \ell \) into the above expression gives us
\[ \log \ell = -\beta \mathbf{v}' \left( (\mathbf{e}_N (\mathbf{e}_N - \mathbf{\eta})') \circ \mathbf{B} \right) (\log a(\phi) + \mathbf{e}_N \log \ell) + \beta \mathbf{v}' \mathbf{B} \log \mathbf{z} \]
\[ + \beta \mathbf{v}' \mathbf{B} \left( \mathbf{\eta} \circ (\log \phi + \log \mathbf{\eta} + \mathbf{\kappa}) \right) + \beta \mathbf{v}' \log \mathbf{v} - \frac{\gamma}{\epsilon + 1} (1 + \beta) \log (1 + \psi (\phi)) \]

Collecting terms,
\[ (1 + \beta \mathbf{v}' \left( (\mathbf{e}_N (\mathbf{e}_N - \mathbf{\eta})') \circ \mathbf{B} \right) \mathbf{e}_N) \log \ell = -\beta \mathbf{v}' \left( (\mathbf{e}_N (\mathbf{e}_N - \mathbf{\eta})') \circ \mathbf{B} \right) \log a(\phi) + \beta \mathbf{v}' \mathbf{B} \log \mathbf{z} \]
\[ + \beta \mathbf{v}' \mathbf{B} \left( \mathbf{\eta} \circ (\log \phi + \log \mathbf{\eta} + \mathbf{\kappa}) \right) + \beta \mathbf{v}' \log \mathbf{v} - \frac{\gamma}{\epsilon + 1} (1 + \beta) \log (1 + \psi (\phi)) \]

Next, using our definitions of \( q_z, q_\phi, d \) and \( K \) in Theorem 1, we rewrite the above expression as follows
\[ (1 + \beta \mathbf{d} \mathbf{e}_N) \log \ell = -\beta \mathbf{d} \log a(\phi) + \beta q_z \log \mathbf{z} + \beta q_\phi \log \phi \]
\[ + \beta K - \frac{\gamma}{\epsilon + 1} (1 + \beta) \log (1 + \psi (\phi)) \]

If we then divide through by the scalar \( (1 + \beta \mathbf{d} \mathbf{e}_N) \) we obtain the following expression for equilibrium labor
\[ \log \ell = \frac{\beta}{1 + \beta \mathbf{d} \mathbf{e}_N} \left[ q_z \log \mathbf{z} + q_\phi \log \phi - \mathbf{d} \log a(\phi) + K \right] - \frac{\gamma}{\epsilon + 1} \frac{1}{1 + \beta \mathbf{d} \mathbf{e}_N} \log (1 + \psi (\phi)) \]

This coincides with equation (32) in the theorem with the scalars \( \Lambda_q \) and \( \Lambda_\psi \) defined in (34).

Finally, we may then substitute (53) into the aggregate production function (22). This results in the following expression for aggregate output
\[ \log GDP = \left( 1 + \beta \frac{1 - \mathbf{d} \mathbf{e}_N}{1 + \beta \mathbf{d} \mathbf{e}_N} \right) \left[ q_z \log \mathbf{z} + q_\phi \log \phi - \mathbf{d} \log a(\phi) + K \right] \]
\[ + \left( 1 - \frac{\gamma}{\epsilon + 1} \frac{1 - \mathbf{d} \mathbf{e}_N}{1 + \beta \mathbf{d} \mathbf{e}_N} \right) \log (1 + \psi (\phi)) \]

This coincides with equation (31) in the theorem with scalars \( \Gamma_q \) and \( \Gamma_\psi \) defined in (33). QED.
Proof of Lemma 4. \( \gamma \in (0, 1) \) implies \( \beta > 0 \). All entries of the vector \( d \) are necessarily positive. Together, this implies

\[
\Gamma_q = \frac{1 + \beta}{1 + \beta d e_N} > 0 \quad \text{and} \quad \Lambda_q = \frac{\beta}{1 + \beta d e_N} > 0
\]

QED.

Proof of Proposition 4. From our definition (37) we have that the labor wedge satisfies

\[
(1 - \tau(\phi)) = \bar{\eta}^{-1} \frac{\ell^{1+\epsilon}}{c^{1-\gamma}}
\]

Taking logs of both sides gives

\[
\log (1 - \tau(\phi)) = (1 + \epsilon) \log \ell - (1 - \gamma) \log e + \text{const}
\]

Next we substitute in our expressions of equilibrium GDP and labor from (31) and (32). This yields

\[
\log (1 - \tau(\phi)) = (1 + \epsilon) [\Lambda_q [q \log z + q \log \phi - d \log a(\phi) + K] - \Lambda_\psi \log (1 + \psi(\phi))] \\
- (1 - \gamma) [\Gamma_q [q \log z + q \log \phi - d \log a(\phi) + K] + \Gamma_\psi \log (1 + \psi(\phi))] - \log \bar{\eta}
\]

Collecting terms, we have that

\[
\log (1 - \tau(\phi)) = ((1 + \epsilon) \Lambda_q - (1 - \gamma) \Gamma_q) [q \log z + q \log \phi - d \log a(\phi) + K] \\
- ((1 + \epsilon) \Lambda_\psi + (1 - \gamma) \Gamma_\psi) \log (1 + \psi(\phi)) - \log \bar{\eta}
\]

Note that

\[
(1 + \epsilon) \Lambda_q - (1 - \gamma) \Gamma_q = 0
\]

and

\[
(1 + \epsilon) \Lambda_\psi + (1 - \gamma) \Gamma_\psi = 1
\]

As a result,

\[
\log (1 - \tau(\phi)) = - \log (1 + \psi(\phi)) - \log \bar{\eta}
\]

as in (38). QED.

B Appendix B. Proofs of Propositions in Section 4

Proof for the Vertical Economy. We may compute the influence vector from equation (26)
\[ q = v' \left[ I_N - \left( (1 - \alpha) e_N' \right) \circ W \right]^{-1} \]

\[
= (0,0,1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \\
= (0,0,1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = (0,0,1) \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}
= (0,0,1) \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = (1,1,1)
\]

Therefore, the influence vector in this economy is given by

\[ q = (1,1,1) \]

As for the profits distortion function, \( \psi(\phi) \), we have that

\[ \psi(\phi) \equiv e_N' \left( (e_N - (\phi \circ \eta)) \circ a(\phi) \right) \]

\[ = (1,1,1) \begin{bmatrix} 1 - \phi_1 \\ 1 - \phi_2 \\ 1 - \phi_3 \end{bmatrix} \circ a(\phi) \]
where \( a(\phi) \) is given by

\[
a(\phi) = [I_N - ((1 - \alpha) e'_N \circ W') \circ (e_N (\phi \circ \eta)) - v (e_N - (\phi \circ \eta))]^{-1} v
\]

\[
= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \circ \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1 & \phi_2 & \phi_3 \\ \phi_1 & \phi_2 & \phi_3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 - \phi_1 & 1 - \phi_2 & 1 - \phi_3 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 - \phi_1 & 1 - \phi_2 & 1 - \phi_3 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 1 & 0 & 0 \\ -\phi_1 & 1 & 0 \\ \phi_1 - 1 & -1 & \phi_3 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \phi_1 & 1 & 0 \\ 1/\phi_3 & 1/\phi_3 & 1/\phi_3 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Therefore

\[
\psi(\phi) = e'_N ((e_N - (\phi \circ \eta)) \circ a(\phi))
\]

\[
= (1, 1, 1) \begin{pmatrix} 1 - \phi_1 \\ 1 - \phi_2 \\ 1 - \phi_3 \end{pmatrix} \circ a(\phi)
\]

\[
= (1, 1, 1) \begin{pmatrix} 1 - \phi_1 \\ 1 - \phi_2 \\ 1 - \phi_3 \end{pmatrix} \circ \begin{pmatrix} 0 \\ 0 \\ 1/\phi_3 \end{pmatrix}
\]

\[
= (1, 1, 1) \begin{pmatrix} 0 \\ 0 \\ 1/\phi_3 \end{pmatrix}
\]

\[
= \frac{1 - \phi_3}{\phi_3} = \frac{1}{\phi_3} - 1
\]

Therefore

\[
1 + \psi(\phi) = 1 + \frac{1}{\phi_3} - 1
\]

\[
1 + \psi(\phi) = \frac{1}{\phi_3}
\]

decreasing in \( \phi_3 \).
Proof for the Horizontal Economy. We may compute the influence vector from equation (26)

\[
q = v' \left[ I_N - (1 - \alpha) e_N' \circ W \right]^{-1}
\]

\[
= (1/3, 1/3, 1/3) \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right]^{-1}
\]

\[
= (1/3, 1/3, 1/3) \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]^{-1} = (1/3, 1/3, 1/3) \left[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]
\]

\[
= (1/3, 1/3, 1/3)
\]

and the psi function is given by

\[
\psi(\phi) \equiv e_N' ((e_N - (\phi \circ \eta)) \circ a(\phi))
\]

\[
= (1, 1, 1) \left( \begin{bmatrix} 1 - \phi_1 \\ 1 - \phi_2 \\ 1 - \phi_3 \end{bmatrix} \circ a(\phi) \right)
\]

where

\[
a(\phi) = \left[ I_N - ((1 - \alpha) e_N' \circ W') \circ (e_N (\phi \circ \eta)') - v (e_N - (\phi \circ \eta)) \right]^{-1} v
\]

\[
a(\phi) = \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 - \phi_1 & 1 - \phi_2 & 1 - \phi_3 \\ 1 - \phi_1 & 1 - \phi_2 & 1 - \phi_3 \\ 1 - \phi_1 & 1 - \phi_2 & 1 - \phi_3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}
\]

\[
= \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 - \phi_1 & 1 - \phi_2 & 1 - \phi_3 \\ 1 - \phi_1 & 1 - \phi_2 & 1 - \phi_3 \\ 1 - \phi_1 & 1 - \phi_2 & 1 - \phi_3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}
\]

\[
= \left( \begin{bmatrix} 1 - \frac{1}{3} (1 - \phi_1) & -\frac{1}{3} (1 - \phi_2) & -\frac{1}{3} (1 - \phi_3) \\ -\frac{1}{3} (1 - \phi_1) & 1 - \frac{1}{3} (1 - \phi_2) & -\frac{1}{3} (1 - \phi_3) \\ -\frac{1}{3} (1 - \phi_1) & -\frac{1}{3} (1 - \phi_2) & 1 - \frac{1}{3} (1 - \phi_3) \end{bmatrix} \right)^{-1} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}
\]
where
\[
\begin{bmatrix}
1 - \frac{1}{3}(1 - \phi_1) & -\frac{1}{3}(1 - \phi_2) & -\frac{1}{3}(1 - \phi_3) \\
-\frac{1}{3}(1 - \phi_1) & 1 - \frac{1}{3}(1 - \phi_2) & -\frac{1}{3}(1 - \phi_3) \\
-\frac{1}{3}(1 - \phi_1) & -\frac{1}{3}(1 - \phi_2) & 1 - \frac{1}{3}(1 - \phi_3)
\end{bmatrix}^{-1} = \frac{1}{1 - \frac{1}{3}[(1 - \phi_3) + (1 - \phi_2) + (1 - \phi_1)]}
\times
\begin{bmatrix}
1 - \frac{1}{3}[(1 - \phi_3) + (1 - \phi_2)] & \frac{1}{3}(1 - \phi_2) \\
\frac{1}{3}(1 - \phi_1) & 1 - \frac{1}{3}[(1 - \phi_3) + (1 - \phi_1)] \\
\frac{1}{3}(1 - \phi_1) & \frac{1}{3}(1 - \phi_2)
\end{bmatrix}
\]

thus
\[
a(\phi) = \frac{1}{1 - \frac{1}{3}[(1 - \phi_3) + (1 - \phi_2) + (1 - \phi_1)]}
\begin{bmatrix}
1 - \frac{1}{3}[(1 - \phi_3) + (1 - \phi_2)] & \frac{1}{3}(1 - \phi_2) \\
\frac{1}{3}(1 - \phi_1) & 1 - \frac{1}{3}[(1 - \phi_3) + (1 - \phi_1)] \\
\frac{1}{3}(1 - \phi_1) & \frac{1}{3}(1 - \phi_2)
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

therefore
\[
\psi(\phi) \equiv e_N^t ((e_N - (\phi \circ \eta)) \circ a(\phi))
\]
\[
= (1, 1, 1) \begin{bmatrix}
1 - \phi_1 \\
1 - \phi_2 \\
1 - \phi_3
\end{bmatrix} \circ a(\phi)
\]
\[
= \frac{1}{1 - \frac{1}{3}[(1 - \phi_3) + (1 - \phi_2) + (1 - \phi_1)]}
\begin{bmatrix}
1 - \phi_1 \\
1 - \phi_2 \\
1 - \phi_3
\end{bmatrix}
\begin{bmatrix}
1, 1, 1
\end{bmatrix}
\]
\[
\psi(\phi) = \frac{1}{1 - \frac{1}{3}[(1 - \phi_3) + (1 - \phi_2) + (1 - \phi_1)]}
\frac{1}{1 - \frac{1}{3}[(1 - \phi_3) + (1 - \phi_2) + (1 - \phi_1)]}
\]

which implies
\[
1 + \psi(\phi) = \frac{1}{1 - \frac{1}{3}[(1 - \phi_3) + (1 - \phi_2) + (1 - \phi_1)]}
\]

**Proof for the Hybrid Economy.** Again we may compute the influence vector from equation (26)
\[ q = v'[I_N - ((1 - \alpha) e'_N) \circ W]^{-1} \]

\[ = (0, 0, 1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 - \alpha_2 & 1 - \alpha_2 & 1 - \alpha_2 \\ 1 - \alpha_3 & 1 - \alpha_3 & 1 - \alpha_3 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \]

\[ = (0, 0, 1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 - \alpha_2 & 0 & 0 \\ 0 & 1 - \alpha_3 & 0 \end{bmatrix} \]

\[ = (0, 0, 1) \begin{bmatrix} 1 - (1 - \alpha_2) & 1 & 0 \\ 0 & - (1 - \alpha_3) & 1 \end{bmatrix} \]

\[ = (0, 0, 1) \begin{bmatrix} 1 & 0 & 0 \\ 1 - \alpha_2 & 1 & 0 \\ (1 - \alpha_2)(1 - \alpha_3) & (1 - \alpha_3) & 1 \end{bmatrix} \]

And the psi function is given by

\[ \psi(\phi) \equiv e'_N((e_N - (\phi \circ \eta)) \circ a(\phi)) \]

\[ = (1, 1, 1) \begin{bmatrix} 1 - \phi_1 \\ 1 - \phi_2 \\ 1 - \phi_3 \end{bmatrix} \circ a(\phi) \]

where
\[
a(\phi) = \left[ I_N - ((1 - \alpha) e_N' \circ W') \circ (e_N (\phi \circ \eta')) - v (e_N - (\phi \circ \eta')) \right]^{-1} v
\]

\[
= \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} - \begin{bmatrix}
0 & 0 & 0 \\
1 - \alpha_2 & 0 & 0 \\
0 & 1 - \alpha_3 & 0
\end{bmatrix} \circ \begin{bmatrix}
\phi_1 & \phi_2 & \phi_3 \\
\phi_1 & \phi_2 & \phi_3 \\
\phi_1 & \phi_2 & \phi_3
\end{bmatrix} - \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} \begin{bmatrix}
1 - \phi_1 & 1 - \phi_2 & 1 - \phi_3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} - \begin{bmatrix}
0 & 0 & 0 \\
(1 - \alpha_2) \phi_1 & 0 & 0 \\
0 & (1 - \alpha_3) \phi_2 & 0
\end{bmatrix} \circ \begin{bmatrix}
\phi_1 & \phi_2 & \phi_3 \\
\phi_1 & \phi_2 & \phi_3 \\
\phi_1 & \phi_2 & \phi_3
\end{bmatrix} - \begin{bmatrix}
0 \\
0 \\
1 - \phi_1 & 1 - \phi_2 & 1 - \phi_3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 \\
- (1 - \alpha_2) \phi_1 & 1 & 0 \\
0 & - (1 - \alpha_3) \phi_2 & 1
\end{bmatrix} - \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 - \phi_1 & 1 - \phi_2 & 1 - \phi_3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 \\
- (1 - \alpha_2) \phi_1 & 1 & 0 \\
- (1 - \phi_1) & - (1 - \alpha_3) \phi_2 & - (1 - \phi_2) & 1 - (1 - \phi_3)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 \\
- (1 - \alpha_2) \phi_1 & 1 & 0 \\
\phi_1 - 1 & \alpha_3 \phi_2 - 1 & \phi_3
\end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\frac{1}{\phi_4} & 0 & 1
\end{bmatrix}
\]

Thus

\[
\psi(\phi) = (1, 1, 1) \begin{bmatrix}
1 - \phi_1 \\
1 - \phi_2 \\
1 - \phi_3
\end{bmatrix} \circ \begin{bmatrix}
0 \\
0 \\
1/\phi_3
\end{bmatrix}
\]

\[
= (1, 1, 1) \begin{bmatrix}
0 \\
0 \\
\frac{1 - \phi_3}{\phi_3}
\end{bmatrix} = \frac{1 - \phi_3}{\phi_3} = \frac{1}{\phi_3} - 1
\]

Therefore

\[
1 + \psi(\phi) = \frac{1}{\phi_3}
\]
C Appendix D. Microfoundations for the Financial Constraint

Furthermore, we now discuss a possible interpretation of the financial constraint imposed on these firms. We consider some multiple interpretations to the model that capture the essence of the financial frictions in the model. This will allow us to decompose the constraint of the firm into limited enforcement constraints and higher financial costs.

Suppose a firm faces a limited enforcement constraint. Firm $i$ maximizes profits,

$$\Pi_i = \max_{\sigma_i, x_i} \pi_i y_i - \bar{p}_i x_i$$

subject to

$$y_i = z_i x_i^{\alpha_i}$$

$$(1 - \sigma_i) \bar{p}_i x_i \leq w_i$$

$$(1 - \theta_i) p_i y_i \leq p_i y_i - \sigma_i \bar{p}_i x_i.$$  \hfill (54)  \hfill (55)

In the expression above $\bar{p}_i$ is the marginal cost of $x_i$ which by the assumption that $\sum_{j \in N_i} \alpha_{ij} = 1$, is constant and independent of $x_i$. The first constraint is the technological constraint of the firm. The second constraint states that a fraction $(1 - \sigma_i)$, chosen by the firm, must be paid up-front with an liquid funds $w_i$, where $w_i$ is given exogenously. Thus, $\sigma_i \bar{p}_i x_i$ is the amount of trade credit obtained from suppliers. Changes in the availability of these liquid funds are the focus of interest in this paper.

In addition we assume, following the contracting literature, that a firm may pledge at most $\theta_i$ of their revenue to pay their suppliers. If the firm chooses to default on its suppliers, they lose a fraction $\theta_i$ of the firm’s income. Hence, upon default, the firm keeps the fraction $(1 - \theta_i) p_i y_i$. Thus, the third constraint is an incentive constraint that states that the fraction of output they get to keep should they choose to default on suppliers must exceed the revenue firm’s expect to make minus the amount it owes after it pays for part of its inputs with liquid funds.

By rearranging this constraint, we obtain an equivalent constraint, $\sigma_i \bar{p}_i x_i \leq \theta_i p_i y_i$. This one reads that the amount that the firm can owe to its suppliers after it paid from a certain fraction in advance must not exceed the pledgeable amount of output, $\theta_i p_i y_i$.

In the specific examples that follow, we let the liquid funds to be some proportion of the firms output $w_i = \omega_i p_i y_i$. Then,

$$ (1 - \sigma_i) c_i x_i \leq \omega_i p_i y_i \hfill (56)$$

$$ c_i x_i \leq \theta_i p_i y_i \hfill (57)$$
so combining both constraints,

$$\omega_i + \theta_i = \chi_i$$

so $\theta_i$ is the fraction of trade credit and $\omega_i$ is the working capital of the firm.

D Appendix D. Construction of Input-Output Tables

Construction of Input-Output Tables. We work with the Use of Commodities by Industries After Redefinitions (Uses AR). The USES AR tables reported by the BEA are organized in the following way: Every row corresponds to commodities and every column to an industry or component of aggregate demand (consumption, investment, etc). An entry of that table reports the expenditures on commodities in the corresponding row by the industry at the corresponding column. for every commodity produced in every industry in billions of US dollars. We use this tables to reconstruct the Input-Output table of the US, but its construction of this tables merits some discussion.

The NIPA categories are based on attributes of the goods and services (commodities) produced by an industry. The Uses AR table is constructed from another table called the the Uses of Commodities by Industries Before Redefinitions (Uses BR). The data in the Uses BR table is constructed from firm level data. However, because several firms produce goods that fall into different product categories. The calibration of Input Output models requires symmetry. The Uses AR reorganizes the data in the Uses BR to reassigns a portion the commodity uses of a given industry to another industry when the firms in the former also produce the product of latter.

Not all commodities correspond to the industries at the five digit level. To construct a symmetric I-O table, we assign the uses of Secondary smelting and alloying of aluminum (331314) sector to merge Alumina refining and primary aluminum production (33131a). We do the same for the Federal electric utilities (S00101) sector which is allocated to Federal general government (nondefense) (S00600) sector and for State and local government passenger transit (S00201) and State and local government electric utilities (S00202) which are both allocated to State and local general government (S00203). Finally, the input output tables also report entries for used goods and scrap. We reassign these rows across all other rows according to their proportion.

Treatment of Finance, Insurance, and Real-Estate Financing (FIRE). We treat the Finance, insurance, real estate, rental, and leasing (FIRE) sectors as a special sector. The reason is that the I-O tables treat the financial sector as an additional production input. The Uses table reports transfers from each sector to the FIRE industries. However, we do not want to treat financial, insurance and real estate payments as intermediate production inputs. The reason for this that we interpret production function as related to the physical of production: if a given sector purchases a given volume of production inputs, production would be the same. In contrast, higher interest rates, insurance premia, or rental rates will only affect the the distribution of the value
Table 4: FIRE Sectors

<table>
<thead>
<tr>
<th>NIPA</th>
<th>Sector Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>52A000</td>
<td>Monetary authorities and depository credit intermediation</td>
</tr>
<tr>
<td>522A00</td>
<td>Nondepository credit intermediation and related activities</td>
</tr>
<tr>
<td>523A00</td>
<td>Securities and commodity contracts intermediation and brokerage</td>
</tr>
<tr>
<td>523900</td>
<td>Other financial investment activities</td>
</tr>
<tr>
<td>524100</td>
<td>Insurance carriers</td>
</tr>
<tr>
<td>524200</td>
<td>Insurance agencies, brokerages, and related activities</td>
</tr>
<tr>
<td>525000</td>
<td>Funds, trusts, and other financial vehicles</td>
</tr>
<tr>
<td>529000</td>
<td>Real estate</td>
</tr>
<tr>
<td>531000</td>
<td>Automotive equipment rental and leasing</td>
</tr>
<tr>
<td>532100</td>
<td>Consumer goods and general rental centers</td>
</tr>
<tr>
<td>532400</td>
<td>Commercial and industrial machinery and equipment rental and leasing</td>
</tr>
<tr>
<td>532500</td>
<td>Lessors of nonfinancial intangible assets</td>
</tr>
</tbody>
</table>

added of each sector —the portion going to the capital share and the rest to their financing. Thus, we assign the commodity share of the FIRE sectors as part of the capital share of production. In turn, the purchases of the financial sector are considered as part of final uses. To avoid double counting, we don’t include this share in GDP accounting. We will use the flows from industries to the financial sector when we investigate sectoral wedges.

At the five digit level, these industries correspond to:

**Treatment of Exports and Imports.** We work with a closed economy model. The IO-tables include a foreign sector. However, even at five digits, most inputs used in the US are already produced within the US. In fact, most commodities in the US are both imported and produced domestically, and used as inputs and final goods. There are some commodities that are imported but For all commodities except oil, that are produced in the US, final uses exceed the imports. Thus, we treat all imports as if they are consumed by households —in the case of oil, we assume ignore oil imports. In the calibration, we leave aside issued of foreign trade. We will think of households as purchasing US produced goods, exporting some of these and importing the rest for final consumption only. Thus, we treat all US production as final uses. When we consider the household’s expenditure shares on domestic production, we exclude imports but include exports as part of the shares.

**Treatment of Inventories.** Part of the final uses of a given sector are considered inventory that will be used in final periods. Our model is static so there is no inventory accumulation. In practice, inventories can be stored as final goods, or as materials. To account for inventories, we compute the proportion of intermediate and final goods use production excluding inventories. Then, we distribute a given year’s inventories according to the intermediate and final use weights, and add those amounts to the intermediate and final use production of a given year.
E Additional Graphs

Figure 9: Regression Fit - Expenditure Shares and $\phi$ vs. Sectoral Output