Notes 10: An Equation Based Model of the Macroeconomy

In this note, I am going to provide a simple mathematical framework for 8 of the 9 major curves in our class (excluding only the labor supply curve). I am doing this not because I believe that these equations are the true representation of the U.S. macroeconomy with respect to functional form. Instead, I am doing it to formalize the intuition that we have already built. There is nothing new in what I am doing in this note. I am just putting simple functional forms on all the curves we already built. The math will just reinforce the intuition that we have already developed (that is exactly what math is supposed to do). Some of you are not comfortable thinking in terms of mathematical equations (you prefer the intuition that we have already built). For others, the math can help to solidify the intuition. By doing this, we can be very specific as to what shifts each curve, why curves slope the way they do, and how the curves related to each other. I will start by putting some functional forms on the demand side curves (particularly the money demand curve - defined below). I will then try to tackle the supply side curves (we have already done a little of this given our assumptions on Cobb-Douglas production functions earlier in the class).

Demand Side Curves

The demand side curves in our class are the curves that are directly related to aggregate demand. Those curves include: the money demand curve, the money supply curve, the LM curve, the IS curve, and the aggregate demand (AD) curve. The money market curves (money demand, money supply, LM) provide the equilibrium interest rate which determines investment.

Real Money Supply

**Real Money Supply** \[ \frac{M}{P} \]  \hspace{1cm} (Equation 1: Real Money Supply)

The intuition behind the curve is that the Fed chooses the money supply as a policy variable (M) and then prices react to changes in the economy (we have a model of P). So, the real money supply is just M/P.

Real Money Demand

Our real money demand curve was expressed as \[ \frac{M}{P} = f(r, Y). \] Let’s put a functional form on our real money demand curve in the following way:

**Real Money Demand** \[ \alpha_1 Y - \alpha_2 r \]  \hspace{1cm} (Equation 2: Real Money Demand)

What have I done? I have just assumed a linear relationship for the money demand curve in term of the determinants of money demand (which are Y and r). In the real world, there is nothing that says that the money demand curve is a linear relationship of Y and r. I have just done this to put a little structure on the curve to provide some intuition. Note: I have set expected inflation equal to zero (such that r = i). I have only done this for simplicity (I am not talking about the expected inflation effect on money demand this year). You could put it back into the money demand curve if you like.

What are the parameters of the money demand curve? \( \alpha_1 \) represents the size of the transaction demand for money. The higher \( \alpha_1 \), the more that increases in Y increases the demand for money. This
is just the intuition we have developed the last few weeks. Agents need money to transact. The money that people want to transact (i.e., the higher are aggregate expenditures, $Y$, the higher is the demand for money).

$\alpha_2$ represents the interest elasticity of money demand. As interest rates go up, holding money (as opposed to another asset such as a bond) gets more expensive. Just like what we have done in class the last few weeks, an increase in $Y$ increases money demand and an increase in $r$ lowers money demand. Higher expenditures ($Y$) means higher demand for money (higher transaction demand of money). Higher interest rates mean lower demand for money (higher opportunity cost of holding money). To reiterate, the way I have expressed things, both $\alpha_1$ and $\alpha_2$ are positive.

**The LM curve**

In equilibrium, money supply (the left hand side of the below equation) must equal money demand (the right hand side of the below equation). Let’s do that now.

$$\frac{M}{P} = \alpha_1 Y - \alpha_2 r$$

Let’s do just a little algebra and solve this equation for $r$ (put only $r$ on the left hand side). Doing so yields the following equation (to get this, just rearrange the above equation):

$$r = \left( \frac{\alpha_1}{\alpha_2} \right) Y - \left( \frac{1}{\alpha_2} \right) \left( \frac{M}{P} \right)$$

(Equation 3: LM curve)

This curve is the LM curve! As $Y$ goes up, $r$ goes up (holding $M$ and $P$ constant). What is the rational for this? It is the transaction demand for money. An increase in $Y$ increases money demand driving up interest rates (holding real money supply constant). The way we have expressed the LM curve, we have two components – the part governing the slope of the LM curve and the intercept of the LM curve.

$$r = \left( \frac{\alpha_1}{\alpha_2} \right) Y - \left( \frac{1}{\alpha_2} \right) \left( \frac{M}{P} \right)$$

(slope) (intercept)

An increase in $M$, all else equal, lowers the intercept of the LM curve (there is a negative sign in front of the $M/P$). We represent this as a shift to the right.
This is exactly what we have been doing all year. An increase in M shifts the LM curve right (down). What is the intuition for this? An increase in M (holding P constant) increases M/P. In the money market (holding the transaction demand of money (Y) constant), this will lower interest rates. That is what the math says and that is what the intuition says. Again, the math is only there to put structure on our intuition. Likewise, a fall in P will also shift the LM curve to the right (shift the intercept down).

To reiterate, the LM curve represents the relationship that money demand = money supply. That is why I repeatedly say in class: “everywhere along the LM curve, the money market is in equilibrium”. To make the LM curve, we just set money demand equal to money supply. In other words, to make the LM curve we impose the equilibrium condition that money demand equals money supply.

The IS Curve

The IS curve is just the graphical representation of \( Y = C + I + G \) (in \( \{Y,r\} \) space). We have models for each of the components.

\[ C = C_0 = C_{PVLR, \text{Consumer Confidence,Taxes,Liquidity Contraints—current Y,type of consumers}} \]

Note: Any time I write \( Z(.) \) for some variable \( Z \), that just means that \( Z \) is a function of other things (some of which are exogenous and some of which are determined by our model).

Consumption is determined by other variables. What determines consumption (this is our lecture from topic 3)? If we have permanent income consumers as specified in class, \( C = \frac{PVLR_{after \ tax}}{LL} \).

As PVLR increases, \( C \) increases. As labor income taxes increase (permanently), \( C \) will fall. As discussed earlier in the class, only permanent changes in PVLR (or taxes) will induce changes in consumption that will be sizeable. We assume temporary changes in PVLR have zero affect on consumption (if households are non-liquidity constrained PIH and LL is large). If households are liquidity constrained, they could respond to changes in current resources (even if the changes are temporary).

Note: Consumption does NOT depend on interest rates. That is by assumption. Changes in interest rates yield both income and substitution effects that move consumption in opposite directions (for net savers, the economy as a whole is a net saver). We will just impose the empirical fact (that we observe in the data) that income effects and substitution effects cancel such that the interest elasticity of consumption equals zero.

\[ I = I_0 - d_I r = I_{(Business \ Confidence,Liquidity \ Contraints,Investment Tax \ Credits)} - d_I r \]

Investment has two components: an autonomous component (i.e., a part that does not depend on interest rates) and a component that depends on interest rates. Like we have done all term, we have already linearized the investment equation that results from the firm optimization decision equating the marginal product of capital equal to the real interest rate. The autonomous part of investment is supposed to represent the marginal product of capital (the extra benefits to the firm from investing one more dollar in a machine (capital)). Again, there is nothing truly linear about this relationship (quite the contrary, as we have seen in Topic 3, the relationship is actually quite non-linear). We are linearizing now simply for tractability in building our intuition.
\( G = G \)

(it is an exogenous variable in our class, the government just increases or decreases \( G \) when it wants).

Putting this together we get:

\[
Y = C(\cdot) + I(\cdot) + G - d_1 r
\]

(IS curve with \( Y \) on vertical-axis).

This just represents everything we have been doing in class so far. An increase in \( r \) (holding \( C(\cdot), I(\cdot), \) and \( G \) fixed) will reduce \( Y \). The reason for this is that firms will invest less as the cost of investing (the real interest rate) increases.

Even though, the above equation provides exactly the intuition we want, we have expressed the above equation as the IS curve drawn in \{\( r,Y \}\} space where \( r \) is on the vertical axis. So, to get the IS curve (that we draw in class), we just rearrange the equation by solving for \( r \). Doing so yields the following:

\[
(\text{Equation 4: IS curve)}
\]

\[
\begin{align*}
r &= \left( \frac{1}{d_1} \right) \left[ C_{\text{PVL}R, \text{Consumer Confidence, taxes, etc.}} + I_{A, \text{Business Confidence, etc.}} + G \right] - \left( \frac{1}{d_1} \right) Y \\
\text{(intercept) (everything in[ ])} &\quad \text{(slope)}
\end{align*}
\]

Notice, like the LM curve, this IS curve has both a slope and an intercept. But, remember, the causal effect goes from changing \( r \) to changing \( Y \) (as we wrote above) not from changing \( Y \) to changing \( r \). An increase in \( r \) (holding \( C(\cdot), I(\cdot), \) and \( G \) constant) will lower \( Y \).

Let’s do an example with this new IS curve. An increase in \( A \) (which increases PVL and consequently \( C(\cdot) \) and which separately increases the autonomous part of \( I(\cdot) \)) will increase the intercept of the IS curve. We represent this as a shift right (shift up) of the IS curve.

**The Aggregate Demand Curve**

Things get a little more complicated now, but only slightly so. The aggregate demand curve is generated by creating equilibrium in the IS-LM market. It is the level of economic activity in the
economy when both the money market clears (money demand = money supply) and when firms optimize (MPK = r).

To solve for the aggregate demand curve, we just equate the IS curve with the LM curve (both equal r):

\[
\left(\frac{\alpha_1}{\alpha_2}\right)Y - \left(\frac{1}{\alpha_2}\right)\left(\frac{M}{P}\right) = \left(\frac{1}{d_I}\right)\left[C(.) + I(.) + G\right] - \left(\frac{1}{d_I}\right)Y
\]

The above equation gives us a relationship between Y and P holding M, C(.), I(.) and G constant. Using some simple algebra, I can solve the above equation for Y (just to get some intuition).

\[
Y = \frac{\left(\frac{1}{d_I}\right)\left[C(.) + I(.) + G\right] + \left(\frac{1}{\alpha_2}\right)\left(\frac{M}{P}\right)}{\frac{\alpha_1}{\alpha_2} + \frac{1}{d_I}}
\]

To simplify notation, I am just going to do the following. Set:

\[
\eta_1 = \left[[\alpha_1/\alpha_2] + (1/d_i)\right]
\]
\[
\eta_2 = (1/d_i)
\]
\[
\eta_3 = (1/\alpha_2)
\]

Note, given the prior assumptions, \(\eta_1\), \(\eta_2\), and \(\eta_3\) are all positive. Cleaning up the notation we get:

\[
Y = \frac{\eta_2[C(.) + I(.) + G] + \eta_3\left(\frac{M}{P}\right)}{\eta_1}
\]  

(AD curve with Y on y-axis)

This is the aggregate demand curve (if we were going to put P on the horizontal axis). It says that an increase in P reduces Y. The reason for this relationship is that an increase in P lowers \((M/P)\) (holding \(M\) constant) which increases interest rates and lowers investment (thereby lowering Y). The size of that effect depends on the parameters in \(\eta_3\) and \(\eta_1\) (which are the responsiveness of firm investment to changes in interest rate \(d_i\) and the slope of the money demand curve \(\alpha_2\)). These parameters determine how much interest rates will change from a real money supply shock \(\alpha_2\) and how much investment will change when interest rates change \(d_i\).

Also, from this specification, you can see why changes in M cause the same percentage increase in P holding C(.), I(.), and G constant. If there is no change in Y, the ratio of \(M/P\) must remain constant. If it didn’t, real money supply would change, changing r, and changing I (meaning Y changes). If Y is fixed in the long run, then Y cannot change.

The way we graph the AD curve, we put Y on the horizontal axis. To do that, we would have to solve the above equation for P.

\[
P = \frac{\eta_3M}{\eta_1Y - \eta_2[C(.) + I(.) + G]}
\]  

(Equation 5: AD Curve)

As above, holding P constant, an increase in M, an increase in C(.), an increase in I(.), or an increase in G will shift the AD curve to the right (holding P constant, Y will increase).


**Supply Side Curves**

We have already done most of the supply side curves. I am going to amend our supply function slightly to incorporate the role of oil. Let’s define a new Cobb-Douglas production function along the following lines:

\[ Y = A(K)^{\gamma_1} (N)^{\gamma_2} (oil)^{1-\gamma_1-\gamma_2} \]

This production function is the same as the one we were using earlier in class in many respects: there are constant returns to scale in \( K \), \( N \), and oil (jointly). That means that if we double (simultaneously) \( K \), \( N \), and oil, \( Y \) will double (holding \( A \) fixed). There is diminishing return to scale in \( K \), \( N \), and oil (separately). There is complementarity in production across all the inputs. Oil is the quantity of oil used in the production of output. The key thing to remember is that firms will equate the marginal product of oil to the price of oil. Because of diminishing returns to scale in oil, an increase in the price of oil will reduce the quantity of oil used. As oil prices go up, \( Y \) will fall.

**The Long Run Aggregate Supply Curve**

If we evaluate the above production function at \( N^* \), we have the long run aggregate supply curve.

\[ Y^* = A(K)^{\gamma_1} (N^*)^{\gamma_2} (oil)^{1-\gamma_1-\gamma_2} \quad \text{(Equation 6: Long run aggregate supply curve).} \]

The only things that increase \( Y^* \) are \( A \), \( K \), \( N^* \) or oil (where oil is negatively related to oil prices per the above discussion).

**The Labor Demand Curve**

The labor demand curve just equates the marginal product of labor equal to the real wage. Taking a derivative of the above production function and equating that to the real wage we get:

\[ \frac{W}{P} = \gamma_2 A(K)^{\gamma_1} N^{\gamma_2} (oil)^{1-\gamma_1-\gamma_2} \quad \text{(Equation 7: Labor demand curve)} \]

This is nearly identical to what we had before where \( \gamma_2 = .7 \) and \( \gamma_1 = .3 \) (prove it to yourself). There is still a positive relationship between \( A \) and \( N \) (holding \( W/P \) fixed), \( K \) and \( N \) (holding \( W/P \) fixed) and quantity of oil and \( N \) (holding \( W/P \) fixed). Remember, there is a negative relationship between the quantity of oil and oil prices.

**The Short Run Aggregate Supply Curve**

Using the equations, it is harder to illustrate the short run aggregate supply curve (given both the labor demand curve and \( Y^* \) are highly non-linear given the Cobb-Douglas production function). To make things easier, I am going to do three things. First, I am going to ignore the labor supply curve. This is not a big assumption given that if we are on the short run aggregate supply curve we are off our labor
supply curve by definition (we define the short run as periods of disequilibrium in the economy where we leave our labor supply curve temporarily). Second, I am going to make the production side of the economy linear by taking logs. The notation will get a little messy, but the intuition is exactly the same as we have seen in class. Lastly, to simplify the notation, I am just going to define $\gamma 3 = 1 - \gamma 1 - \gamma 2$.

Let's take logs of Y using the production side equation:

$$\ln Y = \ln A + \gamma_1 \ln K + \gamma_2 \ln N + \gamma_3 \ln \text{oil}$$

Let's take logs of the labor demand curve and solve for $\ln(N)$

$$\ln N = \frac{-\ln W + \ln P + \ln \gamma_2 + \ln A + \gamma_1 \ln K + \gamma_3 \ln \text{oil}}{1 - \gamma_2}$$

Note: In solving for $\ln(N)$, I multiplied through by $-1/1$ (multiplied the top and bottom by -1). The reason for this is that I wanted the denominator (bottom) to be a positive number ($1 - \gamma_2 > 0$ given that all the $\gamma$'s are less than 1). As you see below, for ease of interpretation, I want all grouping of equation parameters (the $\alpha$'s, $\gamma$'s, etc.) to be positive.

Last step: substitute $\ln(N)$ from the reworked labor demand curve shown above (i.e., the profit maximizing amount of labor that firms will hire if they face a given real wage ($W/P$)) into the production function (also given above). Doing so yields our short run aggregate supply curve (in logs for simplicity):

$$\ln Y = \left(\frac{1}{1 - \gamma_2}\right) \ln A + \left(\frac{\gamma_1}{1 - \gamma_2}\right) \ln K + \left(\frac{\gamma_3}{1 - \gamma_2}\right) \ln \text{oil} - \left(\frac{\gamma_2}{1 - \gamma_2}\right) (\ln W - \ln P - \ln \gamma_2)$$

(short run aggregate supply curve, with Y as the dependent variable (i.e., Y on the vertical axis))

Note, this equation yields exactly the same intuition that I provided to you about the short run aggregate supply curve in class. An increase in P will increase Y on the supply side of the economy (there is a positive relationship between P and Y). As P increases, W/P falls and firms want to hire more labor. As they hire more labor, they produce more output. That is all this equation says. Additionally, holding P constant, an increase in W will lower Y. Lastly, holding W and P constant, an increase in A or K or an increase in oil (an fall in oil prices) will increase Y.

The way we write the short run aggregate supply curve in class, we solve for P (put P on the left hand side by itself):

**Equation 8: Short run aggregate supply curve**

$$\ln P = \psi_1 \ln Y - \psi_2 - \psi_3 \ln A - \psi_4 \ln K - \psi_5 \ln \text{oil} + \psi_6 \ln W$$

slopes  intercept (all other terms except the Y term)

All the $\psi$'s are positive. I just took all the coefficients and reduced them to $\psi$'s. The reason that $\psi_1 > 1$ is because of diminishing marginal product of labor. If Y increases by 1%, the change in P that had to generate that 1% increase in Y had to be greater than 1%.

What shifts the SRAS? An increase in A and an increase in K will shift the SRAS to the right (lower the intercept). An increase in W will shift the SRAS curve to the left (increase the intercept). This is exactly what we have been doing in class. Holding P constant, an increase in W increases W/P making more expensive for firms to hire workers. As a result, they hire less workers and aggregate production (Y) falls.
A note on macroeconomic equilibrium

If I were teaching a first year Ph.D. class in macro, my first lecture would be doing what I just did above (giving a linearized version of the macro economy). I would make my students solve for the equilibrium amount of Y and P in the short run and long run using the above equations. To get the short run level of Y and P, I would have them use equation 8 and equation 5 (where they took logs of equation 5). Notice, the equilibrium level of Y and P (in the short run) would depend on A, M, K, oil prices, G, C(\cdot), I(\cdot), sticky wages (W), and the parameters of the economy (d, y’s, the a’s, etc.). I would then tell them that in the rest of the course we would build models of C(\cdot), I(\cdot) and try to estimate d, y’s, and a’s. For the long run, they would just solve for the equilibrium Y and P using equations 5 and 6 (again taking logs of equation 5). In the long run, Y is only a function of A, K, oil, and N* (which we will have a model for). The AD curve will pin down the level of prices in the long run. The math gives us exactly the intuition that I have been giving you all term.