Capital Market Equilibrium with Personal Tax

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CAPITAL MARKET EQUILIBRIUM WITH PERSONAL TAX\footnote{Earlier versions of this paper were presented at the University of Chicago, University of California at Berkeley, University of British Columbia, NBER Conference on Taxation and Financial Markets, and Purdue University. I thank the workshop participants and in particular Fischer Black, John Cox, Eugene Fama, Jonathan Ingersoll, and Alan Kraus for helpful comments. I am especially grateful to Merton Miller and Myron Scholes who posed this problem to me and were generous of their time in useful discussions. I remain responsible for possible errors and deficiencies. Aid from the Center for Research in Securities Prices of the University of Chicago is gratefully acknowledged.}

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This paper examines the effect of the capital gains tax on investors' optimal consumption and investment behavior and on equilibrium asset prices in an intertemporal economy. It explicitly considers the fact that capital gains and losses on stock are taxed only when the investor sells the stock. Ownership of stock then confers upon the investor a timing option which enables him to realize capital losses immediately and defer capital gains. This option is a large fraction of the total benefit which accrues to the stockholder, and is the prime reason for the novel implications of capital gains taxation, discussed in this paper.

1. INTRODUCTION

CAPITAL GAINS AND LOSSES on stock are taxed when the investor sells the stock—not when gains and losses actually occur. Ownership of stock confers upon the investor a timing option which enables him to realize capital losses immediately and defer capital gains, thereby reducing the present value of the stream of tax payments on capital gains net of tax credits on capital losses. This option has important implications for investors' optimal consumption and investment behavior, for the effective tax rates on capital gains and dividends, and for equilibrium asset prices.

We prove that the optimal liquidation policy is to realize losses immediately and defer gains until the event of a forced liquidation, i.e., until the investor's death, or until an exogenous event beyond his control forces him to sell the asset (see Theorem 1). When the stock price exceeds the stock basis, the investor is locked in his investment. The value of the locked-in position exceeds the after tax proceeds from liquidating the asset. Using the methodology of Black and Scholes [1] we price the locked-in position by deriving conditions which preclude arbitrage. The price is the discounted sum of future benefits, i.e., the after tax realized capital gains and losses, after tax dividends, and return of principal. This calculation enables us to address two issues: the effective tax rate on capital gains and the value of the timing option.

Conventional wisdom suggests that the effective tax rate on capital gains and losses is zero, if the investor is not forced to sell the asset during his lifetime and if (as is the case) capital gains escape taxation upon the investor's death. We prove that, even in this simple case, the effective tax rate is positive, reflecting the fact that the investor voluntarily sells the asset during his lifetime in order to realize capital losses. In the general case we show that the effective tax rate
depends on the variance of stock returns and frequency of forced liquidations
and takes values in the range from zero to the actual tax rate on capital gains
and losses (see Tables I and II).

We also estimate the price of the timing option. This is defined as the wasted
fraction of a dollar invested in stock, if the investor fails to take advantage of his
timing option and realize capital losses whenever they occur. We find that the
timing option is a substantial fraction of the bundle of benefits associated with
stock ownership, at least for high variance stocks (see Table III).

We then study the effect of the capital gains tax on the investor's intertemporal
consumption and investment behavior. Under assumptions which effectively
complete the market, we prove a separation theorem which states that the
investor effectively has full use of the value of his locked-in position, without
voluntarily realizing the capital gains. He achieves this by selling the asset short
or by making compensating adjustments in his other portfolio assets (see Theo-
rem 2). This implies that the investor's optimal liquidation policy is separable
from his optimal consumption and investment policy: in the first stage, he
realizes all losses and defers all gains; in the second stage he makes optimal
consumption and investment decisions, where his wealth is the sum of the values
of all his asset positions.

If the investor were to follow the suboptimal liquidation policy of always
realizing his gains and losses, then the after tax rate of return on an asset would
simply be the before tax rate of return, multiplied by the factor \(1 - \tau\), where \(\tau\) is
the tax rate on capital gains and losses. However, the optimal liquidation policy
dictates that the investor defers the realization of capital gains, and he often finds
himself locked in an investment with unrealized gains. The after tax rate of
return on this position is complex, is nonstationary, and depends on the current
stock price and on the stock basis.

This complication is resolved by introducing a theoretical construct, a taxable
security's tax exempt counterpart. The tax exempt counterpart is a fictitious
security which has perfectly correlated return with the taxable one. We determine
the expected rate of return of the tax exempt counterpart such that there exist no
arbitrage opportunities between a taxable security and its tax exempt counterpart
(see Theorem 3). We then prove that the investment opportunity set remains
unchanged, if each taxable security is replaced by its tax exempt counterpart (see
Theorem 4). Essentially then we transform the consumption and investment
problem with personal tax to an equivalent problem with tax exempt securities,
as in Merton [12, 13]. We then illustrate how the investment in tax exempt
securities is translated into investment in the existing taxable securities and how
this investment is implemented.

We aggregate investors' demand for the securities' tax exempt counterparts,
and, by equating aggregate demand to supply, we derive the capital asset pricing
model, an equilibrium condition between the tax exempt securities' expected
rates of return, as in Merton [14] and Breeden [3]: We then translate this
equilibrium condition to the expected rates of return of the taxable securities.
Thus we derive a capital asset pricing model for the taxable securities (see Theorem 6).

The pricing implications differ in several respects from the implications of the intertemporal models of Merton [14] and Breeden [3], which assume zero tax, and from the single period model of Brennan [4], which assumes positive taxes on capital gains and dividends. First, we find that the appropriate "market portfolio" against which assets' beta coefficients are determined is not the value-weighted market portfolio, but a portfolio with complex weights; these weights depend on the basis of each security and are unobservable. However the market portfolio retains the economically meaningful interpretation as a portfolio which is perfectly correlated with changes in aggregate consumption. Second, the expected rate of return on a security depends on the security's variance of return and on the frequency of forced liquidations (see Table IV). Finally, the effective tax rate on dividends varies across securities and depends on the security's variance of return, the frequency of forced liquidations, and the dividend yield (see Table V).

2. THE MODEL AND ASSUMPTIONS

We consider a single good exchange economy. There exist \( N \) taxable securities, indexed \( j = 1, 2, \ldots, N \), and a riskless, tax exempt bond. We assume the following:

**Assumption 1:** Investors are price takers and trade only at equilibrium prices. Transactions costs are zero. The shares of the securities and the bond are infinitely divisible.

**Assumption 2:** Short sales of the bond are permitted with full use of the proceeds. Effectively there is unlimited short term borrowing and lending at the after tax rate, \( r \), the yield of the tax exempt bond.

The Internal Revenue Code on the personal taxation of capital gains and losses is complex, ever changing and ambiguous, subject to interpretation by the courts. We abstract from these complexities and assume the following:

**Assumption 3:** Unrealized capital gains and losses remain untaxed. Realized capital gains and losses during the investor's lifetime are taxed at the rate \( \tau \), \( 0 \leq \tau < 1 \). Upon the investor's death, capital gains and losses are realized and taxed at the rate \( \tau' \), \( 0 \leq \tau' \leq \tau \). The rates \( \tau, \tau' \) are constant over time. No distinction is made between the short term and long term status of capital gains and losses.

Currently ordinary income of up to $3,000 may be offset with capital losses annually. Unused capital losses are carried forward indefinitely. No limit exists, however, on the amount of capital gains which may be offset with capital losses.
Assumption 3 asserts that the $3,000 deduction limit is nonbinding. There are two reasons which justify this assertion. First, the investor is periodically forced to sell his securities by factors beyond his control. These forced liquidations are discussed and explicitly modeled as Poisson arrivals in Assumption 5. Since portfolio assets on average appreciate in value, the investor realizes on average positive net capital gains. Thus he applies the capital losses against the capital gains. Second, the investor may carry forward indefinitely any unused capital losses. Since it is unlikely that he realizes net capital losses in several consecutive years, he only loses the interest on the tax deduction in those years that the limit is binding.

Capital gains and losses on assets held for one year or less are termed short term and are taxed differently than gains and losses on assets held for longer than one year. Net long term gains are taxed at rate $4\tau_0$ and net long term losses are taxed at rate $5\tau_0$, where $\tau_0$ is the tax rate on ordinary income. Net short term capital gains and losses are taxed at rate $\tau_0$. However, a $1$ long term gain and a $1$ short term loss, both realized in the same tax year, offset each other one to one and result in zero tax. Assumption 3 disregards the distinction between the short term and long term status and asserts that all capital gains and losses are taxed at a uniform rate.

This scenario is plausible if the investor is periodically forced to sell his securities by factors beyond his control and, on average, realizes large long term capital gains. Then short term capital losses offset long term capital gains one to one and are effectively taxed at the long term rate. Also short term capital gains are deferred until they become long term and are effectively taxed at the long term rate also.

**Assumption 4:** Dividends are taxed at the rate $\tau_d$, which is constant over time.

For individual investors, $\tau_d$ is the marginal tax rate on ordinary income and $\tau_d > \tau$. Miller and Scholes [15] describe tax sheltering schemes which imply $\tau_d = \tau$ or even $\tau_d = 0$. For corporate investors, $\tau_d < \tau$. We leave the relation $\tau_d \geq \tau$ unspecified.

Forced liquidation of assets occurs upon the investor's death. Death is mod-

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2 Until 1981, a loophole in the tax code provided yet a third reason to justify this assertion. An investor could employ financial futures contracts to convert capital losses into ordinary losses and deduct them from ordinary income without limit. To illustrate, suppose that the investor holds a nine month Treasury bill in his portfolio. He may replace this investment by a three month futures contract on a six month Treasury bill. Three months later, if the futures price exceeds the Treasury bill price, he takes delivery and sells the bill, thereby realizing an ordinary loss. If, instead, the futures price is below the Treasury bill, he liquidates the futures contract and realizes a capital gain. If the investor engages in several such contracts every year, he will most likely generate both ordinary losses and capital gains. He offsets the capital gains with capital losses and applies the ordinary losses against his ordinary income. Effectively then he applies capital losses against ordinary income and the $3,000 limit is circumvented.

3 Trading strategies, which take into account the differential taxation of short term and long term capital gains and losses, and their equilibrium implications, are discussed in [5]. The corresponding discussion for bonds is in [6].
eled as a Poisson arrival with force $\lambda$ and uncorrelated with the securities' returns. The investor's expected lifetime is $\lambda^{-1}$. Upon death capital gains and losses are realized and the estate is taxed at rate $\tau'$, $0 \leq \tau' \leq \tau$, on net capital gains. Currently $\tau' = 0$ but in the past $\tau'$ has varied in the range $[0, \tau]$.

A different set of phenomena may also be modeled as forced asset liquidations. During his lifetime, the investor may liquidate an asset in order to speculate, rebalance his portfolio, or consume the proceeds. In practice he considers the tradeoff between the capital gains tax incurred upon liquidation and the benefits of speculation, portfolio rebalancing, and consumption. We prove in Theorem 1 that the investor defers the realization of capital gains, in the absence of the need to speculate, rebalance his portfolio, or consume. We prove in Theorem 2 and also argue in Section 4.2 that the investor defers the realization of capital gains even when faced with the ordinary needs of portfolio rebalancing and consumption. Forced asset liquidations represent extraordinary and non-recurrent needs to speculate, rebalance the portfolio, and consume, which are distinguished from routine needs of portfolio rebalancing and consumption. Upon a forced liquidation, capital gains are taxed at rate $\tau$, i.e., $\tau' = \tau$.

We consider only one Poisson process of forced liquidations on a security. This process captures either the phenomenon of death or the phenomena of forced liquidations during the investor's lifetime. Specifically we assume:

**ASSUMPTION 5:** Forced asset liquidations of a security are Poisson arrivals with force $\lambda$, where $\lambda$ may differ across securities. The Poisson process is independent of the process which generates the securities' returns. Furthermore, the Poisson arrivals are independent across investors.

Since the Poisson arrivals are independent across investors and are independent of the securities' returns, an investor who incurs a loss, say $I$, in the event of a forced liquidation, should be able to insure his loss at a cost equal to the expected loss, $\lambda I dt$, over period $[t, t + dt]$. We assume that such an insurance industry exists. Specifically we assume the following:

**ASSUMPTION 6:** A competitive insurance industry exists which insures at the actuarially fair premium, $\lambda I dt$, the loss, $I$, due to a forced liquidation over $[t, t + dt]$.

**ASSUMPTION 7:** The dividend on the $j$th security over $[t, t + dt]$ is $(1 - \tau_d)^{-1}P_j\delta_j dt$ before tax, or $P_j\delta_j dt$ after tax. The before tax capital gain is

$$(1) \quad dP_j = P_j \{ \mu_j dt + \sigma_j dw(t) \} \quad (j = 1, 2, \ldots, N),$$

where $\mu_j$ is a scalar, $\sigma_j$ is a $N$-dimensional vector, and $dw(t)$ is the increment of the Wiener process $w(t)$ in $\mathbb{R}^N$.

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$^4$Feldstein and Yitzhaki [9] discuss various motives of asset liquidation and present econometric estimates of the effect of the capital gains tax on the selling of stock.
In Sections 3 and 4 we focus on just one security. We then drop the subscript $j$ and, without loss of generality, $\sigma$ becomes a scalar and $w(t)$ becomes a Wiener process in $R^1$.

**Assumption 8:** The investment opportunity set is stationary, i.e., $r$, $\delta$, $\mu$, and $\sigma$ are constants.

3. OPTIMAL REALIZATION OF CAPITAL GAINS AND LOSSES

3.1. The Optimal Liquidation Policy

We prove that the optimal liquidation policy is to realize losses immediately and defer gains until the event of a forced liquidation. We compare policies $A$ and $B$ of an investor holding a share of stock, previously acquired at price $\hat{P}$ (i.e., with basis $\hat{P}$), and currently priced at $P = P(t)$, where $P < \hat{P}$. In policy $A$ the investor retains the stock until stopping time $T$, $T > t$. At time $T$ he sells the stock for one of two reasons. At time $T$ there is an arrival of the Poisson process which dictates forced liquidations. Alternatively, at time $T$ the stock sale is dictated by policy $A$. Capital gains and losses at $T$ are taxed at rate $\tau(T) = \tau$ or $\tau'$, the prevailing capital gains rate. The after tax sale proceeds are $(1 - \tau(T))P(T) + \tau(T)\hat{P}$.

In policy $B$ the investor sells the stock at time $t$ and receives $P + \tau(\hat{P} - P)$ after tax. He repurchases one share and invests the remaining proceeds, $\tau(\hat{P} - P)$, in a riskless, tax exempt bond with yield $r$. At time $T$ he sells the stock and receives $(1 - \tau(T))P(T) + \tau(T)P$ after tax. He sells the bond and receives $\tau(\hat{P} - P)e^{\tau(T - t)}$. The net proceeds are $(1 - \tau(T))P(T) + \tau(T)P + \tau(\hat{P} - P)e^{\tau(T - t)}$ and exceed the net proceeds of policy $A$ by $(\hat{P} - P)[\tau e^{\tau(T - t)} - \tau(T)] > 0$. The latter expression is positive because $\tau \geq \tau'$ implies $\tau \geq \tau(T)$ and $\tau e^{\tau(T - t)} - \tau(T) > 0$; also $\hat{P} - P > 0$ by assumption. Thus policy $B$ dominates policy $A$, i.e., the investor optimally realizes losses immediately. This conclusion is independent of the investor's expectations on future stock prices. It is also independent of dividend yield because in both policies, $A$ and $B$, the investor holds one share and receives the same dividend.

We prove next that the investor optimally defers gains until the event of a forced liquidation. We compare policies $C$ and $D$ of an investor holding a share of the stock, where $P > \hat{P}$. In policy $C$ the investor sells the stock and receives $P - \tau(P - \hat{P})$ after tax. He borrows $\tau(P - \hat{P})$ at the after tax rate $r$ and repurchases one share. He retains the stock until stopping time $T'$, $T' > t$. The stock sale at stopping time $T'$ is dictated by either a forced liquidation or by policy $C$. The after tax sale proceeds are $(1 - \tau(T'))P(T') + \tau(T')P - \tau(P - \hat{P})e^{\tau(T - t)}$.

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5The IRS terms the transaction a "wash sale" and disallows the loss deduction if the same stock is repurchased within thirty days. The investor has a high probability of bypassing this regulation by waiting at least thirty days before repurchasing the stock. Alternatively, the investor may purchase a different stock with similar risk and return characteristics.
In policy D the investor defers the gains at time $t$. He sells the stock at time $T'$ and receives $(1 - \tau(T'))P(T') + \tau(T')P'$ after tax. The net proceeds of policy D exceed the net proceeds of policy C by $(P - P')[\tau e^{rT} - \tau(T')] > 0$. Thus policy D dominates policy C, i.e., the investor optimally defers gains until the event of a forced liquidation. These results are summarized below:

**Theorem 1**: If Assumptions 1–3 hold, then the optimal liquidation policy is to realize losses immediately and defer gains until the event of a forced liquidation.

### 3.2. The Value of Holding Stock

An investor holds a share of stock with price $P = P(t)$ and basis $P$. The value of the stock to the investor is determined by both the stock price and basis. If the basis exceeds the stock price, $P \leq P$, the investor optimally sells the stock and receives $(1 - \tau)P + \tau P$ after tax. The value of the stock to the investor is $V(P, P) = (1 - \tau)P + \tau P$, in the sense that he is indifferent between the share or $V(P, P)$ in cash. If the stock price exceeds the basis, $P > P$, the investor optimally defers the gain. The value of the stock to the investor is $V(P, P) > (1 - \tau)P + \tau P$. The sense in which the stock value exceeds the liquidation proceeds was made clear in Section 3.1. Combining the two cases we have

\[ V(P, P) = (1 - \tau)P + \tau P, \quad P \leq P, \]
\[ > (1 - \tau)P + \tau P, \quad P > P. \]

If $P > P$ the investor may not realize the value of his position, $V(P, P)$, by selling the stock because he loses $V(P, P) - (1 - \tau)P - \tau P$ in deviating from his optimal policy. Legally the investor may not transfer the stock and basis to another investor and receive side payment $V(P, P)$. In Section 4 we discuss two ways in which the investor may effectively divest the stock from his portfolio, defer the gain, and have full use of the proceeds, $V(P, P)$.

In the remainder of this section we determine the functional form of $V(P, P)$ by comparing two positions on the same stock but with different bases. Consider two portfolios with the following composition:

**First Portfolio**: (i) $V_P(P, P')$ shares with basis $P$, $P' < P$; (ii) insurance which pays $I = V_P(P, P')(V(P, P') - (1 - \tau)P - \tau P')$ in the event of a forced liquidation over $[t, t + dt]$ of the shares in (i), and costs $\lambda I dt$; (iii) a riskless, tax exempt bond with price $r^{-1}[V_P(P, P')\delta P + (\sigma^2/2)P^2V_{PP}(P, P') + \lambda I]$.

**Second Portfolio**: (i) $V_P(P, P')$ shares with basis $P'$, $P' < P$; (ii) insurance which pays $I' = V_P(P, P')(V(P, P') - (1 - \tau)P - \tau P')$ in the event of a forced liquidation over $[t, t + dt]$ of the shares in (i), and costs $\lambda I' dt$; (iii) a riskless, tax exempt bond with price $r^{-1}[V_P(P, P')\delta P + (\sigma^2/2)P^2V_{PP}(P, P') + \lambda I']$.

Subscript $P$ denotes partial derivatives with respect to $P$. For notational convenience we define $V \equiv V(P, P)$ and $V' \equiv V(P, P')$. We prove that the two
portfolios have equal after tax return over \([t, t + dt]\), irrespective of forced liquidations, and argue that the investor is indifferent between them.

The investor is indifferent to a forced liquidation of the first portfolio's shares because the loss, \(V_p\left[ V - (1 - \tau')P - \tau'\bar{P} \right] \), is fully compensated by the insurance. Likewise he is indifferent to a forced liquidation of the second portfolio's shares. Conditional on no forced liquidations, the first portfolio's value increases over \([t, t + dt]\) by

\[
V_p \left[ \left\{ \delta P + \mu PV_p + \frac{\sigma^2}{2} P^2 V_{pp} \right\} dt + \sigma PV_p dw \right] - \lambda I dt + \left[ V_p \left\{ \delta P + \frac{\sigma^2}{2} P^2 V_{pp} \right\} + \lambda I \right] dt
\]

and equals the second portfolio's value increase, which is

\[
V_p \left[ \left\{ \delta P + \mu PV_p' + \frac{\sigma^2}{2} P^2 V_{pp} \right\} dt + \sigma PV_p' dw \right] - \lambda I' dt + \left[ V_p' \left\{ \delta P + \frac{\sigma^2}{2} P^2 V_{pp} \right\} + \lambda I' \right] dt.
\]

Therefore the investor is indifferent between the two portfolios and we obtain

\[
V_p V + r^{-1} \left[ V_p \left\{ \delta P + \frac{\sigma^2}{2} P^2 V_{pp} \right\} + \lambda I \right] = V_p V' + r^{-1} \left[ V_p' \left\{ \delta P + \frac{\sigma^2}{2} P^2 V_{pp} \right\} + \lambda I' \right].
\]

Rearranging, we obtain

\[
\frac{\lambda \tau' \hat{P} + \left\{ (1 - \tau')\lambda + \delta \right\} P - (r + \lambda)V + (\sigma^2/2) V_{pp}}{V_p}
= \frac{\lambda \tau' \hat{P}' + \left\{ (1 - \tau')\lambda + \delta \right\} P - (r + \lambda)V' + (\sigma^2/2) V_{pp}'}{V_p'} \equiv k_0.
\]

Since the left-hand side is independent of \(\hat{P}'\) and the right-hand side is independent of \(\hat{P}\), \(k_0\) is a function of \(P\) and of parameters \(r, \sigma^2, \delta, \lambda, \tau, \) and \(\tau'\), but is independent of \(\hat{P}\) and \(\hat{P}'\). It is easily shown that \(V\) is homogeneous of degree one in \(P\) and \(\hat{P}\). Then the left-hand side in equation (3) is homogeneous of degree one in \(P\) and \(\hat{P}\). It follows that \(k_0 = -kP\) where \(k = k(r, \sigma^2, \delta, \lambda, \tau, \tau')\). Equation (3) becomes

\[
\lambda \tau' \hat{P} + \left\{ (1 - \tau')\lambda + \delta \right\} P - (r + \lambda)V + kPV_p + \frac{\sigma^2}{2} P^2 V_{pp} = 0,
\]

\(P > \hat{P}\).
The parameter $k$ is thus far undetermined. Continuity of $V$ at $P = \hat{P}$ requires
\begin{equation}
(5) \quad V(\hat{P}, \hat{P}) = \hat{P}
\end{equation}
and continuity of $V_p$ at $P = \hat{P}$ (the "smooth-pasting" condition) requires
\begin{equation}
(6) \quad V_p(\hat{P}, \hat{P}) = 1 - \tau.
\end{equation}
The unique solution to (4) subject to the boundary conditions (5), (6), and the condition $V(P, \hat{P}) \geq 0$, is
\begin{equation}
(7) \quad V(P, \hat{P}) = \frac{\lambda r' \hat{P}}{r + \lambda} + \frac{(1 - \tau')\lambda + \delta}{r + \lambda - k} P + (n - m)^{-1}
\times \left[ 1 - \tau - \left(1 - \frac{\lambda r'}{r + \lambda}\right)m - \frac{(1 - \tau')\lambda + \delta}{r + \lambda - k} (1 - m) \right] p^n \hat{P}^{1-n}
\quad - (n - m)^{-1} \left[ 1 - \tau - \left(1 - \frac{\lambda r'}{r + \lambda}\right)n\right]
\quad - \frac{(1 - \tau')\lambda + \delta}{r + \lambda - k} (1 - n) \right] p^n \hat{P}^{1-n}, \quad P > \hat{P}
\end{equation}
where $m, n, m < 0, n > 0$, are the roots of the quadratic equation
\begin{equation}
(8) \quad \frac{\sigma^2}{2} x^2 + \left(k - \frac{\sigma^2}{2}\right)x - (r + \lambda) = 0.
\end{equation}
We prove by contradiction that the coefficient of $P^n$ in equation (7) equals zero.
If the coefficient of $P^n$ is nonzero, the requirement that $V(P, 0)$ be finite implies $1 - n \geq 0$. The quadratic (8) implies $(\sigma^2/2)(n - 1) n + kn - (r + \lambda) = 0$. If indeed $1 - n \geq 0$, then $r + \lambda - k \leq 0$ and equation (7) implies $V(P, 0) < 0$ which is absurd. Therefore the coefficient of $P^n$ in equation (7) equals zero, i.e.
\begin{equation}
(9) \quad 1 - \tau - \left(1 - \frac{\lambda r'}{r + \lambda}\right)m - \frac{(1 - \tau')\lambda + \delta}{r + \lambda - k} (1 - m) = 0.
\end{equation}
\footnote{If $\lambda = \delta = 0$ the above argument does not hold. The differential equation (4) becomes
\[-rV + kPV_p + \frac{\sigma^2}{2} P^2 V_{pp} = 0.\]
The solution satisfying (5) and (6) is
\[V(P, \hat{P}) = \frac{1 - \tau - n}{m - n} p^n \hat{P}^{1-n} - \frac{1 - \tau - m}{m - n} p^n \hat{P}^{1-n}.\]
Since $V(P, 0)$ must be positive and finite, $n = 1, k = r, m = -2r/\sigma^2$, and
\[V(P, \hat{P}) = \left(1 - \frac{\tau}{1 + 2r/\sigma^2}\right) p^{1+2r/\sigma^2} p^{-2r/\sigma^2}.\]
This is equation (15), derived in the main text in the limit as $\lambda \to 0$ and $\delta \to 0.$}
Then equation (7) simplifies to

\begin{equation}
V(P, \hat{P}) = \frac{\lambda \tau \hat{P}}{r + \lambda} \left[ 1 - \frac{\tau}{1 - m} + \frac{\lambda m \tau'}{(r + \lambda)(1 - m)} \right] P \\
+ \left[ \frac{\tau}{1 - m} - \frac{\lambda \tau'}{(r + \lambda)(1 - m)} \right] P^m \hat{P}^{1 - m}, \quad P > \hat{P}.
\end{equation}

Equations (8) and (9) determine \( m \) and \( k \). We eliminate \( k \) and obtain

\begin{equation}
m^2 + (a - b - c)m - ab = 0
\end{equation}

where

\begin{align}
a &\equiv 2(r + \lambda) / \sigma^2, \\
b &\equiv \frac{1 - \tau}{1 - \lambda \tau'/(r + \lambda)}, \\
c &\equiv \frac{(1 - \tau')\lambda + \delta}{(1 - \lambda \tau'/(r + \lambda)) \sigma^2/2}.
\end{align}

Then \( m \) is uniquely determined as the negative root of equation (11) and \( k \) is uniquely determined by equation (9).

Figure 1 illustrates the function \( V(P, \hat{P}) \). For \( P \leq \hat{P} \), \( V(P, \hat{P}) = (1 - \tau) P + \tau \hat{P} \), a straight line with intercept \( \tau \hat{P} \) and slope \( 1 - \tau \). For \( P \geq \hat{P} \), \( V(P, \hat{P}) \) is given by

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{}
\end{figure}
equation (10). As \( P \to \infty \) the curve approaches an asymptote with intercept \( \lambda \tau' \hat{P} / (r + \lambda) \) and slope \( 1 - \tau / (1 - m) + \lambda m \tau' / (r + \lambda)(1 - m) \).

We consider a number of limiting cases:

\[
(15) \quad \lim_{\lambda, \delta \to 0} V(P, \hat{P}) = \left(1 - \frac{\tau}{1 + 2r / \sigma^2}\right)P + \frac{\tau}{1 + 2r / \sigma^2} \hat{P}^{1 + 2r / \sigma^2}P^{-2r / \sigma^2}, \quad P \geq \hat{P}.
\]

Alternatively

\[
(16) \quad \lim_{\sigma^2 \to \infty} V(P, \hat{P}) = (1 - \tau)P + \tau \hat{P}
\]

which is the after-tax value of position \((P, \hat{P})\) assuming immediate liquidation. With high stock variance, voluntary liquidation is imminent and the position value is effectively the after tax liquidation value.

If \( \sigma^2 \) is small, then \( m \to -\infty \) and

\[
(17) \quad \lim_{\sigma^2 \to 0} V(P, \hat{P}) = P - \frac{\lambda \tau'}{r + \lambda}(P - \hat{P}), \quad P > \hat{P}.
\]

This equation is true for any dividend yield, but has an intuitive interpretation for \( \delta = r \). Then the stock price remains constant forever. \( V(P, \hat{P}) \) is the stock price minus \((\lambda \tau' / (r + \lambda))(P - \hat{P})\), the discounted expected tax liability associated with a forced liquidation.

3.3. The Effective Capital Gains Tax

We define the effective tax rate, \( \tau^e \), on capital gains, by

\[
(18) \quad V(P, \hat{P}) = (1 - \tau^e)P + \tau^e \hat{P}.
\]

When \( P < \hat{P} \), the investor optimally realizes a loss, \( V(P, \hat{P}) = (1 - \tau)P + \tau \hat{P} \), and the effective tax rate equals the capital gain rate. When \( P > \hat{P} \) the investor optimally defers the realization of capital gains. As \( \sigma^2 \to 0 \), \( V(P, \hat{P}) \to P - (\lambda \tau' / (r + \lambda))(P - \hat{P}) \) by equation (17) and \( \tau^e \to \lambda \tau' / (r + \lambda) \). If the event of a forced liquidation is the investor's death whereupon gains escape taxation, \( \tau' = 0 \) and the effective tax rate equals zero. The conventional belief is correct in the case \( \sigma^2 \to 0 \), because gains are deferred until death, escape taxation, and the effective tax rate on capital gains is zero.

The conventional belief fails if the stock return is variable, \( \sigma^2 > 0 \). In the extreme case \( \sigma^2 \to \infty \), \( V(P, \hat{P}) \to (1 - \tau)P + \tau \hat{P} \) by equation (16) and \( \tau^e \to \tau \): even if the capital gains evade taxation upon a forced liquidation, i.e., \( \tau' = 0 \), and even if forced liquidations never occur, i.e., \( \lambda = 0 \), the effective tax rate equals the capital gains rate and differs from zero. The explanation is simple: although the investor could refuse to exercise his option and realize losses, it is in his best interest to realize losses whenever they occur. The probability of exercising this option increases with stock variance, and the effective tax rate, faced by the
TABLE I
THE RATIO OF THE EFFECTIVE TAX RATE TO THE ACTUAL TAX RATE, $\tau^e/\tau$, ON CAPITAL GAINS AND LOSSES$^a$

<table>
<thead>
<tr>
<th>$\hat{P}/P$</th>
<th>$\sigma = .05$/month</th>
<th>$\sigma = .10$/month</th>
<th>$\sigma = .20$/month</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>.13</td>
<td>.13</td>
<td>.71</td>
</tr>
<tr>
<td>.5</td>
<td>.26</td>
<td>.63</td>
<td>.88</td>
</tr>
<tr>
<td>.8</td>
<td>.70</td>
<td>.84</td>
<td>.96</td>
</tr>
<tr>
<td>$\geq 1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

$^a$We assume $\tau = .10$/annum and $\lambda = \delta = 0$.

An investor, is his capital loss rate. The conventional thinking focuses on the tax consequences of realized gains while ignoring the tax consequences associated with the realized losses.

In the case $\lambda = \delta = 0$, equations (15) and (18) imply

$$
\frac{\tau^e}{\tau} = \frac{1}{1 + 2r/\sigma^2} \frac{1 - (\hat{P}/P)^{1 + 2r/\sigma^2}}{1 - \hat{P}/P}, \quad P \geq \hat{P},
$$

$$
= 1, \quad \quad P \leq \hat{P}.
$$

Table I displays the ratio $\tau^e/\tau$. The effective tax rate is increasing in the stock variance and in the ratio of the basis to the stock price.

In the case $\hat{P} = 0$, equations (10) and (18) imply

$$
\frac{\tau^e}{\tau} = \frac{\lambda}{r + \lambda} + \frac{r}{(1 - m)(r + \lambda)}.
$$

Table II displays the ratio $\tau^e/\tau$. The effective tax rate is increasing in stock variance, probability of forced liquidations, and dividend yield. Tables I and II indicate that the ratio $\tau^e/\tau$ lies anywhere in $[0, 1]$ for plausible parameter values.

### 3.4. The Timing Option

Suppose that an investor buys the stock but, instead of following the optimal policy, holds it until a forced liquidation. The value of the stock to him is less than $P$ and is $V(P, 0) + \lambda \tau^e P/(r + \lambda)$. The first component is the value of the

TABLE II
THE RATIO OF THE EFFECTIVE TAX RATE TO THE ACTUAL TAX RATE, $\tau^e/\tau$, ON CAPITAL GAINS AND LOSSES$^a$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\sigma = .05$/month</th>
<th>$\sigma = .10$/month</th>
<th>$\sigma = .20$/month</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta = 0$</td>
<td>$\delta = .10$/annum</td>
<td>$\delta = 0$</td>
<td>$\delta = .10$/annum</td>
</tr>
<tr>
<td>0</td>
<td>.13</td>
<td>.35</td>
<td>.38</td>
</tr>
<tr>
<td>.10/annum</td>
<td>.55</td>
<td>.63</td>
<td>.65</td>
</tr>
<tr>
<td>.20/annum</td>
<td>.70</td>
<td>.74</td>
<td>.76</td>
</tr>
</tbody>
</table>

$^a$We assume $\hat{P} = 0$, $r = .10$/annum, and $\tau = \tau^e = .25$. 
stock, if the stock basis were zero, in which case he would never voluntarily sell the stock. The second component is the discounted expected tax shield which is realized in the event of a forced liquidation, $\tau' P \int_0^\tau e^{-r t} e^{-\lambda t} dt = \lambda \tau' P / (r + \lambda)$. The price of the timing option is defined as the difference between the purchase price of the stock, $P$, and the sum of the benefits, $V(P,0) + \lambda \tau' P / (r + \lambda)$, associated with the suboptimal policy of never voluntarily realizing losses. The price, $v$, of the timing option per dollar invested in the stock is

$$v = \frac{1}{P} \left[ P - V(P,0) - \frac{\lambda \tau' P}{r + \lambda} \right] = \frac{\tau}{1 - m} - \frac{\lambda \tau'}{(r + \lambda)(1 - m)}.$$

The timing option is illustrated in Figure 1. Table III presents values of the timing option in the case $r = .10/\text{annum}$ and $\tau = \tau' = .25$ for a range of $\sigma, \lambda, \delta$ values. The price of the timing option is increasing in the stock variance and dividend yield and is decreasing in the probability of forced liquidations. For high variance stocks and infrequent forced liquidations, the timing option is a sizeable fraction of the bundle of benefits associated with stock purchase.

4. SEPARATION OF LIQUIDATION POLICY AND CONSUMPTION-INVESTMENT POLICY

We prove that the investor's optimal liquidation policy is independent of his consumption and investment policies. The conclusion follows under different assumptions which effectively complete the market. In Section 4.1 we introduce a stylized market for selling stock short and state the equilibrium condition which forbids arbitrage opportunities. In Section 4.2 we prove the separation theorem, based on the existence of the short sale market. We then show that the separation theorem is robust and holds under weaker conditions than the existence of the short sale market.

4.1. The Market for Short Sale Contracts

When an investor sells short one share, he receives a stock certificate from a lending broker. The investor is required to pay to the broker whatever dividend is declared on the stock. At any future time of his choice, the investor delivers one stock certificate to the broker and closes out his position.

The investor is required to put up collateral, consisting partly of stock, bond, and Treasury certificates and partly of cash. The cash collateral earns either zero
interest, or interest which is typically less than the market rate. As the stock price increases, the investor is required to increase his collateral; as the stock price decreases, he may decrease his collateral. The requirement of cash collateral implies that the investor has less than full use of the short sale proceeds: upon the sale of the stock certificate, he surrenders part or all of the sale proceeds to the lending broker as cash collateral. Under our assumption that the investor has no limit on borrowing, the reduced use of the short sale proceeds is of no consequence. The real consequence of the cash collateral is the opportunity cost of the funds earning reduced interest which may be modeled as a rent, $\alpha P \, dt$, on the stock certificate. Formally we assume the following:

**Assumption 9:** In a short sale the investor has full use of the proceeds, $P$. The effect of cash collateral and the broker's premium are summarized by a "rent" $\alpha P(t) \, dt$, over the period $[t, t + dt]$. The parameter $\alpha$ is specified in equation (28) in a way to eliminate arbitrage opportunities.

We may repeat the arguments of Section 3.1 and prove that the investor's optimal liquidation policy is to realize losses immediately (by closing the short position) and defer gains until the event of a forced liquidation. We define $S(P, \hat{P})$ as the value of a short position on a share with price $P$ and basis $\hat{P}$. A short position is a liability and $S(P, \hat{P})$ is negative. If $P \geq \hat{P}$, the investor optimally closes the short position and pays out $(1 - \tau)P + \tau \hat{P}$. If $P < \hat{P}$, he optimally defers the realization of a gain and the value of his position exceeds $- (1 - \tau)P - \tau \hat{P}$. Thus

$$S(P, \hat{P}) = - (1 - \tau)P - \tau \hat{P}, \quad P < \hat{P},$$

$$= - (1 - \tau)P - \tau \hat{P}, \quad P \geq \hat{P}.$$  

Consider a long and a short position on the stock, both with basis, $\hat{P}$. Whenever $P > \hat{P}$, the investor optimally defers the gain on the long position and realizes the loss on the short position. Whenever, $P < \hat{P}$, the investor optimally realizes the loss on the long position and defers the gain on the short position. The cash flows of a short position are not equal, with sign reversed, to the cash flows of a long position. Also the value of a long position is not equal, with sign reversed, to the value of a short position, except in the special case $P = \hat{P}$. In general $V(P, \hat{P}) \neq - S(P, \hat{P})$, if $P \neq \hat{P}$.

We determine the functional form of $S(P, \hat{P})$ by considering a portfolio with a short and long position on the same stock.\(^7\) Specifically the portfolio consists

---

\(^7\)If an investor has a short and long position on the same stock, the IRS considers realized losses to be long term and realized gains to be short term and, therefore, taxed at a higher rate. To avoid this complication, we assume that the shares held long and short are financially identical, i.e., have perfectly correlated prices, but are shares of two distinct firms, so that they are not "substantially identical" assets for tax purposes. This liberal interpretation of market completeness is undesirable and is dispensed with in Section 4.2, following Theorem 2.
of: (i) \(-S_p(P, \hat{P})\) shares long with basis \(\hat{P}, P > \hat{P}\); (ii) \(V_p(P, \hat{P})\) shares short with basis \(\hat{P}, P < \hat{P}\); (iii) insurance which pays \(I = -S_p[V - ((1 - \tau')P + \tau'\hat{P})]\) in the event of a forced liquidation of the long position over \([t, t + dt]\) and costs \(\lambda I dt\); (iv) insurance which pays \(I' = V_p[S + ((1 - \tau')P + \tau'\hat{P})]\) in the event of a forced liquidation of the short position over \([t, t + dt]\) and costs \(\lambda I' dt\); (v) a riskless, tax exempt bond with price

\[
 r^{-1} \left[ S_p \left( \delta P + \frac{\sigma^2}{2} P^2 V_{pp} \right) + \lambda I + V_p \left( \alpha P + \delta P - \frac{\sigma^2}{2} P^2 S_{pp} \right) + \lambda I' \right].
\]

The investor loses \(-S_p[V - ((1 - \tau')P + \tau'\hat{P})]\) in the event of a forced liquidation of the long position and is fully compensated by the insurance. Likewise he is fully compensated in the event of a forced liquidation of the short position. Conditional on no forced liquidations, the portfolio return over \([t, t + dt]\) is

\[
 -S_p \left[ \left( \delta P + \mu PV_p + \frac{\sigma^2}{2} P^2 V_{pp} \right) dt + \sigma PV_p dw \right] \\
 + V_p \left[ \left( -\alpha P - \delta P + \mu PS_p + \frac{\sigma^2}{2} P^2 S_{pp} \right) dt + \sigma PS_p dw \right] \\
 -\lambda I dt - \lambda I' dt + S_p \left( \delta P + \frac{\sigma^2}{2} P^2 V_{pp} \right) dt + \lambda I dt \\
 + V_p \left( \alpha P + \delta P - \frac{\sigma^2}{2} P^2 S_{pp} \right) dt + \lambda I' dt \equiv 0.
\]

Therefore the portfolio value is zero, i.e.,

\[
(23) \quad -S_p V + V_p S + r^{-1} \left[ S_p \left( \delta P + \frac{\sigma^2}{2} P^2 V_{pp} \right) + \lambda I \\
 + V_p \left( \alpha P + \delta P - \frac{\sigma^2}{2} P^2 S_{pp} \right) + \lambda I' \right] = 0.
\]

With the aid of equation (4) this equation simplifies to

\[
(24) \quad -\lambda\tau' \hat{P} - \{(1 - \tau')\lambda + \alpha + \delta\} P - (r + \lambda) S + kPS_p + \frac{\sigma^2}{2} P^2 S_{pp} = 0,
\]

\(P < \hat{P}\).

Continuity of \(S\) at \(P = \hat{P}\) requires

\[
(25) \quad S(\hat{P}, \hat{P}) = -\hat{P}
\]

and continuity of \(S_p\) at \(P = \hat{P}\) requires

\[
(26) \quad S_p(\hat{P}, \hat{P}) = -(1 - \tau).
\]
The unique solution to (24) subject to the boundary conditions (25) and (26) is

\[ S(P, \hat{P}) = -\frac{\lambda \tau P}{r + \lambda} \frac{(1 - \tau')\lambda + \alpha + \delta}{r + \lambda - k} P \]
\[ + (n - m)^{-1} \left[ -(1 - \tau) + \left(1 - \frac{\lambda \tau}{r + \lambda}\right)m \right] P^n \hat{P}^{1-n} \]
\[ - (n - m)^{-1} \left[ -(1 - \tau) + \left(1 - \frac{\lambda \tau}{r + \lambda}\right)n \right] P^n \hat{P}^{1-m} , \]

where \( m, n, m < 0, n > 0 \), are the roots of the quadratic (8). Since \( S(0, \hat{P}) \) is bounded, the coefficient of \( P^m \) must equal zero, i.e.,

\[ -(1 - \tau) + \left(1 - \frac{\lambda \tau}{r + \lambda}\right)n + \left(\frac{(1 - \tau')\lambda + \alpha + \delta}{r + \lambda - k}\right)(1 - n) = 0. \]

Equation (28) uniquely determines \( \alpha \). In the special case \( \lambda = \delta = 0 \), we obtain

\[ \alpha = (r + \sigma^2/2)\tau \]

and

\[ S(P, \hat{P}) = -\left[1 - \tau \ln(P/\hat{P})\right]P, \quad P \leq \hat{P}. \]

Equation (29) states that the lending broker extracts rent \((r + \sigma^2/2)\tau P dt\) by requiring cash collateral \((1 + \sigma^2/2r)\tau P\), paying zero interest. Since \((1 + \sigma^2/2r)\tau \sim 1\), the theory explains the order of magnitude of the empirically observed cash collateral. The cash collateral is not an unfair penalty on the investor who sells short; it is the equilibrium price of his timing option to realize losses and defer gains.

4.2. A Separation Theorem

**Theorem 2:** If Assumptions 1–9 hold, then the investor holding stock with price \( P \) and basis \( \hat{P} \) has use of the proceeds, \( V(P, \hat{P}) \), without voluntarily realizing capital gains.
PROOF: If \( P \leq \hat{P} \), the investor optimally liquidates the position, receives \((1 - \tau)P + \tau\hat{P} = V(P, \hat{P})\) after tax, and has use of the proceeds \(V(P, \hat{P})\).

If \( P > \hat{P} \), the investor sets up the following hedge: he (i) retains the share with price \( P \) and basis \( \hat{P} \); (ii) sells short \((1 - \tau)^{-1}V_{p}(P, \hat{P})\) shares; (iii) buys insurance which pays \( I = V(P, \hat{P}) - ((1 - \tau)P + \tau\hat{P}) \) in the event of a forced liquidation of the long position over \([t, t + dt]\) and costs \( M dt\); (iv) invests \((1 - \tau)^{-1}PV_{p}(P, \hat{P}) - V(P, \hat{P})\) in a riskless, tax exempt bond. The net cash inflow is \(V(P, \hat{P})\). To complete the proof we show that the hedge's return over \([t, t + dt]\) is zero under all possible realizations. In the event of a forced liquidation of the long position, the investor loses \(V(P, \hat{P}) - ((1 - \tau)P + \tau\hat{P})\) and is fully compensated by the insurance. He is also indifferent to a forced liquidation of the short position because the basis equals the stock price. Conditional on no forced liquidations, the hedge's return over \([t, t + dt]\) is

\[
\left\{ \delta P + \mu PV_{p} + \frac{\sigma^{2}}{2} P^{2}V_{pp} \right\} dt + \sigma PV_{p} dw + (1 - \tau)^{-1}V_{p}
\times \left[ \left\{ -\alpha P - \delta P + \mu PS_{p} + \frac{\sigma^{2}}{2} P^{2}S_{pp} \right\} dt + \sigma PS_{p} dw \right]
- \lambda \left[ V - (1 - \tau)P - \tau\hat{P} \right] dt + r(1 - \tau)^{-1}PV_{p} dt - rV dt \equiv 0
\]

where we make use of equations (4), (24). Q.E.D.

The separation theorem also applies to a short position. If the investor has a short position on stock with price \( P \) and basis \( \hat{P} \), where \( P < \hat{P} \), the investor may effectively close his position without realizing a capital gain: he retains the short position; buys \(-(1 - \tau)^{-1}S_{p}(P, \hat{P})\) shares; buys insurance which covers the loss \( S + ((1 - \tau)P + \tau\hat{P}) \) in the event of a forced liquidation of the short position; and lends \( -S + (1 - \tau)^{-1}PS_{p}(P, \hat{P}) \). The net cash outflow is \(-S(P, \hat{P})\). The reader may verify that the return over \([t, t + dt]\) is zero.

The separation theorem is robust. Without recourse to the stylized short sale market, the investor may satisfy the routine needs of consumption and portfolio rebalancing without realizing capital gains. Two observations are pertinent. First, being risk averse, the investor holds a diversified portfolio. An asset is a desirable or undesirable portfolio component to the extent that it contributes to overall portfolio characteristics, e.g., expected portfolio return, variance of portfolio return, and covariance of portfolio return with the market or with inflation. Second, in the normal course of events, the investor receives cash income from a number of sources: wages and salary; cash dividends; the after tax proceeds from the sale of assets, so as to realize capital losses; and the after tax proceeds from the sale of assets, necessitated by forced liquidations.

\(^{8}\) At \( P = \hat{P} \), the derivative \( S_{pp} \) is discontinuous. \( S_{pp} \) stands for \((S_{pp} + S_{pp})/2\), the average of left hand and right hand derivatives.
The investor may oftentimes avoid the realization of a capital gain by satisfying routine consumption needs with cash income. Furthermore, he may rebalance his portfolio by properly reinvesting his cash income net of consumption, thereby avoiding the realization of capital gains. Suppose, for example, that he wishes to sell an asset highly correlated with the market, in order to reduce his portfolio's covariance with the market. He may reduce the covariance by retaining the asset and reinvesting his cash income net of consumption in assets which are uncorrelated with the market.

Naturally there will be instances when consumption needs are high or the investor wishes to drastically rebalance his portfolio. We recognize that these situations arise and we have explicitly modeled them in Assumption 5 as forced asset liquidations generated by a Poisson process. The separation theorem explicitly recognizes these events and states that capital gains are deferred until the event of a forced liquidation.

5. OPTIMAL CONSUMPTION AND INVESTMENT

5.1. Spanning of Asset Returns

The previous discussion focused on a taxable security with price $P$, after tax dividend $\delta P\, dt$, and before tax capital gain $dP = \mu P\, dt + \sigma P\, dw$ over $[t, t+dt]$. We define the tax exempt counterpart of the taxable security as a security with price, $P$, and tax exempt capital gain plus dividend equal to $\mu' P\, dt + \sigma P\, dw$. By definition, the tax exempt's rate of return is perfectly correlated with, and has the same variance, $\sigma^2$, as the taxable security's before tax rate of return. A security's tax exempt counterpart is a useful theoretical construct. Theorem 3 specifies the expected rate of return, $\mu'$, of the tax exempt counterpart so that there exist no arbitrage opportunities between the taxable security and its tax exempt counterpart. Theorem 4 states that the taxable security may be replaced by its tax exempt counterpart, leaving the investment opportunity set unchanged.

**THEOREM 3:** If Assumptions 1–9 hold, then a taxable security and its tax exempt counterpart present no arbitrage opportunities, if and only if the tax exempt's expected rate of return is

$$\mu' = \mu - k + r$$

where $k$ is determined by equations (9) and (11).

**PROOF:** Consider a portfolio with the following composition: (i) $V_p^{-1}(P, \tilde{P})$ taxable shares with basis $\tilde{P}$, $P \geq \tilde{P}$; (ii) insurance which pays $I = V_p^{-1}[V - ((1 - \tau\gamma)P + \tau\gamma\tilde{P})]$ in the event of a forced liquidation over the period $[t, t+dt]$

---

9Endogenizing the forced liquidations increases the realism but also the complexity of the model and adds little to the conclusions of our paper.
and costs $\lambda I\, dt$; (iii) a riskless, tax exempt bond with price $P - V_p^{-1}V$. The portfolio’s price is $V_p^{-1}V + P - V_p^{-1}V = P$. In the event of a forced liquidation the investor loses $V - \{(1 - \tau')P + \tau'\hat{P}\}$ and is fully compensated by insurance. Conditional on no forced liquidations, the portfolio’s after tax return is

$$V_p^{-1}\left[\delta P + \mu PV_p + \frac{\sigma^2}{2} P^2 V_{pp} \right] dt + \sigma PV_p \, dw$$

$$-\lambda V_p^{-1}\left[ V - \{(1 - \tau')P + \tau'\hat{P}\} \right] dt$$

$$+ rP \, dt - rV_p^{-1}V \, dt = \left[ (\mu - k + r) \, dt + \sigma \, dw \right] P$$

where we make use of equation (4). The portfolio’s after tax rate of return is $(\mu - k + r) \, dt + \sigma \, dw$. If $\mu' < \mu - k + r$ or $\mu' = \mu - k + r$ the portfolio dominates (is dominated by) the tax exempt counterpart of the security and presents arbitrage opportunities. $Q.E.D.$

**Theorem 4:** If Assumptions 1–9 hold, then the investment opportunity set remains unchanged if a taxable security is replaced by its tax exempt counterpart with expected rate of return $\mu' = \mu - k + r$.

**Proof Outline:** The security’s tax exempt counterpart may be replicated by the taxable security in a portfolio, as explained in the proof of Theorem 3. Likewise the taxable security’s return may be replicated by a portfolio of the tax exempt counterpart, the riskless bond, and insurance. $Q.E.D.$

This theorem simplifies the investor’s consumption and investment decisions which are considered next: the taxable securities are replaced by their tax exempt counterparts.

### 5.2. Optimal Consumption and Investment

**Assumption 10:** The investor makes sequential consumption and investment decisions with the objective to maximize

$$E\left[ \int_t^T U(c(s), s) \, ds + \hat{U}(W(T), T) \right].$$

$E$ is the expectation operator over the Wiener process $w(t)$, and the Poisson process which generates $T$. $c(s)$ is the consumption flow. $W(T)$ is the stock of wealth at time $T$. $U$ is increasing, strictly concave, and twice differentiable in $c$; also $\partial U(0, s)/\partial c = \infty$. $\hat{U}$ is increasing, strictly concave and twice differentiable in $W(T)$; also $\partial U(0, T)/\partial W = \infty$.

As the result of earlier investments, the investor now holds $a_{jq}$ shares of security $j$ with basis $\hat{P}_j$, $j = 1, 2, \ldots, N$, $q = 1, 2, \ldots, Q$. He has a short
position in $b_{jq}$ shares of security $j$ with basis $\hat{P}_{jq}$. He also holds a riskless, tax exempt bond and cash with total value $B$. Thus the investor's asset holdings are described by the scalar $B$ and four $N \times Q$ matrices $(a_{jq}), (\hat{P}_{jq}), (b_{jq}), (\hat{P}_{jq})$. The investor also receives exogenous income at the rate $y(t)$.

If capital gains tax is zero, $\tau = \tau' = 0$, the investor's wealth is

$$W(t) = B + \sum_{j=1}^{N} \sum_{q=1}^{Q} (a_{jq} - b_{jq})P_j.$$ 

At time $t$, he chooses his consumption rate $c(t)$ and the fraction $\alpha_j(t)$ of his wealth to invest in the $j$th security, $j = 1, 2, \ldots, N$, to maximize his expectation of lifetime consumption, equation (32). This problem was formulated and solved in Merton [12, 13].

Three complications arise with personal tax. First, the investor has to choose not only his consumption rate and the fraction of his wealth to invest in each security, but also decide on which assets to realize capital gains and losses. Theorem 1 provides the answer: the investor optimally realizes all his losses and defers all his gains.

Second, the investor is locked in investments with unrealized capital gains. His holdings are described by the scalar $B$ and four $N \times Q$ matrices $(a_{jq}), (\hat{P}_{jq}), (b_{jq}), (\hat{P}_{jq})$, which are potentially a large number of state variables. Theorem 2 resolves this complication: it asserts that the investor has use of the proceeds of each asset position, without voluntarily realizing capital gains. Specifically, his wealth is given by equation (35). Once the investor calculates his wealth by equation (35), the scalar $B$ and the matrices $(a_{jq}), (\hat{P}_{jq}), (b_{jq}), (\hat{P}_{jq})$ become redundant in solving the consumption and investment problem.

Third, the after tax rate of return on a long position

$$\frac{dV(P, \hat{P})}{V(P, \hat{P})} = \left[ \frac{\mu PV_F}{V} + \frac{\sigma^2}{2} P^2 \frac{V_{PP}}{V} \right] dt + \frac{\sigma PV_F}{V} dw$$

is complex: the expected rate of return and variance are nonstationary, being functions of the stochastic variable $P$. Theorem 4 resolves this complication by means of the securities' tax exempt counterparts, introduced in Theorem 3: the taxable securities are replaced by their tax exempt counterparts, leaving the investment opportunity set unchanged. These observations lead to the following theorem:

**Theorem 5:** If Assumptions 1–10 hold, then the investor's maximized expected utility, $J(W(t), t)$, and optimal consumption rate, $c(t)$, are the solution to the problem

$$J(W(t), t) = \max_{c, \{a_j \}} E \left[ \int_{t}^{T} U(c(s), s) ds + \hat{U}(W(T), T) \right]$$

subject to

$$dW = \sum_{j=1}^{N} \left[ (\alpha_j (\mu_j - r) + r) dt + \alpha_j \sigma_j dw \right] W + (y - c) dt$$
and the initial condition

\[ W(t) = B + \sum_{j=1}^{N} \sum_{q=1}^{Q} \left[ a_{jq} V(P_j, \hat{P}_{jq}) + b_{jq} S(P_j, \hat{P}_{jq}) \right]. \]

Furthermore, the optimal investment in the jth taxable security is such that, combined with an appropriate amount of the bond and insurance, it has the same return as investment \( a_j W \) in its tax exempt counterpart.

**Proof:** Equation (35) follows from Theorem 2. By Theorem 4 the investment opportunity set remains unchanged if the taxable securities are replaced by their tax exempt counterparts. Equation (34) is the change in wealth over \([t, t+dt]\) when wealth \( a_j W \) is invested in the jth security’s tax exempt counterpart, \( j = 1, 2, \ldots, N \). The maximized expected utility is given by (33) subject to (34) and (35). Also, optimal consumption is given by the solution to this problem. Having solved for the optimal investment, \( a_j W \), in the jth security’s tax exempt counterpart, the investor replicates this investment by a portfolio consisting of the taxable security, the bond and insurance. **Q.E.D.**

Merton [12, 13] proved that \( J(W(t), t), c \) and \( \{a_j\} \) are given by the solution to the problem

\[
0 = \max_{c, \{a_j\}} \left\{ U(c, t) + J_t + J_W \left( \sum_{j=1}^{N} a_j(\mu_j - r) + r \right) W + y - c \right. \\
+ \frac{1}{2} J_{WW} \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j \sigma_{ij} W^2 \right\}
\]

with initial condition (35) and boundary condition

\[
J(W(T), T) = \hat{U}(W(T), T)
\]

where the covariance \( a_{ij} \) is defined by \( a_{ij} dt \equiv \text{cov}(\sigma, dw, \sigma, dw) \).\(^{10}\) The solution is discussed in Merton. It suffices here to illustrate by example how the investor translates the investment in tax exempt securities into investment in taxable securities. Suppose that the investor owns one share with basis \( \hat{P}_j \), \( P_j > \hat{P}_j \). In solving equation (36) he finds that his optimal investment in the jth security’s tax exempt counterpart is \( a_j W \). He translates this investment into the following portfolio: he (i) retains the share and defers the capital gain; (ii) purchases (or, sells short, if negative) \((1 - \tau)^{-1}(a_j W / P_j) - V_p(P_j, \hat{P}_j)\) shares; (iii) invests \( a_j W - V(P_j, \hat{P}_j) - (1 - \tau)^{-1}(a_j W - P_j V_p(P_j, \hat{P}_j)) \) in a riskless, tax exempt bond; and (iv) purchases insurance which pays \( V(P_j, \hat{P}_j) - ((1 - \tau)P_j + \tau \hat{P}_j) \) in

\(^{10}\)If \( T \) is stochastic and generated by a Poisson process, Richard [17] has shown that we add the term \( \lambda U(W(t), t) - J(W(t), t) \) to the right-hand side of equation (36). Our discussion and Theorem 6 remain unchanged with the addition of this term.
the event of a forced liquidation. The reader may verify that the investor incurs net cash outflow \( a_j W - V(P_j, P_T) \) and the portfolio return over \([t, t + dt]\) equals the return of \( a_j W \) in the tax exempt counterpart.

6. EQUILIBRIUM IMPLICATIONS

6.1. The Equilibrium Relationship between Asset Returns: The Capital Asset Pricing Model

The investor's demand, \( a_j W \), for the securities' tax exempt counterparts is given by the first order conditions of problem (36) with respect to investment

\[
(\mu'_j - r)I_W + \sum_{j=1}^{N} a_j \sigma_{ij} WJ_{WW} = 0 \\
(j = 1, 2, \ldots, N).
\]

By aggregating demand across investors and setting it equal to aggregate supply, Merton [14] derived the capital asset pricing model

\[
\mu'_j - r = \beta_j(\mu_M - r).
\]

With zero capital gains tax, equation (39) is interpreted as follows: \( \mu'_j - r \) is the expected return of the \( j \)th security less the riskless rate. The market portfolio is a value-weighted portfolio of all outstanding securities. The \( j \)th security's beta coefficient is \( \beta_j = \sigma_{jm}/\sigma_M^2 \) where \( \sigma_{jm} \) is the covariance of the \( j \)th security with the market portfolio return and \( \sigma_M^2 \) is the variance of the market portfolio return. Also \( \mu_M \) is the expected rate of return of the market portfolio. With zero capital gains tax, the empirical significance of the market portfolio is that, in principle at least, it is observable and may be proxied by a portfolio of stocks, bonds, real estate, etc., in their market proportions.

With nonzero capital gains tax, the market portfolio in equation (39) has a cumbersome interpretation: it is the value-weighted portfolio of all outstanding securities' tax exempt counterparts. If there are \( A_j \) outstanding shares of the \( j \)th security, the weight of this security in the market portfolio is not \( A_j P_j \); it is the value of the tax exempt counterparts held by investors and depends on the basis of each security. With the basis of each security unobservable, the market portfolio is also unobservable.

When the capital gains tax is zero, Breeden [3] proved that the market portfolio is perfectly correlated with the change in aggregate consumption rate. This insight bypasses the cumbersome interpretation of the market portfolio when capital gains tax is nonzero and leads to the following theorem:

\[\text{Actually Breeden proved a more general result than implied here: even when the investment opportunity set is nonstationary, the asset pricing equation (39) holds, where the portfolio } M \text{ is most highly correlated with the change in aggregate consumption rate. In this generalized model, the portfolio } M \text{ is no longer identified as the market portfolio, even with zero capital gains tax. When the investment opportunity set is stationary, as we assume in this paper, the portfolio } M \text{ is perfectly correlated with the change in aggregate consumption rate.}\]
THEOREM 6: If Assumptions 1–10 hold, then

\[(40) \quad \left( \mu_j + \frac{\delta_j}{1 - \tau_d} \right) - r = \beta_j(\mu_M - r) + \left( k_j - r + \frac{\delta_j}{1 - \tau_d} \right) \]

\((j = 1, 2, \ldots, N)\)

where \(\beta_j = \sigma_{JM}/\sigma^2_M; \sigma_{JM}\) is the covariance of the jth security's return with the change in aggregate consumption rate; \(\sigma^2_M\) is the variance of the change in aggregate consumption rate; \(\mu_M\) is the expected rate of return of a tax exempt portfolio with return perfectly correlated with the change in aggregate consumption; and \(k_j\) is defined by equations (9) and (11).

PROOF: In equation (39), \(\beta_j = \sigma_{JM}/\sigma^2_M\), where \(\sigma_{JM}\) is the covariance of return of the jth security's tax exempt counterpart with the change in aggregate consumption rate. By definition, a security and its tax exempt counterpart have the same variance of return and their returns are perfectly correlated. Therefore \(\sigma_{JM}\) is also the covariance of the jth security's return with the change in aggregate consumption rate. Also in equation (39), \(\mu_M\) is the expected rate of return of a tax exempt asset with return perfectly correlated with the change in aggregate consumption rate. By Theorem 3, for each security j, \(\mu_j' = \mu_j - k_j + r\). We eliminate \(\mu_j'\) from equation (39), add \(\delta_j/(1 - \tau_d)\) to both sides and obtain equation (40). \(Q.E.D.\)

The expected before tax rate of return, \(\mu_j + \delta_j/(1 - \tau_d)\) on the jth security is the sum of the before tax expected capital gain rate, \(\mu_j\), and the before tax dividend yield, \(\delta_j/(1 - \tau_d)\). Theorem 6 states that the expected before tax rate of return is the sum of three terms: the riskless, tax exempt rate, \(r\), the risk premium \(\beta_j(\mu_M - r)\), which is proportional to the security's beta, \(\beta_j\); and the term \(k_j - r + \delta_j/(1 - \tau_d)\) which we call a tax premium. If all taxes are zero, i.e., \(\tau = \tau' = \tau_d = 0\), then \(k_j - r + \delta_j/(1 - \tau_d) = 0\) and the term is appropriate: it reflects the effect of stock variance, dividend yield and forced liquidations, on expected before tax return, in the presence of taxes. These effects are discussed next.

6.2. The Variance Effect

If dividends are zero, \(\delta_j = 0\), and if there are no forced liquidations, \(\lambda_j = 0\), then the tax premium is zero and equation (40) reduces to \(\mu_j = r + \beta_j(\mu_M - r)\). Given the security's beta, the expected stock return is independent of the stock variance. This is a surprising result because we earlier found (in Table III) that the price of the timing option is sensitive to the stock variance, even in the case \(\lambda = \delta = 0\).

From equations (40) and (9), the tax premium is (suppressing \(j\))

\[(41) \quad k - r + \frac{\delta}{1 - \tau_d} = \lambda + \frac{\delta}{1 - \tau_d} - \frac{c(1 - m)]\sigma^2/2}{b - m} .\]
Table IV displays the tax premium in the case $r = .10/\text{annum}$, $\tau = \tau' = .25$, and $\tau_d = .50$ for a range of $\sigma, \lambda, \delta$ values. Given $\lambda$ and $\delta$, the expected return of a low variance stock ($\sigma = .05/\text{month}$) exceeds the expected return of a high variance stock ($\sigma = .20/\text{month}$) with the same beta by about .01 per year. The higher expected return of the low variance stock compensates the investor for the low value of the timing option associated with the low variance stock. Table IV indicates that the magnitude of the variance effect is small. This conclusion agrees with the empirical findings in Fama and McBeth [8].

Table IV also indicates that the expected before tax stock return is higher, the higher the probability of a forced liquidation: frequent forced liquidations negate the benefit of capital gain deferral, and the investor is compensated by a higher before tax expected return.

### 6.3. The Dividend Yield Effect

We define the marginal effective tax rate on dividends, $\tau_d^e$, as the increase of the expected, before tax stock return per unit increase of the before tax dividend yield, $\delta(1 - \tau_d)^{-1}$. Taking the limit of the tax premium as $\sigma^2 \to 0$, we obtain (suppressing $j$)

$$
\lim_{\sigma^2 \to \infty} \left[ \lambda - \frac{\delta}{1 - \tau_d} - \frac{c(1 - m)\sigma^2/2}{b - m} \right]
$$

$$
= \lambda - \left( \frac{1 - \tau'}{1 - \tau} \right)\lambda + \left( \frac{\tau_d - \tau}{1 - \tau} \right) \left( \frac{\delta}{1 - \tau_d} \right).
$$

For high variance stocks, the effective tax rate on dividends is $(\tau_d - \tau)/(1 - \tau)$ and equals the effective tax rate derived in Brennan's [4] one period model. Taking the limit of the tax premium as $\sigma^2 \to 0$, we obtain

$$
\lim_{\sigma^2 \to 0} \left[ \lambda - \frac{\delta}{1 - \tau_d} - \frac{c(1 - m)\sigma^2/2}{b - m} \right]
$$

$$
= \lambda - \frac{1 - \tau'}{1 - \lambda \tau'/(r + \lambda)} + \left[ 1 - \frac{1 - \tau_d}{1 - \lambda \tau'/(r + \lambda)} \right] \left( \frac{\delta}{1 - \tau_d} \right).
$$
TABLE V

<table>
<thead>
<tr>
<th>$\lambda$</th>
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<th>$\sigma = .10$/month</th>
<th>$\sigma = .20$/month</th>
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<td>.42</td>
<td>.38</td>
</tr>
<tr>
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<td>.40</td>
<td>.39</td>
<td>.37</td>
</tr>
<tr>
<td>.20/annum</td>
<td>.38</td>
<td>.36</td>
<td>.36</td>
</tr>
</tbody>
</table>

*a We assume $r = .10$/annum, $\tau = \tau' = .25, \tau_d = .50.$

For low variance stocks, the effective tax rate on dividends is $1 - (1 - \tau_d)/(1 - \lambda \tau'/(r + \lambda)).$ If forced liquidations are infrequent, $\lambda$ is small and the effective tax rate is $\tau_d,$ the actual tax rate on dividends. If forced liquidations are frequent, $\lambda$ is large and the effective tax rate is $(\tau_d - \tau')/(1 - \tau').$

Table V displays the average effective tax rate on dividends as the before tax dividend yield increases from zero to .20 per year. To illustrate, consider the entry .46 for $\lambda = 0$ and $\sigma = .05$/month in Table V: if the before tax dividend yield increases from zero to .20 per year, the before tax expected stock return increases by .091 and the average effective tax rate on dividends is $.091/.20 = .46.$ Table V demonstrates that the effective tax rate varies across securities and depends on the security's variance of return and frequency of forced liquidations. The effective tax rate on dividends also depends on the dividend yield because the tax premium in equation (40) is nonlinear in the dividend yield. These predictions are consistent with Hess' [10] empirical finding that the effective tax rate varies across securities. The effective tax rate on dividends depends on the assumed relation between the tax rate on capital gains and dividends and is ultimately an empirical question.12 Table V illustrated values of the effective tax rate in the commonly assumed case, $\tau_d > \tau.$

7. CONCLUDING REMARKS

Capital gains and losses on assets held for one year or less are short term and are taxed at a higher rate than gains and losses on assets held for periods longer than one year. With the long term tax rate being about half the short term rate, the investor may optimally choose to realize a long term gain, repurchase the asset and afford himself the opportunity to realize future losses at the short term rate. It is shown in [5] that this is indeed the case for medium and high variance stocks, even with sizeable transactions costs. The optimal trading strategies and their equilibrium implications are discussed.

Finally the assumptions, that investors are in a uniform tax bracket for capital gains and in a uniform tax bracket for dividends, need to be relaxed in order to

12 See the papers by Black and Scholes [2], Litzenberger and Ramaswamy [11], Miller and Scholes [16], and the references therein.
study the equilibrium implications of tax clienteles. With heterogeneous tax brackets, we need to introduce ad hoc restrictions (e.g., limits on borrowing and selling short) in order to eliminate arbitrage opportunities between pairs of investors in different tax brackets.

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REFERENCES


