Strategic Analysis of the Competitive Exercise of Certain Financial Options

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Warrants and convertible bonds are claims on the firm which change the outstanding number of common stock shares when exercised or converted. Exercise of such claims in competitive circumstances is modeled here as a noncooperative game played by a continuum of players. First, equilibria of the game are shown to exist by applying results of Schmeidler. Second, the game's equilibria are compared to outcomes that come about when one individual controls the exercise of all the claims, but is constrained to exercise them in one block. The results are analogues of earlier results by one of the authors on the competitive pricing of such claims. Journal of Economic Literature Classification Number: 521.

1. INTRODUCTION

Warrants and convertible bonds are securities issued by a firm with an option on the common stock of the issuing firm. The holder of a warrant may, within a prespecified period of time, exercise the warrant; i.e., obtain a prespecified number of shares of common stock at a prespecified price, the exercise price. If a warrant expires unexercised, it becomes worthless. The holder of a convertible bond effectively owns an ordinary bond with the additional option to convert it into a prespecified number of shares of stock within a prespecified period of time. (About 10% of all bond issues listed on the New York Bond Exchange in 1979 included convertible provisions.)

The pricing of warrants and convertible bonds (generically referred to hereafter as "convertibles") is complicated by strategic timing factors recently recognized by several authors ([2–4]). The key feature that leads to the strategic complication is that exercising a convertible forces the firm to

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issue new shares of common stock, thereby affecting the value of the existing stock and of the remaining unexercised convertibles.

The main point of the present paper is that the noncooperative game model of Schmeidler [8], in which the player set is a nonatomic continuum, can be effectively used to model the strategic problem when the convertibles are initially held by perfect competitors. In our application of Schmeidler’s model the players’ actions are their exercise decisions. Section 2 of this paper is devoted to a description of Schmeidler’s model applied to a simple (but abstract) class of such situations. Schmeidler’s results about existence of equilibria are seen to apply.

In Section 3 additional financial structure is imposed and results analogous to those in [2] are established, even though in [2] the strategic aspects are not treated as a game. Although the notion of equilibrium is different in our game model, the first three propositions of [2] reappear in a somewhat altered form. These results compare the equilibria (in our model, Nash-type equilibria) of a situation in which no single individual controls the exercise of a significant fraction of the convertibles (competitive case) with the outcomes when one individual controls the exercise of all the convertibles but is constrained to exercise them all at once (block-constrained monopoly case). The results imply that, under certain assumptions, even though in the competitive case there is generally no equilibrium in which all the players exercise their convertibles at the same time, the equilibrium value of each unit of convertible (its price in [2]) equals the price of the convertible that would prevail in the regime of the block-constrained monopolist. The practical relevance of this is that prices of convertibles held by block-constrained monopolists have already been explicitly derived in [3] and [5] as an application of the Black–Scholes ([1]) option pricing theory.

Section 4 is devoted to some remarks about games that arise when the convertibles are not competitively held.

2. A Class of Nonatomic Games

We take as the set of players of our game the unit interval \( I \). The players’ initial holdings of the convertibles are described by a non-negative real-valued function \( w \) on \( I \). The function \( w \) is assumed bounded and measurable, and the total number of units of initially held convertible security is \( W = \int_I w(s) \, ds > 0 \).

Let \( \{1, \ldots, T\} \) be the finite set of times at each element \( t \) of which a player can choose to exercise any fraction of his initial holdings. (Any fraction left unexercised after time \( T \) is, by convention, exercised at time \( T + 1 \).) A player’s income accruing at time \( t \) depends on the actions of all players taken by time \( t \) and on random variables \( \theta_1, \ldots, \theta_k \), which have been realized by time
We assume that Θ, takes its value from a finite subset Θ, of the real numbers and becomes public knowledge as soon as it is realized. For notational convenience we include the trivial random variables Θ, and Θ_{T+1}, taking only the value zero. We define Θ = Θ, × ⋯ × Θ_{T+1}, and Θ|t = Θ, × ⋯ × Θ, for t ∈ {1,..., T}.

We consider only "open-loop" strategies, thereby assuming that players cannot condition their actions at any time t on actions taken by other players before time t, although we do allow them to condition on realizations Θ|t of the random vector Θ|t. The open-loop restriction is made for tractability, but it turns out to be not very restrictive after all (see footnote 2).

Formally, let F denote the set of functions f: Θ → {1,..., T + 1} with the (nonanticipating) property that for any t if f(Θ) = t and Θ' ∈ Θ satisfies Θ'_τ = Θ, ∀τ ≤ t, then f(Θ') = t. Note that F is a finite set with cardinality, say, Q. Associating with each element of F a unit vector in Δ, the unit simplex of Q-dimensional Euclidean space, any element in this simplex may be interpreted as a strategy for an individual, where the qth component of any such element describes the fraction of an individual's endowment to be exercised at the time selected by the element of F corresponding to q. (We shall therefore identify each component by its associated f from now on.) An (open-loop) strategy combination is then a measurable function y from I to Δ. For each strategy combination y, each t ∈ {1,..., T + 1}, and each Θ ∈ Θ, the total amount exercised at t is denoted

\[ \int y(t, \Theta) = \sum_{f: f(\Theta) = t} \int y_f(s) w(s) ds. \]

Let \( y \) denote the \( |\Theta| \times (T + 1) \) vector \((\int y(t, \Theta))\) and

\[ (\int y|t, \Theta) = (\int y(1, \Theta), \int y(2, \Theta),..., \int y(t, \Theta)). \]

To complete the description of the game we must specify each player's payoff for each strategy combination. Since the players' incomes accrue at various times \( t \in {1,..., T} \), we evaluate each of these income streams at time one. First, when the strategy combination is \( y \) and the realization of the random vector \( \Theta \) is \( \Theta \), the value at time one per unit of endowment of the stream of incomes to any player exercising all his endowment according to \( f \) is denoted \( V_f(\int y, \Theta) \). (A special form for \( V_f(\int y, \Theta) \) will be assumed in Section 3.) It is in the dependence of \( V_f(\int y, \Theta) \) on \( y \) only through \( \int y \) that the

1 The measurability restriction here is in some contrast to the spirit of finite-player noncooperative game theory, in which the only restriction on the set of joint strategies is that it be a product of individual strategy sets. In the context of a model with a continuum of players, however, the restriction seems unavoidable and not unduly restrictive.
EXERCISE OF FINANCIAL OPTIONS

n nonatomic endowment assumption affords its key simplification: no single player's action has an effect on any other player. Next, when the strategy combination is \( y \), the payoff per unit of endowment to a player who chooses to exercise all his endowment according to \( f \) is denoted \( U_f(\langle y \rangle) \) and is some function of the vector \( (V_f(\langle y, \theta \rangle))_{\theta \in \Theta} \).

A strategy combination \( y \) is an equilibrium if for (almost) all \( s \in I \) and for all \( z \in \Delta \),

\[
\sum_{f \in F} y_f(s) U_f(\langle y \rangle) \geq \sum_{f \in F} z_f U_f(\langle y \rangle).
\]

(Note that the linear form of the equilibrium condition means that the amount of the initial holdings of player \( s \) does not affect his own incentive to deviate from any strategy combination \( y \). Note also that fractional exercises in this model correspond to mixed strategies in [8].)\(^2\)

**Theorem 1.** If, for every \( f \in F \), \( U_f \) is continuous, then an equilibrium exists; furthermore, an equilibrium exists in which almost all players select unit vectors in \( \Delta \).

The proof of Theorem 1 is an immediate application of the two theorems in [8]. (Strictly speaking, our Theorem 1 only follows from Schmeidler's results when the endowment function \( w \) is constant. It is a simple matter, however, to extend Schmeidler's results to our setting.) Notice that Theorem 1 does not conclude that almost all players select the same unit vector; and, indeed, equilibria with that form will not generally exist. (See the example in Section 3.)

3. Properties of the Equilibria

In this section we specify the financial structure and establish features of the equilibria of the game.\(^3\) We assume that initially there are \( N > 0 \) shares of common stock in the firm outstanding and the measurement units are such

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\(^2\) Returning to the open-loop restriction: suppose now that more general strategies were permitted, with exercises at \( t \) allowed to depend also on other players' aggregated exercises before \( t \), and consider any equilibrium, defined analogously to equilibrium in the open-loop game. The realization of every player's strategy at the equilibrium as a function on \( \Theta \) is itself an open-loop strategy; and, from nonatomicity, the combination of such open-loop strategies must itself be an equilibrium in the open-loop game. On the other hand, any equilibrium in the open-loop game is necessarily an equilibrium in the game in which more general strategies are permitted.

\(^3\) For brevity we only sketch proofs and impose somewhat more structure on the game than is strictly necessary. The reader may wish to consult [2] for more details.
that one unit of convertible becomes one share of stock upon exercise. Thus there are potentially $N + W$ shares of stock if all units of the convertible are exercised. (It will become clear that there is no loss of generality in assuming that the set of initial shareholders is disjoint from the set of players.)

During each period $t \in \{1, \ldots, T\}$: some number of units of convertible are exercised at unit exercise price $b_t$; coupons are paid at the rate of $c_t$ per unit of the convertible not already exercised in periods $1, \ldots, t - 1$; and dividends are paid at the rate of $d_t$ per share to all initially held shares of stock plus all shares created by exercise in periods $1, \ldots, t$. (That both coupons and dividends accrue in the period of exercise is but one of many conventions that are compatible with our results.) The rates $b_t$, $c_t$, $d_t$ may depend on $(\theta \mid t)$ and on the total amounts exercised in each earlier period $1, \ldots, t - 1$. The rate $d_t$ may also depend on the units exercised in period $t$. Hence we write

$$b_t = b_t \left( \left( \int y \mid t - 1, \theta \right), (\theta \mid t) \right),$$
$$c_t = c_t \left( \left( \int y \mid t - 1, \theta \right), (\theta \mid t) \right),$$
$$d_t = d_t \left( \left( \int y \mid t, \theta \right), (\theta \mid t) \right).$$

The functions $(b_t, c_t, d_t)$ constitute the firm’s policy and are assumed fixed and publicly known to the players. (By convention, we set $b_{T+1} = c_{T+1} = d_{T+1} = 0$.)

The firm begins each period $t$ with an amount of capital $K_t$. $K_1$ is fixed, but $K_2, \ldots, K_T$ depend on the players’ actions and on the realization of the random vector. At time $t$, the capital $K_t$ increases by the proceeds from the exercise of convertibles, decreases by the coupons paid to convertibles unexercised by time $t$, and decreases by the dividends paid to stockholders of record at time $t$, including those players that have acquired stock via exercise at time $t$. The capital is then invested by the firm and grows by the factor $(1 + r)$ by the beginning of time $t + 1$. To this is added the random output $\theta_{t+1}$, which is independent of the amount invested. The firm’s capital at time $t + 1$ is therefore

$$K_{t+1} \left( \left( \int y \mid t, \theta \right), (\theta \mid t + 1) \right)$$
$$= \theta_{t+1} + (1 + r) \left[ K_t \left( \left( \int y \mid t - 1, \theta \right), (\theta \mid t) \right) \right]$$
$$+ \left( \int y(t, \theta) \right) b_t \left( \left( \int y \mid t - 1, \theta \right), (\theta \mid t) \right).$$
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\[- \left( \sum_{t=1}^{T+1} \left( \int y(t, \theta) \right) c_t \left( \left( \int y | t-1, \theta \right), (\theta | t) \right) \right) \]

\[- \left( N + \sum_{t=1}^{T} \int y(t, \theta) \right) d_t \left( \left( \int y | t, \theta \right), (\theta | t) \right) \].

The firm's policy \{b_t, c_t, d_t\} is limited to be within the class of functions that yield \(K_{T+1} = 0\) for every possible strategy combination and \(\theta\); i.e., the firm pays a liquidating dividend at time \(T\).

To specify the players' payoff functions, we assume that unlimited borrowing and lending are possible at the riskless rate of interest \(r\) (compare the firm's growth factor). Thus, when the strategy combination is \(y\) and the realization of the random vector is \(\theta\), the value at time one of the income stream to a player endowed with one unit of the convertible and exercising according to \(f\) is

\[V_f \left( \int y, \theta \right) = -\frac{1}{(1+r)^{r(\theta)-1}} b_{r(\theta)} \left( \left( \int y | f(\theta), \theta \right), (\theta | f(\theta)) \right) \]

\[+ \sum_{t \leq f(\theta)} \frac{1}{(1+r)^{t-1}} c_t \left( \left( \int y | t-1, \theta \right), (\theta | t) \right) \]

\[+ \sum_{t > f(\theta)} \frac{1}{(1+r)^{t-1}} d_t \left( \left( \int y | t, \theta \right), (\theta | t) \right) \].

We assume further that before the game begins a contingent claim that pays one unit of money at time \(T\) if \(\theta\) occurs (and nothing otherwise) is traded for each \(\theta\) on a competitive market. Let \(p(\theta)\) denote the ratio of the market price of this claim to that of a claim that pays one unit of money with certainty at \(T\) (alternatively think of \(p\) as a publicly available probability distribution on \(\theta\)) and assume that \(p\) is unaffected by what occurs in the game.

Under these assumptions

\[U_f \left( \int y \right) = \sum_{\theta} p(\theta) V_f \left( \int y, \theta \right) .\]

(If the functions \(b_t, c_t,\) and \(d_t\) are continuous in their respective first arguments, then \(U_f\) is continuous as assumed in Theorem 1.)

For any strategy \(y\), let \(m_y\) denote the function that selects the first exercise time under \(y\); i.e., for each \(\theta\)

\[m_y(\theta) = \min \left\{ t : \int y(t, \theta) > 0 \right\} .\]

The first lemma follows directly from the definitions.
Lemma 1. Suppose \( y \) is an equilibrium with \( \int y_j(s) w(s) \, ds > 0 \). Then for every \( g \in F \)

\[
U_f \left( \int y \right) = U_{m_y} \left( \int y \right) \geq U_x \left( \int y \right).
\]

Since at equilibrium each unit of endowment generates the same payoff, \( U_{m_y} \left( \int y \right) \) admits interpretation as the market price of a unit of convertible at the equilibrium \( y \).

It is worthwhile to go through a deterministic example at this point; i.e., \( \theta_i \) are all identically zero and are suppressed, and the \( f_i \)'s are identified with their respective exercise times. Let \( T = 2; N = 1; w(s) = 1 \forall s \in [0, 1]; b_i(\cdot) \equiv 1 \) for \( i = 1, 2; c_i(\cdot) \equiv 0 \) for \( i = 1, 2; K_1 = 10 \). For every strategy combination \( y \),

\[
U_1 \left( \int y \right) = -1 + d_1 \left( \int y_1 \right) + \frac{1}{(1 + r)(1 + \int y_1 + \int y_2)} \times \left( \int y_2 + (1 + r) \left[ 10 + \int y_1 - \left( 1 + \int y_1 \right) d_1 \left( \int y_1 \right) \right] \right)
\]

\[
U_2 \left( \int y \right) = -\frac{1}{1 + r} + \frac{1}{(1 + r)(1 + \int y_1 + \int y_2)} \times \left( \int y_2 + (1 + r) \left[ 10 + \int y_1 - \left( 1 + \int y_1 \right) d_1 \left( \int y_1 \right) \right] \right)
\]

\[
U_3 \left( \int y \right) = 0.
\]

Obviously, either \( U_1(\int y) > 0 \) or \( U_2(\int y) > 0 \) or both. Hence, from Lemma 1, at equilibrium:

\[
\int y_2 = 0 \text{ and } U_1 \left( \int y \right) \equiv U_2 \left( \int y \right) \equiv d_1 \left( \int y_1 \right) \equiv \frac{r}{1 + r}.
\]

\[
\int y = (1, 0, 0) \text{ at equilibrium } \Leftrightarrow d_1(1) \geq \frac{r}{1 + r};
\]

\[
\int y = (0, 1, 0) \text{ at equilibrium } \Leftrightarrow d_1(0) \leq \frac{r}{1 + r};
\]

\[
\int y = (\alpha, 1 - \alpha, 0) \text{ at equilibrium for } 0 < \alpha < 1 \Leftrightarrow d_1(\alpha) = \frac{r}{1 + r}.
\]

Assuming \( d_1 \) continuous guarantees (via the intermediate value theorem or
Theorem 1) that at least one equilibrium exists. Furthermore, it should be obvious that by manipulating \( d_i \) we can produce any number (\( \geq 1 \)) of equilibria.

Returning to our development we come to

**Lemma 2.** For every strategy combination \( y \), and for every \( \theta \in \Theta \),

\[
K_1 + \sum_{t=1}^{T} \frac{1}{(1+r)^{t-1}} \theta_t = N \sum_{t=1}^{T} \frac{1}{(1+r)^{t-1}} d_t \left( \left( \int y | t, \theta \right), (\theta | t) \right) + \sum_{f \in F} V_f \left( \int y, \theta \right) \int y_f(s) w(s) ds.
\]  

(1)

Lemma 2 states that for every state \( \theta \) the game is constant-sum when the initial stockholders are included as dummy players. Its proof is immediate from the definitions and the requirement that \( K_{T+1} = 0 \). (If, in addition, \( y \) is an equilibrium and we multiply (1) by \( p(\theta) \) and sum across \( \theta \), we then obtain one version of the Modigliani–Miller [7, 8] theorem: namely, that the sum of the market values of all the stock and the convertibles outstanding at time one (the market value of the firm) equals the sum of the firm’s capital in place, \( K_1 \), plus the market value of the stream \( \theta_1, ..., \theta_T \).

Before proceeding to our results concerning the block-constrained monopolist, one last bit of notation is necessary to describe his strategic possibilities. If \( f \in F \), denote by \( e(f) \) the exercise vector arising when all players use \( f \).

**Theorem 2.** For any equilibrium \( y \),

\[
U_{m_y}(e(m_y)) = U_{m_y} \left( \int y \right).
\]

Since \( e(m_y) \) is a possible strategy for a block-constrained monopolist, Theorem 2 states that the value of each unit of convertible to the block-constrained monopolist who exercises according to \( m_y \) is the same as the market price of each unit at the equilibrium \( y \).

A sketch of the proof is as follows. Suppose the strategy combination changed from \( y \) to \( e(m_y) \). The payoff to each unit of endowment thereby changes from \( U_{m_y}(y) \) (Lemma 1) to \( U_{m_y}(e(m_y)) \), hence all (active) players' payoffs change in the same direction. The change affects \( U_{m_y} \) only through the dividends from \( m_y(\theta) \) on (for each \( \theta \)), however, hence the original stockholders' dividends are also changed in the same direction. From Lemma 2, therefore, the changes must be zero.

Nothing so far implies that the time of first exercise at equilibrium corresponds to the time at which a block-constrained monopolist would
choose to act. To explore this, let $t^*: \Theta \to \{1, \ldots, T+1\}$ select the last time at which a block-constrained monopolist would choose to act; i.e.,

$$U_i(e(t^*)) \geq U_j(e(f^*)) \forall f \neq t^*$$

with the inequality being strict for all $f$ satisfying $f(\theta) > t^*(\theta)$ for some $\theta$. (Such a $t^*$ clearly exists.)

**Theorem 3.** If $y$ is any equilibrium, then for every $\theta$,

$$\sum_{t=1}^{t^*(\theta)} \int y(t, \theta) > 0. \tag{2}$$

The proof is a consequence of Lemma 2 and Theorem 2. If (2) failed for some $\theta$ and if $\int y(t)(s) w(s) ds > 0$, then by defining the pure strategy $g$ by $g(\theta) = \min(f(\theta), t^*(\theta))$, the contradiction $U_g(\int y) > U_j(\int y)$ could be obtained. Theorem 3 states that by the last time at which a block-constrained monopolist would choose to act, at least some convertibles have been exercised at every equilibrium.

**Theorem 4.** Under the hypothesis of Theorem 1 there is an equilibrium $y$ with the property that

$$\sum_{t=1}^{t^*(\theta)-1} \int y(t, \theta) = 0 \text{ and } \int y(t^*(\theta), \theta) > 0$$

for all $\theta$. At this equilibrium,

$$U_{m_y}(\int y) = U_i(e(t^*)). \tag{3}$$

This theorem states the main result of this section: whereas there may exist more than one equilibrium, the highest convertible market price among these equilibria equals the convertible price in a block-monopolist’s regime; and the block-monopolist exercises the entire block of convertibles at the same time at which exercise begins in this equilibrium.

It is also possible, however, that equilibria exist with times of first exercise earlier than any times at which a block-constrained monopolist would act. To illustrate, consider the example again with $d_i(1) \geq r/(1+r) > d_i(0)$. In this case we know that equilibria exist both with all exercises at time 1 and with all exercises at time 2, yet

$$U_i(0, 1, 0) > 9/2 = U_i(1, 0, 0).$$

The proof of Theorem 4 involves constructing an auxiliary game in which
no exercise is permitted before $t^*(\theta)$ for each $\theta$. Modifying Theorems 1 and 3 appropriately produces an equilibrium in the auxiliary game with positive exercise at the time selected by $t^*(\theta)$. It is then tedious but not difficult to check that this equilibrium of the auxiliary game is also an equilibrium of the original game. Equation (3) follows from Theorem 2.

4. Concluding Remarks

We have considered only the competitive case and the block-constrained monopoly case. One can equally well imagine an unconstrained monopolist, a finite collection of oligopolists, or a collection of oligopolists together with a competitive set of players (with the oligopolists represented as atoms in the initial-endowment measure). In [3] an example of a duopoly is described, each player holding one (indivisible) unit of a warrant which can either be exercised or held. The example has a unique Nash equilibrium at which neither player exercises, but a monopolist holding both warrants would choose to exercise one and hold one.

To illustrate some additional possibilities, consider again the example in Section 3 with $r = \frac{3}{4}$ and $d_1(\alpha) = \max(0, \frac{1}{2} - \alpha)$ for $\alpha \in [0, 1]$. From our earlier analysis, since $d_1(0) > r/(1 + r) > d_1(1)$, the only equilibria in the competitive case occur at values of $\alpha$ such that $d_1(\alpha) = \frac{3}{5}$. Hence there is a unique (up to permutation of the players) equilibrium in the competitive case with $\alpha = \frac{1}{10}$. Now suppose instead that the convertible blocks are in two indivisible blocks, each of size $\frac{1}{2}$, and each block is held by a different individual, neither of whom owns any of the stock initially. Again we have a game in which each individual must choose whether to exercise his block in the first or second period (the option of no exercise is clearly dominated). The payoff per unit of security is $\frac{9}{7}$ if they both exercise in the first period and $\frac{85}{26}$ if they both exercise in the second. If one player chooses the first period and the other the second period, the player choosing the second period receives $\frac{96}{26}$ per unit and the other player receives $\frac{85}{26}$. Thus the game has the structure of a “prisoner’s dilemma,” the only equilibrium being that both players exercise in the second period. We know then that the returns per unit at the unique duopoly equilibrium are worse than the returns per unit at the unique equilibrium of the competitive case. To summarize, in this example the players fare equally well in the competitive case and the block-constrained monopoly case but worse than both of these in the block-constrained duopoly case.
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