ADMISSIBLE UNCERTAINTY IN THE INTERTEMPORAL ASSET PRICING MODEL

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Received October 1978, final version received January 1980

We embed the Sharpe–Lintner, two-parameter asset pricing theory in an intertemporal general equilibrium model. The investment opportunity set changes stochastically over time, in general the short-term and long-term interest rates and the distribution of the rate of return of the market portfolio are non-stationary. This non-stationarity, which is admissible in the Sharpe–Lintner model, has two implications. First, it may bias econometric methods which fail to explicitly take into account the non-stationarity. Second, the sequential application of the Sharpe–Lintner model in the discounting of stochastic cash flows becomes computationally complex and of little practical use.

1. Introduction

We develop a general equilibrium theory of capital asset pricing which is simple yet accommodates many stylized empirical facts. The advantages of this theory are threefold. First, being a general equilibrium theory, it is defined in terms of fundamentals, which are production technologies and consumers' resources, beliefs, and preferences. Thus, for example, shifts in interest rates, in production levels, and in risk aversion are endogenously determined at the equilibrium.

Second, the theory is simple. Although the theory allows for an investment opportunity set which changes stochastically over time, risk premia on asset returns are explained solely in terms of the covariance of asset returns with the market portfolio. This feature of the theory contrasts with the theories of Merton (1973) and Long (1974) wherein risk premia on asset returns depend on the covariance of asset returns not only with the market portfolio, but also with a number of hedging portfolios. The identification of the hedging

*Earlier versions of this paper were presented in seminars at Carnegie-Mellon University, the University of Chicago, Columbia University, Bell Laboratories, and the Western Finance Association meetings. I thank the participants and, in particular, Sudipto Bhattacharya for their helpful comments. I also thank the editor of this Journal and the referee, John B. Long, Jr., for detailed and helpful comments. I remain responsible for errors.
portfolios and the estimation of asset covariances with these portfolios are nontrivial empirical issues.

Third, the theory accommodates the following stylized empirical facts:

(a) The probability distribution of asset returns and of the market portfolio is non-stationary. Indeed, rates of return of assets and of the market portfolio may be serially correlated. Also, the beta coefficients of assets and portfolios relative to the marked portfolio are non-stationary.

(b) The term structure of interest rates evolves stochastically over time. In particular, specific hypotheses on the term structure, e.g., various forms of the expectations hypothesis, may or may not be true.

(c) The empirical evidence with regard to the validity of the two-parameter asset pricing equation is inconclusive. The failure of some tests to confirm the two-parameter asset pricing equation may be attributed to an incorrect proxy for the (as yet unobservable) market portfolio which was used in those tests. Given the empirical difficulty at this stage to reject the two-parameter asset pricing equation in favor of a multi-parameter version, and to identify the various hedging portfolios, there is an advantage in theoretically developing a two-parameter asset pricing model which is in agreement with the empirical evidence cited in (a) and (b).

(d) Consumers are heterogeneous and hold portfolios which are not identical in composition.

The theory has an important implication which relates to the empirical testing of the SLM asset pricing model and to the application of the model in empirical studies of market efficiency. Even if the SLM model is the true description of the economy, as opposed to the multi-factor extensions of the model, and even if there are no substantive difficulties in the identification of the true market portfolio, yet the distribution of the market portfolio return, the return on the riskless asset, the security betas, and the market price of risk may well be non-stationary and may bias econometric methods which fail to explicitly take into account these non-stationarities.

The theory also illustrates the limitations of the SLM asset pricing model in the discounting of stochastic cash flows of multi-period projects. Non-stationarity in the distribution of the market portfolio return, the return on the riskless asset, the security betas, and the market price of risk is admissible in the context of the SLM model. Therefore, the sequential

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1 The theory and empirical findings on the asset pricing model are reviewed in Jensen (1972). An extensive critique of the empirical work appears in Roll (1977), where the sensitivity of the empirical results on the chosen proxy for the market portfolio is stressed.

2 Similar comments apply to multi-factor extensions of the SLM model and to Breeden's (1979) consumption CAPM.
application of the SLM model in the discounting of stochastic cash flows of multi-period projects becomes computationally complex and of little practical use, unless one can produce convincing evidence to the effect that these non-stationarities are unimportant in practice.

A preliminary and informal discussion of our main ideas serves as an introduction to the formal development which begins in section 2. The single-period capital asset pricing theory (SLM) of Sharpe (1964), Lintner (1965), Mossin (1966), and Black (1972) rests on the assumption that each consumer's utility is solely a function of consumption at the beginning of the period and of wealth at the end of the period, but is otherwise independent of the state of the economy at the end of the period. Fama (1970) embedded the SLM theory in an intertemporal economy where each consumer sequentially makes consumption and investment decisions over his lifetime. Fama noted that the SLM theory requires each consumer's derived utility in every period to be a function of his wealth at that period (and possibly of his past consumption) but otherwise to be independent of the state of the economy.

State dependent derived utility has three possible sources: (a) state dependent tastes, (b) stochastic relative prices of consumption goods, and (c) investment opportunities which depend on preceding events. Our model eliminates the first source of state dependence by assuming that tastes are state independent. It eliminates the second source of state dependence by focusing on a single-good economy wherein relative asset prices of consumption goods do not exist. Fama (1970) and Merton (1973) eliminated the third source of state dependence by assuming that investment opportunities do not depend on preceding events.

The following example illustrates that there exist economies where the SLM asset pricing equation may hold true even if investment opportunities depend on preceding events. Consider a single-good economy with homogeneous consumers, where each consumer's utility function is state independent. Assume that there is only one investment opportunity in positive net supply and that the distribution of the rate of return on this investment opportunity is independent of preceding events. Trade amongst the homogeneous consumers is redundant and therefore each consumer consumes a fraction of his wealth in every period and invests the remaining fraction of his wealth in the single investment opportunity which exists in positive net supply. It can be shown that each consumer's derived utility function in every period is state independent and therefore the SLM asset equations hold true.

Similar comments apply to multi-factor extensions of the SLM model and to Breeden's (1979) consumption CAPM. Banz and Miller (1978) and Breeden and Litzenberger (1978) proposed an alternative computational procedure which circumvents some of these difficulties. The prices of state contingent claims are first estimated through the option pricing model and are used in discounting future cash flows. Bhattacharya (1979) proposed yet another procedure, the four asset hedge, which is an extension of the Black–Scholes option hedge.
pricing equation holds true under appropriate additional assumptions. Yet the equilibrium rates for short-term and long-term borrowing and lending amongst consumers may well depend on the level of aggregate wealth and therefore may depend on preceding events.

Whereas the assumption of homogeneous consumers is restrictive, we obtain the same kind of simplification even if consumers are heterogeneous, as long as prices are determined as if consumers were homogeneous. Formally, an economy of heterogeneous consumers is said to have the aggregation property if equilibrium prices are determined as if consumers were homogeneous in terms of resources, beliefs, and preferences. Since we are primarily interested in asset pricing, if an economy possesses the aggregation property, we may derive the asset prices in a simplified economy where all consumers are homogeneous. We shall refer to the latter consumers as aggregated consumers. Contracts amongst aggregated consumers (i.e., financial contracts) are in zero net supply. As long as the distribution of the rates of return on assets in positive net supply is independent of preceding events, the derived utility of aggregated consumers is state independent.

Under appropriate additional assumptions the SLM pricing equation still holds true. These arguments are formalized in section 3 and stated as Propositions 1, 2, and 3. An example in section 4 illustrates these ideas and discusses their implications on the theory, testing, and applications of the asset pricing model.

There have been earlier attempts to enrich the economic environment in which the SLM asset pricing equation holds true. Fama and MacBeth (1974) allowed both the investment opportunity set and consumers' utility functions to be state dependent but forbade the consumers from engaging in contracts which hedged against changes in the investment opportunity set and in consumers' tastes. The latter assumption essentially requires that markets be incomplete. Stapleton and Subrahmanyam (1978) assumed that consumers have additive exponential utility of consumption and that the interest rate is non-stochastic and is determined exogenously to the model. A more promising approach was taken by exploiting the myopic properties of logarithmic utility, which were noted by Mossin (1968) and Hakansson (1971). Under the assumption that consumers have logarithmic utility, Rubinstein (1976b) derived the SLM equation in an economy where the investment opportunities depend on preceding events. Rubinstein (1976a)

Rubinstein (1976b) formally assumed that the aggregated consumers alone have logarithmic utility and that the actual consumers have generalized logarithmic utility. By generalized logarithmic utility we mean that the $i$th consumer's utility of consumption $c_i'$ at date $t$ is $\ln(c_i' - c_i^*)$, where $c_i^*$ is a parameter specific to the $i$th consumer. It can be shown that the aggregated consumers' utility of consumption $c_i'$ at date $t$ is $\ln(c_i' - \sum_{t=1}^I c_i'/I)$, where $I$ is the number of consumers in the economy. If the aggregated consumers have logarithmic utility, i.e., $\sum_{t=1}^I c_i'/I = 0$, yet at least some consumers have non-logarithmic utility, there must exist a consumer for whom $c_i'^* < 0$. 

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also derived the SLM equation in an economy where the consumers have power utility and the rate of growth of aggregate consumption follows a random walk. More recently, Breeden and Litzenberger (1978) and Breeden (1979) showed that risk premia on asset returns can be explained in terms of the covariance of asset returns with aggregate consumption alone, if consumers have additive power utility functions or asset prices are governed by a diffusion process. In contrast to the above-mentioned theories, our theory does not rely on market incompleteness, it does not rely on a non-stochastic interest rate which is exogenously determined; it does not rely on logarithmic or power utility, which has the implication that consumers hold portfolios with identical composition, and it does not critically rely on the diffusion process. Section 4 illustrates a model which is free from all of the abovementioned assumptions.

The theory developed in this paper is related to the recent work by Brock (1978), Cox, Ingersoll and Ross (1978), and Prescott and Mehra (1978), wherein the financial theory of capital asset pricing is embedded in a general equilibrium framework. The goal of our paper, however, differs significantly from the goals of the abovementioned three papers. We seek sets of minimal conditions on technologies and on consumers' resources, beliefs and preferences, which retain the simplicity of the two-parameter asset pricing equation.

2. Consumption–investment behavior and equilibrium in the securities market

We employ the following notation

\( t = \) time subscript,
\( \phi = \) state of the economy at \( t \), it summarizes all relevant available information,

\( F_t(\phi_t | \phi_{t-1}) = \) distribution function of \( \phi_t \) conditional upon \( \phi_{t-1} \),
\( N_t = \) number of investment opportunities available at date \( t-1 \), if a riskless asset exists, it is included in the set \( N_t \),

\( R_t(\phi_t) = \{ R^1_t(\phi_t), R^2_t(\phi_t), \ldots, R^N_t(\phi_t) \} = \) one plus real rate of return on investment opportunities over \( (t-1, t) \),

\( I = \) unit vector,

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For this consumer the marginal utility of consumption is finite at zero consumption level, i.e., \( (d/dC)\ln(C - C_i)_t | C_t = 0 = -\frac{1}{C^2_t} < \infty \). Thus the non-negativity of consumption constraint, \( C_i \geq 0 \) may well be binding. Rubinstein's theory does not explicitly consider this constraint and is therefore consistent with an economy where \( \lambda_i \geq 0 \) for all \( i \). Then if the aggregated consumers have logarithmic utility, all consumers must have logarithmic utility, i.e., \( \lambda_i = 0 \), for all \( i \). See also related comments in Grauer (1976) and Rubinstein (1977, p. 30).

Bhattacharya (1979) also derived the consumption CAPM by considering the appropriate limit as the time interval tends to zero in Rubinstein's (1976a) valuation equation.
\[ T = \text{consumer's lifetime}, \]
\[ C_t \equiv \{c_0, c_1, \ldots, c_T\} = \text{consumption vector until date } t, \]
\[ U(C_T | \phi_T) = \text{utility of lifetime consumption}, \]
\[ w_t \equiv \{w_{t}^1, w_{t}^2, \ldots, w_{t}^{N_{t+1}}\} = \text{consumer's wealth invested in the } N_{t+1} \text{ investment opportunities at date } t \text{ and after consumption at date } t, \]
\[ W_t = \text{consumer's wealth at date } t \text{ and prior to consumption at date } t \]

There is no presumption that consumers are identical and, therefore, \( T, C_t, w_t, W_t, U(C_T | \phi_T) \) are in general different across consumers

We make the following assumptions, (A 1)–(A 3), which are maintained throughout the paper, unless otherwise stated

(A 1) **Perfect markets** Investment opportunities are traded in perfect markets. In particular, there are no transactions costs or taxes, assets are infinitely divisible, consumers are price takers, the market is in equilibrium, if a riskless asset exists, consumers can borrow and lend at the same interest rate, and short sales of all assets, with full use of proceeds, are allowed

(A 2) **Homogeneous expectations** Consumers' expectations are homogeneous. Alternatively, consumers have heterogeneous expectations, but the aggregation problem is solved so that equilibrium prices are determined as if consumers were homogeneous

(A 3) **State independent utility** Each consumer's utility of lifetime consumption does not explicitly depend upon the state of the economy, i.e.,
\[ U(C_T | \phi_T) = U(C_T) \]

A family of probability distributions, defined by Ross (1978), and labeled 'separating distributions', plays an important role in the theory of asset pricing, as will be shortly explained. Commonly employed distributions which are separating distributions include multivariate normal distributions of returns, stable Paretoian distributions of returns, market factor models, and diffusion processes, where each decision period is infinitesimal. These distributions are labeled separating because in a one-period economy, where consumers have state independent utilities, these distributions imply two-fund portfolio separation

Our purpose in this section is to relax the following assumption which is typically made in the derivation of the SLM asset pricing equation

**Constant investment opportunity set** \( N_t \) is at most a function of time, and the probability distribution of \( R_t \) is independent of \( \phi_{t-1} \) for all \( t \)

*The set of separating distributions is defined in Ross (1978a, thm 2)*
Typically, the theory assumes (A 1)–(A 3), a constant investment opportunity set and either (a) the distribution of \( R_t(\phi_t) \) conditional upon \( \phi_{t-1} \) is a separating distribution, or (b) each consumer's utility function is additive quadratic. Under these assumptions one derives the SLM asset pricing equation\(^7\)

\[
E_{t-1}[R_t^1 - R_t^Z] = \frac{\text{cov}_{t-1}(R_t^1, R_t^M)}{\text{var}_{t-1}(R_t^M)} E_{t-1}[R_t^M - R_t^Z], \tag{1}
\]

where \( R_t^M \) is the return on the market portfolio and \( R_t^Z \) is the return on the zero-beta portfolio, i.e., \( \text{cov}_{t-1}(R_t^M, R_t^Z) = 0 \).

We shall prove that the SLM asset pricing equation can be derived even if we partly relax the assumption that the investment opportunity set is constant. Consider a consumer who consumes a fraction of his wealth in every period and invests the remaining fraction of his wealth in the \( N_t \) investment opportunities. Following Fama (1970) we formulate the consumer’s sequential optimization problem as

\[
J_{t-1}(C_{t-1}, W_{t-1} | \phi_{t-1}) \equiv \max_{\{c_{t-1, w_{t-1}} \in \Omega_{t-1}\} \phi_{t}} \left\{ J_t(C_{t-1}, w_{t-1}, R_t(\phi_t) | \phi_t) \right\} dF_t(\phi_t | \phi_{t-1}), \tag{2}
\]

for \( t = 1, 2, \ldots, T \). \( \Omega_{t-1} \) is the set of feasible consumption and investment decisions such that

\[
\begin{align*}
0 &\leq c_{t-1} \quad \text{(non-negative consumption),} \\
c_{t-1} + w_{t-1} I &\leq W_{t-1} \quad \text{(budget constraint),} \\
0 &\leq w_{t-1} R_t(\phi_t), \quad \forall \phi_t \quad \text{(no personal bankruptcy)}
\end{align*} \tag{3}
\]

The boundary condition is

\[
J_T(C_{T-1}, W_T | \phi_T) = U(C_{T-1}, W_T | \phi_T) \tag{4}
\]

The objective (2) subject to constraints (3) and (4) completely defines the consumer’s optimization problem. The following lemma can be proved by straightforward induction.

**Lemma** Consider a consumer’s optimization problem defined by eqs (2)–(4). Assume that the consumer optimally invests a non-zero amount of wealth in the subset of the first \( n_t \) investment opportunities out of the total \( N_t \) investment opportunities which are available at date \( t-1 \), where \( t = 1, 2, \ldots, T \), and optimally invests zero wealth in the remaining \( N_t - n_t \) investment opportunities.

Assume also that the probability distribution of the rate of return on each one of the first \( n \) investment opportuni- ties is independent of \( \phi_{t-1} \), for all \( t \). Finally, assume (A 3), \( U(C_T | \phi_T) = U(C_T) \). Then the consumer's derived utility function is state independent, i.e., \( J_{t-1}(C_{t-2}, W_{t-1} | \phi_{t-1}) = J_{t-1}(C_{t-2}, W_{t-1}) \), for all \( t \).

In the next section the lemma is incorporated in a general equilibrium theory of asset pricing.

3. Technical uncertainty and admissible uncertainty in the asset pricing model

We first consider a production sector with constant returns to scale technologies. At date \( t-1 \) there exist \( n \) competitive firms. In general, the number of investment opportunities exceeds the number of firms, i.e., \( n_i \leq N_i \). These include bilateral contracts amongst consumers or groups of consumers, such as borrowing-lending contracts of various maturities, option contracts and insurance policies. Furthermore, a levered firm offers to consumers more than one type of investment opportunities such as equity and bonds with different indenture provisions. Without restricting the investment opportunity set we assume that firms are unlevered, since equity and bonds of a levered firm can be manufactured through the writing of options on the equity of an unlevered firm which has the same technology.

The \( j \)th firm is endowed with a linear production function \( f(K_{t-1}^j) = K_{t-1}^j R_t^j \). \( K_{t-1}^j \) is the input at date \( t-1 \) and \( K_{t-1}^j R_t^j \) is the output at date \( t \). Since the firm is competitive, the market value of the firm at date \( t-1 \) and after new capital is raised at \( t-1 \) equals \( K_{t-1}^j \), and the value of the firm at date \( t \) and before new capital is raised at \( t \) equals \( K_{t-1}^j R_t^j \). By assumption, firms are unlevered and therefore, the one plus rate of return on the equity of the \( j \)th firm equals \( R_t^j \). We denote the vector of returns of the \( n_i \) competitive firms by \( \tilde{R}_i = \{ R_1^i, \ldots, R_{n_i}^i \} \). We note the distinction between \( \tilde{R}_i \) and \( R_n \), where \( R_n \) is the vector of returns of the \( n_i \) firms and of the remaining \( N_i - n_i \) investment opportunities.

Production inputs are nonnegative, i.e., \( K_{t-1}^j \geq 0 \) for all \( j, t \). At some \( t, j \), this constraint may well be binding, in which case the \( j \)th technology is inactive at date \( t-1 \).

The probability distribution of \( \tilde{R}_i \) is known and is independent of preceding events, i.e., is independent of \( \phi_{t-1} \). By contrast, the distribution of returns of the \( N_i - n_i \) investment opportunities may well depend on preceding events. For example, the equilibrium value of the short-term interest rate may depend on preceding events.

We shall collectively refer to the assumptions which we have made so far.
on the production sector as assumption (A 4) We now prove the following proposition

**Proposition 1**\(^8\) Assume perfect markets (A 1), homogeneous expectations (A 2), state independent utility (A 3), and production technologies and firms conforming to assumption (A 4) Also assume that the aggregation problem is solved, i.e., equilibrium prices are determined as if consumers were identical in terms of resources, beliefs and preferences\(^9\) And the distribution of \(\hat{R}_r\) is a separating distribution Then the asset pricing equation (1) holds for the rates of return on the equity of the \(n_f\) firms Eq (1) also holds for the rates of return of the remaining \(N_0 - n_f\) investment opportunities if \(\hat{R}_r\) is generated by a diffusion process

**Proof** A relative asset pricing relationship will be determined in the aggregated economy, wherein consumers have homogeneous resources,

\(^8\)Two variations of the assumptions of Proposition 1 are stated. These variations lead to the following proposition

**Proposition 1’** Assume perfect markets (A 1), homogeneous expectations (A 2), state independent utility (A 3), and production technologies and firms conforming to assumption (A 4) Also assume that one technology is riskless and consumers are sufficiently risk averse that the riskless technology is always active. Finally, make one of the following two assumptions

(i) Decisions are made continuously in time and the vector \(\hat{R}_r\) is generated by a diffusion process

(ii) Consumers have additive quadratic utility functions

Then the asset pricing equation (1) holds for the rates of return on all investment opportunities

Proposition 1’, unlike Proposition 1 in the main text, implies that the term structure of interest rates evolves deterministically over time.

The proof of Proposition 1’ is similar to the proof of Proposition 1 and is not presented here.

The proof is available from the author upon request.

\(^9\)Rubinstein (1974) provides sufficient conditions for aggregation in a two-date economy where the \(i\)th consumer maximizes \(U_i(c_o) + \rho_i E V(c_1)\) and \(-V^r(W)/U^r(W) = A_i + B_i W, -V^r(W)/V^r(W) = A_i + B_i W\) Rubinstein’s conditions are listed below

(i) All consumers have the same resources, beliefs and tastes \(\rho_i, U_i, V_i\)

(ii) All consumers have the same beliefs, rates of patience \(\rho_i\), and taste parameters \(B_i \neq 0\)

(iii) All consumers have the same beliefs and taste parameters \(B = 0\)

(iv) All consumers have the same resources, beliefs, and taste parameters \(A = 0, B = 1\)

(v) A complete market exists and all consumers have the same taste parameter \(B = 0\)

(vi) A complete market exists and all consumers have the same resources and tastes \(\rho_i, A_i = 0\), and \(B = 1\)

Furthermore, these conditions easily generalize to an economy with more than two dates. Brennan and Kraus (1978) demonstrate the necessity of these conditions for aggregation.

Some qualifying remarks are called for. Rubinstein’s proof assumes that the conditions on non-negative consumption, \(c_o \geq 0\) and \(c_1 \geq 0\) are not binding. In order to ensure this, we shall further assume that \(B > 0\) and \(A \leq 0\). These conditions imply that marginal utility of consumption becomes infinite at some non-negative level of consumption, i.e., \(U'(W) = (A + B W)^{-1}\) becomes infinite at \(W = -A/B \geq 0\). Conditions \(B > 0\) and \(A \leq 0\) eliminate Rubinstein’s cases (iii) and (v). Furthermore, in order to ensure that there exist feasible consumption paths for the \(i\)th consumer, we shall assume that his endowment at date zero exceeds \(-B \Sigma_{i=0}^{(K')} \hat{A}_i\), where \(K'\) is the one plus rate of return on the riskless technology.
beliefs and preferences. The assumed aggregation property implies that the 
same relative asset pricing relationship holds in the original economy also, 
because the aggregation property implies that the original economy and the 
aggregated economy are supported by the same prices.

Consider the aggregated economy. Bilateral contracts amongst 
homogeneous consumers are redundant. Given an equilibrium where the 
homogeneous consumers engage in bilateral contracts, there exists another 
equilibrium supported by the same prices, wherein consumers have the same 
consumption schedules, yet they do not engage in bilateral contracts. This 
implies that equilibrium prices are determined as if consumers invest only in 
the \( n \) firms at date \( t - 1 \). But \( \hat{R}_t \) is independent of \( \phi_{t-1} \). The lemma implies 
that the derived utility of each aggregated consumer is state independent. 
This observation, along with the other assumptions made in the proposition, 
implies that the asset pricing eq (1) holds for the rates of return on the equity 
of the \( n \) firms.

If \( \hat{R}_t \) is generated by a diffusion process, then the rates of return on the 
remaining \( N_t - n \) investment opportunities satisfy eq (1) also. To see this, 
note that Merton's (1973) two-parameter asset pricing equation applies to 
the rates of return of contracts in zero net supply as well as to the rates of 
return of firms. The proof is complete.

Proposition 1 is illustrated in the next section through an example. In the 
remaining part of this section, three generalizations of Proposition 1 are 
considered. In the first generalization, we relax the requirement that 
production technologies have constant returns to scale. Specifically, we 
assume:

\[(A.5) \text{ Production technologies and firms} \]

The output of the \( j \)th firm at date 
\( t \) is \( f_j(K_{t-1}^j, \tilde{\xi}_t, t) \) where \( K_{t-1}^j \) is the input and \( \tilde{\xi}_t \) is a vector of random shocks 
which is independent of \( \phi_{t-1} \). Each firm is competitive and profit-
maximizing.

In the particular case where production occurs continuously in time and 
the production uncertainty is generated by a diffusion process, the output of 
the \( j \)th firm at date \( t + dt \) is

\[K_t^j + g^j(K_t^j, t)[\mu^j(\phi_t, t)dt + \sigma^j(\phi_t, t)dw(t)],\]

where \( K_t^j \) is the input at date \( t \), \( g^j(K_t^j, t) \) is a function of the input and time, 
\( \mu^j(\phi_t, t) \) is a function of the state and time, \( \sigma^j(\phi_t, t) \) is an \( M \)-dimensional 
vector whose elements are functions of the state and time, and \( dw(t) \) is the 
increment of the Wiener process \( w(t) \) in \( R^M \).
We now state

**Proposition 2** Assume perfect markets (A.1), homogeneous expectations (A.2), state independent utility (A.3), production technologies and firms conforming to assumption (A.5), the aggregation problem is solved. Also production occurs continuously in time and the production uncertainty is generated by a diffusion process. Then the asset pricing eq (1) holds for the rates of return on all investment opportunities.

**Proof outline** Unlike the proof of Proposition 1, we may not assert that the rate of return on the securities on the jth firm equals the rate of return $f_j(K_{t-1}, \hat{e}_t, t)/K_{t-1} - 1$ on the investment of the jth firm. Since, however, the increment $d\mu(t)$ of the Wiener process is independent of preceding events, the state of the aggregated economy is summarized by the level of aggregate wealth. The composite consumer’s derived utility is a function of aggregate wealth alone. Arguments similar to those used in the proof of Proposition 1 complete the proof of Proposition 2.

The line of reasoning which we employed in the proof of Proposition 2 also serves to simplify the asset pricing equation derived by Brock (1978, eq. (2.31)). Brock assumed that consumers are homogeneous and that production technologies and firms conform to assumption (A.5). We conclude that the state of the economy at every point in time is summarized by just one state variable, namely aggregate wealth. Thus, Brock’s K-factor asset pricing equation may be collapsed to a single-factor asset pricing equation. In Brock’s model there exist, not K ‘Ross risk prices’, but just one such price which is identified as the price of market risk.

In the second generalization of Proposition 1 we allow the tastes of consumers and the production technologies to be state dependent and obtain the $(m+2)$-factor model of Merton (1973) and Long (1974) (A similar generalization applies to Proposition 2).

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10 Two variations of the assumptions of Proposition 2 are noted. These variations lead to the following proposition.

**Proposition 2'** Assume perfect markets (A.1), homogeneous expectations (A.2), state independent utility (A.3), production technologies and firms conforming to assumption (A.5). Also assume that one technology is riskless and has constant returns to scale, and consumers are sufficiently risk averse that this technology is always active. Finally, make one of the following two assumptions,

(i) Production occurs continuously in time and the production uncertainty is generated by a diffusion process.

(ii) Consumers have additive quadratic utility functions.

Then the asset pricing equation (1) holds for the rates of return on all investment opportunities.

The proof is similar to earlier proofs and is left to the reader.
Proposition 3  Assume perfect markets (A.1), homogeneous expectations (A.2), state dependent utility $U(C_T \mid \phi_T) = U(C_T \mid x_T)$, where $x_t$ is an $m$-vector which is a subset of the information set $\phi_t$ such that $F(x_t \mid \phi_{t-1}) = F(x_t \mid x_{t-1})$, production technologies and firms described by (A.4), where the technology depends on $x_{t-1}$, i.e., $F(\tilde{R}_t \mid \phi_{t-1}) = F(\tilde{R}_t \mid x_{t-1})$, and the aggregation problem is solved. Also, make one of the following assumptions

(i) $\tilde{R}_t$ is multivariate normal

(ii) Production occurs continuously in time and the production uncertainty is generated by a diffusion process

(iii) Consumers have additive quadratic utility functions

Then, in case (i) the $(m+2)$-factor asset pricing equation holds for the rates of return on the equity of the $n_t$ firms. In cases (ii) and (iii) the $(m+2)$-factor asset pricing equation holds for the rates of return on all investment opportunities.

The proof is similar to the proof of Proposition 1 and is omitted. The proposition is easily generalized to the case where consumers aggregate to $L$ homogeneous classes. Then an asset pricing equation with $m+L+1$ factors holds.

4. An illustration of admissible uncertainty and some concluding remarks

We consider an economy where consumers have homogeneous beliefs and the $i$th ($i=1, 2, \ldots, I$) consumer has additive HARA utility

$$U^i(C_T) = \sum_{t=0}^{T} \rho^{-t} \left( \frac{1-\gamma}{\gamma} \right) \left( \frac{c^i_t}{1-\gamma} - \tilde{c}^i_t \right)^{\gamma},$$

(7)

where $\rho, \gamma, \tilde{c}^i_t$ are parameters such that $\rho > 0$, $\gamma < 1$ and $\tilde{c}^i_t \geq 0$ for all $i, t$. In order to ensure that there exist feasible consumption paths for the $i$th consumer, we assume that his endowment $W_0$ at date zero satisfies $W_0 > (1-\gamma) \sum_{t=0}^{T} (R^1)^{-t} \tilde{c}^i_t$, where $R^1 > 0$ is the one plus rate of return on a riskless technology which may or may not be active. Under these conditions equilibrium prices are determined as if consumers were identical and the composite consumer had resources $W_0 = \frac{\sum_{i=1}^{I} W_0^i}{I}$, utility (7) with parameter $\tilde{c} = \sum_{i=1}^{I} \tilde{c}^i / I$, and beliefs the same as the homogeneous beliefs of all consumers. We further assume that markets are perfect. The production sector is described by assumption (A.4) and the distribution of $\tilde{R}_t$ is a separating distribution. We note that the assumptions of Proposition 1 are met and, therefore, the two-parameter asset pricing equation holds. Yet we shall demonstrate that the interest rate and the rate of return on the market portfolio are state dependent.
The aggregation property ensures that the set of prices, which supports the aggregated economy, also supports the original economy. Therefore, the interest rate and the rate of return on the market portfolio in the aggregated economy equal the interest rate and the rate of return on the market portfolio in the original economy. Without loss of generality, then we shall study the behavior of these rates in the aggregated economy. In the aggregated economy, the composite consumer invests in technologies with one plus rates of return \( \hat{R}_i \), subject to the constraint that production inputs are nonnegative. In general, technologies may be active or inactive. Without loss of generality, we assume that the composite consumer does not invest in any assets in zero net supply, i.e., does not engage in bilateral contracts, because such contracts are redundant in an economy with homogeneous consumers. The efficient frontier is illustrated in Fig. 1, under the assumption that \( \hat{R}_i \) is multivariate normal and the riskless technology is inactive.

![Efficient Frontier Diagram](image)

**Fig. 1** Illustration of the efficient frontier and the derived utility function of the composite consumer at date \( t-1 \). The tangency point \( M \) designates the market portfolio held at date \( t-1 \).

By construction, the equilibrium short-term interest rate \( r \) is such that the composite consumer does not borrow or lend. \( R_i^1 \) is the rate of return on a riskless technology which in general may or may not be active. In this particular illustration, the riskless technology is inactive. \( R_p \) is the rate of return on the portfolio of assets held by the composite consumer.

Fig. 1 also illustrates a utility indifference curve of the composite consumer for pairs of expected portfolio return, \( E(R_p) \), and standard deviation of portfolio return, \( \text{std}(R_p) \), which is tangent to the efficient frontier. The composite consumer chooses portfolio \( M \) which is the point of tangency of his utility indifference curve and the efficient frontier. Portfolio \( M \) is the value-weighted portfolio of the securities of the \( n \) firms. This portfolio is identified as the market portfolio. Note that the remaining \( N_1 - n \) investment opportunities (bilateral contracts, options, insurance contracts) are in zero net supply and therefore are not included in the market portfolio. The tangent at \( M \) to the utility indifference curve and the efficient frontier meets
the vertical axis at intercept \( r_t > R^1 \), \( r_t \) is the shadow interest rate. If we allow the possibility of borrowing/lending amongst the homogeneous consumers, \( r_t \) is the equilibrium interest rate at which the borrowing/lending market clears.

In general, the slope of the composite consumer's utility indifference curves is a function of the composite consumer's wealth, \( \hat{W}_{t-1} = \sum_{i=1}^{t-1} \hat{W}_i / I \). Only in the special case of power utility, i.e., \( \hat{c}_i = \sum_{i=1}^{t-1} \hat{c}_i / I = 0 \), are the composite consumer's indifference curves independent of his wealth. In general, however, \( \hat{c}_i \geq 0 \) and the point of tangency \( M \) depends on \( \hat{W}_{t-1} \). The composition of the market portfolio, the distribution of the rate of return on the market portfolio and the interest rate are all functions of \( \hat{W}_{t-1} \). At dates prior to \( t - 1 \), the composite consumer's wealth \( \hat{W}_{t-1} \) is a stochastic variable because it depends on the realizations of the production processes. Therefore, the distribution of the market portfolio and the interest rate at date \( t - 1 \), are state dependent, in particular, they are functions of the state variable \( \hat{W}_{t-1} = \sum_{i=1}^{t-1} \hat{W}_i / I \). Despite this state dependence, we have shown that the SLM asset pricing equation is satisfied.

The economy described in this section illustrates the first implication of our theory, that the central role which the market portfolio occupies in the modern theory of portfolio selection and relative asset pricing may not be fully warranted. In this economy, the covariance matrix of the rates of return of the production processes is stationary. Therefore, the covariance matrix of the rates of return of the subset of firms, which are active in production, is also stationary. Yet the excess return on the stock of any particular firm will be non-stationary because the interest rate will, in general, be non-stationary. Furthermore, the covariance of the return and the beta coefficient of the stock of any particular firm with respect to the market portfolio will be non-stationary. In contrast to the above, the arbitrage theory of asset pricing developed in Ross (1976) and tentatively tested in Roll and Ross (1979) is free from the above criticism. Ross's theory postulates a \( k \)-factor, linear stationary market model in which the market portfolio plays no special role. Referring again to the economy described in this section, our theory implies that the market portfolio return may be an inappropriate factor in the market model, yet the stationary rates of return of the production processes may be adequately described by a \( k \)-factor, stationary market model, as suggested by Ross.

The economy described in this section also illustrates the second implication of our theory. The market price of risk and the covariance of the cash flows of a project with the market return will, in general, depend on

\[ ^{11} \text{Likewise in Breeden's (1979) consumption CAPM the expected return on a portfolio which is perfectly correlated with changes in aggregate consumption, and the beta coefficient of the stock of any particular firm with respect to changes in aggregate consumption will, in general, be non-stationary.} \]
preceding events Therefore, the sequential application of the SLM model in the discounting of stochastic cash flows becomes computationally complex and of little practical use

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