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MARKET RISK ADJUSTMENT IN PROJECT VALUATION

GEORGE M. CONSTANTINIDES*

I. INTRODUCTION

This paper develops a rule which reduces the problem of valuation in the presence of market risk to the problem of valuation in a world where the market price of risk is zero. Given a valuation problem in an intertemporal, continuous time framework, the rule is applied in two steps as follows: We first replace one or more of the model parameters by their “effective values” in a specified way. Then we discount all expected cash flows at the riskless rate of return as if the market price of risk were zero. The broad applicability of the valuation rule is illustrated through the diverse examples of asset valuation, option pricing, determination of the optimal capital structure of a firm, and cash management.

We motivate the discussion of this paper by briefly examining a forerunner to our valuation rule, the certainty equivalence approach to evaluating a stream of cash flows. In a single period model, the risk-adjusted net present value RANPV(\(\bar{X}\)) of the cash flow \(\bar{X}\), realized at the end of the period, is given by the single period Sharpe-Lintner capital asset pricing model (CAPM) as

\[
\text{RANPV}(\bar{X}) = \frac{\bar{X} - (\bar{R}_M - R_F)\text{cov}(\bar{R}_M, \bar{X})/\sigma_M^2}{1 + R_F}.
\]

Essentially this formula states that the expected cash flow \(\bar{X}\) is adjusted to its certainty equivalent \(\bar{X} - (\bar{R}_M - R_F)\text{cov}(\bar{R}_M, \bar{X})/\sigma_M^2\), which is subsequently discounted at the riskless rate of return, \(R_F\). The certainty equivalence approach may also be viewed as follows: The expected cash flow \(\bar{X}\) is first adjusted to its “effective value” \(\bar{X} - (\bar{R}_M - R_F)\text{cov}(\bar{R}_M, \bar{X})/\sigma_M^2\) and is then discounted at the riskless rate of return \(R_F\), as if the market price of risk were zero. Viewed in this way, the certainty equivalence approach resembles our valuation rule. Indeed it will be apparent later in the paper that the certainty equivalence approach is a special case of our valuation rule.

The paper is organized as follows: Section II states the assumptions which lead to Merton’s (1973) intertemporal CAPM. In section III we define the problem and derive the valuation rule in the simplest case. Three examples are considered in section IV where the rule is applied to the valuation of an asset, the pricing of a European call option, and the determination of the optimal capital structure of a firm. In section V we generalize the rule to controlled processes and apply it to a cash management problem. In section VI we provide concluding remarks.

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II. THE CAPITAL ASSET PRICING MODEL

The assumptions which lead to the intertemporal CAPM, as developed by Merton (1973), are stated below.

A. Perfect Markets

The securities of firms are traded in a perfect capital market. In particular there are no transactions costs in trading securities; there are no taxes; securities are infinitely divisible; investors can borrow and lend at the same interest rate; short sales of all securities, with full use of the proceeds, are allowed; and trading in securities takes place continuously in time.

B. Securities

The prices of securities are lognormally distributed. For each security, the expected rate of return per unit time, $\alpha_i$, and the variance of return per unit time, $\sigma_i^2$, exist and are finite with $\sigma_i^2 > 0$. The opportunity set is non-stochastic in the sense that $\alpha_i$, $\sigma_i^2$, the covariance of returns per unit time $\sigma_{ij}$, and the riskless borrowing-lending rate $r$, are all nonstochastic functions of time.

C. Investor Preferences

Each investor maximizes his strictly concave and time-additive utility function of consumption over his lifespan. Investors have homogeneous expectations regarding the investment opportunity set.

Under these assumptions, Merton (1973) proved that the equilibrium security returns satisfy the CAPM relationship

$$\alpha_i - r = \lambda \sigma_{im} / \sigma_M$$

where the subscript $M$ refers to the market portfolio and $\lambda \equiv (\alpha_M - r) / \sigma_M$.

We shall be studying projects which are traded by firms in a perfect market, but which are not necessarily traded in the capital market. We now prove that the equilibrium return of any project, which is held by a firm, also satisfies the CAPM relationship (1). Hamada (1969) showed that a firm undertakes a project only if $\alpha_P - r > (\alpha_M - r) \sigma_{PM} / \sigma_M^2$, where the subscript $P$ designates the project. The firm realizes a windfall gain if the inequality is strict. Under the assumption of free

1. This result follows from Merton's (1973) theorem 1. Some minor differences between Merton's assumptions and the assumptions made here are noted. These differences do not affect the derivation of the theorem:
   a) Unlike Merton, we make a distinction between the terms "project" and "security". Depending on the interpretation of the project, in specific applications investors in the capital market may or may not be allowed to invest in the project directly.
   b) Merton assumed that firms issue only equity. We assume that firms may also issue debt.
   c) Merton assumed that firms pay no dividends and accomplish the transfer of cash to shareholders through share repurchase. We allow dividends as well as share repurchase.

In the next section we shall be considering projects with state dependent returns and which, therefore, violate assumption B, that the opportunity set is non-stochastic. It has been proven in Constantinides (1977) that the assumption that the opportunity set is non-stochastic may be replaced by the weaker assumption that the efficient frontier (and not necessarily the entire investment opportunity set) is state independent. Projects with state dependent returns are in agreement with the latter assumption.
entry by new firms, the equilibrium price of the project is sufficiently high so that no firm realizes a windfall gain through undertaking the project. Then the equilibrium return of this project satisfies (1).

Assumption A may be relaxed to allow for the case where a riskless asset does not exist. It is a simple matter to repeat Merton’s (1973) derivation in the absence of a riskless asset, and obtain the continuous-time version of Black’s (1972) CAPM. Equation (1) still obtains, where now \( r \) is interpreted as the expected return of the zero-beta portfolio. For the sake of concreteness, in the discussion of the following sections we shall assume that a riskless asset exists and we shall be referring to \( r \) as the riskless rate. However, our entire discussion applies also to the case where there does not exist a riskless asset. One would simply have to replace our references to the riskless rate by the expected return on the zero-beta portfolio.

III. THE GENERAL MODEL

We consider a project of market value \( V(x, t) \) which is completely specified by the state variable \( x \), and time \( t \). In specific applications which appear in later sections, the project will be interpreted as being an investment, an option, a claim on a firm, or a subsystem of the firm. The project generates a stream of cash flows and the market value of the project represents the time- and risk-adjusted value of these cash flows.

We assume that the state variable \( x \) changes in the time interval \((t, t+dt)\) by

\[
dx = \mu dt + \sigma dw
\]

where \( \mu = \mu(x, t) \), \( \sigma = \sigma(x, t) \) and \( dw \) is the increment of a Wiener process \( w(t) \).\(^2\) We also assume that the cash return generated by the project in the time interval \((t, t+dt)\) is \( c dt \), where \( c = c(x, t) \).

The return on the project in the time interval \((t, t+dt)\) is the sum of capital appreciation \( dV(x, t) \) and cash return \( c dt \). Assume that the function \( V(x, t) \) is twice continuously differentiable w.r.t. \( x \) and once continuously differentiable w.r.t. \( t \).\(^3\) We may then write \( dV(x, t) \) through Ito’s Lemma\(^4\) as

\[
dV(x, t) = \left( V_t + \mu V_x + \frac{\sigma^2}{2} V_{xx} \right) dt + \sigma V_x dw
\]

2. For a heuristic discussion of the Wiener process see Kushner (1971), chap. 10. For a rigorous discussion, see Breiman (1968).

A Wiener process is defined as follows: For any partition \( t_0 < t_1 < t_2 < \cdots \) of the time interval \([t_0, \infty)\), the random variables \( w(t_1) - w(t_0), w(t_2) - w(t_1), w(t_3) - w(t_2), \ldots \), are independent and normally distributed with mean zero and variance \( t_1 - t_0, t_2 - t_1, t_3 - t_2, \ldots \), respectively. Therefore, for any infinitesimal time interval, the random disturbance of the state is normally distributed with mean \( \mu(x, t) dt \) and variance \( \sigma^2(x, t) dt \).

We assume that the functions \( \mu(x, t) \) and \( \sigma(x, t) \) satisfy the necessary regularity conditions, so that the stochastic process defined by (2) exists.

3. The differentiability assumption must be verified in every specific application, once the function \( V(x, t) \) has been explicitly evaluated.

where \( V_t = \frac{\partial V(x,t)}{\partial t} \) etc. The rate of return on the project is

\[
d\frac{V(x,t)}{V(x,t)} + c dt = \frac{1}{V} \left( c + V_t + \mu V_x + \frac{\sigma^2}{2} V_{xx} \right) dt + \frac{\sigma V_x}{V} dw
\]

(4)

with expected value per unit time \( \alpha_p \), and covariance with the market per unit time \( \sigma_{pm} \) given by

\[
\alpha_p = \left( c + V_t + \mu V_x + \frac{\sigma^2}{2} V_{xx} \right) / V
\]

(5)

and

\[
\sigma_{pm} = \rho \sigma M \sigma V_x / V
\]

(6)

where \( \rho = \rho(x,t) \) is the instantaneous correlation coefficient between \( dw \) and the market return.

Under the assumptions made in section II, the project return satisfies the intertemporal CAPM of equation (1). We substitute (5) and (6) in (1) and upon simplification obtain

\[
c - rV + V_t + (\mu - \lambda \rho \sigma) V_x + \frac{\sigma^2}{2} V_{xx} = 0.
\]

(7)

The solution to this partial differential equation which satisfies the relevant boundary conditions gives the market value of the project. For reasons that will be apparent later, we rewrite (7) as

\[
c - r\hat{V} + \hat{V}_t + \mu^* \hat{V}_x + \frac{\sigma^2}{2} \hat{V}_{xx} = 0
\]

(8)

where

\[
\mu^* = \mu^*(x,t) \equiv \mu(x,t) - \lambda \rho(x,t) \sigma(x,t).
\]

(9)

Consider, for the moment, a capital market which pays no premium for market risk, \textit{i.e.} \( \alpha_M = r = 0 \). Denote by \( \hat{V}(x,t) \) the value of the above project in this market.

We set \( \alpha_M = r = 0 \) in (7) and obtain the following differential equation for \( \hat{V}(x,t) \)

\[
c - r\hat{V} + \hat{V}_t + \mu \hat{V}_x + \frac{\sigma^2}{2} \hat{V}_{xx} = 0.
\]

(8')

Since the boundary conditions are independent of the market risk premium,\(^6\) the boundary conditions imposed on \( \hat{V} \) are identical to those imposed on \( V \).

Comparison of equations (8) and (8') indicates that \( V(x,t) \) may be considered as the market value of the project in a capital market which pays no premium for market risk, provided the drift \( \mu(x,t) \) of the state variable is replaced by \( \mu^*(x,t) \).

5. This procedure was first used by Merton (1970) in deriving the differential equation which is satisfied by the price of a European call option.

6. This point will be clarified in the next section when applications are discussed and the boundary conditions are explicitly stated.
This observation suggests the following rule for determining the market value of a project.

**Rule**

Step one: Replace the drift $\mu(x,t)$ by

$$
\mu^*(x,t) = \mu(x,t) - \lambda \rho(x,t) \sigma(x,t).
$$

Step two: Evaluate the stream of cash flows as if the market price of risk were zero, i.e., discount expected cash flows at the riskless rate.

The rule is easily generalized to the case where the state variable $x$ is a vector $x = [x_1, x_2, \ldots, x_n]$ rather than a scalar. Changes in the elements of $x$ satisfy a joint diffusion process such that $E(dx_i) = \mu_i \, dt$ and $\text{cov}(dx_i, dx_j) = \sigma_{ij} \, dt$, where $\mu_i$ and $\sigma_{ij}$ are functions of $x$ and $t$. Proceeding as before we derive the following differential equation which is analogous to equation (7)

$$
c - rV + V_i + \sum_{i=1}^{n} (\mu_i - \lambda \rho_{iM} \sigma_i) V_i + \frac{1}{2} \sum_{i,j=1}^{n} \sigma_{ij} V_{ij} = 0.
$$

The rule still holds, that is to say, we first replace the drift parameters $\mu_i$ by $\mu_i^* = \mu_i - \lambda \rho_{iM} \sigma_i$, $i = 1, \ldots, n$, and ignore market risk thereafter.

The rule is also easily generalized to the case where the cash flow generated by the project in the time interval $(t, t+dt)$ is stochastic and is given by $c dt + sdz$ instead of $c dt$, where $c = c(x,t)$, $s = s(x,t)$ and $dz$ is the increment of a Wiener process $z(t)$ which in general is correlated with $dw$ of equation (2). Equation (4) is replaced by

$$
\frac{dV(x,t) + c dt + sdz}{V(x,t)} = \frac{1}{V} \left[ c + V_i + \mu V_x + \frac{\sigma^2}{2} V_{xx} \right] dt
$$

$$
+ \sigma V_x dw + sdz
$$

and equation (7) is replaced by

$$
(c - \lambda \rho_c s) - rV + V_i + (\mu - \lambda \rho \sigma) V_x + \frac{\sigma^2}{2} V_{xx} = 0
$$

where $\rho_c = \rho_c(x,t)$ is the instantaneous correlation coefficient between $dz$ and the market return. The latter equation suggests a slightly generalized version of the rule: The drift parameter $\mu(x,t)$ is replaced by $\mu^*(x,t)$ of equation (9) as before. Furthermore the expected cash flow rate $c(x,t)$ is replaced by $c^*(x,t)$ where

$$
c^*(x,t) = c(x,t) - \lambda \rho_c(x,t) s(x,t).
$$

Note that the adjustment for market risk of the drift parameter $c(x,t)$ is formally identical to the corresponding adjustment of the drift parameter $\mu(x,t)$. Upon adjustment of the drift parameters $\mu$, $c$ we evaluate the stream of cash flows as if the market price of risk were zero. It is worthwhile to note that the correlation
between the Wiener processes \( w(t) \) and \( z(t) \) does not influence the valuation procedure.

Examples which illustrate the usefulness and general applicability of the rule are presented in the next two sections.

IV. APPLICATIONS OF THE RULE

A. Valuation of an Asset

Consider an asset with lifetime \( t \). In any time interval \( (t, t + dt) \) the asset earns \( dx \) where \( dx = \mu dt + \sigma dw \) and \( \mu, \sigma, \rho \), are constant (as before, \( \rho \) is the correlation coefficient of \( dw \) and market return). All earnings are payable in cash to the owner of the asset at time \( T \). No interest accrues on the earnings from the time that they are earned until the time that they are paid. The latter assumption is made in order to simplify the example.

We denote the cumulative earnings of the asset at time \( t \) by \( x(t) \) and the value of the asset by \( V(x, t) \). The boundary condition is \( V(x, T) = x \), which says that at time \( T \), all cumulative earnings are paid in cash and the value of the asset at \( T \) equals the cumulative earnings. Notice that this boundary condition does not involve the market price of risk. Also \( c(t) = 0 \). We now apply the rule developed in the last section as follows:

Step one: Replace the expected earnings rate \( \mu \) by \( \mu^* = \mu - \lambda \rho \sigma \).

Step two: Evaluate the asset as if the market price of risk were zero. The expected cumulative earnings at time \( T \) are \( \mu^* T \), and the net present value of \( \mu^* T \) at time zero is \( \mu^* T e^{-rT} \). Thus the value of the asset at time zero is \( (\mu - \lambda \rho \sigma) T e^{-rT} \).

More generally, if \( \mu, \rho, \sigma \) are functions of time, the value of the asset at time zero is

\[
e^{-rT} \int_0^T (\mu - \lambda \rho \sigma) dt.
\]

The procedure which we have followed may now be related to the certainty equivalence approach: \( \mu - \lambda \rho \sigma \) represents the certainty equivalent of the expected earnings rate \( \mu \), which is subsequently discounted at the riskless rate. However in other applications, where the drift parameter \( \mu \) may not be interpreted as an earnings rate, the analogy with the certainty equivalence approach is no longer valid.

B. Valuation of an Option

Consider a European call option with market value \( V(x, t) \) at time \( t \), where \( x \) is the market price of the security on which the option is written. The security price

\[
7. We do not assume that the security on which the option is written is necessarily priced according to the CAPM (1). A feasible scenario is as follows: An option is written in the U.S. on a foreign security. Institutional constraints, such as taxes, transactions costs, and special restrictions prohibit trading of the foreign security in the U.S. capital market. In general, this security is not priced according to the U.S. CAPM. Upon exercise of the option, the holder of the option receives the dollar value which the foreign security commands in the foreign capital market in lieu of the certificate of the foreign security. For simplicity we disregard fluctuations in the exchange rate.

More generally the option may be written on some index \( x \), such as a consumer price index, where again the dynamics of \( x \) are given by \( dx / x = \alpha dt + \sigma' dw \), with \( \alpha, \sigma' \) constants.
is lognormally distributed, i.e., \(dx/x = \alpha dt + \sigma' dw\), where \(\alpha, \sigma'\) are constants. In terms of our earlier notation, \(dx = \mu dt + \sigma dw\), where \(\mu = \alpha x\) and \(\sigma = \sigma' x\). The maturity date of the option is \(T\) and the exercise price is \(E\). No dividends are paid.

Since no cash flows are received before time \(T\), we set \(c(t) = 0\). The boundary condition is \(V(x, T) = \max[0, x - E]\) and is independent of the market price of risk. Following our rule, we proceed as follows:

Step one: Replace \(\mu\) by \(\mu^* = \mu - \lambda \sigma = (\alpha - \lambda \sigma') x\). Price changes are now described by \(dx/x = \alpha^* dt + \sigma' dw\), where \(\alpha^* = \alpha - \lambda \rho \sigma'\).

Step two: Ignore market risk and discount expected cash flows at the riskless rate. Thus

\[
V(x_0, 0) = e^{-rT} \int_{x_0}^{\infty} (x - E) dF(x_T|x_0)
\]

where \(F(x_T|x_0)\) is the probability distribution of \(x(T)\) conditional on \(x(0) = x_0\). The solution of this integral gives

\[
V(x_0, 0) = x_0 e^{\alpha^* - r} N \left( \frac{\ln(x_0/E) + (\alpha^* + \sigma^2/2)T}{\sigma \sqrt{T}} \right) - E e^{-rT} N \left( \frac{\ln(x_0/E) + (\alpha^* - \sigma^2/2)T}{\sigma \sqrt{T}} \right)
\]  

(10)

where \(N(\cdot)\) is the standard error function.

In the special case when the security on which the option is written, is traded in the capital market without market imperfections, by the CAPM (1) we obtain \(\alpha - r = \lambda \rho \sigma'\) which implies \(\alpha^* = r\). Then (10) becomes the familiar Black-Scholes (1973) formula for the pricing of an option

\[
V(x_0, 0) = x_0 N \left( \frac{\ln(x_0/E) + (r + \sigma^2/2)T}{\sigma \sqrt{T}} \right) - E e^{-rT} N \left( \frac{\ln(x_0/E) + (r - \sigma^2/2)T}{\sigma \sqrt{T}} \right)
\]  

(11)

C. Theory of the Firm and Optimal Capital Structure

We consider a firm with equity \(V_e\) and a discount bond \(V_b\) at time \(t = 0\). The bond matures at time \(t = 1\) and the firm liquidates at the same time. We assume that the firm may take no action at times within the time interval \((0, 1)\) such as issuing or buying back equity and debt. Whereas the latter assumption is restrictive, it is typical of one-period models of the firm where the firm makes decisions only at the points in time \(t = 0, 1\), and is made here solely for the purpose of illustrating the application of our rule to a particular class of models of the firm.

8. This integral is explicitly evaluated in Sprengle (1964), the appendix.
We denote by \( x(1) \) the value of the firm assets at \( t = 1 \), before taxes and before the repayment of the bond. If the firm is able to meet its obligation to bondholders, the bondholders are paid in full and the equityholders receive the remaining wealth after taxes are paid. If the firm is unable to meet its obligation to bondholders, then the firm is declared bankrupt. Depending on the specific model assumptions the firm is liquidated or reorganized, the equityholders receive nothing and the bondholders receive the residual wealth \( x(1) \) net of bankruptcy costs and possible tax payments. Without discussing the details of these models we may say that, given some \( x(1) \), the stockholders and bondholders receive at time one \( Y_e(x(1)) \) and \( Y_b(x(1)) \) respectively, as functions of \( x(1) \).\(^9\) The values of equity and debt at time zero are then the expected values of \( Y_e(x(1)) \) and \( Y_b(x(1)) \), respectively, discounted for time and risk. Whereas earlier models had ignored the adjustment of the discount rate for market risk, more recently Rubinstein (1972) and Kim have adjusted the expected value of \( Y_e \) and \( Y_b \) for both time and market risk through application of the one-period CAPM. With costly bankruptcy this adjustment is computationally complex and it typically involves the determination of partial moments. On theoretical grounds this procedure is objectionable for the following reason: Even if the distribution of \( x(1) \) is normal, the distributions of \( Y_e \) and \( Y_b \) are not normal (at best they are truncated normal), and a one-period CAPM is applicable only under the assumption that investors have quadratic utility functions. Gonzalez, Litzenberger and Rolfo (1977) have recently analyzed the absurdity of the application of the one-period CAPM to cash flows which are not normally distributed. Specifically they showed that the implicit assumption of quadratic utility function leads to an optimal capital structure of the firm, for which the firm returns are stochastically dominated by the returns of a firm with a non-optimal capital structure.

We shall demonstrate that the adjustment of \( Y_e \) and \( Y_b \) for market risk through application of our rule is straightforward and free from the fallacies discussed by Gonzalez et al. (1977). We make two assumptions. First we assume that the equity and debt are traded continuously over the time interval \([0, 1]\). Second we assume that the random variable \( x(1) \) is distributed as follows: Let the random variable \( z(t) \) be defined by

\[
dz(t) = \mu(z,t)dt + \sigma(z,t)dw
\]

\[z(0) = 1\]

where \( dw \) is the increment of a Weiner process. We define \( x(1) \) by

\[x(1) = z(1)\]

This definition of \( x(1) \) covers a wide class of distributions. In particular, if \( \mu, \sigma \) are constant, \( x(1) \) is normally distributed with mean \( \mu \) and variance \( \sigma^2 \); if \( \mu = \hat{\mu}z \), \( \sigma = \hat{\sigma}z \), where \( \hat{\mu}, \hat{\sigma} \) are constant, \( \ln x(1) \) is normally distributed with mean \( \hat{\mu} - \hat{\sigma}^2/2 \) and variance \( \hat{\sigma}^2 \).

We apply our rule to this problem as follows.

\(^9\) For detailed specifications of \( Y_e(x(1)) \) and \( Y_b(x(1)) \) see for example Kraus and Litzenberger (1973) and Baron (1975) for one period models and Scott (1976) for a multiperiod model.
Step 1: We replace \( \mu \) by \( \mu^* = \mu - \lambda \rho \sigma \).

Step 2: We discount \( Y_e \) and \( Y_b \) as if the market price of risk were zero, i.e.,

\[
Y_e = e^{-r} E_e(x(1))
\]

and

\[
Y_b = e^{-r} E_b(x(1)).
\]

This procedure is free from the criticism of Gonzalez et al. because nowhere is it assumed that investors have quadratic utilities. The procedure is also computationally simpler than the procedure involved in the application of the single-period CAPM. Furthermore, our rule provides the following insight to problems in corporation finance: At least in those cases where the cash flows are normally or lognormally distributed, considerations of market risk may shift some of the model parameters, but leave the structure of the model and the qualitative implications regarding optimal capital structure unchanged.

V. Valuation of Controlled Projects

In general the manager of a project may influence the cash flows of the project. As an illustration, we assume that the project is a machine which is characterized by a single state variable \( x \), where \( x \) is some measure of efficiency. Assume that the machine deteriorates in a random fashion. Let \( u(t) \) be the expenditure rate for maintenance of the machine. The manager of the machine may choose the time path of \( u(t) \) and in this sense he controls the project. Maintenance influences the deterioration of the machine as \( dx = \mu(x, u, t) dt + \sigma(x, u, t) dw \). Maintenance also influences the cash flow \( c(x, u, t) dt \) generated over \( (t, t + dt) \). The problem then is to determine the maintenance policy \( u(t) \) which maximizes the value of the stream of cash flows, \( V(x, t) \).

Under the assumption that \( |c| < \infty, |u| < \infty \), it is shown in the appendix that \( V \) satisfies

\[
\max_u \left[ c - rV + V_t + (\mu - \lambda \rho \sigma) V_x + \frac{\sigma^2}{2} V_{xx} \right] = 0 \tag{12}
\]

which is an extension of equation (7). The argument which follows equation (7) is applicable to equation (12) also. Therefore our valuation rule applies to controlled projects as well.

As another application to a controlled project, we consider \( V(x, t) \) to be the value of a firm which is characterized by a single state variable, the working capital of the firm, \( x \). The control variable is the rate at which dividends are paid, if \( u > 0 \); it is the rate at which the firm raises capital, if \( u < 0 \). Then \( c = u \), that is to say, \( u \) stands for the cash flows of the equityholders of this firm. Assume that the change of the working capital level, \( dx = \mu(x, u, t) dt + \sigma(x, u, t) dw \) is the only stochastic element of this model. Then our rule specifies that we may replace \( \mu \) by \( \mu^* = \mu - \lambda \rho \sigma \) and ignore market risk thereafter, thus greatly simplifying the discussion.

We consider next a different type of control process and the applicability of our rule. For purposes of illustration we assume that the "project" is the cash
management system of a firm: it generates a stream of cash flows which are the holding cost of cash, penalty cost of negative cash balances and transactions costs in transferring cash in and out of interest bearing bonds. The firm controls this project, in the sense that it follows a cash management policy with the goal to minimize the present value of the stream of these costs. This model has been extensively studied in the literature\textsuperscript{10} under the assumption that the objective of the firm is to minimize the time-adjusted costs of the system. The latter objective ignores the adjustment of expected costs for market risk. We show that a straightforward application of our rule bridges the discrepancy between the two objectives.

Let $x$ be the level of cash reserves at time $t$ and $V(x,t)$ be the market value of all future costs incurred in the operation of the system. The demand for cash in time $(t, t+dt)$ is $dx = \mu dt + \sigma dw$. A fixed cost is incurred each time that bonds are liquidated to increase the level of cash reserves and each time that cash is converted into bonds. Assume that the cash policy is described by three time dependent parameters $\bar{x}(t) > x_0(t) > x(t)$ such that the cash level is adjusted to $x_0(t)$ whenever $x \geq \bar{x}(t)$ or $x < x(t)$; and the cash level is not adjusted as long as $x(t) < x < \bar{x}(t)$. It can be shown\textsuperscript{11} that the boundary conditions on the function $V(x,t)$ do not involve the market price of risk: At those well defined instances ($x \geq \bar{x}(t)$ or $x < x(t)$) when it is optimal to adjust the cash level, the adjustment cost is deterministic and nothing in this process involves the market price of risk. We now apply the rule as follows:

Step one: Replace the drift parameter $\mu(x,t)$ of the demand for cash by $\mu^*(x,t) = \mu(x,t) - \lambda \rho(x,t) \sigma(x,t)$.

Step two: Evaluate the stream of costs as if the market price of risk were zero. Then the market value costs coincide with the time-adjusted expected costs.

In other words the effect of market risk on the market value costs and on the parameters $x(t)$, $x_0(t)$, $\bar{x}(t)$ of the cash policy is fully captured by replacing the actual drift $\mu$ by the effective drift $\mu^*$, and ignoring market risk thereafter. For example, high positive correlation between the demand for cash and market return has the same effect on the parameters of the optimal policy as low expected demand for cash does, other things equal.

VI. CONCLUDING REMARKS

We have limited our discussion to diffusion processes (2) and have ignored discontinuous stochastic processes in a continuous-time framework, such as Poisson or Pareto-Levy processes. The reason for this omission is that Merton's

\textsuperscript{10} See Constantinides (1976) for a review of cash management models.

\textsuperscript{11} The boundary conditions on the functions $V(x,t)$ are

$$V(x(t),t) = K_1 + V(x_0(t),t)$$

$$V(x(t),t) = K_2 + V(x_0(t),t)$$

and

$$V_x(x(t),t) = V_x(x_0(t),t) = V_x(\bar{x}(t),t) = 0.$$

See Constantinides and Richard (1978) for the derivation of these conditions.
intertemporal CAPM (1) is derived under the assumption that the rate of return is generated by a diffusion process. We do not, as yet, have an intertemporal CAPM for discontinuous stochastic processes. It must be stressed, though, that the family of diffusion processes (2) is quite rich, since \( \mu \) and \( \sigma \) are in general functions of \( x \) and \( t \).

We have also limited our discussion to a continuous time framework and have excluded a discrete time framework. After all, in a discrete time framework a CAPM is applicable to every time period, and the market value of a stream of cash flows may, in principle, be evaluated by the sequential application of the one-period CAPM. Indeed this procedure has been outlined by Bogue and Roll (1974). Theoretical objections to the application of the discrete-time CAPM were discussed in our section IV-C. Additional difficulties in the derivation of our rule are illustrated when we attempt to duplicate the development of section III in discrete time. Assume that time periods are of length \( \delta \) and that the state equation is \( \bar{x}(t+\delta) - x(t) = \bar{z} \) where the distribution of \( \bar{z} \) is known. The rate of return on the project is

\[
\frac{V(x(t) + \bar{z}, t+\delta) - V(x(t), t) + c}{V(x(t), t)\delta}.
\]

(13)

In section III we are justified in expanding (13) in terms of a Taylor series and deriving (3), because \( \delta \) is small (\( \delta \to 0 \) in continuous time) and the stochastic increments \( \bar{z} \) follow the stochastic process (2). In discrete time \( \delta \) is not necessarily small and the derivation of section III is not valid.

It is helpful to place our rule in perspective with related developments in finance. The Modigliani-Miller arbitrage and the CAPM are two powerful financial methodologies, each however with limitations: The arbitrage approach requires that the securities under consideration belong to the same risk class whereas the CAPM approach requires that the assumptions, which lead to the CAPM, are satisfied. Table 1 summarizes some developments in finance through the two methodologies.

**TABLE 1**

**The Arbitrage Approach and the CAPM Approach in Some Developments in Finance**

<table>
<thead>
<tr>
<th>Arbitrage Approach (Requires the existence of a risk class)</th>
<th>CAPM approach (Conditional on the assumptions which lead to the CAPM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modigliani-Miller propositions (no tax)</td>
<td>Modigliani and Miller, Hamada (1969), Mossin (1966) and others</td>
</tr>
<tr>
<td>Generalized project valuation: Reduction to a riskless world</td>
<td>Cox and Ross (1976), This paper</td>
</tr>
</tbody>
</table>
Black and Scholes (1973) observed that over an infinitesimal time interval an option and its underlying security (i.e. the security on which the option is written) belong to the same risk class. They applied an arbitrage argument and evaluated the option in terms of the underlying security. Cox and Ross (1976) generalized this approach and noted that, if a financial claim is spanned by securities which are traded in the market, the financial claim may be evaluated through the arbitrage approach, where investors' preferences only play the indirect role of determining some equilibrium parameter values. In particular the claim may be evaluated in a risk neutral world, thus considerably simplifying the valuation.

A parallel development has taken place through the CAPM and without assuming the existence of a risk class. Hamada (1969), Mossin (1966) and others derived the Modigliani-Miller propositions through the CAPM. Black and Scholes (1973) and Merton (1970) derived the option pricing differential equation through the CAPM. However, if the underlying security is traded in the market, then the security and the option belong to the same risk class, and the option pricing differential equation is valid, even when the assumptions, leading to the CAPM, do not hold. For this reason the derivation of the option pricing equation through the arbitrage argument dominates the derivation through the CAPM, provided that the underlying security is traded in the market.

The rule developed in this paper bears some similarity to the Cox-Ross rule in the sense that both rules reduce a valuation problem under market risk to a valuation problem in a risk neutral world. Neither of the two rules dominates the other: Our rule is limited to situations where the CAPM is valid, while the Cox-Ross procedure is limited to situations where a risk class exists. Thus our example in section IV-B on option pricing may not be solved by the Cox-Ross procedure, as long as the underlying security is not traded in the market; and our example in section V on cash management may not be solved by the Cox-Ross procedure as long as a security or portfolio does not exist which perfectly hedges against the variability in the demand for cash.

APPENDIX

Let \( V(x,t) \) be the market value of the project, if the project is optimally controlled. Assume that the project is controlled by \( ud\tau \) in time \((t,t+dt)\) and is optimally controlled thereafter. Since the control \( u \) is not necessarily optimal, the market value of the project at \( t \) is \( J(x,t) \), where \( J(x,t) \leq V(x,t) \). The value of the project at \( t+dt \) is \( J(x+dx,t+dt) \), where \( J(x+dx,t+dt) = V(x+dx,t+dt) \). The rate of return on the project is

\[
\frac{V(x+dt,t+dt) + c\,dt - J(x,t)}{J(x,t)} = \frac{V(x,t) - J(x,t)}{J(x,t)} + \frac{1}{J(x,t)} \left[ \frac{c + V_t + \mu V_x + \frac{\sigma^2}{2} V_{xx}}{J(x,t)} \right] \, dt + \frac{\sigma V_x}{J(x,t)} \, dw \quad (A1)
\]

with expected value per unit time \( \alpha_p \), and covariance with the market per unit time...
\[ \sigma_{PM} \text{ given by} \]
\[ \alpha_p = \frac{V(x,t) - J(x,t)}{J(x,t)dt} + \frac{1}{J(x,t)} \left( c + V_x + \mu V_x + \frac{\sigma^2}{2} V_{xx} \right) \tag{A2} \]

and

\[ \sigma_{PM} = \rho \sigma_M \sigma x / J(x,t). \tag{A3} \]

If the project is in equilibrium in the capital market, it satisfies the intertemporal CAPM of equation (1). We substitute (A2) and (A3) in (1) and obtain

\[ \frac{V(x,t) - J(x,t)}{J(x,t)dt} + \frac{1}{J(x,t)} \left( c + V_x + \mu V_x + \frac{\sigma^2}{2} V_{xx} \right) - r = \frac{\lambda \rho \sigma V_x}{J(x,t)}. \tag{A4} \]

But \( V(x,t) - J(x,t) > 0 \) and the above simplifies into

\[ \frac{1}{J(x,t)} \left( c + V_x + \mu V_x + \frac{\sigma^2}{2} V_{xx} \right) - r \leq \frac{\lambda \rho \sigma V_x}{J(x,t)}, \quad \text{or} \]

\[ c - r J(x,t) + V_t + (\mu - \lambda \rho \sigma) V_x + \frac{\sigma^2}{2} V_{xx} \leq 0, \quad \text{or} \]

\[ c - r V + V_t + (\mu - \lambda \rho \sigma) V_x + \frac{\sigma^2}{2} V_{xx} \leq 0. \tag{A5} \]

If \( u \) is optimal then \( V(x,t) - J(x,t) = 0 \) and (A5) is satisfied as an equality. Thus the optimal \( u \) is given by

\[ \max_u \left[ c - r V + V_t + (\mu - \lambda \rho \sigma) V_x + \frac{\sigma^2}{2} V_{xx} \right] = 0. \tag{A6} \]

REFERENCES


