STOCHASTIC CASH MANAGEMENT WITH FIXED
AND PROPORTIONAL TRANSACTION COSTS*†

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A continuous time model of cash management is formulated with stochastic demand and
allowing for positive and negative cash balances. The form of the optimal policy is assumed
to be of a simple form \((d, D, U, u)\). The parameters of the optimal policy are explicitly
evaluated and the properties of the system are discussed.

1. Introduction

Baumol [4] and Tobin [18] discussed the cash management problem with de-
terministic demand. The qualitative form of the optimal cash policy with stochastic
demand for cash was discussed by Whistler [21], Eppen and Fama [10], Girgis [12],
form of the optimal policy, analytically derived the values of the policy parameters,
and discussed the effect of the model parameters on the cash policy and the
transactions demand for cash. Eppen and Fama [9] discussed the same implications
using numerical methods.

Bather [2] and Antelman and Savage [1] formulated the inventory problem in a
continuous-time framework, the demand being generated by a Wiener process. Vial
[19] extended their approach and formulated the cash management problem in
continuous time with fixed and proportional transaction costs, linear holding and
penalty costs and the demand for cash generated by a Wiener process. Vial proved
that, if an optimal policy exists, it is of a simple form \((d, D, U, u)\), given by equations
(4), (5).

In this paper we assume that an optimal policy exists and it is of a simple form.† We
determine the parameters \(d, D, U\) and \(u\) of the optimal policy and discuss the
properties of the optimal solution.

2. The Model and Notation

The state of the cash management system at time \(t\) is defined by the cash level \(x(t)\),
or simply \(x\). The cost rate of keeping a positive or negative cash balance \(x\) is
\(C(x) = \max\{hx - px\}\), where \(h\) and \(p\) are the holding cost rate and penalty cost rate,
respectively, and \(h, p\) are positive constants. The holding cost of cash is primarily the
opportunity cost of holding cash rather than investing the wealth in interest-bearing
bonds. Clearly this cost is not subtracted from the cash balance. The penalty cost may
be considered as the interest payment on negative cash balances. We assume that the
penalty cost is not subtracted from the cash balance. This approximation hardly
affects the validity of our results.

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† Because of the presence of fixed transaction costs the optimal policy is one of “impulse control” where
the control is applied at some stopping times and changes the cash level instantaneously by a finite amount.
The theory of “impulse control” was introduced by Bensoussan and Lions [5], [6] and extended by Richard
[17]. This technique is applied by Constantinides and Richard [8] in determining sufficient conditions for
the existence of an optimal policy.
We assume that the cost of transferring funds in or out of the cash balance and changing the cash level from $x_0$ into $x_1$ is given by

$$B(x_1 - x_0) = K^+ + k^+ (x_1 - x_0), \quad x_1 > x_0,$$

$$= 0, \quad x_1 = x_0,$$

$$= K^- + k^- (x_0 - x_1), \quad x_1 < x_0,$$  \hspace{1cm} (1)

where $k^+, k^-, K^+$ and $K^-$ are positive constants. We further assume that these costs do not decrease the cash balance.

Denote the cumulative demand for cash in the time interval $[t, s]$ by $D(t, s)$. The demand for cash is defined to be such that, for any partition $t_0 < t_1 < \cdots < \infty$ of the time interval $[t_0, \infty)$, the random variables $D(t_0, t_1)$, $D(t_1, t_2)$, $\ldots$, $D(t_{n}, t_{n+1})$, $\ldots$ are independent and normally distributed; the mean and variance of the demand $D(t_1, t_{i+1})$ are $ED(t_1, t_{i+1}) = (t_{i+1} - t_i) \mu$ and var $D(t_1, t_{i+1}) = (t_{i+1} - t_i)^2 \sigma^2$ where $\mu$ and $\sigma^2$ are constants independent of time. The demand function may be rigorously defined in terms of a Wiener process $\omega(t)$ as $D(t, s) = (s - t) \mu + (\omega(s) - \omega(t)) \sigma$. With probability one, $\omega(t)$ and therefore $D(t, s)$ are continuous functions.

The decision-maker continuously and costlessly observes the cash level and intervenes when necessary to adjust the cash balance. We define a continuous control $w(t)$, $|w(t)| < \infty$, such that the cash balance increases (decreases) by $w(t) dt$ in the time interval $[t, t + dt]$ as the result of transactions. We also define impulse control as follows: Let $0 = \tau_0 < \tau_1 < \cdots < \tau_i < \cdots$ be a sequence of stopping times and $\xi_0, \xi_1, \ldots, \xi_i, \ldots$ a sequence of controls such that at time $\tau_i$ the cash level is instantaneously increased by the amount $\xi_i$. Denote the cash level at the stopping time $\tau_i$, but before the impulse control has been applied, by $x(\tau^-_i)$ and denote the cash level at the stopping time $\tau_i$ and after the impulse control has been applied by $x(\tau_i)$. Then the state equations of the cash level may be written as

$$dx(t) = w(t) dt - D(t, t + dt), \quad \tau_i < t < \tau_{i+1}, \forall i > 0,$$

$$x(\tau_i) = x(\tau^-_i) + \xi_i, \quad \forall i > 0.$$  \hspace{1cm} (2)

In the presence of only proportional transaction costs we will assume that the optimal transaction policy is continuous, except perhaps for an initial adjustment of the cash level. In the presence of fixed transaction costs we will assume that the optimal policy is one of impulse control. This assumption is intuitively appealing because a continuous adjustment policy in the presence of fixed transaction costs would incur infinite costs.

We treat the cases of only proportional and fixed and proportional transaction costs separately in the following sections. In the remainder of this section we complete the model description. We allow for both continuous and impulse control so that the model formulation and Theorem 1 of this section are applicable to both the cases with and without fixed transaction costs.

The total cost in the time interval $[0, T]$ is made up of the holding cost, penalty cost, and the transaction cost of continuous and impulse control. A transaction policy, $p$, is defined as the set of rules which tell the decision-maker how much to transact at every instant $t$, given the state description at time $t$. The objective is defined as the minimization of the limit of the expected cost rate in the time interval $[0, T]$, as the horizon $T$ extends to infinity

$$\gamma = \inf_p \lim_{T \to \infty} T^{-1} \mathbb{E}_{x(0), 0} \left[ \sum_{i=0}^{N} B(\xi_i) + \int_0^T \left[ C(x(s)) + B(w(s)) \right] ds \right]$$  \hspace{1cm} (3)

For a discussion of the Wiener process as well as an introduction to the stochastic calculus, see Kushner [13, chapter 10]. Informally, the reader who is unfamiliar with the stochastic calculus may think of the demand for cash in the time interval $(t, t + dt)$ as given by $D(t, t + dt) = \mu dt + \sigma \xi dt^{1/2}$ where $E(\xi) = 0$, $\text{var}(\xi) = 1$.
where \( E_{x(0),0}^p \) is the expectation operator at time zero, cash level \( x(0) \) and policy \( p \). \( N \) is the index of the last stopping time in the interval \([0, T]\).

We demonstrate that the objective as defined in (3) is meaningful in the sense that there exists at least one policy for which the limit of the expected cost rate exists as the horizon extends to infinity.

Consider the policy, \( p_\alpha \), of transacting and adjusting the cash balance to the zero level by impulses applied at stopping times \( 0 = \tau < \tau_1 < \tau_2, \ldots, \tau_N \) where the stopping times and \( N \) are defined in terms of an arbitrary positive number \( \alpha \) as \((N - 1)\alpha < T < N\alpha\), \( \tau_i = i\alpha, i = 1, \ldots, N - 1 \) and \( \tau_N = T \). Then the limit of the average expected cost for this policy as the horizon extends to infinity is given by

\[
\gamma_0 = \lim_{T \to \infty} T^{-1} E_{x(0),0}^p \left[ \sum_{i=0}^{N} B(\xi_i) + \int_0^T C(x(s))ds \right]
\]

\[
= \lim_{T \to \infty} T^{-1} \left\{ B(x(0)) + N E_{0,0} \left[ B(x(\alpha)) + \int_0^\alpha C(x(s))ds \right] \right\}.
\]

But \( B(x(0)) \), and \( E_{0,0} [B(x(\alpha)) + \int_0^{\alpha} C(x(s))ds] \) are well defined nonnegative constants independent of \( T \). Also \( N/T \to 1/\alpha \) as \( T \to \infty \). Therefore

\[
\gamma_0 = \alpha^{-1} E_{0,0} \left[ B(x(\alpha)) + \int_0^\alpha C(x(s))ds \right],
\]

finite.

Thus we have established that at least for some policies the limit in (3) is bounded. If an optimal policy exists, the limit of the average expected cost for this policy also is bounded.

Vial [19] considered a model in all respects like ours, but the objective was defined as the expected discounted total cost and showed that, if an optimal policy exists, it is such that if the cash level is \( x \), we transact instantaneously to the level \( y(x) \) where

\[
y(x) = \begin{cases} 
D, & x \leq d, \\
x, & d < x < u, \\
U, & u \leq x, \quad \text{and} \\
d < D < U < u. & 
\end{cases}
\]

(4)

In the following sections and for the objective defined by (3) we assume that the optimal policy is given by (4) and proceed to examine the properties of the parameters \( d, D, U, u \). For the case of zero fixed transaction costs we set \( d = D \) and \( u = U \) as the optimal policy.

We prove a theorem which will be useful in our subsequent discussion.

**Theorem 1.** Assume that a cash policy \( p \) is described by (4) and assume that there exists a twice continuously differentiable function \( V(x) \) and a constant \( \gamma_p \) such that, for a cash management system which extends to infinity, the total cost in the time interval \([0, T]\) may be represented by

\[
E_{x(0),0}^p \left\{ \sum_{i=0}^{N} B(\xi_i) + \int_0^T \left[ C(x(s)) + B(w(s)) \right] ds \right\}
\]

\[
= \gamma_p T + V(x(0)) - E_{x(0),0}^p V(x(T)).
\]

(6)

\footnote{The inequalities are strict because Vial assumed \( K^+, K^-, k^+, k^- \neq 0 \).}
Then the function $V$ satisfies the following differential equation in the range $x \in (d, u)$:

$$
-\gamma_p + C(x) - \mu V_x(x) + (\sigma^2/2)V_{xx}(x) = 0.
$$

(7)

**Proof.** Consider separately the cases with and without fixed transactions costs:

(a) With fixed transaction costs the policy $p$ is assumed to be given by (4) with the inequalities $d < D < U < u$ strict. Then the cash level is controlled only at some stopping times by a finite amount of cash. Therefore we set $w(t) = 0$. Assume that at time $t$ the cash level lies within the region of no transactions, i.e., $x(t) \in (d, u)$. In the time interval $[t, t + \Delta]$ we may write (6) as

$$
E_{x(t), t}^p \left[ \sum_{i=1}^{N} B(\xi_i) + \int_t^{t+\Delta} C(x(s)) ds \right] ds
= \gamma_p \Delta + V(x(t)) - E_{x(t), t}^p V(x(t + \Delta))
$$

(8)

where $i = 1, \ldots, N$ is the index of the stopping times $t < \tau_1 < \tau_2 \cdots < \tau_N < t + \Delta$ in the interval $[t, t + \Delta]$.

By Itô's Lemma we obtain

$$
E_{x(t), t}^p V(x(t + \Delta)) - V(x(t)) = E_{x(t), t}^p \int_t^{t+\Delta} \left[ -\mu V_x(x(s)) + (\sigma^2/2)V_{xx}(x(s)) \right] ds.
$$

(9)

Substituting in (8), rearranging and dividing by $\Delta$

$$
-\gamma_p + \Delta^{-1}E_{x(t), t}^p \int_t^{t+\Delta} \left[ C(x(s)) - \mu V_x(x(s)) + (\sigma^2/2)V_{xx}(x(s)) \right] ds
+ \Delta^{-1}E_{x(t), t}^p \sum_{i=1}^{N} B(\xi_i) = 0.
$$

(10)

It may be shown that the probability of one or more stopping times occurring in the time interval $(t, t + \Delta)$ tends to zero as $\Delta \to 0$ and that

$$
\lim_{\Delta \to 0} \Delta^{-1}E_{x(t), t}^p \sum_{i=1}^{N} B(\xi_i) = 0.
$$

(11)

Then by letting $\Delta \to 0$ in (10) we obtain (7).

(b) Consider next the case of only proportional transaction costs. Since we assume that the policy is given by (4) with $d = D$ and $u = U$ there are no impulses in the control after the initial control and we may set $\xi_1, \xi_2, \ldots, \xi_N = 0$. If $x(t) \in (d, u)$ then $\xi_0 = 0$ also, that is, there is no initial impulse control. From (6) we obtain

$$
E_{x(t), t}^p \int_t^{t+\Delta} \left[ C(x(s)) ds + B(w(s)) \right] ds
= \gamma_p \Delta + V(x(t)) - E_{x(t), t}^p V(x(t + \Delta)).
$$

(12)

4 For a discussion of Itô's Lemma see Kushner [13, chapter 10].

5 The proof is based on the following result: If changes in $y$ are governed by a Wiener process, if $y(0) = 0$, and if $m_x$ denotes the first passage time of $y$ though $a > 0$, then $P(m_x < \Delta) = a\Delta e^{-a^2/2\Delta}/(2\pi\Delta)^{1/2} ds$. It follows that $\lim_{\Delta \to 0} P(m_x < \Delta) = 0$. This result has to be modified for drift and the existence of two reflecting boundaries but the result remains the same.

To prove (11) we note that $B(\xi)$ is bounded by $B(u - d)$. Then $E_{x(t), t}^p \sum_{i=1}^{N} B(\xi_i) < B(u - d) \sum_{i=1}^{N} P_i$ where $P_i$ is the probability of $i$ stopping times occurring in the time interval $(t, t + \Delta)$. But $\sum_{i=1}^{N} P_i < P_1 \sum_{i=1}^{N} P_i^{-1} = P_1/(1 - P)$ where $P$ is the maximum of the probability of hitting one of the boundaries $u$ or $d$ in the time interval $(t, t + \Delta)$ if $x(t) = D$ or $U$. Since $P < 1$, $P_1/(1 - P)$ is finite. From our previous expression for $P(m_x < \Delta)$ it follows that $\lim_{\Delta \to 0} P(m_x < \Delta) = 0$ and this result may be adapted to the case of drift and two reflecting barriers to show that $\lim_{\Delta \to 0} \gamma_\Delta = 0$. (11) follows directly.
Substituting (9) in (12), rearranging and dividing by $\Delta$

$$- \gamma_p + \Delta^{-1} E^p_{\xi(t), t} \int_t^{t+\Delta} \left[ C(x(s)) - \mu V_x(x(s)) + (\sigma^2/2) V_{xx}(x(s)) + B(w(s)) \right] ds = 0.$$  

Taking the limit as $\Delta \to 0$

$$- \gamma_p + C(x) - \mu V_x(x) + (\sigma^2/2) V_{xx}(x) + B(w) = 0. \quad (13)$$

By assumption $w = 0$ if $x \in (d, u)$. Thus for $x \in (d, u)$, (13) becomes (7). Q.E.D.

The constant $\gamma_p$ associated with policy $p$ and the function $V(x)$ are interpreted through the following corollaries.

**Corollary 1.** $\gamma_p$ is the limit of the expected cost rate of policy $p$ as the horizon extends to infinity

$$\gamma_p = \lim_{T \to \infty} T^{-1} E^p_{\xi(0), 0} \left[ \sum_{i=1}^{N} B(\xi_i) + \int_0^T \left[ C(x(s)) + B(w(s)) \right] ds \right]. \quad (14)$$

**Proof.** Divide (6) by $T$ and take the limit as $T \to \infty$. $V(x(0))$ is a constant, therefore $\lim_{T \to \infty} T^{-1} V(x(0)) = 0$. Also since we assumed a policy of the form (4) $E^p_{\xi(0), 0} V(x(T)) \leq \sup_{d<x<u} V(x)$, a constant, and $\lim_{T \to \infty} T^{-1} E^p_{\xi(0), 0} V(x(T)) = 0$. Equation (14) follows directly. Q.E.D.

**Corollary 2.** $V(x(t))$ is a potential cost function in the sense that $V(x(t)) - V(x'(t))$ is the difference in total cost from time $t$ to infinity of two cash management systems both governed by policy $p$, with cash levels $x(t)$ and $x'(t)$ respectively at time $t$ and otherwise identical.

**Proof.** From (6) we obtain

$$V(x(t)) - V(x'(t)) = E^p_{\xi(t), t} \left[ \sum_{i=1}^{N} B(\xi_i) + \int_t^T \left[ C(x(s)) + B(w(s)) \right] ds \right]$$

$$- E^p_{\xi(t), t} \left[ \sum_{i=1}^{N} B(\xi_i) + \int_t^T \left[ C(x(s)) + B(w(s)) \right] ds \right]$$

$$+ E^p_{\xi(t), t} V(x(T)) - E^p_{\xi(t), t} V(x(T)). \quad (15)$$

It may be shown that for any policy $p$ of the form (4) $x(T)$ is ergodic, that is to say, as the time $T$ tends to infinity the distribution of $x(T)$ tends to a stationary distribution $G_p(x)$, independent of the value of $x$ at time $t$.\(^6\) Also $x(T)$ is bounded in the interval $(d, u)$ and since $V$ is a twice continuously differentiable function of $x$, $V(x(T))$ is bounded also. Thus the limits $E^p_{\xi(t), t} V(x(T))$ and $E^p_{\xi(t), t} V(x(T))$ as $T \to \infty$ exist. We obtain

$$\lim_{T \to \infty} \left[ E^p_{\xi(t), t} V(x(T)) - E^p_{\xi(t), t} V(x(T)) \right] = \lim_{T \to \infty} E^p_{\xi(t), t} V(x(T)) - \lim_{T \to \infty} E^p_{\xi(t), t} V(x(T))$$

$$= \int V(y) dG_p(y) - \int V(y) dG_p(y) = 0.$$  

Taking the limit in (15) as $T \to \infty$ the last term vanishes and the right hand side of (15) becomes the limit as $T \to \infty$ of the difference in total cost from time $t$ to $T$ of two cash management systems both governed by policy $p$ with cash levels $x(t)$ and $x'(t)$ respectively at time $t$ and otherwise identical. Q.E.D.

\(^6\) The proof of the ergodic theorem for one reflecting barrier is given in Gihman and Skorohod [11, Theorem 1, p. 211]. The proof may be generalized to the case of two reflecting barriers.
3. Optimal Policy with Zero Expected Demand for Cash

In this section we assume that the expected demand for cash is zero, that is \( \mu = 0 \), \( \sigma^2 \neq 0 \). Consider first the case of nonzero fixed transaction costs.

By the assumed policy (4), if \( x < d \) we incur the fixed transaction cost and transact until the marginal benefit of increasing the cash balance equals the proportional transaction cost rate arising from cash increases:

\[
V_x(D) = -k^+.
\]

The total cost incurred in transacting from \( x = d \) to \( x = D \) is \( K^+ + (D - d)k^+ \). If \( x = d \) is the maximum cash balance at which we are justified to incur the fixed transaction cost, the reduction in the cost function in transacting from \( x = d \) to \( x = D \) must equal the transaction cost:

\[
V(d) - V(D) = K^+ + (D - d)k^+.
\]

By similar arguments we obtain

\[
V_x(U) = k^-
\]

and

\[
V(u) - V(U) = K^- + (u - U)k^-.
\]

For \( \mu = 0 \) the solution to (7) is

\[
V(x) = xV_x(0) + (1/\sigma^2)(\gamma x^2 - hx^3/3), \quad 0 \leq x \leq u, \\
= xV_x(0) + (1/\sigma^2)(\gamma x^2 + px^3/3), \quad d \leq x \leq 0,
\]

where \( \gamma, V_x(0) \) are yet to be determined. It is easily verified that \( V(x) \) is twice continuously differentiable at \( x = 0 \).

Assume that \( d < D \leq 0 < U < u \). For \( x > 0 \), condition (18) is a quadratic, with roots \( U_1, U_2 \), \( U_1 < U_2 \) (Figure 1). However at \( x = U_2 \), the marginal reduction of the potential function increases with decreasing \( x \). Thus, the level \( x = U_2 \) cannot represent the boundary \( U \), since further cost reduction may be effected by reducing the cash level below \( U_2 \). The level \( x = U_1 \) uniquely represents the boundary \( U \).

From (19) it follows that the vertical distance of the point \( (u, V(u)) \) from the line through \( (U, V(U)) \) and slope \( k^- \), is \( K^- \). Two such values of \( u \) are possible and are shown in Figure 1 as \( u_1 \) and \( u_2 \). From the figure it also appears that the maximum vertical distance between a point on the curve and the line through \( (U, V(U)) \) with slope \( k^- \) occurs at \( x = U_2 \). This maximum distance is

\[
V(U_2) - V(U) - (U_2 - U)k^- = (4/3h^2\sigma^2)[\gamma^2 - h\sigma^2(k^- - V_x(0))]^{3/2}.
\]

If the latter is to be satisfied, the following must be true:

\[
\gamma^2 > h\sigma^2(k^- - V_x(0)) + (3h^2\sigma^2K^-/4)^{2/3}.
\]
By a similar argument for $x < 0$, the following must be true also:

$$\gamma^2 \gtrsim \rho a^2 (k^+ + V_x(0)) + (3\rho^2 a^2 K^+/4)^{2/3}. \quad (21)$$

$V_x(0)$ is yet undetermined and we choose $V_x(0)$ in order to minimize the expected cost rate, which by Corollary 1 is equivalent to minimizing $\gamma$. Therefore we choose $V_x(0)$ to minimize $\gamma$ consistent with (20) and (21). We then obtain

$$V_x(0) = \left((hk^- - pk^+)(h + p)^{-1} + (h + p)\right)^{-1}\left[(3h^2k^-/(4\sigma))^{2/3} - (3p^2k^+/4\sigma)^{2/3}\right]$$

and

$$\gamma^2 = (\sigma^2hp/(h + p))(k^+ + k^-) + (p/(h + p))(3h^2\sigma^2k^-/4)^{2/3}$$

$$+ (h/(h + p))(3p^2\sigma^2K^+/4)^{2/3}. \quad (23)$$

In choosing $V_x(0)$ and $\gamma^2$ as in (22) and (23), the condition (19) is satisfied and we conclude that $u = U_2$. Thus

$$U = (\gamma/h) - (1/h)(3h^2\sigma^2K^-/4)^{1/3} \quad \text{and,} \quad (24)$$

$$u = (\gamma/h) + (1/h)(3h^2\sigma^2K^-/4)^{1/3}. \quad (25)$$

Similarly we derive,

$$D = -(\gamma/p) + (1/p)(3p^2\sigma^2K^+/4)^{1/3} \quad \text{and} \quad (26)$$

$$d = -(\gamma/p) - (1/p)(3p^2\sigma^2K^+/4)^{1/3}. \quad (27)$$

![Figure 1](image_url)  

**Figure 1.** The Potential Function $V(x)$. 

From equations (23)–(27) we conclude that the assumption \( d < D < 0 < U < u \) is satisfied provided

\[
- \sigma^2 h (k^+ + k^-) < (3h^2 \sigma^2 K^- / 4)^{2/3} - (3p^2 \sigma^2 K^- / 4)^{2/3} < \sigma^2 p (k^+ + k^-). \tag{28}
\]

These results are summarized below:

**Proposition 1.** If the expected demand for cash is zero and if the problem parameters satisfy (28), the optimal policy of the form (4) is given by equations (24)–(27) and the expected cost rate is given by (23).

If condition (28) is not satisfied we investigate separately the cases \( d \leq 0 < D < U < u \) and \( d < D < U < 0 < u \). Clearly the cases \( 0 < d < D < U < u \) and \( d < D < U < 0 < u \) never arise since they imply a permanent positive and negative cash balance, respectively, which is suboptimal.

If \( d \leq 0 < D < U < u \) we may repeat the procedure of this section. The only difference is that \( V(D) \) and \( V_x(D) \) are given now by the expression for \( V(x) \) as defined in the range \( 0 < x < u \). Although this procedure is straightforward and amenable to numerical calculation, unfortunately we do not obtain explicit algebraic expressions for the policy parameters.

As the fixed transaction costs \( K^+ \) and \( K^- \) tend to zero, condition (28) is always satisfied and the parameters of the optimal policy satisfy \( d = D = -\gamma/p < 0 \) and \( u = U = \gamma/h > 0 \). Although this procedure of obtaining the optimal values of the policy parameters in the absence of fixed transaction costs is not rigorous, it does give the correct result. The correct procedure of solving for the policy parameters in the absence of fixed transaction costs is illustrated in §4 where we treat the more general case of nonzero expected demand for cash. The procedure of §4 may also be applied in the case of zero expected demand for cash. The result is summarized below:

**Proposition 2.** If the expected demand for cash is zero, and if there are no fixed transaction costs, then the optimal policy of the form (4) is given by \( u = U = \gamma/h > 0, \, d = D = -\gamma/p < 0 \) and \( \gamma = [(hp/(h+p)\sigma^2)(k^+ + k^-)]^{1/2} \).

The properties of the cash management policy based on Propositions 1 and 2 are discussed in §5. In the next section we examine the parameters of the optimal policy when the expected demand for cash is nonzero.

### 4. Nonzero Expected Demand for Cash

We repeat the procedure of the last section to examine the parameters of the optimal policy when the expected demand for cash is nonzero. If we allow for both fixed and proportional transaction costs we cannot, in general, obtain closed-form expressions for the policy parameters, although the general procedure is a straightforward extension of the methodology of the previous section. In this section we confine our attention to the case of only proportional transaction costs. The solution to (7) is expressed in terms of \( \lambda = 2\mu / \sigma^2 \) as

\[
V(x) = \lambda^{-1} \{ V_x(0) + \gamma / \mu - h / \mu \lambda \} (e^{\lambda x} - 1) + \mu^{-1} (hx^2 / 2 + hx / \lambda - \gamma x), \quad 0 < x < U,
\]

\[
= \lambda^{-1} \{ V_x(0) + \gamma / \mu + p / \mu \lambda \} (e^{\lambda x} - 1) - \mu^{-1} (px^2 / 2 + px / \lambda + \gamma x), \quad D < x < 0,
\]

where \( \gamma, \, V_x(0) \) are yet to be determined. It is easily verified that \( V(x) \) is twice continuously differentiable at \( x = 0 \).

Assume that the optimal policy is given by (4) with \( d = D \) and \( u = U \). If \( D > 0 \) or \( U < 0 \), (4) prescribes a policy with a permanent positive or permanent negative cash
balance respectively, which is clearly suboptimal. Therefore we take \( D < 0 \) and \( U > 0 \). Conditions (16) and (18) become

\[
\left\{ V_x(0) + \frac{\gamma}{\mu - h/\mu} \right\} e^{\lambda U} + \mu^{-1}(h + h/\lambda) = k^- \quad \text{and} \quad (29)
\]

\[
\left\{ V_x(0) + \frac{\gamma}{\mu + p/\mu} \right\} e^{\lambda D} - \mu^{-1}(pD + p/\lambda + \gamma) = -k^+. \quad (30)
\]

We proceed to show that the parameters \( D \) and \( U \) of the optimal policy satisfy also the conditions

\[
V''(U) = 0 \quad \text{and} \quad (31)
\]

\[
V''(D) = 0. \quad (32)
\]

If the cash level is higher than \( U \), the optimal policy according to (4) is to transact and decrease the cash level to \( U \). At \( x = U \) the marginal rate of improving the total future cost equals the marginal cost of transacting to decrease the cash balance. This relationship is expressed by (18). Also unless \( V''(U) > 0 \), the marginal rate of improving the total future cost will increase if the cash balance is decreased below the level \( x = U \). Thus we require \( V''(U) > 0 \). Consider next the slope \( V'(U + \delta), 0 < \delta \ll 1 \). At \( x = U + \delta \) the cash management system is suboptimal and \( V'(U + \delta) < V(U) + \delta k^- \), or \( V(U + \delta) < k^- \). But we argued before that \( V'(U) = k^- \). It follows that \( V''(U) < 0 \). Combining the requirements \( V''(U) > 0 \) and \( V''(U) < 0 \) and bearing in mind that \( V \) is twice continuously differentiable we obtain (31). Similarly we derive (32).

The relationships (31) and (32) may be better understood by reference to Figure 1. In the presence of a fixed transaction cost, i.e., \( K^- > 0 \), the levels \( u \) and \( U \) are distinct and both correspond to a slope \( k^- \) of the function \( V(x) \). As \( K^- \) tends to zero, the levels \( u \) and \( U \) approach each other. At \( K^- = 0 \), the function \( V(x) \) has an inflection point at \( x = U \). In essence (31) and (32) replace (17) and (19) in the absence of a fixed transaction cost.

Substituting \( V \) in (31) and (32) we obtain

\[
\lambda \left( V_x(0) + \frac{\gamma}{\mu - h/\mu} \right) e^{\lambda U} + \frac{h}{\mu} = 0 \quad \text{and} \quad (33)
\]

\[
\lambda \left( V_x(0) + \frac{\gamma}{\mu + p/\mu} \right) e^{\lambda D} - \frac{p}{\mu} = 0. \quad (34)
\]

(29), (30), (33) and (34) are four equations with four unknowns \( U, D, \gamma, V_x(0) \). Solving we obtain

\[
U = \frac{\gamma}{h} + \frac{\mu k^-}{h} > 0, \quad (35)
\]

\[
D = -\frac{\gamma}{p} + \frac{\mu k^+}{p} < 0 \quad \text{and} \quad (36)
\]

\[
h \left( 1 - \exp(-\lambda\gamma/h - \lambda\mu k^-/h) \right) = p \left( -1 + \exp(\lambda\gamma/p - \lambda\mu k^+/p) \right). \quad (37)
\]

We show that \( \gamma \), as defined by (37) is positive as required and is uniquely determined. If \( \mu > 0 \), \( h \left( 1 - \exp(-\lambda\gamma/h - \lambda\mu k^-/h) \right) \) is positive at \( \gamma = 0 \), is monotonically increasing at a decreasing rate and tends asymptotically to the value \( h \) as \( \gamma \to \infty \). Also \( p \left( -1 + \exp(\lambda\gamma/p - \lambda\mu k^+/p) \right) \) is negative at \( \gamma = 0 \), is monotonically increasing at an increasing rate and tends to infinity as \( \gamma \to \infty \). Thus (37) has a unique solution at a positive value of \( \gamma \). If \( \mu < 0 \) a similar argument shows that (37) has a unique solution at a positive value of \( \gamma \).

Next we show that \( D \), as defined by (36) is indeed negative. If \( \mu > 0 \) then
\[ \gamma/h + \mu k^-/h > 0 \] and
\[ p \left( -1 + \exp(\lambda \gamma/p - \lambda \mu k^+/p) \right) = h \left( 1 - \exp(-\lambda \gamma/h - \lambda \mu k^-/h) \right) > 0. \]
Therefore \( \lambda \gamma/p - \lambda \mu k^+/p < 0 \) and \( D < 0 \). If \( \mu < 0 \), from (36) we obtain \( D < 0 \). Similarly we show that \( U > 0 \) for \( \mu \geq 0 \). These results are summarized below.

**Proposition 3.** If the expected demand for cash is nonzero and if there are no fixed transaction costs then the optimal policy of the form (4) is given by (35)–(37) with \( u = U, \ d = D \).

It may be easily verified that the optimal policy and expected cost rate (35)–(37) tend to the corresponding expressions of Proposition 2 as \( \mu \to 0 \). Consider, instead, the case \( \mu \to \infty \): from (37) we obtain \( \gamma \to \mu k^+ \) which implies that for large expected demand the expected cost rate is primarily determined by the cost of converting bonds into cash to satisfy the demand. Also \( D \to 0 \), that is to say the lower control limit tends to zero. Because of the large expected demand, the probability of negative demand in any time interval is exceedingly small, and therefore it is suboptimal to refrain from adjusting a negative cash balance upwards in the hope that the (negative) demand for cash will shift the cash balance upwards. Similarly we show that as \( \mu \to -\infty, \ \gamma \to -\mu k^- \) and \( U \to 0 \).

### 5. Properties of the Cash Management Policy

If the expected demand for cash is zero and if the problem parameters satisfy (28), from (23) we observe that the expected cost rate squared, is also the sum of the squares of the expected cost rates of two systems, possessing the characteristics of the original system, with one system having only fixed and the other having only proportional transaction costs.

Increasing the proportional transaction cost rate \( k^+ \) or \( k^- \) or both, increases both \( U \) and \( u \) and decreases both \( D \) and \( d \) but leaves the intervals \( u - U \) and \( D - d \) unchanged.

Increasing the fixed transaction cost \( K^+ \), increases \( U, \ u, \ D - d \), decreases \( d \) but leaves \( u - U \) unchanged. Increasing the fixed transaction cost \( K^- \), decreases \( D, \ d \), increases \( u - U, \ u \) but leaves \( D - d \) unchanged.

The transactions demand for money is given in terms of the parameters \( d, \ D, \ U, \ u \) for \( \mu = 0 \) as \( (u^2 + uU + U^2 - d^2 - DD - D^2)/3(u + U - D - d) \) and the effect of cost changes on the transactions demand for money may be studied through this expression.

If \( h = p, \ k^+ = k^- = k \) and \( K^+ = K^- = K \) we obtain \( D = -U \) and \( d = -u \), that is, the policy is symmetrical. Also the relative effect of fixed and proportional transaction costs depends on \( P \) defined as
\[ P = (s^2hk)/(3h^2s^2K/4)^{2/3} = (16s^2k^3/9hK^2)^{1/3}. \]

If \( P \gg 1 \) the effect of proportional transaction costs is negligible as compared to the effect of fixed transaction costs. The reverse is true, if \( P \ll 1 \). Consider the following range of parameters: \(^7\) Holding cost/day \((h) \sim 10^{-4} - 10^{-3}; \) fixed transaction cost

\(^7\) This expression is derived in a slightly different form in Miller and Orr [15], p. 754, footnote 30.

\(^8\) The range of parameters is similar to that used by Miller and Orr [15].
(K) \sim 1 - 10^2; \text{ proportional transaction cost } (k) \sim 10^{-3} - 10^{-2}; \text{ cash demand variance/day } (\sigma^2) \sim 1 - 10^6. \text{ For this range } P \sim 10^{-3} - 10^2. \text{ We conclude that under reasonable assumptions either the fixed or the proportional transactions costs may dominate the optimal policy.}

6. Extensions and Concluding Remarks

Various generalizations and extensions of the procedure of §3 and §4 have already been indicated. A case of particular interest, considered by Miller and Orr [14], [15], forbids negative cash balances. In terms of our approach, we add the constraint \( d = 0 \). Also we may make the holding and penalty cost rates functions of the cash level. These extensions are discussed in Constantinides [7].

The holding and penalty cost function may be generalized as \( \max((x - x_0)h, (x_0 - x)p) \), where \( x = x_0 \) is the most desirable cash level. Through the transformation \( x \to x + x_0 \) it is easily shown that the only change of Propositions 1–3 is a shift of the policy parameters by the amount \( x_0 \).

References

