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MULTIPERIOD CONSUMPTION AND INVESTMENT BEHAVIOR WITH CONVEX TRANSACTIONS COSTS*

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The effect of convex transactions costs on consumers' derived utility functions and on optimal consumption and investment decisions is examined in a general multiperiod framework. The extent to which multiperiod consumption-investment behavior and capital market equilibrium may be studied in a single period framework is discussed. Optimal investment policy, in terms of a region of no transactions, is shown to be of a particularly simple form. (FINANCE—INVESTMENT CRITERIA; DECISION ANALYSIS—SEQUENTIAL; UTILITY/PREFERENCE—THEORY)

1. Introduction

The paper examines the effect of convex transactions costs on consumers' derived utility functions and on optimal consumption-investment decisions. In the absence of transactions costs and under otherwise weak assumptions, Fama [7], [8] proved that the observable behavior in any period of a consumer who has a multiperiod horizon is indistinguishable from the behavior of some other consumer who has a single period horizon and maximizes his expectation of a concave, state dependent (derived) utility function. Furthermore, under plausible assumptions the derived utility function is state independent. These observations by Fama make possible the study of consumption-investment behavior and capital market equilibrium in a single period framework rather than in a cumbersome intertemporal framework.

The first issue addressed in this paper is the extent to which multiperiod consumption-investment behavior and capital market equilibrium may be studied in a single period framework in the presence of convex transactions costs. We prove that derived utility is monotone increasing and concave in each of the assets (Proposition 1). Derived utility is concave in total wealth also, provided that asset proportions remain unchanged (Proposition 2). These properties suggest that consumption-investment behavior may be studied in a single period framework, provided that assets are treated as distinct goods (which may be transformed to one another at some transactions cost). Under plausible assumptions the derived utility function is also shown to be state independent.

The second issue discussed in this paper is the properties of optimal investment decisions. Locally optimal investment decisions are shown to be globally optimal (Proposition 3). With proportional transactions costs optimal investment policy is conveniently described in terms of a region of no transactions (Proposition 4). Under some homogeneity assumptions, the region of no transactions is a cone and optimal investment is homogeneous of degree one in the asset holdings (Proposition 5). If there exist only two assets (or mutual funds) optimal investment policy is simply described in terms of two parameters, \( a \leq a \): The consumer refrains from transacting so long as portfolio proportions lie in the interval \([a, a]\); and transacts to the closer boundary \( a \) or \( a \) whenever the portfolio proportions lie outside this interval (Proposition 7). These properties not only increase our understanding of optimal investment behavior but also simplify numerical estimation of these decisions.

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The third issue concerns the properties of optimal consumption behavior. Locally optimal consumption decisions are shown to be globally optimal (Proposition 3). Under some homogeneity assumptions, optimal consumption is homogeneous of degree one in the asset holdings (Proposition 5), and marginal propensity to consume lies between zero and one (Proposition 6).

2. The Model

A consumer makes sequential consumption and investment decisions at dates \( t = 0, 1, 2, \ldots, T \). There are \( n+1 \) goods, or "assets" at each date. At date \( t \) the consumer owns \( x_t \) units of account (e.g., dollars) of the \( i \)th asset, before he makes investment decisions and before he consumes at date \( t \). We define vectors \( x_t = [x_t^1, x_t^2, \ldots, x_t^n] \) and \( X_t = [x_t^0, x_t] \). The zeroth asset plays a special role, which will be shortly explained. The consumer makes investment decisions at date \( t \) denoted by \( u_t = [u_t^1, u_t^2, \ldots, u_t^n] \). \( u_t^i \) denotes the number of units of account by which the consumer increases (decreases) his holding of the \( i \)th asset; after the transaction the consumer's holdings of the \( n \) assets \( i = 1, 2, \ldots, n \) are \( x_t + u_t \). The total amount transferred to the \( n \) assets is \( u_t I \) where \( I \) is a unit column vector. This amount is subtracted from the zeroth asset. The consumer incurs transactions costs, \( T(u_t) \), which are charged to the zeroth asset. After the transaction the consumer's holding of the zeroth asset is \( x_t^0 = u_t I - T(u_t) \). We assume that \( T(0) = 0 \), \( T(u) \) is convex in \( u \), and \( |T(u) - T(v)| < \| u - v \|, \forall u, v, u \neq v \).

After the transactions are completed, the consumer consumes \( c_t^i \geq 0 \) units of account of the \( i \)th asset, \( i = 0, 1, \ldots, n \). We define vectors \( c_t = [c_t^1, c_t^2, \ldots, c_t^n] \) and \( C_t = [c_t^0, c_t^1, c_t^2, \ldots, c_t^n] \). Consumption depletes the \( n+1 \) assets. After consumption the consumer holds \( y_t^0 = x_t^0 - u_t I - T(u_t) - c_t^0 \) units of account of the zeroth asset and \( y_t = x_t + u_t - c_t \) units of account of the remaining \( n \) assets.

The consumer's holdings \( [y_t^0, y_t] \) are transformed at date \( t+1 \) to \( X_{t+1} = F_t(y_t^0, y_t, \phi_{t+1}) \), \( f_t \) is a monotone increasing and concave function of \( y_t^0, y_t \), and \( F_t(0, 0, \phi_{t+1}) > 0 \). \( \phi_{t+1} \) is a state variable which is realized at date \( t+1 \). This state variable will be further discussed shortly.

The simplest kind of transformation occurs when \( (y_t^0, y_t) \) represent investments in the stock or bond market. Let \( r_t^i \) be the capital gain rate and \( d_t^i \) be the dividend rate for the \( i \)th asset. The transformation on the \( i \)th asset is \( x_{t+1}^i = (1 + r_{t+1}^i) y_t^i \); also, the cash asset grows by \( d_{t+1}^i y_t^i \). There is no need, however, to limit the interpretation of the transformation function to simple investments. Consider, for example, investment in a European call option which matures at date \( t+1 \); if the option is exercised, the option is transformed to the underlying stock and the cash asset is decreased by the exercise price of the option plus commission fee. The transformation function also allows for the possibility that the consumer receives stochastic endowments of the \( n+1 \) assets at date \( t+1 \), i.e., receives exogenous income. More generally, the transformation may be interpreted as a stochastic production function where \( y_t^0, y_t \) are the inputs and \( X_{t+1} \) is the vector of outputs. For the general results discussed in this and the next section it is unnecessary to specify the function \( f_t \) in further detail.

At date \( t \) the state of the consumer is summarized by his asset holdings \( X_t \), by his stream of consumption \( C_{t-1} \) at all dates prior to \( t \), and by a vector \( \phi_t \). The vector \( \phi_t \) is of minimal dimension and is observable by the consumer. \( \phi_t \) summarizes the consumer's beliefs at date \( t \) regarding the state variable \( \phi_{t+1} \), i.e., \( F_t(\phi_{t+1} | \phi_t, X_t, C_{t-1}) = F_t(\phi_{t+1} | \phi_t) \); \( F_t \) is a probability distribution function at date \( t \). Since the transformation \( f_t \) of asset holdings over \((t, t+1)\) is a function of \( \phi_t \), the state variable \( \phi_t \) summarizes the consumer's beliefs regarding the transformation \( f_t \). For example, \( \phi_t \) summarizes the consumer's probability distribution of the rates of return on assets
over \((t, t + 1)\). As we shall shortly explain, \(\phi_t\) also summarizes the consumer’s tastes at date \(t\).

The consumer’s utility of lifetime consumption is \(U(C_T, \phi_T)\). \(U\) is monotone increasing and concave in \(C_T\). Note that the consumer’s tastes are state dependent. The consumer makes sequential consumption-investment decisions with the objective to maximize the expectation of his utility of lifetime consumption.

We define the function \(V_t(C_{t-1}, X_t, \phi_t)\) as the derived utility of consumption \(C_{t-1}\) and assets \(X_t\) at time \(t\) and state \(\phi_t\), before consumption-investment decisions have been made at date \(t\) and assuming that optimal consumption-investment policies are followed from date \(t\) to \(T\). The dynamic program is stated as:

\[
V_t(C_{t-1}, X_t, \phi_t) \equiv \max_{(\phi_t', c_t, u_t') \in \Omega_t} \int_{\phi_{t+1}} V_{t+1}(C_t, f_t(x_t^0 - u_t I - T(u_t) - c_t^0, x_t + u_t - c_t, \\
\phi_{t+1'), \phi_{t+1})} dF_t(\phi_{t+1} | \phi_t), \quad t = 0, 1, \ldots, T,
\]

with boundary condition

\[
V_{T+1}(C_T, X_{T+1}, \phi_{T+1}) = U(C_T, \phi_T).
\]

At date \(t\) the feasible set \(\Omega_t\) of consumption-investment decisions \(c_t^0, c_t, u_t\) is a function of the state. The set \(\Omega_t\) ensures that consumption at date \(t\) is nonnegative and that there exist feasible future consumption paths. We may define \(\Omega_t\) as:

\[
\begin{align*}
  c_t^0 &> 0, \quad c_t > 0 \quad \text{(nonnegative consumption)}, \\
  x_t^0 - u_t I - T(u_t) - c_t^0 &> 0, \\
  x_t + u_t - c_t &> 0 \quad \text{(nonnegative investment)}.
\end{align*}
\]

If the transformation function \(f_t\) is such that nonnegative inputs at date \(t\) ensure nonnegative outputs at \(t + 1\) (limited liability), then the set \(\Omega_t\) defined by (3) ensures feasible consumption paths.

A more general definition of \(\Omega_t\) which allows negative investment, i.e., borrowing and selling assets short, is given below:

\[
\begin{align*}
  c_t^0 &> 0, \quad c_t > 0 \quad \text{(nonnegative consumption)}, \\
  x_{t+1}^0 + x_{t+1} I - T(-x_{t+1}) &> 0, \text{ for all } \phi_{t+1} \quad \text{(nonnegative net worth)}.
\end{align*}
\]

The latter definition ensures that the consumer is solvent at date \(t + 1\) onwards and that there exists at least one feasible consumption plan, namely \(c_t^0 = c_t = 0\) for \(t = t + 1, t + 2, \ldots, T\).

The set \(\Omega_t\), is convex whether it is defined by (3) or by (3'). This property is stated and proved as a lemma for future reference.

**Lemma 1.** \(\Omega_t\) is a convex set.

**Proof.** Consider first the case where \(\Omega_t\) is defined by (3). \(x_t^0 - u_t I - T(u_t) - c_t^0\) is a concave function of \(u_t, c_t^0\) and therefore \(x_t^0 - u_t I - T(u_t) - c_t^0 \geq 0\) defines a convex set. Also \(c_t^0 > 0, c_t > 0\) and \(x_t + u_t - c_t > 0\) define convex sets. \(\Omega_t\) is the intersection of these sets and is therefore convex.

Consider next the case where \(\Omega_t\) is defined by (3'). Since \(y_t^0 = x_t^0 - u_t I - T(u_t) - c_t^0\) and \(y_t = x_t + u_t - c_t\) are concave functions of \(c_t^0, c_t, u_t\), and since \(f_t(y_t^0, y_t, \phi_{t+1})\) is a
monotone increasing and concave function of $y^0_r, y_r$, it follows that $X_{i+1} = f_i$ is a concave function of $c^0_i, c_i, u_i$, for given $\phi_{i+1}$. By assumption, $|T(u) - T(v)| < ||u - v||$, \(\forall u, v, u \neq v\). Therefore $x^0_{i+1} + x_{i+1}I - T(-x_{i+1})$ is a monotone increasing and concave function of $X_{i+1}$ and $x^0_{i+1} + x_{i+1}I - T(-x_{i+1})$ is a concave function of $c^0_i, c_i, u_i$. For each $\phi_{i+1}, x^0_{i+1} + x_{i+1}I - T(-x_{i+1}) \geq 0$ defines a convex set. The intersection of these sets defined by all $\phi_{i+1}$, and of the sets $c^0_i \geq 0$ and $c_i \geq 0$ is the convex set $\Omega_i$. The proof is complete.

The consumer's initial state $X_0, \phi_0$, the set of feasible consumption-investment decisions $\Omega$, defined by (3) or (3'), the sequential optimization defined by (1) and the boundary condition (2) completely specify the consumer's decision problem. It is assumed that the initial state and the model specification are such that the set $\Omega_0$ is nonempty. In the next section we discuss properties of the derived utility function and of the optimal consumption-investment decisions.

3. Properties of Derived Utility, Consumption and Investment

PROPOSITION 1. For all $\phi_i$ and $t = 0, 1, \ldots, T$, derived utility $V_i(C_{i-1}, X_t, \phi_t)$ is monotone increasing and concave in $C_{i-1}, X_t$.

PROOF. By the boundary condition (2), $V_{T+1}(C_T, X_{T+1}, \phi_{T+1}) = U(C_T, \phi_T)$. Therefore $V_{T+1}$ is monotone increasing and concave in $C_T, X_{T+1}$. The proof proceeds by induction. We assume that $V_{i+1}(C_i, X_{i+1}, \phi_{i+1})$ is monotone increasing and concave in $C_i, X_{i+1}$.

We first prove monotonicity of $V_i(C_{i-1}, X_t, \phi_t)$ in $C_{i-1}, X_t$. Let $\bar{c}_i^0, \bar{c}_i$ and $\bar{u}_i$ be the optimal decisions corresponding to state $C_{i-1}, X_t, \phi_t$. For any vectors $\delta C_{i-1} \geq 0, \delta X_t \geq 0$, the decisions $\bar{c}_i^0, \bar{c}_i$ and $\bar{u}_i$ are feasible given state $C_{i-1} + \delta C_{i-1}, X_t + \delta X_t, \phi_t$. Therefore:

$$V_i(C_{i-1} + \delta C_{i-1}, X_t + \delta X_t, \phi_t)$$

$$\geq \int_{\phi_{i+1}} V_{i+1}(C_{i-1} + \delta C_{i-1}, \bar{c}_i^0, \bar{c}_i, f_i(x^0_t - \bar{u}_tI - T(\bar{u}_t))$$

$$- \bar{c}_i^0 + \delta x^0_t, x_t + \bar{u}_t - \bar{c}_i + \delta x_t, \phi_{i+1}) dF_i(\phi_{i+1} | \phi_t)$$

$$\geq \int_{\phi_{i+1}} V_{i+1}(C_{i-1}, \bar{c}_i^0, \bar{c}_i, f_i(x^0_t - \bar{u}_tI - T(\bar{u}_t))$$

$$- \bar{c}_i^0, x_t + \bar{u}_t - \bar{c}_i, \phi_{i+1}) dF_i(\phi_{i+1} | \phi_t)$$

$$\geq V_i(C_{i-1}, X_t, \phi_t).$$

We next prove concavity of $V_i(C_{i-1}, X_t, \phi_t)$ in $C_{i-1}, X_t$. Consider the recursive equation (1), $y^0_r = x^0_r - u_rI - T(u_r) - c^0_r$ and $y_r = x_r + u_r - c_r$ are concave in $x^0_r, x_r, c^0_r, c_r$ and $u_r$. Also $f_i(y^0_r, y_r, \phi_{i+1})$ is monotone increasing and concave in $y^0_r, y_r$. Therefore $f_i$ is concave in $x^0_r, x_r, c^0_r, c_r$ and $u_r$. Therefore $V_{i+1}(C_i, f_i, \phi_{i+1})$ is monotone increasing and concave in $C_i, f_i$, given $\phi_{i+1}$. Therefore $V_{i+1}$ is concave in $x^0_r, x_r, C_{i-1}, c_r$ and $u_r$. The concavity is preserved under integration (addition) and therefore the integral is concave in $x^0_r, x_r, C_{i-1}, c_r$ and $u_r$. The concavity is also preserved under the operation of maximization since $\Omega_i$ is a convex set, by Lemma 1. (See, for example, Rockafellar [16, Theorem 5.3].) Therefore $V_i(C_{i-1}, X_t, \phi_t)$ is concave in $C_{i-1}, X_t$. The proof is complete.

Some variations of Proposition 1 are stated without proof. If $U(C_T | \phi_T)$ is strictly monotone increasing in $C_T$ then $V_i(C_{i-1}, X_t, \phi_t)$ is strictly monotone increasing in
$C_{t-1}, X_t$ for all $\phi_t, t$. Alternatively, if $U(C_T | \phi_T)$ is monotone increasing in $C_T$ and
strictly monotone increasing in $c^0, c^1, \ldots, c^T$ then $V_t(C_{t-1}, X_t, \phi_t)$ is strictly monoton
creasing in $c^0, c^1, \ldots, c^0$ and $X_t$, for all $\phi_t, t$.

In interpreting Proposition 1, we first assume that the $n + 1$ assets are different
consumption or production goods. Then the recursive equation (1) and Proposition 1
state that the consumer's intertemporal optimization problem reduces to a single
period problem: At each date $t$, and given the state $C_{t-1}, X_t, \phi_t$, the consumer
maximizes his expectation of a monotone increasing and concave (derived) utility function of consumption $c_t^0, c_t^1$ in the $n + 1$ goods at date $t$ and of wealth $X_{t+1}$ in the
$n + 1$ goods at date $t + 1$. Furthermore, if tastes and relative asset prices are state
independent, i.e., $U(C_T, \phi_T) = U(C_t)$ for all $\phi_t$, and if investment opportunities are
state independent, i.e., $f_t(y^0_t, y_t, \phi_{t+1}) = f_t(y^0_t, y_t)$ for all $\phi_{t+1}$, then the consumer's derived
utility function is state independent, i.e., $V_{t+1}(C_t, X_{t+1}, \phi_{t+1}) = V_{t+1}(C_t, X_{t+1})$
for all $\phi_{t+1}$. The state variable $\phi_{t+1}$ becomes superfluous in the sense that the
consumer's optimal consumption-investment decisions are independent of this vari
able. This discussion essentially generalizes earlier results by Fama [7] which were
derived under the assumption of zero transactions costs.

It is useful to express the consumer's holdings $X_t$ in terms of total wealth,
$W_t = x^0_t + x_t I$, and fractional holdings in the $n$ assets, $\xi_t = x_t / W_t$. Clearly $x_t = W_t \xi_t$
and $x^0_t = (1 - \xi_t) W_t$. Derived utility $\tilde{V}_t(C_{t-1}, W_t, \xi_t, \phi_t)$ in terms of state variables
$C_{t-1}, W_t, \xi_t, \phi_t$ is defined by

$$\tilde{V}_t(C_{t-1}, W_t, \xi_t, \phi_t) \equiv V_t(C_{t-1}, (1 - \xi_t) W_t, W_t \xi_t, \phi_t). \tag{4}$$

We now prove:

**Proposition 2.** For all $\xi_t, \phi_t, t = 0, 1, \ldots, T$, derived utility $\tilde{V}_t(C_{t-1}, W_t, \xi_t, \phi_t)$ is
monotone increasing and concave in $C_{t-1}, W_t$; for all $W_t, \phi_t, t = 0, 1, \ldots, T$, derived
utility $\tilde{V}_t(C_{t-1}, W_t, \xi_t, \phi_t)$ is concave in $C_{t-1}, \xi_t$.

**Proof.** Given $\xi_t, \phi_t, t$, then $X_t$ is monotone increasing and concave in $W_t$. By
Proposition 1, $V_t$ is monotone increasing and concave in $C_{t-1}, X_t$. Therefore $V_t$ is
monotone increasing and concave in $C_{t-1}, W_t$. The second part is similarly proved.

Before we further study the effect of transactions costs on consumption-investment
decisions, we briefly review the implications of the model in the absence of transac
tions costs. We set $T(0) = 0$ and obtain Fama's [7] model:¹ State $(C_{t-1}, X_t, \phi_t)$ is
concisely represented by $(C_{t-1}, W_t, \phi_t)$, where $\xi_t$ becomes a superfluous state variable.
At date $t$ the consumer makes optimal consumption-investment decisions with the
objective to maximize his expectation of a monotone increasing and concave (derived)
utility function of consumption at date $t$ and of wealth $W_{t+1}$ at date $t + 1$.² These
observations by Fama provide a general theoretical justification for the study of
consumption-investment behavior and capital market equilibrium in a single period
model rather than in a cumbersome intertemporal model.

In the presence of transactions costs $\xi_t$ becomes a relevant state variable at date $t$.
At date $t$ the consumer maximizes the expectation of $\tilde{V}_{t+1}(C_t, W_{t+1}, \xi_{t+1}, \phi_{t+1})$. This
function is not in general concave in $W_{t+1}$ unless $\xi_{t+1}$ is taken as fixed at date $t$. These
qualifications appear in [4].

¹Our transformation function $f_t$ is slightly more general than the transformation function in Fama's
model.
²Strict concavity of the derived utility function is proven in Fama [8]. See also the discussion in Ziemba
[22].
In characterizing the consumer's optimal consumption-investment decisions, we state:

**Proposition 3.** Locally optimal consumption-investment decisions are globally optimal.

**Proof.** In the proof of Proposition 1, we showed that the expectation at $t$ of $V_{t+1}$ is concave in $x_t^0, x_t, C_{t-1}, c_t^0, c_t, u_t$. The consumer's objective at date $t$ is the maximization of the expectation of $V_{t+1}$, given $\phi_t, x_t^0, x_t, C_{t-1}$. The objective is concave in $c_t^0, c_t, u_t$ and the result follows.

Thus numerical procedures which search for local optima yield globally optimal consumption-investment decisions.

4. Proportional Transactions Costs

We sharpen the characterization of optimal consumption-investment behavior under the additional assumption that the transactions costs function is positively homogeneous of degree one, i.e., $T(\lambda u) = \lambda T(u)$, $\lambda > 0$. The joint assumption of convexity and homogeneity of the transactions costs function implies $T(u + v) < T(u) + T(v)$ (see, for example, Rockafellar [16, Theorem 4.7]): The cost of transaction $u + v$ is less than the sum of the costs of transactions $u$ and $v$. This property is crucial in the subsequent discussion.

A transactions costs function of considerable practical importance is the proportional transactions costs function defined by $T(u) \equiv \sum_{i=1}^{n} \max[k_{2i} u_i, -k_{2i} u_i]$, where $0 < k_{2i} < 1$, $0 < k_{2i} < 1$, and $k_{2i}, k_{2i}$ are given constants. Note that this function has the desirable properties of convexity and positive homogeneity of degree one; also $T(u) = 0$ and $|T(u) - T(v)| < ||u - v||$, $\forall u, v, u \neq v$.

We define the function $g_t$ as

$$g_t(u_t, X_t; C_{t-1}, \phi_t) = \max_{(c_t^0, c_t)} \int_{\phi_t+1} V_{t+1}(C_t, f_t(x_t^0 - u_t I - T(u_t) - c_t^0, x_t + u_t - c_t, \phi_{t+1}) dF_t(\phi_{t+1}, \phi_t)$$

where $(c_t^0, c_t)$ is such that $(c_t^0, c_t, u_t) \in \Omega_t$. We shall abbreviate $g_t(u_t, X_t; C_{t-1}, \phi_t)$ by $g(u_t, X_t)$ when $t, C_{t-1}, \phi_t$ are easily understood. We also define the subset $\omega_t$ as $\omega_t = \{u_t | (c_t^0, c_t, u_t) \in \Omega_t, (c_t^0, c_t) \text{ are optimal}\}$. Clearly, the consumer makes investment decisions $u_t \in \omega_t$ at date $t$ with the objective to maximize $g_t(u_t, X_t; C_{t-1}, \phi_t)$.

Optimal investment policy is conveniently described in terms of a region of no transactions, $\Psi_t = (X_t | g(u_t, X_t) < g(0, X_t), u_t \in \omega_t)$. If the consumer enters date $t$ with assets $X_t \in \Psi_t$, an optimal investment policy is to carry no transactions, i.e., $u_t = 0$. We note that the region of no transactions $\Psi_t$, in general depends upon past history, $C_{t-1}, \phi_t$. When $X_t$ lies outside the region of no transactions, optimal investment is described as follows:

**Proposition 4.** Assume homogeneous of degree one transactions costs. The consumer enters date $t$ with assets $X_t$. An optimal investment decision $u_t$ is such that:

(a) After the transaction, the resulting asset holdings lie in the region of no transactions, i.e., $(x_t^0 - u_t I - T(u_t), x_t + u_t) \in \Psi_t$.

(b) If there exists a $0 < \lambda < 1$ such that $(x_t^0 - \lambda u_t I - T(\lambda u_t), x_t + \lambda u_t) \in \Psi_t$, then $\lambda u_t$ is an optimal investment decision also.

**Proof.** (a) The proof is by contradiction. Assume that $(x_t^0 - u_t I - T(u_t), x_t + u_t)$
is not in \( \Psi_i \). Then there exists some \( v \neq 0 \) such that:

\[
g(0, x_i^0 - u_i I - T(u_i), x_i + u_i)
\]
\[
< g(v, x_i^0 - u_i I - T(u_i), x_i + u_i)
\]
\[
< g(0, x_i^0 - u_i I - T(u_i) - vI - T(v), x_i + u_i + v)
\]
\[
< g(0, x_i^0 - (u_i + v)I - T(u_i + v), x_i + (u_i + v))
\]

where we used the property \( T(u_i) + T(v) \leq T(u_i + v) \). Therefore the investment decision \( u_i \) is inferior to the investment decision \( u_i + v \), given state \( X_i \).

It remains to show that investment \( u_i + v \) is feasible, given state \( X_i \). Consider first the case where the set \( \Omega_i \) is defined by equation (3). By assumption, the investment \( v \) is feasible, given state \( (x_i^0 - u_i I - T(u_i), x_i + u_i) \); i.e., there exists a consumption vector \( c_i^0 \geq 0, c_i \geq 0 \) such that \( x_i^0 - u_i I - T(u_i) - c_i^0 - vI - T(v) \geq 0 \) and \( x_i + u_i - c_i + v_i \geq 0 \). Since \( x_i^0 - (u_i + v)I - T(u_i + v) - c_i^0 > x_i^0 - u_i I - T(u_i) - c_i^0 - vI - T(v) \), investment \( u_i + v \) is feasible, given state \( X_i \). A similar argument applies when \( \Omega_i \) is defined by (3).

Since the investment \( u_i \) is inferior to the feasible investment \( u_i + v \), given state \( X_i \), the assumption is contradicted.

(b) We break up the investment \( u_i \) into two parts \( \lambda u_i \) and \( (1 - \lambda)u_i \), \( 0 \leq \lambda < 1 \). The total transactions cost remains unchanged because \( T(\lambda u_i) + T((1 - \lambda) u_i) = T(u_i) \).

Also transaction \( (1 - \lambda) u_i \) is feasible given state \( (x_i^0 - \lambda u_i I - T(\lambda u_i), x_i + \lambda u_i) \). Therefore

\[
g_i(u_i, X_i) = g_i((1 - \lambda) u_i, x_i^0 - \lambda u_i I - T(\lambda u_i), x_i + \lambda u_i)
\]
\[
< g_i(0, x_i^0 - \lambda u_i I - T(\lambda u_i), x_i + \lambda u_i).
\]

The second step follows from the fact that \( (x_i^0 - \lambda u_i I - T(\lambda u_i), x_i + \lambda u_i) \in \Psi_i \). Also

\[
g_i(0, x_i^0 - \lambda u_i I - T(\lambda u_i), \lambda u_i) = g_i(\lambda u_i, X_i)
\]

since the resulting asset holdings are equal. Combining the above two equations we obtain \( g_i(u_i, X_i) < g_i(\lambda u_i, X_i) \); i.e., the investment decision \( \lambda u_i \) is at least as good as the investment decision \( u_i \). Since \( u_i \) is optimal, \( \lambda u_i \) is optimal also.

The proof is complete.

The region of no transactions, \( \Psi_i \), plays a double role. First (by definition) it is the set of asset holdings \( X_i \) such that an optimal investment policy is to carry no transactions. Second (by Proposition 4) it is the set of asset holdings \( (x_i^0 - \bar{u}_i I - T(\bar{u}_i), x_i + \bar{u}_i) \) after optimal transaction \( \bar{u}_i \), given any initial holdings \( X_i \) (\( X_i \) may or may not lie in the region of no transactions). Without the assumption of homogeneous transactions costs the region of no transactions will not, in general, play the second role: An optimal policy might be to transact to a point which lies outside the region of no transactions.

Part (b) of Proposition 4 may be explained as follows. Let \( \bar{u}_i \) be an optimal investment decision. The points \( (x_i^0 - \lambda \bar{u}_i I - T(\lambda \bar{u}_i), x_i + \lambda \bar{u}_i), 0 \leq \lambda \leq 1 \) describe the path of the asset holdings from the initial position \( (\lambda = 0) \) to the final position \( (\lambda = 1) \). If some point for which \( 0 < \lambda < 1 \), lies in the region of no transactions, then \( \lambda \bar{u}_i \) is an optimal investment also. Informally, whenever there exists an optimal investment decision such that the resulting asset holdings lie within the region of no transactions, there exists another optimal investment decision such that the resulting asset holdings lie on the boundary of the region of no transactions. Thus, whenever the initial asset
holdings lie outside the region of no transactions, an optimal investment decision is to transact to the boundary of this region. In contrast, with concave transactions costs we would expect all optimal (nonzero) investment decisions to be such that the resulting asset holdings lie within the region of no transactions.

If the utility function is additively separable, i.e., \( U(C_t, \phi_t) = \sum_{i=0}^{T-1} \hat{U}_i(c_i^0, c_t, \phi_t) \), then the consumer's objective at date \( t \) is to maximize his expectation of \( \sum_{i=0}^{T-1} \hat{U}_i(c_i^0, c_t, \phi_t) \) which is independent of past consumption. Also \( \Omega_t \) is independent of \( \zeta_{t-1} \), \( \zeta_{t-1} \) is a superfluous state variable and the state at date \( t \) is simply \((X_t, \phi_t)\). The recursive equation (1) is simplified to

\[
J_t(X_t, \phi_t) = \max \left[ \hat{U}_i(c_i^0, c_t, \phi_t) + \int_{\phi_{i+1}} J_{t+1}(f_i(x_{i+1} - u, I - T(u) - c_i^0, x_i + u - c_t, \phi_{i+1}, \phi_{i+1}) dF_i(\phi_{i+1} | \phi_t) \right],
\]

\( t = 0, 1, \ldots, T, \)

with boundary condition

\[
J_{T+1}(X_{T+1}, \phi_{T+1}) = 0.
\]

If the utility function is multiplicatively separable, i.e., \( U(C_t | \phi_t) = \prod_{i=0}^{T-1} \hat{U}_i(c_i^0, c_t, \phi_t) \), the state at date \( t \) is simply \((X_t, \phi_t)\). Also the recursive equation (1) simplifies accordingly. In the subsequent discussion we shall assume additively separable utility and leave the corresponding development in the case of multiplicatively separable utility as an exercise to the reader.

We now state:

**PROPOSITION 5.** Assume that the transactions costs function \( T(u) \) is positively homogeneous of degree one in \( u \); the utility function \( U(C_t, \phi_t) \) is additively or multiplicatively separable; \( \hat{U}_i(c_i^0, c_t, \phi_t) \) is positively homogeneous of degree \( \alpha \) in \( c_i^0, c_t \); and the transformation function \( f_i(Y_t, \phi_t) \) is positively homogeneous of degree one in \( Y_t \). Then

(a) The optimal policy functions \( \tilde{c}_i^t(X_t, \phi_t), \tilde{c}_i^t(X_t, \phi_t), \tilde{u}_i(X_t, \phi_t) \) are homogeneous of degree one in \( X_t \).

(b) The region of no transactions is a cone, i.e., \( X_t \in \Psi_t \) implies \( \lambda X_t \in \Psi_t \), for all \( \lambda > 0 \).

**PROOF.** (a) Inspection of (3) or (3') indicates that \((c_i^0, c_t, u_t) \in \Omega_t(X_t, \phi_t) \) implies \( \lambda c_i^0, \lambda c_t, \lambda u_t \in \Omega_t(\lambda X_t, \phi_t), \lambda > 0 \). If \( J_{t+1}(\lambda X_{t+1}, \phi_{t+1}) = \lambda^\alpha J_{t+1}(X_{t+1}, \phi_{t+1}) \), inspection of (1') and the above result indicates that \( J_t(\lambda X_t, \phi_t) = \lambda^\alpha J_t(X_t, \phi_t) \) and optimal \( c_i^0, c_t, u_t \) are homogeneous of degree one in \( X_t \). To complete the induction proof we note that (2') implies that \( V_{t+1}(\lambda X_{t+1}, \phi_{t+1}) = \lambda^\alpha V_{t+1}(X_{t+1}, \phi_{t+1}) \).

(b) The above results imply that, if \( u_t \in \omega_t(X_t, \phi_t) \) then \( \lambda u_t \in \omega_t(\lambda X_t, \phi_t) \). They also imply that \( g_t(\lambda u_t, \lambda X_t, \phi_t) = \lambda^\alpha g_t(u_t, X_t, \phi_t) \). If \( g_t(u_t, X_t) < g_t(0, X_t) \) and \( u_t \in \omega_t(X_t, \phi_t) \), then \( g_t(\lambda u_t, \lambda X_t) < g_t(0, \lambda X_t) \) for all \( \lambda > 0 \). But \( u_t \in \omega_t(X_t, \phi_t) \). Therefore, if \( X_t \in \Psi_t \) then \( \lambda X_t \in \Psi_t \), i.e., \( \Psi_t \) is a cone.

The proof is complete.

We express state \((X_t, \phi_t)\) as \((W_t, \xi_t, \phi_t)\) where \( W_t = x_t^0 + x_t I \) is total wealth and \( \xi_t = x_t / W_t \) are portfolio proportions. We define the optimal consumption function of the \( i \)th asset in terms of state variables \((W_t, \xi_t, \phi_t)\) as \( \tilde{c}_i^t(W_t, \xi_t, \phi_t) = \tilde{c}_i^t(X_t, \phi_t) \). The marginal propensity to consume out of wealth is described as follows:

\(^3\text{If the utility function is not homogeneous in consumption, it may sometimes be converted to a homogeneous function by a linear transformation of the asset variables. See [3] for details.}\)
PROPOSITION 6. Under the homogeneity assumptions of Proposition 5, the marginal propensity to consume each asset and the sum of the assets out of wealth lies between zero and one, i.e.,

(a) \( 0 < \frac{\partial \hat{c}_i^t(W_t, \xi_t, \phi_t)}{\partial W_i} < 1, \ i = 0, 1, \ldots, n; \)

and

(b) \( 0 < \sum_{i=0}^{n} \frac{\partial \hat{c}_i^t(W_t, \xi_t, \phi_t)}{\partial W_i} < 1. \)

PROOF. (a) By the homogeneity of optimal policy functions, \( \frac{\partial \hat{c}_i^t(W_t, \xi_t, \phi_t)}{\partial W_t} = \hat{c}_i^t / W_t. \) Also \( 0 < \hat{c}_i^t / W_t < 1, \) by the conditions (3) or (3'). The result follows.

(b) By homogeneity,

\[
\sum_{i=0}^{n} \frac{\partial \hat{c}_i^t(W_t, \xi_t, \phi_t)}{\partial W_i} = \frac{\sum_{i=0}^{n} \hat{c}_i^t}{W_t}.
\]

Also \( 0 < \sum_{i=0}^{n} \frac{\hat{c}_i^t}{W_t} < 1. \) The result follows.

This Proposition was earlier proved by Zabel [21] and Magill and Constantinides [12] under stronger assumptions.

5. The Two Asset Case with Proportional Transactions Costs

The region of no transactions attains a particularly simple form under the assumption that there exist only two assets. One may conveniently think of the two assets as money (or a bank account) and a mutual fund of all other assets; or, as a mutual fund of bonds and a mutual fund of stocks. Clearly the two-asset case is of considerable practical importance.

The investment decision \( u \) is a scalar. The assumptions \( T(\lambda u) = \lambda T(u) \) for \( \lambda > 0, \) \( T(0) = 0 \) and \( |T(u) - T(v)| < ||u - v|| \) for \( u \neq v, \) imply that \( T(u) = \max[ku, -k'u], \) where \( 0 < k < 1, \) \( 0 < k' < 1 \) and \( k, k' \) are given constants; i.e., transactions costs are proportional.

We now prove:

PROPOSITION 7. Under the homogeneity assumptions of Proposition 5 and the assumption that there exist only two assets, i.e., \( i = 0, 1, \) the region of no transactions is convex.

PROOF. It will suffice to prove that, if two rays \( A, B \) are in \( \Psi, \) so is any point \( C \) which lies in the cone defined by rays \( A, B. \)

Consider some point \( C \) which lies in the cone defined by rays \( A, B. \) Point \( C \) stands for \( (x^0_C, x^1_C), \) where we have suppressed the time subscript for convenience. Reference to Figure 1 is helpful. From point \( (x^0_C, x^1_C) \) we draw a line with slope \( -(1 + k)^{-1}. \) This
line meets ray $A$ at point $(x_A^0, x_A^1)$. Define $u \equiv x_c^1 - x_A^1 > 0$. Then $x_c^0 = x_A^0 - u - T(u)$, $x_c^1 = x_A^1 + u$ and
\[ g(u, x_A^0, x_A^1) = g(0, x_c^0, x_c^1). \] (6)

We need to prove that the line through $(x_A^0, x_A^1)$ always intersects ray $A$. If $\Omega_t$ is defined by (3), rays $A, B$ lie in the positive orthant and the slope of $A$ is nonnegative. If $\Omega_t$ is defined by Equation (3'), the slope of ray $A$ equals or exceeds $-(1 + k)^{-1}$. So long as the slope of ray $A$ exceeds $-(1 + k)^{-1}$, the line through $C$ intersects ray $A$ at some finite point, i.e., $u$ is finite. (If the slope of ray $A$ is $-(1 + k)^{-1}$, the point of intersection is at infinity.)

Since $(x_A^0, x_A^1) \in \Psi_t$,
\[ g(u, x_A^0, x_A^1) \leq g(0, x_0^0, x_1^1). \]

By the concavity of $g$ in $u$ and the above, for any $\lambda > 0$,
\[ g((1 + \lambda)u, x_A^0, x_A^1) \leq g(u, x_A^0, x_A^1). \] (7)

Thus
\[ g(\lambda u, x_c^0, x_c^1) = g((1 + \lambda)u, x_A^0, x_A^1) \leq g(u, x_A^0, x_A^1) \quad \text{(by (7))} \]
\[ \leq g(0, x_c^0, x_c^1) \quad \text{(by (6))}. \]

By a similar argument, for some $u < 0$ and for any $\lambda > 0$,
\[ g(\lambda u, x_c^0, x_c^1) \leq g(0, x_0^0, x_1^1). \]

Therefore $(x_c^0, x_c^1) \in \Psi_t$. The proof is complete.

Zabel [21] proved Proposition 7 under the following additional assumptions, none of which were necessary in our proof: The decision horizon is limited to two periods, i.e., $T = 2$; the zeroth asset is riskless; the returns on the risky asset are uncorrelated in the two periods; borrowing and selling short are prohibited; single period utility is of power form, is state independent and is identical in the two periods; no dividends are paid. Kamin [10] also proved Proposition 7 under the following assumptions, none of which were necessary in our proof: The consumer consumes only at the end of the horizon; borrowing and selling short are prohibited; rates of return are independent of earlier realizations; utility is state independent and is of power or logarithmic form; no dividends are paid.

Propositions 4, 5 and 7 imply that at any point in time optimal investment policy is determined in terms of two control limits $\underline{a} \leq \bar{a}$. These limits are in general functions of the state variable $\phi$. If $x_i^1/x_i^0 \in [\underline{a}, \bar{a}]$, an optimal investment policy is to refrain from transacting. If $x_i^1/x_i^0 < g(\underline{a})$ an optimal investment policy is to transact to level $g(\bar{a})$. That the optimal policy is of the control limit type is not surprising. What is far from obvious is that the region of no transactions is convex and that there is only one connected interval $[\underline{a}, \bar{a}]$ of portfolio proportions in which it is optimal to refrain from transacting.

6. Concluding Remarks

A comprehensive discussion of the properties of derived utility and optimal consumption-investment decisions has been presented under fairly weak assumptions. Recently, building upon the results of this paper, Abrams and Karmarkar [1] have presented conditions under which the derived utility function is differentiable.
A promising direction of future research is the derivation of closed-form expressions for the optimal decisions. Magill and Constantinides [12] made a first step in this direction under the assumption that asset returns are lognormally distributed. Another promising direction is the development of efficient algorithms for the numerical estimation of optimal decisions. The general properties of derived utility, optimal decisions and the region of no transactions discussed in this paper should prove useful in the development of such algorithms.4

4This paper is a revised version of [5]. I would like to thank two anonymous referees for several helpful comments. As usual, I remain responsible for errors.

References


