I. INTRODUCTION

The reprinted articles and contributed essays gathered in this volume present valuation theories put forth and tested in finance. An important class of valuation theories, pioneered by Modigliani and Miller (1958) and by Black and Scholes (1973), are based on the equilibrium condition that there exist no arbitrage opportunities. These essentially preference-free theories generate results of great generality without necessitating the specification of the equilibrium in its full detail.

A second important class of valuation theories makes assumptions on preferences and derives more specific pricing restrictions than do the preference-free theories. This essay provides an overview of these latter preference-dependent valuation theories. The unifying theme is that the various preference-dependent valuation theories are specializations (or sometimes generalizations) of the fundamental valuation equation, which equates the price of a claim to the expectation of the product of the future payoff and the marginal rate of substitution of the representative investor. Our goal is to stress the similarities rather than the differences of these theories and thereby provide the reader with a conceptual framework for some of the theories in this book.

Within the space limitations of one essay, I address a subset of the interesting issues, in particular, the filtering of information in an intertemporal economy of homogeneously informed agents, and the role of conditioning information on the empirically testable implications of these theories. In this essentially theoretical essay I have attempted to bridge the theoretical and empirical research in finance by referencing representative empirical papers.

Papers referenced with an asterisk (*) are included in this and the companion volume. I would like to thank Sudipto Bhattacharya, Wayne Ferson, Chi-fu Huang, and Ken Singleton for constructive criticism on an earlier draft.
The essay steers away from several important topics, such as economies with heterogeneously informed agents, existence and other technical aspects of an equilibrium, the application of valuation theory to corporate finance, corporate and personal taxes, and transaction costs. Some of these issues are dealt with elsewhere in these volumes. In particular, Sudipto Bhattacharya’s overview and the articles and essays in the second volume address the economics of information and its application in finance.

To convey the economics to a broad readership, including the beginning student of finance and economics, I sometimes sacrifice mathematical rigor. In specific applications I rely upon the simplifying assumption that prices and per capita consumption are multivariate normal. This implies that prices may be negative and that utility may be undefined if it is unbounded from below. These technical problems are often resolved in a continuous-time framework in which prices and consumption are modeled as an Itô process. Thus whenever I invoke the simplifying property that prices and per capita consumption are multivariate normal, it is to be understood that in a more rigorous treatment I would be modeling prices and consumption as an Itô process.

II. THE BASIC MODEL

Consider an exchange economy that extends over $T + 1$ periods, $t = 0, 1, 2, \ldots, T, T < \infty$. The assumptions on the discreteness of the trading interval and on the finiteness of the time horizon are intended to keep the discussion simple. I make references, when appropriate, to the continuous-trading economies, such as in Merton (1973a), and to the infinite-horizon recursive economies, such as in Lucas (1978).

There is one nonstorable consumption good only. The numeraire at date $t$ is the consumption good at date $t$.

Information at date $t$ is the vector $\phi^t \in E$, a Markov process. The cumulative probability distribution of $\phi^{t+1}$ is $F^t(\phi^{t+1} | \phi^t)$. Consumers know the functions $F^t(\cdot | \cdot)$ at all times and know the vector $\phi^t$ at date $t$. Thus consumers have homogeneous information.

There are $n$ firms, indexed $i = 1, 2, \ldots, n$. The $i$th firm pays net dividend $D_i^t(\phi^i)$ at date $t$, where the functions $D_i^t(\cdot)$ are known at all times. In this exchange economy the net dividend functions $D_i^t(\phi^i)$ are primitives. If storage of the consumption good is permitted, it may be modeled with a firm that provides the storage service.

That the functions $D_i^t(\phi^i)$ are primitive does not imply that firms are passive in their investment and financing decisions. Production economies, such as those in Brock (1982), Cox, Ingersoll, and Ross (1985a), and Prescott and Mehra (1980), endogenously determine the firms’ investment and financing policy functions, which in turn map into net dividend functions, $D_i^t(\phi^i)$. Thus production economies impose additional restrictions on the observables of the economy, and these restrictions involve the net dividend functions. If an
empirical test rejects the restrictions imposed by an exchange economy, it also rejects some of the restrictions imposed by the corresponding production economy.

I simplify the presentation with the assumption that the market is complete, or at least “effectively” complete. Although the assumption that the market is complete requires the existence of a large number of traded securities, Arrow (1964) and Kreps (1982) prove that the requirements on the number of traded securities may be considerably reduced. In economies where the information revelation is modeled by a finite-dimensional Brownian motion or diffusion process, Duffie and Huang (1985*) prove that the market can be effectively completed by a finite number of long-lived securities equal in number to the dimension of the information process plus one. See also the insightful discussion by Scheinkman (1988*).

When the market is complete, equilibrium prices and aggregate consumption remain unchanged when we replace the heterogeneous consumers with one consumer (the “representative” consumer) with chosen preferences and with endowment equal to the sum of the heterogeneous consumers’ endowments. An equilibrium in a heterogeneous-consumer economy is observationally equivalent to the equilibrium in some homogeneous-consumer economy. Hereafter I refer to the representative consumer as simply the “consumer.” This result is foreshadowed by Prescott and Mehra (1980). Existence of a representative consumer is shown by Constantinides (1982, Lemma 1). In the same paper Lemma 2 states that, if the heterogeneous consumers have time-additive, von Neumann–Morgenstern utility functions, the representative consumer’s utility function inherits these properties. Note that the conditions for the existence of a representative consumer are less stringent than the conditions for the aggregation of preferences, yet they are sufficient for our purposes. See also the related papers by Bhattacharya (1981), Breeden and Litzenberger (1978), Constantinides (1980), Dybvig and Ingersoll (1982), Dybvig and Ross (1982), and Rubinstein (1974).

I assume that the consumer receives no labor or other exogenous income. In equilibrium the consumer must hold all the stock of each firm and consume the aggregate dividend \( c^t = \sum D_i^t(\phi^t) \) at date \( t \). The consumer has von Neumann–Morgenstern preferences and expected utility at \( t \) given by

\[
E \left[ \sum_{t=1}^{T} u^t(c^t)|\phi^t \right] = E \left[ \sum_{t=1}^{T} u^t \left( \sum_{i=1}^{n} D_i^t(\phi^t) \right) |\phi^t \right],
\]

where the direct utility function \( u^t(\cdot) \) is monotone increasing and strictly concave.

In equilibrium, the ex dividend (shadow) price, \( P_i^t \), of the \( i \)th firm at date \( t \) is such that the consumer holds all the stock and is indifferent (at the margin) to buying or selling a fraction \( \alpha \) of the stock; that is, the objective function

\[
\max_{\alpha} E \left[ u^t(c^t - \alpha P_i^t) + \sum_{t+1}^{T} u^t(c^{t+1} + \alpha D_i^t) |\phi^t \right]
\]

is maximized at \( \alpha = 0 \). Since the objective function is strictly concave in \( \alpha \), the
first-order necessary condition of optimality is also sufficient. With simple manipulation this condition yields the fundamental valuation equation

\[ P_t^i = E \left[ \sum_{t'=t+1}^{T} \frac{u_t^i(c_{t'})}{u_t^{i-1}(c_{t'-1})} \cdot D_t^i | \phi^i \right], \]

where the subscript \( c \) denotes a derivative. Since the right side is a known function of \( \phi^i \), the equilibrium stock price is a known function of the information \( \phi^i \) and is sometimes denoted by \( P_t^i(\phi^i) \).

Since some of the net dividends may be negative, there is nothing in equation (1) to guarantee that a firm’s price is nonnegative. I impose the restriction that the net dividends are nonnegative for all \( \phi^i \). This restriction is actually too strong. In production economies the nonnegativity of prices is guaranteed by the stockholders’ limited liability.

The fundamental valuation equation is derived by Rubinstein (1976*), Lucas (1978), Breeden and Litzenberger (1978), and undoubtedly by others. It serves as the starting point of the valuation theories discussed in this essay. For future reference I rewrite it in terms of a firm’s rate of return from date \( t - 1 \) to date \( t \), defined as \( R_t^i \equiv (P_t^i + D_t^i)/P_{t-1}^i \). Since

\[ P_{t-1}^i = E \left[ \sum_{t'=t}^{T} \frac{u_t^i(c_{t'})}{u_t^{i-1}(c_{t'-1})} \cdot D_t^i | \phi_{t-1}^i \right] \]

\[ = E \left[ \frac{u_t^i(c_{t'})}{u_t^{i-1}(c_{t'-1})} \cdot \left( D_t^i + \sum_{t'=t+1}^{T} \frac{u_t^i(c_{t'})}{u_t^{i-1}(c_{t'-1})} \cdot D_t^i \right) | \phi_{t-1}^i \right] \]

\[ = E \left[ \frac{u_t^i(c_{t'})}{u_t^{i-1}(c_{t'-1})} \cdot D_t^i + E \left( \sum_{t'=t+1}^{T} \frac{u_t^i(c_{t'})}{u_t^{i-1}(c_{t'-1})} \cdot D_t^i | \phi_{t-1}^i \right) \right] | \phi_{t-1}^i \]

(by the law of iterated expectations)

\[ = E \left[ \frac{u_t^i(c_{t'})}{u_t^{i-1}(c_{t'-1})} \cdot (D_t^i + P_t^i) | \phi_{t-1}^i \right] \]

I divide both sides by \( P_{t-1}^i \) and use the definition of the rate of return to obtain

\[ E \left[ \frac{u_t^i(c_{t'})}{u_t^{i-1}(c_{t'-1})} \cdot \frac{R_t^i}{\phi_{t-1}^i} \right] = 1. \]

The riskless rate of interest from date \( t - 1 \) to date \( t \), denoted by \( R_0^i \), may be shown to satisfy the valuation equation (2). The excess rate of return on the \( i \)th firm, defined as \( r_t^i \equiv R_t^i - R_0^i \), satisfies

\[ E \left[ \frac{u_t^i(c_{t'})}{u_t^{i-1}(c_{t'-1})} \cdot r_t^i | \phi_{t-1}^i \right] = 0, \]

which simplifies to

\[ E[u_t^i(c_{t'}) \cdot r_t^i | \phi_{t-1}^i] = 0. \]

Equations (2) and (3) are restatements of the valuation equation (1).

The rate of return in equations (2) and (3) is defined between adjacent consumption periods, but it need not be. For example, the compound rate of
return of the \(i\)th firm from date \(t-2\) to date \(t\) is easily shown to satisfy the following generalized version of equation (2):

\[
E \left[ \frac{u_t^i(c^t)}{u_t^{i-2}(c^{t-2})} \cdot \frac{(P_t^i + D_t^i)(P_{t-1}^{i-1} + D_{t-1}^{i-1})}{P_t^{i-1} P_t^{i-2}} \cdot \phi^{i-2} \right] = 1.
\]

The same reasoning that led to equation (1) shows that the price \(V(X')\) at date \(t\) of any claim or contract (which may or may not be the stock of a firm) with cash flows \(X' = (X^t X^{t+1} \cdots X^T)\) is

\[
V(X') = E \left[ \sum_{t=1}^T u_t^i(c^t) \cdot X^t | \phi^t \right].
\]

The fundamental valuation equation, as stated in equations (1)–(4), is conditional on the information \(\phi^{t-1}\). By the law of iterated expectations I may take the expectation of the valuation equation with respect to any information subset of \(\phi^{t-1}\). In particular, I take the expectation of equation (3) with respect to the subset \(\phi^0\) to obtain the unconditional valuation equation

\[
E[u_t^i(c^t) \cdot r_t^i | \phi^0] = 0.
\]

Examples that display the role of conditioning information are in Grossman and Shiller (1982), Hansen and Richard (1985), and Hansen and Singleton (1982, 1983).

In the following sections I specialize the conditional valuation equation (3) or the unconditional valuation equation (5) and derive versions of the consumption-based asset pricing model (CCAPM), the capital asset pricing model (CAPM), the intertemporal capital asset pricing model (ICAPM), the arbitrage pricing model (APT), and the intertemporal arbitrage pricing model. Typically, I relate the conditional (or unconditional) expected excess return to the conditional (or unconditional) covariance of the asset’s return with consumption, the market portfolio return, state variables, or economywide factors.

III. CONSUMPTION-BASED ASSET PRICING MODELS

Consumption-based asset pricing models are developed in Rubinstein (1976*), Breeden and Litzenberger (1978), and Breeden (1979*). These models make assumptions on the form of the utility function and/or the joint probability distribution of consumption and excess asset returns with the objective of simplifying the fundamental valuation equation and relating expected excess returns to the covariance of excess returns with consumption. Subsequent related work includes Bhattacharyya (1981), Grossman and Shiller (1982), and Kraus and Litzenberger (1983). See also Ross (1988*).

One way to simplify the valuation equation (3) is to set \(u_t(c^t) = A^t \cdot c^t - B^t \cdot (c^t)^2 / 2\), where \(A^t = A^t(\phi^{t-1})\), \(B^t = B^t(\phi^{t-1})\). Then we obtain

\[
E[(A^t - B^t \cdot c^t) \cdot r_t^i | \phi^{t-1}] = 0,
\]
which simplifies to

\[ E[r'_t | \phi^{t-1}] = \left( \frac{B'_{\lambda}}{A' - B'_{\lambda} \tilde{c}'} \right) \text{cov} (r'_t, c' | \phi^{t-1}) \]

where \( \tilde{c}' = E[c' | \phi^{t-1}] \). The expected excess return is proportional to the asset’s covariance with consumption or, equivalently, to the asset’s consumption beta, defined as \( \text{cov}(r'_t, c' | \phi^{t-1}) / \text{var}(c' | \phi^{t-1}) \). In the definition of the consumption beta, the consumption may be replaced by the change in consumption, \( c' - c^{t-1} \), or by the consumption growth rate, \( c' / c^{t-1} \), without affecting the basic result. If \( A' \) and \( B' \) are independent of \( \phi^{t-1} \), I substitute the quadratic utility in the unconditional valuation equation and obtain the unconditional version of equation (6), where \( \phi^{t-1} \) is replaced by \( \phi^0 \). Note that in the definition of the consumption beta I may not replace consumption by the change in consumption or by the consumption growth rate because \( c^{t-1} \) is not in the information set \( \phi^0 \).

An alternative way to simplify the valuation equation is to invoke Stein’s (1973) lemma. Suppose that \( (x, y) \) are bivariate normally distributed, \( g(y) \) is everywhere differentiable, and \( E[g'(y)] < \infty \); then \( \text{cov} (x, g(y)) = E[g(y)] \text{cov} (x, y) \). (See also Rubinstein (1976*), the appendix.) Thus, if \( c' \) and \( r'_t \) are bivariate normally distributed, conditional on \( \phi^{t-1} \), and the marginal utility satisfies the differentiability and boundedness condition, the valuation equation simplifies to

\[ E[r'_t | \phi^{t-1}] = - \frac{E[u'_x(c') | \phi^{t-1}]}{E[u'_x(c') | \phi^{t-1}]} \text{cov} (r'_t, c' | \phi^{t-1}). \]

If \( c' \) and \( r'_t \) are bivariate normally distributed, conditional on \( \phi^0 \), the unconditional version of equation (7) holds, where \( \phi^{t-1} \) is replaced by \( \phi^0 \). Furthermore, I may replace consumption by the consumption growth rate if the relative risk aversion is constant.

In Breeden’s (1979*) continuous-time model, the assets’ rates of return and consumption are driven by a joint diffusion process. Breeden proves that the assets’ expected excess returns are proportional to their consumption betas. An intuitive interpretation of the result is that an asset’s rate of return and the consumption rate are locally bivariate normal. The assumption of a diffusion process (or, more generally, of an Itô process) bypasses the undesirable property that multivariate normal variables are unbounded from below.


### IV. THE CAPITAL ASSET PRICING MODEL

The CAPM, originated by Sharpe (1964) and Lintner (1965), has played an important role in finance and has been a focal point in the empirical finance
literature. The CAPM is a single-period specialization of the fundamental valuation equation. At the end of the period, $t = 1$, firms pay a liquidating dividend, and consumption equals aggregate wealth; hence $c^t = \sum D_t^t = W^t$. The fundamental valuation equation becomes

$$E[u^0_t(W^t) \cdot r^t \mid \phi^0] = 0,$$

thereby relating the assets' excess return to the marginal utility of aggregate wealth.

If the end-of-period dividends are multivariate normal, then the end-of-period returns and the end-of-period wealth are multivariate normal, and the fundamental valuation equation, with the aid of Stein's lemma, simplifies to

$$E[r^t \mid \phi^0] = -\frac{E[u^1_{w,w}(W^t) \mid \phi^0]}{E[u^1_w(W^t) \mid \phi^0]} \cdot \text{cov}(r^t, W^t \mid \phi^0).$$

Thus a firm's expected excess return is proportional to the covariance of the excess return with the end-of-period wealth or, equivalently, to the covariance of the excess return with the market portfolio return, defined as $W^t / \sum \rho^t_i$.

The contribution of the CAPM is that it relates the expected excess returns to the market portfolio return. The model's reliance on the market portfolio return is criticized by Fama (1976), Roll (1977), Ross, and others, who point out that the model is testable only if the market portfolio return is observable. The only empirically testable implication of the CAPM is that the market portfolio is mean-variance efficient.

Multivariate normality is sufficient but not necessary for the derivation of the CAPM. Chamberlain (1983b) and Owen and Rabinovitch (1983) characterize the elliptical distributions, which imply that expected utility of wealth is completely specified by its mean and variance, hence they imply the CAPM. Ross (1978*) provides a complete characterization of the class of distributions that imply the CAPM. Another sufficient condition is that utility be quadratic.


V. THE INTERTEMPORAL CAPITAL ASSET PRICING MODEL

Whereas I may think of the state at date $t$ as being the information at date $t$, there is a parsimonious way to define the state. Suppose there exists an $m$-vector $s^t(\phi^t)$ and a function $\mathcal{C}^t(\cdot, \cdot)$ such that

$$\mathcal{C}^t(\phi^{t-1}, s^t) \equiv \mathcal{C}^t(\phi^t) \text{ for all } \phi^t.$$

Although $s^t = \phi^t$ satisfies the definition, the definition is useful if there exists a vector $s^t$ that is of small dimension even when $\phi^t$ is of large dimension. The definition in equation (10) asserts that $(\phi^{t-1}, s^t)$ is a sufficient statistic of $\phi^t$ in determining consumption at date $t$. There is no presumption that $(\phi^{t-1}, s^t)$ is a
sufficient statistic of $\phi^t$ in determining the asset prices $P_t(\phi^t)$. Therefore $(\phi^{t-1}, s^t)$ is not generally a sufficient statistic of $\phi^t$ to describe the investment opportunity set available to consumers at date $t$—that is, the probability distribution of returns, $R_t^{t+1}$, conditional on $\phi^t$. The identification of state variables and sufficient statistics for an agent’s dynamic choice problem in the case of diffusion information are discussed by Huang (1986). Without loss of generality I assume that the elements of $s^t$ are uncorrelated with each other.

The ICAPM relates an asset’s expected excess return to the covariance of the asset’s excess return with each component of the state. I write the valuation equation (3) as

$$E[u_t(\hat{\varepsilon}(s^t, \phi^{t-1})), r_t^i | \phi^{t-1}] = 0.$$  

Under the assumption that the assets’ excess returns and the elements of $s^t$ are multivariate normal, conditional on the information $\phi^{t-1}$ and that marginal utility satisfies the differentiability and boundedness conditions, I obtain the conditional ICAPM:

$$E[r_t^i | \phi^{t-1}] = \sum_{j=1}^{m} -E[u_{t^j} \cdot \hat{\varepsilon}_j | \phi^{t-1}] \cdot \text{cov}(s^t, r_t^i | \phi^{t-1}).$$

where $\hat{\varepsilon}_j$ denotes the derivative of $\hat{\varepsilon}$ with respect to the $j$th element of $s^t$. In Merton’s (1973a) original derivation of the ICAPM, the state variables and prices are assumed to follow a joint diffusion process. The diffusion process implies that the assets’ excess returns and the elements of $s^t$ are locally multivariate normal—hence the result. See also the diffusion models of Breeden (1979*), Chamberlain (1985), and Cox, Ingersoll, and Ross (1985a) and the discrete-time models of Long (1974) and Stapleton and Subrahmanyam (1978).

The ICAPM attains economic significance once the state variables are identified. In a single-period framework the end-of-period consumption equals the end-of-period wealth; hence wealth is identified as the sole state variable. In an intertemporal context, Fama (1970) identifies three sources of state variables. The first source is state-dependent direct utility; that is, $u^t = u^t(c^t; s^t)$. The second and closely related source of state dependence is the relative prices of consumption goods in a multigood economy. Then the direct utility of consumption of the different goods implies an indirect utility of consumption of a basket of goods, where the indirect utility also depends on the state variables, which stand for the relative prices of these goods. Note that neither one of these sources of state variables exists in the economy presented in this essay.

The third source of state dependence is the changing investment opportunity set. Candidate state variables are the distribution of wealth among consumers in a heterogeneous-consumer economy. As I argued earlier, if the market is complete, or at least effectively complete, the equilibrium is observationally equivalent to an equilibrium in a homogeneous-consumer economy; hence the distribution of wealth among consumers is described by redundant state variables. Other candidate state variables describe the
evolution of dividends and of the aggregate endowment.

Beyond the definition in equation (10), the theory does not provide an operational procedure for identifying the state variables. Chen, Roll, and Ross (1986) propose macroeconomic variables as proxies for state variables. They test whether these variables relate to contemporaneous stock returns and whether they explain part of the contemporaneous expected excess return of the assets.

The conditional ICAPM reduces to the conditional CAPM under the assumption that utility is logarithmic; that is, \( u'(c') = \rho' \cdot \ln c' \), and \( \rho' \) is independent of \( \phi' \). I substitute the logarithmic utility in the valuation equation (1) and sum across firms to obtain the wealth at time \( t \):

\[
W_t = \sum_{i=1}^{n} (P_{it} + D_i) = \sum_{i=1}^{n} E \left[ \sum_{t=1}^{T} \rho^i c^i \cdot D_i^t | \phi' \right] 
\]

(13)

\[
= c' \cdot \sum_{t=1}^{T} \rho^t, \quad \text{since } c' = \sum_{i=1}^{n} D_i.
\]

I compare equation (13) with the definition of the state in equation (10) and conclude that the state may be represented by the wealth alone. Hence, the ICAPM in equation (12) states that an asset’s expected excess return is proportional to \( \text{cov}(r'_t, W_t | \phi^{t-1}) \). Since the market return is \( W'_t / \sum P'_t^{-1} \), and \( \sum P'_t^{-1} \) is an element of the information set \( \phi^{t-1} \), an asset’s expected excess return is proportional to the covariance of the asset’s excess return with the market return, conditional on \( \phi^{t-1} \).

In the conditional ICAPM of equation (12), the moments are conditional on the information \( \phi^{t-1} \). I may derive the ICAPM in unconditional form by augmenting the state as the vector \( (s'(\phi'), q^{-1}(\phi^{t-1})) \) such that

\[
\hat{c}'(\phi^0, s'(\phi'), q^{-1}(\phi^{t-1})) \equiv c'(\phi') \quad \text{for all } \phi'
\]

by analogy with equation (10). I substitute (14) in the unconditional valuation equation (5) and proceed as before to relate the unconditional excess return of an asset with the unconditional covariance of the return with each element of \( s'(\phi') \) and \( q^{-1}(\phi^{t-1}) \). The unconditional ICAPM involves covariances of the asset’s excess return with the elements of both \( s' \) and \( q^{-1} \), unlike the conditional ICAPM, which involves covariances with only the elements of \( s' \). Empirical studies by Fama (1981, 1984a, b, c), Fama and French (1986), Fama and Schwert (1977), Keim and Stambaugh (1986), Merton (1980), Nelson (1972), and others find evidence that variables derived from the information set \( \phi^{t-1} \) have forecasting power on the assets’ returns over the period \((t-1, t)\). These variables are candidates for the augmenting vector \( q^{-1}(\phi^{t-1}) \).

With logarithmic utility the unconditional ICAPM may be reduced to the unconditional CAPM. I substitute \( c' \) from equation (13) in equation (3), multiply by \( \sum_{i=1}^{n} P_i^{-1} \), and use the definition of the market’s rate of return, \( R'_m = W'_t / \sum_{i=1}^{n} P_i^{-1} \), to obtain \( E[r'_t/R'_m | \phi^{t-1}] = 0 \). Taking the expectation with respect to \( \phi^0 \), I obtain \( E[r'_t/R'_m | \phi^0] \), which leads to the unconditional CAPM, if \( r'_t \) and \( R'_m \) are unconditionally (locally) bivariate normal.
An important application of the intertemporal valuation theory is in the study of the term structure of interest rates. Campbell (1986), Cox, Ingersoll, and Ross (1981, 1985b*), and Donaldson, Johnsen, and Mehra (1985) present term structure theories. Singleton (1988*) surveys the theories and empirical evidence and, in particular, discusses the role of money. Constantinides and Ingersoll (1984*) present a term structure theory that incorporates personal and corporate taxes.


VI. THE ARBITRAGE PRICING THEORY

The arbitrage pricing theory (APT), put forth by Ross (1976, 1977), addresses the criticism on the observability of the market portfolio return leveled against the CAPM. I first discuss the APT and its variations in a single-period context and then explore the implications of the theory in an intertemporal context.

The APT asserts that the excess returns of a given subset, \( i = 1, 2, \ldots, n' \), of the \( n \) firms and/or the zero-net-supply assets have a factor structure

\[
 r_i = \bar{r}_i + \sum_{k=1}^{K} \beta_{ik} \delta_k + \epsilon_i, \quad i = 1, 2, \ldots, n',
\]

where the noise terms \( \epsilon_i \) have bounded variance and are uncorrelated with each other and with the \( K \) factors \( \delta_k \). Time subscripts and the conditioning information \( \phi^0 \) are suppressed in this single-period framework. Without loss of generality, the factors and noise terms have zero mean so that \( \bar{r}_i \) is the expected excess return of the \( i \)th asset. Although there always exists a factor structure with \( K = n' \), the spirit of the APT is that the number of factors is small (though unspecified) compared to the number of assets.

I substitute equation (15) in the fundamental valuation equation (3) and, upon simplification, obtain

\[
 \bar{r}_i = \sum_{k=1}^{K} \left\{ E\left[ -u_i(c) \delta_k \right] \right\} \beta_{ik} + \frac{E\left[ -u_i(c) \epsilon_i \right]}{E[u_i(c)]}.
\]

As a motivation for the APT, Ross (1977) points out that if the assets' returns are spanned by the factors (i.e., if \( \epsilon_i \equiv 0 \) for \( i = 1, 2, \ldots, n' \)), then the second term on the right side of equation (16) is zero and each asset's expected excess return is linear in its factor loadings \( \beta_{ik} \), \( k = 1, \ldots, K \). This result also follows directly from no-arbitrage arguments without the need to invoke the fundamental valuation equation.

The spirit of the APT is that the residuals, \( E\left[ -u_i(c) \epsilon_i \right]/E[u_i(c)] \), are small even when the noise terms \( \epsilon_i \) are nonzero; hence each asset's expected excess return is approximately linear in its factor loadings. Ross (1976) introduces the concept of asymptotic no-arbitrage and argues that the sum of the squared residuals is bounded as the number of assets in the subset \( n' \) tends to infinity.
Huberman (1982*) defines asymptotic no-arbitrage and provides a simple proof of the result that the sum of the squared residuals is bounded. Chamberlain (1983a) and Connor (1984) provide conditions that lead to the asymptotic APT. Chamberlain and Rothschild (1983) and Ingersoll (1984) derive the APT when asset returns only have an approximate factor structure. Chamberlain (1983a), Chen and Ingersoll (1983), Connor (1984), Grinblatt and Titman (1987), and Ross (1978*) relate the exact APT to the CAPM. Some of this work is reviewed in Connor's (1988*) essay.

I present a variant of the asymptotic APT wherein the number of assets in the economy remains finite, and seek conditions under which the residuals are small. Brock (1982) and Cragg and Malkiel (1982) initiate this approach, and Dybvig (1983) and Grinblatt and Titman (1983) derive bounds for the residuals.

I write consumption in the second period as

\[ c^1 = \sum_{i=1}^{n} D_i^1 = \sum_{i=1}^{n} \left( \frac{D_i^1}{P_i^0} \right) P_i^0 = \sum_{i=1}^{n} (r_i^1 + R_0^1) \cdot P_i^0. \]

I assume that aggregate consumption and the noise terms \( e_i \) are bivariate normal. I also assume that the returns of all \( n \) positive-net-supply assets have the factor structure given by equation (15). I substitute \( c^1 \) from equation (17) and \( r_i^1 \) from equation (15) in the residual term, invoke Stein's lemma and, upon simplification, obtain

\[ \frac{E[-u_c(c^1) \cdot e_i]}{E[u_c(c^1)]} = \left\{ - \left( \sum_{i=1}^{n} P_i^0 \right) \frac{E[u_c(c^1)]}{E[u_c(c^1)]} \right\} \left( \frac{P_i^0}{\sum_{i=1}^{n} P_i^0} \right) \cdot \text{var} (e_i). \]

The term in braces is some measure of the relative risk aversion coefficient. The second term is the weight of the \( i \)th asset in the market portfolio in the first period. This term illustrates the intuition of the asymptotic APT: as the number of assets grows such that the weight of the \( i \)th asset in the market portfolio tends to zero, the residual also tends to zero. Reasonable estimates of these parameters for the U.S. stock market indicate that the residual is small in a finite-asset economy.

Under the maintained hypotheses that the economy is single period and that all positive-net-supply assets have a factor structure with factors extracted from the observable subset of assets, the APT is testable. The second maintained hypothesis may be weakened by stating that only the subset of observable assets has a factor structure with noise terms uncorrelated with the return of all positive-net-supply unobservable assets. The second maintained hypothesis makes an assertion on the unobservable assets and is the counterpart of the maintained hypothesis in the testing of the CAPM that a weighted portfolio of the observable assets is a proxy for the market. The testability of the APT is discussed by Dybvig and Ross (1985) and Shanken (1982, 1985b). Examples of APT tests are in Brown and Weinstein (1983), Chen (1983), Cho (1984), Connor and Korajczyk (1986), Conway and Reingenam (1986), Dhrymes, Friend, and Gultekin (1984), Huberman and Kandel (1985), Jobson (1982), Lehman and MODEST (1985), and Roll and Ross (1980).
VII. THE INTERTEMPORAL ARBITRAGE PRICING THEORY

In this section I explore the implications of embedding Ross’s single-period APT in an intertemporal economy. Related work includes Connor and Korajczyk (1986), Ohlson and Garman (1980), Reisman (1986), and Stambaugh (1983).

As in the discussion of the ICAPM, suppose that there exists an \( m \)-vector of state variables \( s^t(\phi^t) \) defined by equation (10). In the intertemporal context, the assertion that the returns of the \( n \) firms have a factor structure is expressed as follows:

\[
\begin{align*}
    r_i &= \tilde{r}_i + \sum_{k=1}^{K} \beta_{ik}^t \delta_k^t + e_i^t, \quad i = 1, 2, \ldots, n,
\end{align*}
\]

(19)

where the noise terms \( e_i^t \) have bounded variance and are uncorrelated with each other and with the \( K \) factors \( \delta_k^t \), conditional on \( \phi^t \). The parameters \( \tilde{r}_i \) and \( \beta_{ik}^t \) are understood to be known functions of \( \phi^t \). Without loss of generality, the factors and noise terms have zero mean, conditional on \( \phi^t \).

I substitute (19) in the valuation equation (11) and obtain

\[
E \left[ u^t_i(\tilde{e}^t(s^t, \phi^t)) \left( \tilde{r}_i + \sum_{k=1}^{K} \beta_{ik}^t \delta_k^t + e_i^t \right) | \phi^t \right] = 0.
\]

(20)

I keep the notation tractable by considering the special case that the state \( s^t \) consists of two components, the one of which is the wealth at date \( t \), \( W^t = \sum_{t=1}^{T} (P_i^t + D_i^t) \), and the other of which is denoted by the scalar \( s^t \). Although it is not necessarily the case that wealth may proxy for one of the state variables, I develop this special case because I wish to illustrate the relation between the ICAPM and the intertemporal APT.

Under the assumption that \( W_t, s_t, \{ \delta_k^t \} \), and \( \{ e_i^t \} \) are multivariate normal, conditional on \( \phi^t \), I invoke Stein’s lemma and simplify the valuation equation to

\[
E[r_i^t|\phi^t] = \sum_{k=1}^{K} \beta_{ik}^t \frac{1}{E[u_i^t|\phi^t]} \left\{ - \text{cov} (\delta_k^t, W^t|\phi^t) \cdot E[\tilde{e}^t_w \cdot u_i^t|\phi^t] \right. \\
- \text{cov} (\delta_k^t, s^t|\phi^t) \cdot E[\tilde{e}^t_s \cdot u_i^t|\phi^t] \right\} \\
- \text{cov} (e_i^t, W^t|\phi^t) \cdot \frac{E[\tilde{e}^t_w \cdot u_i^t|\phi^t]}{E[u_i^t|\phi^t]}
\]

(21)

This equation states that the conditional expected excess return of the \( i \)th asset is linear in the asset’s conditional factor loadings, \( \beta_{ik}^t \), as in the asymptotic single-period APT. However, the expected excess return also depends on two residual terms that are proportional to the covariances \( \text{cov} (e_i^t, W^t|\phi^t) \) and \( \text{cov} (e_i^t, s^t|\phi^t) \).
The first residual term may be simplified by invoking the same arguments that allow me to write the residual in the single-period approximate APT as in equation (18):

\[
\text{cov}(\epsilon_t^i, W^i|\phi^{t-1}) \cdot \frac{E[\hat{c}_w^i u_{c_i}^i|\phi^{t-1}]}{E[u_t^i|\phi^{t-1}]}
\]

(22)

\[
= \left\{ \left( \sum_{i=1}^{n} P_i^{t-1} \right) \frac{E[\hat{c}_w^i u_{c_i}^i|\phi^{t-1}]}{E[u_t^i|\phi^{t-1}]} \right\} \left\{ \frac{P_i^{t-1}}{\sum_{i=1}^{n} P_i^{t-1}} \right\} \text{var}(\epsilon_t^i|\phi^{t-1}).
\]

The intuition of the asymptotic single-period APT carries over to this residual term of the intertemporal APT: as the number of assets in the economy grows such that the weight of the \(i\)th asset in the market portfolio tends to zero, this residual also tends to zero.

The second residual term that is proportional to \(\text{cov}(\epsilon_t^i, s^t|\phi^{t-1})\) is specific to the intertemporal APT. Note that in the single-period APT end-of-period consumption equals end-of-period wealth, and we set \(s^t \equiv 0\), thereby eliminating this residual term. But in the intertemporal APT no assumption has yet been made to lead to the conclusion that the terms \(\text{cov}(\epsilon_t^i, s^t|\phi^{t-1})\) and/or \(E[\hat{c}_w^i u_{c_i}^i|\phi^{t-1}]\) are small, even in the asymptotic case when the weight of the asset in the market portfolio tends to zero. Thus the intertemporal APT is subject to the same limitation as the ICAPM in that the theory does not provide an operational procedure for identifying the state variables.

Under the plausible maintained hypotheses that (1) the observable subset of assets has a factor structure, (2) the noise terms of the observable assets are uncorrelated with the returns of the unobservable assets, and (3) the factors span the state variables that influence the rates of return of the observable assets, then the intertemporal APT becomes testable. Specifically, the noise terms of the observable assets are uncorrelated with the state variables, and the second residual term in equation (21) vanishes. Note that under the same maintained hypothesis the state variables are identified and the ICAPM also becomes testable. See also the discussion in Chen, Roll, and Ross (1986).

An alternative way to make the intertemporal APT testable is to introduce the maintained hypothesis that utility is logarithmic. Then equation (13) implies that \(s^t \equiv 0\), the second residual term vanishes, and the intertemporal APT is reduced to the APT.

Finally, I derive the intertemporal APT in unconditional form. I assume that the returns of the \(n\) firms have the factor structure given by equation (19), where the noise terms have bounded variance and are uncorrelated with each other and with the \(K\) factors, conditional on \(\phi^0\) (or on any other subset of \(\phi^{t-1}\)). I also augment the state as \((s^t(\phi^t), q^{t-1}(\phi^{t-1}))\) satisfying equation (14). I start with the unconditional valuation equation (5) and proceed as before to obtain the unconditional counterpart of equation (21). It differs from equation (21) in two respects. First, \(\phi^{t-1}\) is replaced everywhere by \(\phi^0\). Second, there is one more term on the right side, namely

\[
- \text{cov}(\epsilon_t^i, q_i^{t-1}|\phi^0) \cdot \frac{E[\hat{c}_{q_i} u_{c_i}^i|\phi^0]}{E[u_t^i|\phi^0]},
\]
in the special case that $q^{-1}$ is scalar. As in the conditional equation, I argue that no assumption has yet been made that leads to the conclusion that the two terms proportional to the covariances $\text{cov}(e_i, s'(\phi^0))$ and $\text{cov}(e'_i, q^{-1}(\phi^0))$ are small. These terms vanish if utility is assumed to be logarithmic, or if the factors span the state variables $(s'(\phi^0)q^{-1}(\phi^{-1}))$.

**VIII. INCOMPLETE MARKET**


The fundamental valuation equation and its derivative restrictions such as the consumption CAPM and the CAPM are often rejected in empirical tests. Grossman and Shiller (1981) find that the representative consumer must have an implausibly high degree of risk aversion if the valuation equation is to fit the data. Hansen and Singleton (1983) and Mehra and Prescott (1985) find that the difference in mean returns on risky assets cannot be explained by differences in consumption risk. Mehra and Prescott (1985) suggest that the problem may lie with the complete market/representative consumer economic paradigm. Bewley (1982), Mankiw (1986), Scheinkman (1988*), and Scheinkman and Weiss (1986) present incomplete market models and address some of these issues.

I focus on a subset of the issues that arise in an incomplete market. Specifically, I explore the robustness of the valuation theories presented in the earlier sections to a particular kind of market incompleteness. I do so in the context of a modified version of the two-date, incomplete-market economy presented by Mankiw (1986). In the next section I review the pricing restrictions imposed by a necessary condition of equilibrium—the absence of arbitrage.

At the first date consumers are identical in terms of their endowments, beliefs, and tastes. At the second date each consumer receives a dividend $c$ plus an endowment. A fraction $\lambda$ of the consumers receive endowment $-x/\lambda$, and a fraction $1-\lambda$ receive endowment $x/(1-\lambda)$. The aggregate endowment is zero; hence per capita consumption is $c$. At the first date $c$ and $x$ are random variables with known joint distribution. Each consumer does not know which of the two types of endowment he or she will receive; furthermore the type of endowment is independent of $c$ and $x$.

In a Pareto optimal allocation each consumer consumes $c$. This allocation is attainable in a competitive economy in which each consumer’s endowment is observable by everybody ex post and in which an insurance market exists that pools the endowments. Market incompleteness is introduced in this model by banning the insurance of endowments.

The counterpart of the fundamental valuation equation (3) is

\begin{equation}
E[y \cdot r] = 0,
\end{equation}
where

(24) \[ y = (1 - \lambda)u(c + \frac{x}{1 - \lambda}) + \lambda u(c - \frac{x}{\lambda}). \]

The fundamental valuation equation (23) may be tested directly, where now marginal utility is replaced by the weighted sum of marginal utilities.

In the special case that utility is quadratic, marginal utility is linear in its argument and the weighted sum of marginal utilities becomes \( y = u(c). \) The consumption CAPM in terms of the per capita consumption remains unaltered, despite the fact that the market is incomplete and the allocation is not Pareto optimal. This result is presented in Grossman and Shiller (1982). In another special case, Breeden (1979) assumes that the consumption and endowment are a diffusion process. This implies that utility is (locally) quadratic and, as in the previous special case, the consumption CAPM remains unaltered, even though the market is incomplete and the allocation is not Pareto optimal. Bewley (1982), Mankiw (1986), and Scheinkman (1988*) present models in which the third derivative of the utility function is positive and show that market incompleteness increases the consumption risk premium.

In the discussion below I steer away from the assumption of quadratic (or locally quadratic) utility and highlight the effect of market incompleteness on valuation theories. I assume that \( c, x, \) and \( r_i \) are multivariate normal, invoke Stein’s lemma, and obtain the following:

(25) \[ E[r_i] = \text{cov}(r_i, c) \cdot E\left[(1 - \lambda)u(c + \frac{x}{1 - \lambda}) + \lambda u(c - \frac{x}{\lambda})\right] / E[y] + \text{cov}(r_i, x) \cdot E\left[ u(c + \frac{x}{1 - \lambda}) - u(c - \frac{x}{\lambda})\right] / E[y]. \]

If \( x \) is constant, the second term on the right side vanishes. The consumption CAPM and the CAPM hold despite the fact that the allocation is not Pareto optimal.

If the size of the endowment shock, \( x, \) is a random variable, the consumption CAPM in terms of per capita consumption and the CAPM generally do not hold. Equation (25) states that an asset’s expected excess return is linear in both the per capita consumption risk, \( \text{cov}(r_i, c), \) and the endowment shock risk, \( \text{cov}(r_i, x). \) The coefficient of \( \text{cov}(r_i, x) \) may be of either sign. If \( x \geq 0 (x \leq 0) \) the coefficient is nonpositive (nonnegative). In special cases, such as quadratic utility, the coefficient of \( \text{cov}(r_i, x) \) is zero. In general, however, the consumption CAPM and CAPM are not robust to market incompleteness. In contrast, the single-period, equilibrium APT remains valid with \( y \) replacing \( u, \) in equation (16). However, the magnitudes of the factor risk premia and of the residual term change.

One source of market incompleteness is transaction costs. Constantinides (1986*) studies a two-asset intertemporal consumption-investment model with proportional transaction costs. The demand for assets is shown to be
sensitive to these costs. However, transaction costs have only a second-order effect on the liquidity premia implied by equilibrium asset returns because investors accommodate large transaction costs by drastically reducing the frequency and volume of trade. Hence, in intertemporal models in which the only motivation to trade is consumption or rebalancing of the portfolio, transaction costs have a second-order effect on the theory of valuation. By contrast, transaction costs may play an important role in models in which there is some other reason to trade, such as private information.

**IX. PRICING IMPLICATIONS OF THE NO-ARBITRAGE RESTRICTION**

The justly famous Modigliani-Miller (1958) theorem represents a brilliant application of the no-arbitrage restriction and marks the beginning of modern finance. The key concept introduced by Modigliani and Miller is that of a risk class or spanning. The power of the theorem is that it is preference free, except to the extent that arbitrageurs prefer more to less. The theoretical developments of the no-arbitrage restriction proceed on three fronts.

First, the theory of the firm is extended by Modigliani and Miller and others to incorporate taxes and institutional restrictions. This literature is represented in the second volume with Miller’s (1977*) influential paper and the essays by Kim (1988*) and Scholes and Wolfson (1988*).

Second, in his important paper Ross (1976) proposes an arbitrage pricing theory under the assumption that asset returns are generated by a $K$-factor model. In an asymptotic sense, as the number of assets tends to infinity, or in an approximate sense in a finite-asset economy, the $K$ factors determine $K$ risk classes. This literature is represented in this volume with Huberman’s (1982*) paper and Connor’s (1988*) essay.

The third theoretical development of the no-arbitrage restriction is pioneered by Black and Scholes (1973) in their celebrated option pricing model. The use of arbitrage in the Modigliani-Miller theory is a static, one-shot exploitation by the arbitrageur of a potential relative price discrepancy. Although the stock and the option do not belong to the same risk class in a static sense, when the stock price follows a geometric Brownian motion a dynamically adjusted portfolio of the stock and a bond does belong to the same risk class as the option and makes possible the pricing of the option relative to the stock and the bond. The Black and Scholes paper has had a profound influence on the theory of valuation.

In the first part of his paper, Merton (1973b*) employs the Modigliani-Miller concept of a risk class and through static no-arbitrage arguments develops distribution-free and preference-free restrictions on the price of an option relative to a bond and the underlying security. In the second part Merton employs the Black-Scholes concept of a risk class and through dynamic no-arbitrage arguments extends the Black-Scholes theory. As a historical note, footnote 43 of Merton’s paper contains the germ of term structure theories.
Cox and Ross (1976) develop the concept of "as if risk neutral economy" and apply it in the valuation of options under different stochastic processes. The concept that state probabilities, suitably normalized, may be viewed as prices in an "as if risk neutral" economy is due to Arrow (1964) and is formalized in a dynamic hedging context in the influential paper of Harrison and Kreps (1979). The theory is reviewed by Cox and Huang (1988*) and in the remainder of this section.

Define by \( B' \), \( B' = B'(\phi') \), the price at date \( t \) of a discount bond which pays one unit of the consumption good at date \( T \). Normalize the firms' dividends and prices with \( B' \) as the numeraire at date \( t \). The normalized dividend of firm \( i \) at date \( t \) is \( \hat{D}^i_t = D^i_t / B' \) and the price is \( \hat{P}^i_t = P^i_t / B' \). In terms of the representative-consumer economy of Section II, define \( \xi^i_t = \lambda \xi^i_t(c') B' \), where \( \lambda \) is an arbitrary constant. Eliminate \( u^i_t(c') \) from the valuation equation (1) and obtain

\[
\hat{P}^i_t = E \left[ \sum_{t'=t+1}^{T} \xi^i_t \hat{D}^i_{t'|} \phi' \right] / \xi^i_t.
\]

I now relax the assumptions that the market is complete and that consumers have homogeneous, time-additive and von Neumann–Morgenstern utility functions. Under the assumption of no-arbitrage opportunities and some other weak technical assumptions, it may be shown that a random variable \( \xi^i_t \) exists such that the generalized valuation equation (26) applies. This is an important result because the absence of arbitrage opportunities is a necessary condition of any equilibrium.

The generalized valuation equation states that \( \xi^i_t \) is a martingale. To prove this, consider firm \( i \) which pays dividend \( \hat{D}^i_t = 1 \) at date \( \tau \), \( t + 1 \leq \tau \leq T \), and zero dividend at other dates. Note that the dividend may be reinvested at date \( \tau \) in a discount bond which has unit price (with \( B' \) as the numeraire) and one unit of the consumption good at date \( T \). The price of firm \( i \) at date \( t \) is \( \hat{P}^i_t = B'/B' = 1 \). Equation (26) becomes

\[
1 = E[\xi^i_t | \phi^i_t] / \xi^i_t
\]

and states that \( \xi^i_t \) is a martingale.

By analogy with equation (10) I may define the state at date \( t \) as an \( m \)-vector \( s' = s'(\phi') \) if there exists a function \( \tilde{\xi}^i(\cdot, \cdot) \) such that

\[
\tilde{\xi}^i(\phi'^{-1}, s') = \xi^i(\phi').
\]

One may proceed as in Section V to derive an intertemporal capital asset pricing model. Related results are presented by Chamberlain (1985) and Reisman (1986).

X. FUTURE DIRECTIONS

An important theme of this essay is the role of conditioning information in valuation theory. The intertemporal asset pricing model requires the identifi-
cation of the state variables in order to become an economic theory with empirically testable implications. Likewise the equilibrium version of the intertemporal arbitrage pricing theory requires the identification of the state variables, or at least requires the maintained hypothesis that the factors span the space of the state variables. Furthermore, the state vector needs to be augmented in both theories if the testable economic implications are to be stated in terms of unconditional moments.

The interface of valuation theory and the empirical identification of instruments of the state variables is an exciting direction for future research. There is substantial empirical research underway to identify the time-series and cross-sectional properties of asset returns. This includes the identification of factors, asset portfolios, and macroeconomic variables as instruments of the state variables. A challenge to the theory is to explain why a particular set of state variables is important in the determination of prices.

A second exciting direction for future research lies in the development of an equilibrium valuation theory when the market is incomplete. As I illustrated in Sections VIII and IX, this theory has richer implications than the valuation theory based on the complete market/representative consumer economic paradigm and already provides an explanation to some empirical puzzles. However, the full development and the empirical testing of the incomplete market theory remain a task for the future.

REFERENCES


