Comment on “An unbiased measure of realized variance” and “Realized variance and market microstructure noise” by Peter Hansen and Asger Lunde

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1 INTRODUCTION

If efficient asset prices follow a semi-martingale and are perfectly observed, their quadratic variation can be measured accurately from the sum of a large number of squared returns sampled over very finely spaced intervals, i.e., realized variance (Andersen et al., 2003, and Barndorff-Nielsen and Shephard, 2002). With the emergence of high-frequency data, it seems that we should be able to identify volatility rather easily. However, this identification hinges on being able to observe the true (or efficient) price process. Unfortunately, observed asset prices are affected by market microstructure effects, such as discreteness, different prices for buyers and sellers, “price-impact” of trades, and so forth. If we think of observed prices as efficient prices plus market microstructure noise contaminations, then we face an interesting as well as complex econometric task in using high-frequency data to estimate quadratic variation from noisy observed asset price data.

Hansen and Lunde make several nice contributions to this growing literature. First, they document empirical evidence regarding the dynamic features of the noise including evidence about the nature of the dependence between the efficient price and microstructure noise. Second, they propose a clever procedure to remove microstructure-induced biases from realized variance estimates. We start with the former.

2 THE DEPENDENCE BETWEEN THE NOISE AND THE EFFICIENT PRICE

Hansen and Lunde are among the first to study the relation between efficient price and noise. As they discuss, this relation has important theoretical and
empirical implications. It is also a difficult relation to assess since the equilibrium price and the noise component embedded in asset prices are inherently unobserved. One can in fact write an observed (logarithmic) price \( \tilde{p} \) as the sum of an equilibrium (logarithmic) price \( p^e \) and a residual component \( \eta \), i.e., market microstructure noise:

\[
\tilde{p} = p^e + \eta. \tag{1}
\]

Both \( p^e \) and \( \eta \) are not observed to the econometrician. Equivalently, in terms of returns,

\[
\tilde{r} = r^e + \varepsilon, \tag{2}
\]

where \( \tilde{r} = \tilde{p} - \tilde{p}_{-1}, \ r^e = p^e - p^e_{-1} \) and \( \varepsilon = \eta - \eta_{-1} \).

By examining variance signature plots Hansen and Lunde provide evidence for a strong negative correlation between \( \eta \) and \( p^e \) as indicated by signature plots that are not upward sloping as the sampling frequency increases. This result largely relies on NYSE mid-quote price data for a sample of Dow Jones stocks.

In this Section of our Comment we expand on Hansen and Lunde’s empirical findings. If the efficient price constantly evolves, factors inducing stickiness in the observed price mechanically determine a negative correlation between the unobserved components of the observed prices, \( p^e \) and \( \eta \). This is simply due to the fact that if the observed price is stable, then any change in the efficient price must be exactly off-set by a movement in the opposite direction in the noise component. Some important aspects of the dependence between the efficient price and the noise, as well as aspects of the conditional and unconditional properties of the noise, can therefore be understood by considering the factors inducing stickiness in the observed prices, primarily \( i) \) the market structure, \( ii) \) how prices are defined, and \( iii) \) the method used to sample prices.

### 2.1 MARKET STRUCTURE

The market structure, or rules of trade, will impact the features of the noise. Assume \( \tilde{p} \) refers to quotes. If agents arrive in a random fashion with differing perceptions of the value of an asset and different motives, then continual random adjustments to the posted prices are expected to be made. These random price adjustments should induce noise components that are roughly uncorrelated with the underlying efficient price. This structure is consistent with foreign exchange markets, for instance, where banks around the world post quotes. Alternatively, specialist-driven markets, such as the NYSE, are more centralized with a single agent posting quotes. Without prices being set by multiple agents with dissenting views, the observed prices can remain fixed over longer durations. This results in stickiness in the posted quotes (the \( \tilde{p} \)’s in our framework), thereby contributing to explain the negative correlation between efficient returns and noise returns as documented in Hansen and Lunde’s study. However, if one combines
quotes from multiple exchanges, then the random price setting behavior of a more decentralized market can be replicated.

Here we consider the foreign exchange market and NASDAQ as examples of fairly decentralized markets. The NYSE is used as an example of a relatively centralized market.

Fig. 1 contains calendar-time mid-quote variance signature plots for the Deutchmark/Dollar and Yen/Dollar exchange rates. The data is 5 minute data from all Tuesdays, Wednesdays, and Thursdays from the year 1996.

Fig. 2 contains calendar-time mid-quote variance signature plots for Cisco and Microsoft.

Fig. 3 contains calendar-time mid-quote variance plots for the Deutchmark/Dollar and Yen/Dollar exchange rates. The data is 5 minute data from all Tuesdays, Wednesdays, and Thursdays from the year 1996.

Fig. 4 presents calendar-time mid-quote variance plots for Cisco and Microsoft.

Fig. 5 presents calendar-time mid-quote and transaction variance plots for Cisco and Microsoft.

Fig. 6 presents calendar-time transaction variance plots for IBM using i) NYSE and ii) NYSE and MIDWEST, iii) NYSE and NASDAQ, and iv) the consolidated market. This figure should be compared to Fig. 3 and 4.

2.2 PRICE MEASUREMENT

The noise associated with bid-ask mid-quotes is different from the noise associated with transaction prices. Even within centralized markets, the price formation mechanism leading to observed trade prices differs from the mechanism leading to posted quotes. The former change more randomly than the latter in that they are more affected by the random arrival of agents in the marketplace. Hansen and Lunde recognize that the negative correlation between \( p \) and \( \eta \) in the case of mid-quotes appears not to be a first-order effect when dealing with transaction prices. This result is easily understandable in the context of our assumed price formation mechanism. The observed price \( \tilde{p} \) is relatively stickier when measured using mid-quotes rather than transaction prices regardless of i) the market structure and ii) the method used to sample prices.

Fig. 5 presents calendar-time mid-quote and transaction variance plots for Cisco and Microsoft.

Fig. 6 presents calendar-time transaction variance plots for IBM using i) NYSE and ii) NYSE and MIDWEST, iii) NYSE and NASDAQ, and iv) the consolidated market. This figure should be compared to Fig. 3 and 4.
Fig. 7 presents calendar-time transaction variance plots for GE using i) NYSE and ii) NYSE and MIDWEST, iii) NYSE and NASDAQ, and iv) the consolidated market. This figure should be compared to Fig. 3 and 4.

2.3 SAMPLING METHOD

Sampling returns in calendar time induces properties of the noise that are different from the properties of the noise obtained by sampling returns in trade time. (See Oomen, 2004a, for an interesting treatment of business-time sampling in the context of realized variance estimation). At very high frequencies, calendar time sampling will inevitably result in sampling between quote updates, thereby artificially inducing stickiness in the $\tilde{p}$’s. This stickiness, again, leads to an artificially negative correlation between noise returns and efficient returns. Here we compare the variance signature plots obtained from event time sampling to those previously obtained from calendar time sampling.

Fig. 8 contains event-time mid-quote variance signature plots for Cisco and Microsoft. This figure should be compared to Fig. 2.

Fig. 9 presents event-time mid-quote variance plots for IBM and GE using i) NYSE and ii) NYSE and MIDWEST. This figure should be compared to Fig. 3.

Fig. 10 presents event-time mid-quote variance plots for IBM and GE using i) NYSE and NASDAQ, and ii) the consolidated market. This figure should be compared to Fig. 4.

2.4 REMARKS

Remark 1. The negative correlation between efficient price and noise component may be explained by a mechanical relationship induced by a combination of sluggishness in the adjustments to the observed prices and an ever-evolving efficient price.

Remark 2. Is this negative correlation a first-order effect? The signature plots in Hansen and Lunde’s paper nicely speak to the strong negative dependence between efficient price and noise component when the observed price ($\tilde{p}$) is particularly sticky, i.e., in the case of calendar-time mid-quotes on the NYSE. However, different market structures, different sampling methods, and different price measurements can have drastically different noise features and a different nature of dependence between noise and efficient price. When the observed prices are relatively less sticky (i.e., in the case of foreign exchange markets, NASDAQ and combinations of markets, for example, but also when using transaction prices and/or sampling in event time rather than in calendar time), the evidence in favor of a negative correlation between efficient price and noise component at high frequencies is weaker in general.
Remark 3. Should the realized variance literature be concerned with the inevitable negative correlation between noise and efficient price process? It depends on the data as well as on the sampling method used.

Effective separation of the volatility components of the observed prices entails estimation of the moments of the noise and estimation of the integrated variance of the price process. The former is a necessary input for the latter. As we argue in a recent paper (Bandi and Russell (2005a)), high-frequency observed return data sampled at the highest frequency contain a considerable amount of information about the unobservable noise component of the observed returns. Provided the estimated moments of the noise can be purged of their first-order efficient price-induced contaminations, much can be learned about the features of the unobserved noise component in the observed prices by sampling at very high-frequency. In the case of the second moment of the noise, the variance of the underlying efficient price process, rather than the correlation between efficient price and noise, represents a first-order effect in many cases.

Remark 4. What data do we suggest using?

If interest is placed on the integrated variance of the underlying (as in Hansen and Lunde’s work) than it is sensible to use high-frequency data that are less contaminated by residual market microstructure components, i.e., mid-quotes. However, sampling in calendar time can generate severe distortions at high-frequencies. The presence of stale quotes due to sampling between quote updates can bias the estimates downward substantially. Hence, sampling at high-frequencies for the purpose of noise moment estimation should be conducted in event-time. Finally, if the properties of the noise are not expected to change wildly across exchanges, information provided by multiple exchanges can be employed. Bandi and Russell (2005a,b) use event-time mid-quotes from the NYSE and the MIDWEST to estimate noise moments for the purpose of integrated variance estimation. Fig. 11 presents event-time mid-quote volatility plots for IBM and GE using NYSE and the MIDWEST. At high-frequencies the IBM variance doubles while the GE variance is five times as large as the corresponding value at low frequencies. These increases provide valuable information about the second moment of the noise.

If interest in placed on the noise variance and η is interpreted as a transaction cost, then transaction prices sampled in event-time should be used.

Remark 5. Naturally, the properties of the noise are changing over time. The second moment of the noise component in the observed mid-quotes and transaction prices has been decreasing throughout the years. Markets are becoming more liquid and more decentralized. Both effects will contribute to render the dependence between the efficient returns and the noise in return components less important. On the one hand, more liquid markets will be characterized by a larger number of transactions. The increase transaction rate will likely make the moments and cross-moments of the efficient return process less important over any time interval. On the other hand, the increased decentralization will
induce less stickiness in the observed prices.

Clearly, the dynamic features of the noise will change as well. Incidentally, it is not a coincidence that, as reported by Hansen and Lunde, the noise is more persistent when the dependence between noise and efficient price is more negative. Stable observed returns can induce persistent noise returns in the presence of unpredictable efficient returns. As a consequence, the convenient MA(1) market microstructure model (i.e., the \( \eta \)'s are IID) can be a poor approximation when using mid-quotes from highly centralized markets. However, it can be a valid approximation in other circumstances. For example, simply adding MIDWEST mid-quotes to NYSE mid-quotes increases the first-order autocorrelation of the recorded returns substantially.

3 THE METHODOLOGY

In this section we provide some intuition and discussion for the method suggested by Hansen and Lunde. We start with the former.

3.1 INTUITION

In its more general form, the estimator that Hansen and Lunde suggest is in the tradition of robust covariance estimators such as that of Newey and West (1987). Assume availability of \( M \) equispaced observation over a fixed time span \((0, h)\) so that the distance between observations is \( \delta = \frac{h}{M} \). Write

\[
\hat{V} = \sum_{j=1}^{M} r_{j\delta}^2 + 2 \sum_{h=1}^{qM} \frac{M}{M-h} \sum_{j=1}^{M-h} r_{j\delta} r_{(j+h)\delta},
\]

where \( q_M \) is a frequency-dependent number of covariance terms. The first term in the right-hand side of Eq. (3) is the standard realized variance estimator, the second term is a correction intended the make the estimator unbiased in the presence of correlated noise. If the correlation structure of the noise return is such that the covariances of the noise terms of order higher than \( q_M \) are equal to zero and the efficient returns are local martingales, then the estimator in Eq. (3) is unbiased for the underlying integrated variance \( \int_0^h \sigma_s^2 ds \) of the efficient price process over the period, i.e., \( \mathbb{E}_M(\hat{V}) = \int_0^h \sigma_s^2 ds \). Interestingly, the finite sample unbiasedness of Hansen and Lunde’s estimator is robust to the presence of dependence between the underlying local martingale price process and market microstructure noise.

Consider, for simplicity, the MA(1) market microstructure noise case with noise independent of the underlying price process. In the MA(1) case, the estimator becomes
\[
\hat{V} = \sum_{j=1}^{M} r_{j\delta}^2 + 2 \frac{M}{M-1} \sum_{j=1}^{M-1} r_{j\delta} r_{(j+1)\delta}.
\] (4)

The covariance between \(r_{j\delta}\) and \(r_{(j+1)\delta}\), i.e., \(E_M(r_{j\delta} r_{(j+1)\delta})\), is the same at all frequencies and equal to \(-E(\eta^2)\). Hence, \(E \left( 2 \frac{M}{M-1} \sum_{j=1}^{M-1} r_{j\delta} r_{(j+1)\delta} \right) = -2ME(\eta^2)\). However, the bias of the classical realized variance estimator \(\sum_{j=1}^{M} r_{j\delta}^2\) is equal to \(ME(\eta^2) = 2ME(\eta^2)\). Therefore, the second term in Eq.(4) provides the required bias-correction.

Assume \(E(r_{j\delta}\eta_{j-s}) = 0\) for \(s \geq 1\), i.e., microstructure noise does not predict future efficient returns. If the noise depends on the price process, then the bias of the standard realized variance estimator is equal to

\[
2 \sum_{j=1}^{M} E(r_{j\delta}^2 \delta_{j\delta}) + 2M E(\eta^2) = 2M E(r_{j\delta}^2 \eta_j) + 2M E(\eta^2).
\] (5)

But

\[
E \left( \frac{M}{M-1} \sum_{j=1}^{M-1} r_{j\delta} r_{(j+1)\delta} \right) = E \left( \frac{M}{M-1} \sum_{j=1}^{M-1} \left( r_{j\delta}^2 + \varepsilon_{j\delta} \right) \left( r_{(j+1)\delta}^2 + \varepsilon_{(j+1)\delta} \right) \right)
\] (6)

\[
= \frac{M}{M-1} \sum_{j=1}^{M-1} E(r_{j\delta}^2 \varepsilon_{(j+1)\delta}) + \frac{M}{M-1} \sum_{j=1}^{M-1} E(\varepsilon_{j\delta} r_{(j+1)\delta}^2) + \frac{M}{M-1} \sum_{j=1}^{M-1} E(\varepsilon_{j\delta} \varepsilon_{(j+1)\delta})
\] (7)

\[
= -ME(r_{j\delta}\eta_j) - ME(\eta^2)
\] (8)

which, again, provides the necessary bias-correction in the case of noise depending on the price process.

### 3.2 REMARKS

**Remark 1** The proposed estimator is theoretically interesting and empirically useful. The cancelation that gives rise to unbiased realized variance estimates, even in the presence of a noise component correlated with the underlying efficient price process, is clever.

**Remark 2** In general, the frequency \(\delta\) at which to sample continuously-compounded returns for the purpose of integrated variance estimation through nonparametric estimates can be chosen optimally. This point has been made in recent research and estimable MSE expressions for a variety of nonparametric
variance estimators have been provided (see Bandi and Russell, 2005a,b, and Zhang et al., 2005). 

Hansen and Lunde expand on this framework by considering an optimal (in an MSE-sense) frequency $\delta^*$ in the case of their first-order bias corrected estimator. They assume (i) an $MA(1)$ microstructure noise structure and (ii) microstructure noise independent of the price process.

The authors might wish to emphasize a nice feature of their procedure, namely that their bias-corrected estimator is robust to deviations from Assumption (ii). In fact, their estimator is unbiased even in the presence of noise correlated with the price process. Assumption (ii) is adopted to make the MSE criterion implementable.

Remark 3 As we discuss in Section 1, the complex dynamic properties of the noise component can mechanically derive from the economics of market microstructure and the adopted sampling schemes. However, applied researchers might decide to impose sensible restrictions on the properties of the noise should these restrictions be empirically justifiable and convenient. The realized variance literature has always placed considerable emphasis on the empirical applicability of its theoretical contributions.

Having made this point, parts of Hansen and Lunde’s contribution might be interpreted as slightly contradictory. On the one hand, much emphasis is placed on the complex properties of the noise. On the other hand, MSE-based optimal sampling for the purpose of integrated variance estimation is conducted under Assumptions (i) and (ii) in Remark 2 above. If the authors believe, as they stress repeatedly, that noise dependence and dependence between noise and efficient price are first-order effects in their data, then their MSE criterion should account for both effects.

Remark 4. Bandi and Russell (2005a) discuss MSE-based sampling in the presence of correlated noise. Dependence between the noise and the efficient price is a more delicate issue. While the MSE of the bias-corrected estimator in the presence of dependence between the noise and the efficient price can likely be expressed, empirical estimation of the additional terms that would arise by virtue of this dependence is certainly a very complicated matter.

The authors could argue that at the frequency at which one samples in practice, the dependence between noise and efficient price is probably a second-order effect. This is likely to be true in the case of the realized variance estimator. However, the bias-correction that the authors introduce appears to induce considerably higher optimal frequencies that in the standard (biased) case. At these higher frequencies, the independence between noise and efficient price might be more questionable.

Remark 5. The authors show that bias-correcting the realized variance estimates induces MSE gains. This is an interesting result which has recently
induced further work. Oomen (2004b), for example, consider bias-correcting in the context of both calendar time sampling and business time sampling.

Now that statistical refinements have been made for realized variance estimation in the presence of microstructure noise, we think it is important to ask what are the economic implications of these refinements. One way to assess these implications is to use the utility-based approach advocated by Fleming et al. (2001, 2003) in order to evaluate alternative variance forecasts. This is the approach taken by Bandi and Russell (2005b).

However, there are numerous alternative economic metrics that could (and should) be considered. For example, given an option pricing model linking option prices to integrated variance, the forecasting power of alternative realized variance measures can be assessed. While proposing and implementing sensible economic metrics is a difficult task in general, this is, we believe, an important hurdle to overcome for the realized variance literature.

References


Figure 1 presents midquote variance signature plots for Deutchmark/Dollar and Yen/Dollar Exchange rate data for all Tuesdays, Wednesdays, and Thursdays from 1996.

Figure 2 presents calendar time midquote variance signature plots for NASDAQ stocks Cisco Systems and Microsoft.

Figure 3 presents calendar time midquote variance signature plots for IBM and GE using i) NYSE and ii) NYSE and Midwest.

Figure 4 presents calendar time midquote variance signature plots for IBM and GE using i) NYSE and NASDAQ and ii) consolidated market.
Figure 5 presents calendar time variance signature plots for Cisco Systems and Microsoft using i) transaction prices and ii) midquotes. Figure 6 presents calendar time transaction price variance signature plots for IBM using i) NYSE transactions only and ii) NYSE and Midwest transactions iii) NYSE and NASDAQ transactions and iv) the entire consolidated market. Figure 7 presents calendar time transaction price variance signature plots for GE using i) NYSE transactions only and ii) NYSE and Midwest transactions iii) NYSE and NASDAQ transactions and iv) the entire consolidated market. Figure 8 presents event time midquote realized variance plots for NASDAQ stocks Cisco Systems and Microsoft.
Figure 9 presents event time variance signature plots for IBM and GE stocks using i) NYSE quotes only and ii) NYSE and Midwest quotes. Figure 10 presents event time midquote variance signature plots for IBM and GE stocks using i) NYSE and NASDAQ quotes only and ii) the entire consolidated market. Figure 11 presents NYSE and Midwest midquote variance signature plots for IBM and GE.