True or Spurious Long Memory in Volatility:
Does it Matter for Pricing Options?

Arek Ohanissian, Jeffrey R. Russell and Ruey S. Tsay
Graduate School of Business, University of Chicago
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ABSTRACT

We examine the effects on option pricing of assuming an incorrect data generating process (DGP) for the long memory characteristics found in asset return volatilities. Using a new estimation procedure which simultaneously utilizes spot and option data of differing length and frequency, we document mispricings as large as 67% from mis-specification of the long memory in volatility. We also generate implied volatility plots as a function of maturity and moneyness using estimated parameter values and document the slower flattening of the skew for the true but not the spurious nor no long memory models.

We would like to thank Charles Cao for generously providing the option dataset used in this paper.
Understanding and explaining the dynamics of asset prices is an extremely challenging, yet potentially very fruitful task. In this paper, we aim to examine a piece of this difficult puzzle, namely, the option pricing implications of the observed long memory property of asset return volatilities. Recent evidence documents the long memory property in various volatility measures (such as absolute returns, squared returns, and realized volatility) for various asset prices (such as equity and foreign exchange rates).\footnote{See Taylor (1986), Ding, Granger, and Engle (1993), Dacorogna, Müller, Nagler, Olsen, and Pictet (1993), and Andersen, Bollerslev, Diebold, and Labys (2001) among others for such evidence.} Mere observation of the long memory property, however, does not imply that the true data generating process is a long memory process. Diebold and Inoue (2001) provide both theoretical and numerical evidence that various structural breaks models are able to exhibit long memory behaviour. Furthermore, unlike in the case of returns where theory implies that the price process must be a semi-martingale to preclude arbitrage opportunities,\footnote{See Harrison and Pliska (1981) for a standard model of this theory.} there is no theory to guide one in the specification of volatility dynamics. Thus, accommodating the observed long memory property in option pricing requires a choice between a true or spurious specification for it. In related work,\footnote{See Ohanissian, Russell, and Tsay (2003).} we provide a statistical methodology to distinguish between true and spurious long memory. Specifically, for the high frequency foreign exchange rate data considered in that paper, we are unable to reject the null hypothesis of a true long memory process for the volatility. In this paper, however, we examine the effects of mis-specifying the long memory behaviour of volatility, i.e., assuming it is true long memory when in actuality it is spurious long memory and vice-versa.

We examine the effects of mis-specification of the long memory in volatility on option prices. The option pricing arena is a natural environment to study the effects of mis-specification of volatility since option prices are closely linked to volatility dynamics. Thus, we would expect that any significant economic consequences from mis-specifying the long memory in volatility could translate into significant option mispricings.

Another reason to consider the long memory issue in the option pricing environment in particular is the recent focus on the dependence of the volatility smile effect on the maturity structure of options. In a recent survey, Sundaresan (2000) points out that the remaining puzzle in option pricing is the so-called term structure of volatility smiles. Bollerslev and Mikkelsen (1999) document the fact that stochastic volatility effects are significant for very long term options.\footnote{This persistence in the volatility smile has been documented, at least qualitatively, by others as well such as Aït-Sahalia and Lo (1998).} But, under an assumption of short memory for the volatility process, the randomness of the volatility should decrease with maturity and eventually disappear, thus removing any volatility smile. This puzzle may be related to the observed long memory property of volatility and our work will clarify further if the true or spurious specification is more relevant for explaining and understanding this
puzzle. In particular, we specifically examine the ability of each model to generate Black-Scholes implied volatility dynamics which correspond to those seen in actuality.

Our work is a related complement to that of Taylor (2000) which explores the consequences for option prices of introducing long memory into the volatility specification. In that paper, both short memory and long memory (GARCH) specifications are simulated and then the (Black-Scholes) implied volatilities from the resulting options prices are examined. He finds that the introduction of long memory in volatility does not really impact the smile effects but does impact the term structure of implied volatilities. Our work, however, focuses on the effects of mis-specifying the cause of the observed long memory property.

We specifically consider three affine models: a short memory model, a switching mean for volatility between a low and high state which introduces spurious long memory, and a fractional stochastic volatility model which introduces true long memory. We focus on affine structures in order to utilize recently developed option pricing techniques.\(^5\) Specifically, we derive semi-closed form option pricing equations for both the regime switching model and the fractionally integrated volatility model which, when combined with Heston’s original semi-closed form pricing, provides us with easily and efficiently calculated option prices.\(^6\) Using simulations, we examine what mispricings occur when assuming the wrong model and using its pricing equations. Specifically, we simulate under one model and get a time series of data with which we estimate all three models. Then we compare the corresponding option prices generated from these estimates. We first take the spurious long memory specification as the (null) model to simulate under and then do the same with the true long memory specification.

In order to accomplish our objective of examining the option pricing implications of mis-specifying the long memory in volatility, we introduce a new methodology to estimate these models. Estimation for these types of models is inherently very difficult due to various factors such as: lack of closed form likelihood functions / transition densities for the specifications, estimation of continuous time models using discrete time data, the need for both objective and risk neutral parameter estimation, and unobservability of volatility. Our approach is a combination of various aspects of the current literature to produce a simple, relatively fast, estimation procedure.\(^7\) Essentially, we use the notion of realized volatility to deal with the unobservability issue. We use GMM instead of a likelihood based approach due to the existence of analytic moments for our models. We are able to simultaneously use both spot and option data within this framework to recover

\(^5\)These techniques, which are elaborated on in Appendix A, have been used by Heston (1993), Bates (1996), Bakshi, Cao, and Chen (1997), and Scott (1997) among others and are more generally explained in Duffie, Pan, and Singleton (2000).

\(^6\)The previous studies of long memory volatility in option pricing of Bollerslev and Mikkelson (1999) and Taylor (2000) rely on slower Monte-Carlo techniques for pricing.

\(^7\)We provide a review of existing methods and compare our approach to them later in this paper.
objective and risk neutral parameters which are consistent with each other. Furthermore, we are able to exploit the longer length and higher frequency of the spot data even though we only have daily option data. To the best of our knowledge, our methodology is the only one which simultaneously uses both spot and option data and accommodates differing length and frequency of the two datasets.

Our main finding is that ignoring or mis-specifying the nature of observed long memory characteristics in volatility leads to serious option mispricings. Specifically, when the true DGP is spurious long memory, using either a no long memory model or a true long memory model leads to general underpricing of call options, by as much as two thirds actually. On the other hand, when the true DGP is true long memory, using either a no long memory model or a spurious long memory model leads to general overpricing of call options. These mispricings extend out to the entire horizon we examine which is two years. The magnitude of the mispricings is in general larger in the case with risk premia. In a related issue, we also document Black-Scholes implied volatility plots as a function of maturity and moneyness for the three models using parameter values estimated from real data. The plots indicate that each model is able to generate a volatility skew/smile. Furthermore, the skew is much less steep for the true long memory model than the other two models at the short term, yet all three skews have very similar steepness at the longest horizon. Thus, we empirically document the slower flattening out of the skew for the true long memory model, but that does not apply for the spurious long memory model.

The remainder of this paper is organized as follows. In Section I, we present in detail the three specifications considered in this paper. Section II contains a discussion of the data, estimation, and simulation techniques used and provides the results from estimation of the three models with real data. These estimated parameter values are then used in Section III to compute Black-Scholes implied volatilities and in Section IV to simulate data for the pricing exercise. Finally, Section V summarizes our conclusions along with a discussion of potential extensions.

I Specifications

All three of the specifications implemented in our study are in the continuous-time affine model class.8 Our pricing exercise is done both with and without risk premia. This section lists each specification in both objective and risk-neutral form. The without risk premia case is simply a degenerate version where the risk premia are turned off and the objective and risk neutral forms are identical. The risk premia structure implemented coincides with common parameterizations used in the option pricing literature.

The first specification is the most conventional specification in that there is no accommodation of the long memory property of volatility. The second specification introduces a switching mean for volatility to the conventional specification to thus generate spurious long memory in the volatility. The third specification uses a fractional integration operator to generate true long memory in the volatility. We try to keep the notation as consistent as possible across the models for simplicity in understanding and comparison. The specific option pricing equations are derived in Appendix A.

A No Long Memory (NLM)

Objective:

\[
\begin{align*}
\frac{dS_t}{S_t} &= \mu S_t dt + \sqrt{V_t} S_t dW_t^{(1)} \\
\frac{dV_t}{V_t} &= \kappa (\bar{V} - V_t) dt + \sigma \sqrt{V_t} \left( \rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)} \right)
\end{align*}
\]

Risk Neutral:

\[
\begin{align*}
\frac{dS_t}{S_t} &= r S_t dt + \sqrt{V_t} S_t dW_t^{(1)_r} \\
\frac{dV_t}{V_t} &= \left[ \kappa (\bar{V} - V_t) - \eta \sqrt{V_t} \right] dt + \sigma \sqrt{V_t} \left( \rho dW_t^{(1)_r} + \sqrt{1-\rho^2} dW_t^{(2)_r} \right)
\end{align*}
\]

This specification is taken directly from Heston (1993). \( S_t \) represents the price process while \( V_t \) represents the variance process and \( W_t^{(1)} \) and \( W_t^{(2)} \) are two independent standard Wiener processes which represent the risk in this model. This specification (as well as the following two specifications) allows for non-zero correlation between the price and volatility processes via \( \rho \) which thus accommodates the well-documented leverage effect. The variance process in this specification is the well known square root (or Feller) process. Focusing on the objective specification, we note that \( \bar{V} \) represents the mean for the variance process while \( \kappa \) represents the speed of adjustment. As long as \( 2\kappa \bar{V} \geq \sigma^2 \), the zero boundary is not attainable and thus as long as the variance begins at a non-negative point, it can never become negative subsequently. We next introduce the market price of risk \( \eta V \) in the risk neutral dynamics corresponding to the linear volatility risk premium used by Heston (1993).\(^9\) Under the risk neutral dynamics, the volatility diffusion remains a square root process with new mean of \( \frac{\kappa \bar{V}}{\kappa + \eta} \) and new speed of adjustment of \( \left( \kappa + \eta V \right) \).

\(^9\)Heston references the consumption based asset pricing model of Breeden (1979) and the general equilibrium model of Cox, Ingersoll, and Ross (1985) for justification of this risk premium specification. Bates (1996, 2000) uses a log utility based framework to justify the same volatility risk premium. For further discussion of the volatility risk premium, see the discussion in Garcia, Ghysels, and Renault (2003).
B Spurious Long Memory (SLM)

Objective:

\[ dS_t = \mu S_t dt + \sqrt{V_t} S_t dW^{(1)}_t \]
\[ dV_t = \kappa (v_t - V_t) dt + \sigma \sqrt{V_t} \left( \rho dW^{(1)}_t + \sqrt{1 - \rho^2} dW^{(2)}_t \right) \]
\[ dv_t = [v^l + v^h - 2v_t] dq_t \]

Risk Neutral:

\[ dS_t = r S_t dt + \sqrt{V_t} S_t dW^{(1)*}_t \]
\[ dV_t = [\kappa (v_t - V_t) - \eta V_t] dt + \sigma \sqrt{V_t} \left( \rho dW^{(1)*}_t + \sqrt{1 - \rho^2} dW^{(2)*}_t \right) \]
\[ dv_t = [v^l + v^h - 2v_t] dq^*_t \]

This specification modifies specification A by introducing a third state variable \( v_t \) which represents the (conditional) mean level of the variance process which in our specification can only take on two values (i.e., switches between a high and a low state). Hence, model B denotes a regime switching Heston model. Here, \( q_t \) is a Poisson jump process with random arrival times and (affine) state dependent intensity \( \lambda^V(v_t) \in \{ \lambda^l, \lambda^h \} \). The risk neutral specification contains an additional risk premium (\( \eta^R \)) which represents the regime switching risk. This risk premium comes in through the risk neutral jump process for which the jump probabilities are \( \lambda^V^*(v_t) = \eta^R(v_t) \lambda^V(v_t) \). As this is a two regime model, there is only one free market price of regime switching risk, specifically: \( \eta^R(v^l) \eta^R(v^h) = 1 \). Thus, we define \( \eta^{R,l} \equiv \eta^R(v^l) \) and \( \eta^{R,h} \equiv \eta^R(v^h) \) and note that we estimate and report only \( \eta^{R,l} \) as \( \eta^{R,h} \) is correspondingly calculated as \( \eta^{R,h} = 1/\eta^{R,l} \). Finally, note that the Heston model is exactly recovered when \( v^l = v^h \).

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10Our specification is a generalized continuous time version of spurious long memory Markov switching models analyzed by Diebold and Inoue (2001) and Granger and Hyung (1999). Markov switching models have recently become popular in finance for learning models (see David and Veronesi (2002) and references therein) and for term structure models (see Dai and Singleton (2003) and references therein). Naik (1993) is an early study of regime switching in volatility and option pricing.

11Specifically, \( \lambda^V(v_t) = \left( \frac{\lambda^l v^h - \lambda^h v^l}{v^h - v^l} \right) + \left( \frac{\lambda^h - \lambda^l}{v^h - v^l} \right) v_t \equiv \lambda_0 + \lambda_1 v_t \).

12This type of risk premia is used by Naik (1993) and discussed in Dai and Singleton (2003).

13See Dai and Singleton (2003) for a discussion of risk premia for regime switching models.
C True Long Memory (TLM)

Objective:

\[
\begin{align*}
    dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW_t^{(1)} \\
    V_t &= \bar{v} + I^{(\alpha)}(\tilde{V}_t - \bar{v}) \\
    d\tilde{V}_t &= \kappa(\bar{v} - \tilde{V}_t) dt + \sigma \sqrt{\tilde{V}_t} \left( \rho \sqrt{W_t^{(1)}} + \sqrt{1 - \rho^2} W_t^{(2)} \right)
\end{align*}
\]

Risk Neutral:

\[
\begin{align*}
    dS_t &= rS_t dt + \sqrt{V_t} S_t dW_t^{(1)\ast} \\
    V_t &= \frac{\kappa \bar{v}}{\kappa + \eta^V} + I^{(\alpha)} \left( \tilde{V}_t - \frac{\kappa \bar{v}}{\kappa + \eta^V} \right) \\
    d\tilde{V}_t &= \left[ \kappa(\bar{v} - \tilde{V}_t) - \eta^V \tilde{V}_t \right] dt + \sigma \sqrt{\tilde{V}_t} \left( \rho dW_t^{(1)\ast} + \sqrt{1 - \rho^2} dW_t^{(2)\ast} \right)
\end{align*}
\]

Our third specification, based on that of Comte, Coutin, and Renault (2003), introduces long memory into the volatility process while keeping the affine structure of the model. Here, \( I^{(\alpha)} \) is the fractional integration operator and \( \alpha \) is the long memory parameter. The \( \tilde{V}_t \) process is the familiar square root (short memory) process. Long memory is introduced into the volatility process using the fractional integration operator. Note that as \( I^{(\alpha)}(\tilde{V}_t - \bar{v}) \) is not lower bounded, the positivity of the volatility is not ensured and thus, to be proper we should write \( V_t = |\bar{v} + I^{(\alpha)}(\tilde{V}_t - \bar{v})| \). Comte et al. (2003), however, find that positivity is indeed preserved in simulations for relevant parameter values and the absolute value can be ignored for practical purposes.

Note that the Heston model is exactly recovered from this specification when \( \alpha = 0 \).

II Data, Estimation, and Simulation Techniques

In this section, we describe the data, estimation, and simulation techniques. Finally, we present model estimates from real (not simulated) data. The model estimates will form the basis for the simulations in Section IV.

\[14\] We provide details on this fractional integration operator in the next section when discussing the discretization approach. Further details, however, are available in Comte et al. (2003).

\[15\] See page 8 of Comte et al. (2003) for a complete discussion.
A Real Data

In order to estimate our models, which include both objective and risk neutral parameters, we need both spot and option data. The impact of mis-specifying the long memory property may depend on the maturity. It is possible that both short memory and true and spurious long memory models could forecast similar over short time spans but differ over longer time spans. We therefore use options of various (including long) maturity lengths. In particular, we would like to use the relatively new long term options which have maturities of up to 3 years. Specifically, we use the S&P 500 index and options on the S&P 500 index which include both regular (SPX) and long term (LEAPS) European options. The options dataset contains daily closing prices from September 2, 1993 to August 31, 1994. The spot dataset, however, contains CME tick data from 1990-1998 which specifically lists the time and price of each transaction. Our estimation strategy allows us to use differing time scale and frequency for the two datasets. This affords us the opportunity to take advantage of both the available intra-day and longer length features of the spot dataset.

We use the intra-day spot data to calculate realized volatilities based on equally spaced five minute returns. Although there is no strict guideline on which frequency to use, we select five minute returns to balance the trade-off of using more data versus the data being dominated by microstructure effects. Bandi and Russell (2003) consider this optimal sampling issue in detail and empirically demonstrate that five minute returns are a good choice for this data. We ignore the first 10 minutes of every trading day and any overnight returns for each day. This selection results in 79 intervals per day, and we use the last (transacted) price of each five minute interval to calculate the corresponding interval returns. Finally, we note that the prices we have are for the S&P 500 futures and we convert the returns (difference in log prices) for these futures into the returns for the index itself using the spot-futures parity by adding in a \((r - d) \times 5\) min adjustment.

\[
Fut_{t,\tau} = Spot_t \times e^{(r_{t,\tau} - d_{t,\tau})\tau}
\]

which implies that

\[
\log(Spot_{t+5\min}/Spot_t) = \log(Fut_{t+5\min,\tau-5\min}/Fut_{t,\tau}) + (r - d) \times 5\text{min}
\]

Note, this assumes that the relevant interest and dividend rates are constant over the five minute interval. In fact, we use a single interest rate and dividend rate for each day and the rates that we use are from the

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16 The option dataset used in this paper was provided by Charles Cao and was used in Bakshi, Cao, and Chen (2000).
17 The underlying dataset is from the Futures Industry Institute.
18 These periods involve rather different dynamics than the remaining period and thus are beyond the scope of our paper. Martens, Kofman, and Vorst (1998) remove these periods as well when using the same dataset.
daily series for the 3-month U.S. T-bill and the S&P 500 dividend yield provided by Datastream. Given
these five minute returns calculated for the S&P 500 index, we extract a daily volatility measure based on
the notion of realized volatility, which is discussed in detail in Andersen, Bollerslev, and Diebold (2002) and
the references therein. Essentially, the sum of the squared high frequency returns is taken as a measure of
the volatility and, in the limit as the interval goes to zero, the measure converges to the integrated volatility.
In our case of transaction price data and high frequency returns, however, we adjust the simple sum to
correct for the large negative first order autocorrelation common to this data. Thus, we calculate the daily
volatility using the following equation

\[ \text{DailyVol} = \sum_{t=1}^{79} R_t^2 + 2 \sum_{t=2}^{79} R_t R_{t-1} \]

where the second summation term is the adjustment for the first order autocorrelation of the returns.

Our option dataset consists of put options on the S&P 500 index traded on the CBOE. Put options are
used because they are generally more heavily traded (especially in the case of LEAP contracts) and hence
less likely to be affected by illiquidity. We convert each put price to a call price using put-call parity since
we derive option prices for calls. The daily price used for each put is the average of the bid and ask price
from the final put quote for that day. Furthermore, the corresponding S&P 500 index price at that quote
time is used as the relevant spot price instead of the closing price to ensure synchronicity of prices. For each
day, we use nine options which correspond to a three by three grid of moneyness vs. maturity. Our short
term options are those closest to 1 month in maturity, medium term options are those closest to 6 months in
maturity, and long term options are those closest to 18 months in maturity. Given the selected maturity,
the out of the money (put) option is the one with K/S closest to but not greater than 0.97, the at the money
option is the one with K/S closest to 1, and the in the money option is the one with K/S closest to but not
less than 1.03.

B Estimation

In this subsection, we describe in detail the estimation procedure for the models presented in Section I.
Our specifications are continuous time stochastic processes and although there has been great progress
in estimation for such processes, there are two main issues which complicate the estimation. The first
problem is that transition densities (and hence likelihood functions) are only known in analytic form for
a few special (and rather restrictive) cases. Secondly, volatility is unobservable and hence must either be
extracted or treated as a latent variable. Furthermore, as we are dealing with option pricing, our problem
is further complicated as we need to recover both objective and risk neutral parameters. Some studies of option pricing avoid this issue by assuming there are no risk premia, however this is not a very palatable assumption. Other studies recover both sets of parameters by sequentially using the underlying data and then the option data. These studies inherently allow for a comparison of whether the estimates from underlying data are consistent with the estimates from option data. Recently, there have been a few studies which simultaneously use underlying and option data. In particular, Chernov and Ghysels (2000) use EMM to first estimate parameters and then they subsequently filter volatility using a reprojection technique. Pan (2002) uses Implied State-GMM on affine specifications where (implied state) volatility is extracted from option prices for each period to generate a time series of (now observable) volatility used in the GMM estimation. Eraker (2003) and Polson and Stroud (2003) use MCMC techniques and treat volatility as a latent variable to estimate parameters and simultaneously generate a posterior distribution for volatility at each time period. For further details, the interested reader is referred to Chernov and Ghysels (1999) which provides a good discussion of estimation of stochastic volatility models for option pricing and the (attempts of optimal) use of both underlying and option data.

Our approach is most similar to the IS-GMM methodology of Pan (2002). Like Pan, we take advantage of the affine nature of our specifications which provides us with semi-closed form option pricing and moments of the state variables. Our approach differs from Pan, however, by using realized volatility in place of an implied volatility for our time-series of volatility. This choice makes our approach computationally much simpler. Pan requires extraction of implied volatilities for each set of parameter values for each time period, whereas our approach requires a simple summation for each time period done only once. Our time series of volatility is model independent whereas Pan’s volatility path will be different for alternative models. Given that we estimate three different models, we feel that a single volatility path is much more sensible than having three different volatility series. Furthermore, our approach allows us to make use of the high-frequency spot data even though we do not have high-frequency option data. Similarly, our approach also allows for different lengths of the datasets. In particular, we have nine years of spot data but only one year of option data. Thus, each of the moment equations used in the GMM procedure will use either the longer or the shorter number of observations. The differing number of observations used for each moment equation complicates the GMM procedure somewhat, so we explain the necessary adjustments below.

Prior to explaining the adjustments for the differing sample size and frequency of the data sets, we first mention the selection of moments. As mentioned above, the affine structure of our specifications provides us with closed form moment conditions. Our first set of moment conditions is based on the moments of our state variables. Specifically, we use the first two conditional moments of the (daily) return and the first
conditional moment of the (daily) integrated volatility. Unlike Pan, however, we introduce a second set of moments based on the option prices. Pan uses the option prices only to infer an implied volatility for each time period. We, on the other hand, use the option prices in the moment conditions, and specifically have a moment condition for each type of option, i.e., one for each of the three by three grid. We use the option price itself for the moment and thus consider the option pricing error, or the difference between the calculated (based on the assumed model and estimated parameter values) option price and the actual (based on the true model and true parameter values) option price. The moment condition is that in expectation this difference is zero. Bakshi, Cao, and Chen (2000) use the same moment conditions with the same option structure (three by three moneyness vs. maturity grid). However, they do not use any other conditions (based on the moments of the state variables) as we do. One further note, following Bakshi et al., we normalize the option prices by the strike price because of the non-stationarity of the underlying’s price.

Having a set-up for our moment conditions, we now discuss the details of the GMM procedure. Let us denote the set of spot-based moment conditions with the superscript \( s \) and the set of option-based moment conditions with the superscript \( o \), i.e., \( g^s(\Theta) \) and \( g^o(\Theta) \). The specific moment conditions used for each specification are listed in Appendix B. We note that we have 9 years of daily data for the first set of moments and 1 year of daily data for the second set of moments and thus are using samples of unequal length. The asymptotic theory for GMM estimation in this case is based on neither data set dominating the other in length asymptotically, i.e., \( \omega \equiv \frac{T_o}{T_s} \) with \( \omega \) being a constant between 0 and 1, \( T \) represents to the length of the data and the superscript specifies either the options or spot dataset. Thus, the estimation is based on minimizing the quadratic form \( G_T^T W_T G_T \) where \( W_T \) is a weighting matrix and

\[
G_T = \begin{pmatrix}
g^s_T \\
g^o_T
\end{pmatrix} = \begin{pmatrix}
\frac{1}{T_s} \sum_{t=1}^{T_s} g^s_t(\Theta) \\
\frac{1}{T_o} \sum_{t=1}^{T_o} g^o_t(\Theta)
\end{pmatrix}
\]

A first stage estimate of \( \Theta \) is obtained by choosing the identity matrix for the weighting matrix, i.e.,

\[
\hat{\Theta}_1 = \arg \min_{\Theta} G'_T G_T
\]

which is subsequently used to generate an efficient weighting matrix for a second stage estimate as in Hansen (1982). Specifically, using the inverse of a consistent estimator of the variance-covariance matrix for the

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19 At this point we note that for the true long memory specification, the long memory parameter is not identified with the GMM procedure. Instead it is estimated separately. This point is discussed further at the end of this subsection.

20 Lynch and Wachter (2004) demonstrate the consistency and asymptotic normality of various GMM estimators in the case of unequal sample sizes. The specific estimator we use is what they refer to as “long” which incorporates all of the data in the most straightforward way.
set of moment conditions achieves this efficiency. In our case with unequal sample lengths, however, this variance covariance matrix must be adjusted as in Lynch and Wachter (2004). Defining

\[ S_{ab} = \sum_{j=-\infty}^{\infty} E[g_a^j(\Theta^{true})g_b^{j-1}(\Theta^{true})'], \quad a, b \in \{s, o\} \]  

the adjusted variance-covariance matrix is given by

\[
S_T = \begin{pmatrix}
\omega S_{ss} & \omega S_{so} \\
\omega S_{os} & S_{oo}
\end{pmatrix}
\]  

which is consistently estimated using the first stage estimates. The second stage estimates are given by

\[
\hat{\Theta}_2 = \arg \min_{\Theta} G_T' \hat{S}_T^{-1} G_T
\]

with the following asymptotically normal distribution

\[
\sqrt{\omega T}(\hat{\Theta}_2 - \Theta^{true}) \rightarrow N[0, (D S_T^{-1} D')^{-1}]
\]

where \(D' \equiv \frac{\partial G_T}{\partial \Theta} |_{\Theta = \Theta^{true}}\) is consistently estimated by \(\hat{D}' = \frac{\partial G_T}{\partial \Theta} |_{\Theta = \hat{\Theta}_2}\). Additionally, our estimation is based on more moment conditions than parameters estimated resulting in over-identifying restrictions. We use these over-identifying restrictions to provide a measure of the overall fit of these models on the real data by calculating the test statistic suggested by Hansen (1982). Specifically,

\[
J_T \equiv \omega T G_T' \hat{S}_T^{-1} G_T \sim \chi^2(#\text{moments} - #\text{parameters})
\]

and we provide the test statistic and corresponding p-value for the estimation on the real data.

There are two further points to mention with our estimation. First, the long memory parameter for the true long memory model is not identified with this GMM procedure. Thus, we first estimate it using the log-periodogram regression framework of Geweke and Porter-Hudak (1983).\(^{21}\) Specifically, we use the realized volatility series for this first step estimation as Comte et al.(2003) demonstrate that the integrated volatility will have the same degree of long memory as the instantaneous volatility. Second, we avoided the unobservability problem for the volatility by using the realized volatility in place of the instantaneous volatility. Our second specification, however, has another unobservable state variable, namely the mean of the volatility process which switches between a low and a high state. As there are only two states, we

\(^{21}\)Comte et al. (2003) suggest a similar two step procedure in which the long memory parameter is estimated first and then a method of moments is used to estimate the remaining parameters.
deal with this unobservability relatively simply using the EM algorithm of Hamilton (1990). Essentially,
given a set of parameter values, we calculate the filtered probabilities of being in each state at each time
period. Given these filtered probabilities, the moments/option prices are calculated as a weighted (by the
corresponding probabilities) average of the moment/option prices conditional on each state. We provide the
details of the filtering procedure in Appendix C.

C Simulation and Discretization

The main discretization technique we use for the stochastic differential equations in this paper is the Euler
scheme. In theory, the volatility (square root) processes could be exactly simulated using the known
transition densities of the non-central $\chi^2$ distribution. In order to accommodate the leverage effect, however,
we need to correlate the Wiener process from the volatility equation with the Wiener process from the price
equation. And in order to do this, we discretize each Wiener process and thus do not take advantage of
the known transition density for the square root process. As far as the jump processes are concerned, we
assume that at most one jump is possible within each interval which is justified by using small enough time
intervals.

The fractional integration and derivation operators are discretized via the recursive strategy of Comte et al.
(2003). Specifically, the fractional integration operator of order $\alpha$, $I^{(\alpha)}$, is defined for $0 < \alpha < 1$ as

$$I^{(\alpha)}(f)(t) = \int_{-\infty}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) ds.$$  

Obviously, we cannot integrate to the infinite past and instead must use a truncated version. Furthermore,
the time dimension is discretized and thus, in practice, the long memory volatility is calculated from the
short memory volatility using the following scheme

$$V^{r,n,\Delta}(t_i) = \bar{v} + \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \sum_{j=-n}^{n-1} c_j \Psi^\Delta(\xi_j, t_i, \tilde{V} - \tilde{v})$$

where one chooses $r \in [1, 2]$, $n$ is an integer, and $\Delta$ is a step,\textsuperscript{22} with $c_j$ and $\xi_j$ given by

$$c_j = \left( \frac{r^{1-\alpha} - 1}{1 - \alpha} \right)
     \left( \frac{r^{1-\alpha} - 1}{r^{1-\alpha} - 1} \right)$$

$\xi_j = \left( \frac{1 - \alpha}{2 - \alpha} \right) \left( \frac{r^{2-\alpha} - 1}{r^{1-\alpha} - 1} \right) r^j$

\textsuperscript{22}Note, the effectiveness of this scheme is rather dependent on the choice of $r$ and $n$ for a given $\Delta$. In particular, the results
vary greatly for varying levels of $r$, a fact discussed in Comte et al. (2003). Using simulations, we found that $r = 1.3$ and
$n = 1000$ generated the most consistent fractionally integrated series for our specific $\Delta$ and dataset length.
and $\Psi^\Delta$ given by

$$
\Psi^\Delta(x, t_{i+1}, f) = \Psi^\Delta(x, t_i, f)e^{-x\Delta} + f(t_i)\frac{1 - e^{-x\Delta}}{x}e^{-x\Delta} + f(t_i)1 - e^{-x\Delta}
$$

$$
\Psi^\Delta(x, t_0, f) = 0.
$$

Thus, the short memory volatility process is simulated with step $\Delta$ and then long memory volatility with the same step $\Delta$ is generated using the above scheme. A corresponding scheme exists for the fractional derivation operator.

## D Estimation Results from Real Data

We provide the estimation results from the Real data for all three models. These parameter values will be used in the following two sections in the simulations. For each of the three models, we estimate both a general version with risk premia and a restricted version with no risk premia. The resulting estimates are listed in Table I. Also listed are the $J_T$ test statistics for over-identifying conditions and the corresponding p-values for each case. Before considering the estimated values, we note that there is a difference between the estimation for the real data and the simulated data. Specifically, our above specifications ignore any dividend payments, however, dividends are a part of the real data. Thus, we add $d$ into the models to represent the dividend yield and discount the current stock price by the dividend yield for the maturity of the option in this case. Both $r$ and $d$ are known on a daily basis so we do not estimate these and instead use their actual values to estimate the other parameter values.

\{insert table I here\}

The two long memory models we use are non-standard and thus, have not been previously estimated in the literature, however, our first specification is the well-known Heston model which has indeed been studied and estimated in the literature. Our estimates for this model match well with these alternative estimates. For instance, our estimate for $\kappa$ is 1.83 in the no risk premia case and 3.27 in the risk premia case. Compare these values to the 1.12 reported in Bakshi, Cao, and Chen (2000), values from 0.87 to 1.89 reported in David and Veronesi (2002), and the 2.75 reported in Shu and Zhang (2003). Our estimate for $\sigma$ is 0.51 in the no risk premia case and 0.76 in the risk premia case. Compare these values to the 0.19 of Bakshi et al.,

\(^{23}\) Following Comte et al. (2003), we apply the (discretized) fractional integration operator to the short memory volatility centered by its empirical mean as opposed to its theoretical mean.

\(^{24}\) For the sake of brevity, we do not list the details of this scheme. The interested reader is referred to p.21 of Comte et al. (2003).
values from 0.27 to 0.51 of David and Veronesi, and 0.43 of Shu and Zhang. The mean level of the volatility is much more dependent on the timing of the dataset, however, even these estimates are quite similar in our paper and these other studies at roughly 0.03. The correlation coefficient is rather widely estimated in these other studies and ranges from -0.25 to -0.78 and our estimates of -0.52 and -0.38 are consistent with these other studies.

Given the estimates for the standard Heston model, it is interesting to compare the estimates from the two non-standard (long memory) models with it. For instance, it seems that the estimates for the volatility of volatility coefficient ($\sigma$) decrease when the standard model is generalized to incorporate long memory. This seems interesting given the finding by Bates (2000) which estimates standard models (including stochastic volatility and jump models) using spot and options data separately and finds that the two datasets imply very different volatility of volatility coefficients. Specifically, the volatility of volatility implied from options data are much too high to be consistent with the volatility of volatility implied from the spot data. Thus, our finding of lower volatility of volatility for the long memory models may imply that these models are indeed better suited for the joint spot and options data. Another interesting comparison is that the risk premium coefficient ($\eta^V$) has a larger magnitude in the long memory models compared to the Heston specification. The risk neutral mean volatility is still roughly equal across the models, which implies that in the long memory model cases the mean (objective measure) volatility are much lower than the no risk premia case relative to the difference for the Heston specification. Examining such differences in greater detail would be interesting but unfortunately is beyond the scope of this paper.

### III Black-Scholes Implied Volatility

With the few exceptions noted earlier, the relationship between long memory and option prices has not been studied in detail. In this section we examine the impact of long memory on implied volatility. In particular, we consider plots of the Black-Scholes implied volatility function of maturity and moneyness for each model. For each of the models estimated in Section II, we generate implied volatilities for various maturities and strike prices. Specifically, we provide these sets of plots for three specific starting volatility values: a low volatility which corresponds to a variance level at 80% of the long term mean, a medium volatility which corresponds exactly to the long term volatility, and a high volatility which corresponds to a variance level at 120% of the long term mean. Figures 1 and 2 correspond to these individual plots for the no risk premia and risk premia case, respectively while figures 3 and 4 replot some of the same curves so that we can more easily compare the three models on the same plot and again correspond to the no risk premia.

---

and risk premia case, respectively. The no long memory model and the spurious long memory model both result in very steep volatility skews while the true long memory model results in a much less steep volatility skew at the short maturities. For each model, the volatility skew “flattens out” as the maturity is extended. Furthermore, we clearly see that the flattening out is indeed much slower for the true long memory model than the other models as the starting steepness is much smaller yet the final steepness is roughly the same. As explained in Comte, Coutin, and Renault (2003), short memory models which are able to reproduce the slow decay of the smile/skew with maturity must have a very large level of persistence which in turn results in short term options with smiles/skews which are too pronounced relative to reality. Thus, short memory models are able to accommodate either the short term options or the decay but not both at the same time and our plots in figures 1-4 clearly show this difference between short memory and long memory models.

IV Effects on Option Pricing

In this section, we present the main pricing results of this paper. Our goal is to see what option pricing implications exist from mis-specifying the long memory property in volatility. Thus, we sequentially take each of two long memory models as the true model and examine what happens when we assume the alternative models. We begin by first assuming that the data generating process is a spurious long memory process. Using this model, we simulate spot prices and volatility and calculate option prices. Given these simulated values, we estimate all three models. Hence estimation error is accounted for in all three models (including the true model). Finally, given the estimated parameter values, we calculate (out of sample) option prices and compare them to the option prices under the assumed true model. This procedure is then repeated by assuming that the data generating process is a true long memory process and that the alternative models are no long memory and spurious long memory. As previously mentioned, we use semi-closed form option pricing equations for each specification (see Appendix A). We use nine (simulated) options on each day to estimate the model and then use a larger structure of thirty options (five moneyness levels and six maturities) on the final day to examine the option mispricings. The observation scale and frequency correspond exactly to the real data we use (and discuss in detail in Section II), namely, we simulate 2220 days of spot data with 79 intervals per day and we use 252 days of (calculated true) option data beginning on the 918th observation day of the underlying data. Each simulated series is initialized with starting spot price of $1000 and starting volatility taken to be the mean of the stationary distribution. The parameter values used in the simulations
of the true DGP’s are the estimates from the real data estimation for each model. Furthermore, we consider both cases of with and without risk premia and correspondingly have two sets of results for each true DGP.

A Parameter Estimates for True and Alternative Models

In this subsection, we list and briefly discuss the parameter estimates for each of the two sets of simulations. Specifically, Table II corresponds to the case where the spurious long memory model is taken to be the true DGP and Table III corresponds to the case where the true long memory is taken to be the true DGP. For each table, we provide the mean and standard error for each estimated parameter value from the 100 simulated series. Each true model is estimated along with the false models which allows us to examine the estimation error effect on option pricing for the true model as well as distinguish option pricing errors resulting from model mis-specification as opposed to estimation error. In Tables II and III, there appear to be some finite sample bias involved in the true model estimates. For instance, in Table II, the $\kappa$ parameter seems to be underestimated for the spurious long memory model, as it is estimated at 1.52 for a true value of 1.87 in the no risk premia case and it is estimated at 2.62 for a true value of 3.26 in the risk premia case. A similar underestimation for $\kappa$ is evident in Table III for the true long memory model, although in that case the difference is not statistically significant. Overall, however, given the small sample size we are using due to the sparsity of option data, we find the estimation procedure to perform quite well. Furthermore, it is interesting to note that the alternative model estimates seem to be relatively close to the corresponding true values which further indicates the closeness of the models. Next, however, we find that the seemingly small inherent model differences lead to large option pricing differences.

B Discussion

The main pricing error and Black-Scholes implied volatility tables from these simulations are Tables IV-IX. We provide pricing errors in both dollar terms and relative terms and we list both the mean and the absolute deviation for these errors with a corresponding listing of the standard errors for these statistics. We also provide the Black-Scholes implied volatilities which are essentially non-linear transformations of the prices that allow a different comparison across models. For instance, in the case of a call option which is very much in the money and hence has a very large intrinsic value, there are still differences between the time values of the option across the models which may seem negligible when compared with the intrinsic value yet are still important for our understanding of the option pricing differences implied by the models. Tables IV

{insert tables II and III here}
and \( V \) provide the pricing errors in the case of the DGP being the spurious long memory model without risk premia and with risk premia, respectively, while Tables VI and VII provide those results for the DGP case of true long memory without and risk premia, respectively. Tables VIII and IX then provide an alternative characterization of the pricing errors by listing the mean and standard error of the ratio of the Black-Scholes implied volatility for these models relative to the Black-Scholes implied volatility of the true DGP model of spurious long memory and true long memory, respectively. Note, in these tables, we use boldface type to represent the true (i.e., DGP) model.

These tables document significant mispricing of options when the long memory property of volatility is ignored or mis-specified. We begin by considering the case where the true DGP is spurious long memory which corresponds to Tables IV, V, and VIII. We discuss results from both the no risk premia and the risk premia cases, however, we note that the results are relatively consistent over the inclusion of risk premia, yet seem stronger in the case with risk premia. Nearly every mean value listed in Panel A of Table IV is negative, which implies that the mean of our simulated option prices resulted in underpricing of options relative to the actual option price, no matter which model we use to calculate option prices. Comparing the mean values to their corresponding standard errors, however, we see that these mean values are only significantly different from zero in the case of the two mis-specified models. In particular, the estimation error for the true model of spurious long memory does not result in any significant option mispricing whereas assuming no long memory or true long memory instead results in significant underpricing. The magnitude of this mispricing is much stronger in the risk premia case, reaching nearly $100 for the incorrect true long memory model at the longest maturity which in relative terms is roughly a third of the actual option value. In fact, in relative terms, the smallest underpricing across moneyness levels at the longest maturity for the incorrect true long memory model is over 20% as seen in the last column of Panel C of Table V. In that same panel, we see that ignoring long memory altogether also results in underpricing, particularly for the middle maturities which range from 3 months to 1.5 years. This underpricing is not as severe as in the incorrect true long memory case, however, it is statistically significant. Comparing these values with the absolute deviation of the relative pricing errors listed in Panel D of the same table indicates that not only does the mean of the simulations result in underpricing but so does nearly every individual simulation as the mean and absolute deviation of the relative pricing errors are exactly equal for almost all of the entries in the table. Finally, we note the importance of Table VIII which lists the mean and standard error of the Black-Scholes implied volatilities for the true option prices and the ratio of the implied volatilities from the

\( \{ \text{insert tables IV-IX here} \} \)
(estimated model) generated prices to those from the true prices. As previously mentioned, this measure is quite useful when the scale of the option can be misleading. For instance, in the bottom left hand corner of Panel C of Table IV, we see a listing of 155.71%. This seemingly excessive number is exaggerated because in this corner of the table, call options are very low in price. Considering implied volatilities, however, avoids such an issue. In particular, the same bottom left hand corner of Panel A of Table VIII indicates that this excessive relative pricing error translates into an 11% overstatement of the implied volatility, another large mispricing but not a particularly excessive value.

We now consider the DGP of true long memory and thus move to Tables VI, VII, and IX. As before, we find significant mispricings which again seem to be stronger in the case with risk premia. Panel A of both Tables VI and VII demonstrates that the only significant dollar pricing errors are from the mis-specified models of no long memory and spurious long memory. Specifically, both incorrect models result in overpricing relative to the true call option prices, with the overpricing ranging from 10-20% at the longest horizon for the risk premia case. And again, we see that the relative pricing errors and the relative pricing absolute deviations are very close in magnitude, particularly at the longer maturities, for the mis-specified models which indicates that most (and sometimes all) of the simulations resulted in overpricing. An interesting difference in the case of this DGP, however, is that the estimation error for the true long memory model seems to have a larger effect, particularly for the shortest maturity and largest moneyness category. For instance, Panel C of Tables VI and VII indicates that the relative pricing error for the estimated true model is significantly different from zero at the 1 month maturity and 1.03 moneyness level for both the with and without risk premia cases. It is interesting to note though, that this estimation error effect seems to decrease with maturity and the same tables indicate that there is no mispricing for the true model at the longer maturities.

Thus, whether the observed long memory property in volatility is generated by an inherent or spurious mechanism, ignoring it causes significant underpricing. Furthermore, incorporating long memory in the incorrect form also results in significant mispricing. Finally, for the sake of completeness, we mention the case where the true DGP is taken as the no long memory case. Our focus in this paper is on the observed long memory property of volatility and specifically the pricing implications from confusing spurious and true long memory or even not considering the long memory. We have, however, run simulations for the case where the true DGP is no long memory and although we do not provide tables as in the other two cases for the sake of brevity, we do now briefly summarize these results.\textsuperscript{26} Essentially, we find that using the spurious long memory model when the true case is no long memory results in general overpricing of call options. This

\textsuperscript{26}These additional tables are available on request.
overpricing is again more pronounced in the case with risk premia. On the other hand, using the true long memory model when the true case is no long memory results in general underpricing of call option. The underpricing is strongest at the longest horizon where it ranges between 10-15%.

V Conclusions

In this paper, we examine the effects on option prices of mis-specifying the long memory property in volatility. Our study involves simulating under an assumed true model and estimating and pricing under the true and alternative models to generate option prices for comparison. In the process, we introduce a new estimation methodology which jointly uses spot and option data and also allows for differing length and frequency of the separate datasets. We consider three specifications: one without long memory, one with spurious long memory, and one with true long memory. We first estimate the models using real data to get relevant parameter values to use for the simulations. Using these parameter values, we then generate plots of the Black-Scholes implied volatility in terms of both maturity and moneyness. These plots reveal the ability of all three models to generate skews. We also document a sharp difference between these plots, namely, although all three models generate relatively flat skews at the long horizon, the true long memory model’s short term skew is much flatter than those of the other two models. This finding is directly related to the slower flattening of a skew for a long memory model which we find here does not apply to a spurious long memory model. We then sequentially simulate using each of the two long memory models as the true data generating process, and estimate all of the models and generate option prices. Given these option prices, we document the direction and magnitude of mispricings from mis-specification. We find that there are indeed serious mispricings from mis-specifying the long memory property. For the case of the true DGP being the spurious long memory model, the two alternative models result in general underpricing of call options. This underpricing is both statistically and economically significant and reaches as high as 67% of the true option price. For the case of the true DGP being the true long memory model, the two alternative models result in general overpricing of call options. Again, this mispricing is statistically and economically significant and in general is in the range of 5-20%. For both DGP’s, the magnitude of the mispricings was generally higher for the case with risk premia. Overall, the main implication is that where option pricing is concerned, observed long memory characteristics should not only be considered, but the cause of the observed characteristics, either true or spurious, should be determined to try to minimize any pricing errors.

There are a number of ways in which to extend our work in this paper. We have only looked at out of sample options at one point in time. In the future, we could consider both in sample and out of sample
options as well as the temporal properties of these options and pricing errors. This temporal analysis will be closely related to an analysis of option hedging which again may be more relevant for practitioners. Also, an important improvement would be to consider how to deal with the issues introduced by using realized volatility in place of instantaneous volatility. Using model specific implied volatilities such as Pan (2002) is a potential alternative, however, as previously mentioned, it is much more computationally intensive. A possible alternative to this type of approach would be to use Taylor series expansion based approximations for the implied volatilities which are similar in nature to the volatility of volatility expansion scheme option pricing put forth by Lewis (2000). This alternative model consistent implied volatility approach would be much less computationally burdensome than the alternatives of simulation based filtering or Pan’s implicit inversion. Finally, another extension would be to consider more complicated models by adding in additional volatility factors and/or jumps which potentially correspond better to real data then the Heston (1993) based models we use in this paper, however in order to do so, we would likely lose the efficiency of semi-closed form option pricing.

Appendix A: Option Pricing Equations

In this appendix, we present the derivation of the option pricing equations for the spurious and true long memory specifications used in this paper. The no long memory specification is simply the Heston (1993) for which the option pricing equations are well documented in his original paper and elsewhere in the literature, whereas the other two specifications are non-standard and thus we provide the corresponding details here. We consider both no risk premia and risk premia cases in the paper, however, in this appendix we focus on the risk premia case as the no risk premia case solution is simply a degenerate form of the risk premia case solution.

A Spurious Long Memory

Objective:

\[
\begin{align*}
    dS_t &= \mu S_t dt + \sqrt{V_t} S_t dW^{(1)}_t \\
    dV_t &= \kappa (v_t - V_t) dt + \sigma \sqrt{V_t} \left( \rho dW^{(1)}_t + \sqrt{1 - \rho^2} dW^{(2)}_t \right) \\
    dv_t &= [v^l + v^h - 2v_t] dq_t
\end{align*}
\]
Risk Neutral:

\[
\begin{align*}
\frac{dS_t}{S_t} &= r_s dt + \sqrt{V_t} dW^{(1)}_t \\
\frac{dV_t}{V_t} &= \left[ \kappa (v_t - V_t) - \eta V_t \right] dt + \sigma \sqrt{V_t} \left( \rho dW^{(1)}_t + \sqrt{1 - \rho^2} dW^{(2)}_t \right) \\
\frac{dv_t}{v_t} &= \left[ v^h - v^l \right] dq^s_t + \left[ v^l - v^h \right] dq^h_t
\end{align*}
\]

- Note, the jump probabilities in the risk neutral case are now \( \lambda^{V^*}(v_t) = \eta^R(v_t)\lambda^V(v_t) \).

Our spurious long memory specification is a generalization of the no long memory specification in that there are two (Markov switching) regimes for the volatility process. The process is still a square root process in each regime, however, the unconditional mean of the volatility switches between a low and a high state. This generalization requires a new pricing equation which we derive now. We begin by rewriting the (risk neutral) specification, under a restatement where we deal with the log stock price process instead of the price process directly, i.e., \( X_t = \log(S_t) \), and another restatement where the jump for the regime shifts is broken up into two separate jump types: one up and one down.\(^{27}\) In this case we have,

\[
\begin{align*}
\frac{dX_t}{X_t} &= (r - \frac{1}{2} V_t) dt + \sqrt{V_t} dW^{(1)}_t \\
\frac{dV_t}{V_t} &= \left[ \kappa (v_t - V_t) - \eta V_t \right] dt + \sigma \sqrt{V_t} \left( \rho dW^{(1)}_t + \sqrt{1 - \rho^2} dW^{(2)}_t \right) \\
\frac{dv_t}{v_t} &= \left[ v^h - v^l \right] dq^s_t + \left[ v^l - v^h \right] dq^h_t
\end{align*}
\]

Here, \( q^s_t \) and \( q^h_t \) are Poisson jump counters with (affine) state dependent intensities.\(^{28}\)

\[
\begin{align*}
\lambda^{s}_t &= \lambda^{s}_0 + \lambda^{s}_1 v_t = \left( \frac{\lambda^{s}_1 v^h}{v^h - v^l} \right) + \left( \frac{-\lambda^{s}_1 v^l}{v^h - v^l} \right) v_t \in \{0, \lambda^{s}_t\} \\
\lambda^{h}_t &= \lambda^{h}_0 + \lambda^{h}_1 v_t = \left( \frac{-\lambda^{h}_1 v^l}{v^h - v^l} \right) + \left( \frac{\lambda^{h}_1 v^h}{v^h - v^l} \right) v_t \in \{0, \lambda^{h}_t\}
\end{align*}
\]

Keeping the affine structure in this generalization provides us with the same (general) form for option pricing as in the Heston model. The difference is that the following pricing PDE\(^{29}\) is slightly different from the one

\(^{27}\)This separation into two jumps is performed in order to fit our specification exactly into the affine structure and hence utilize the option pricing framework provided for this affine structure. The jump size distribution is degenerate and of size 1, or more correctly, the size of the coefficient \([v^h - v^l]\) when in the low state and \([v^l - v^h]\) when in the high state.

\(^{28}\)Note that although we call these intensities “state dependent” as they depend on which regime you are in, other papers (such as Dai and Singleton (2003)) would call these intensities as state-independent since the intensities are not dependent on any of the other state variables. We use this terminology because we explicitly label the regime as a state variable and use that variable separately in the option pricing whereas other papers’ option pricing approach merges the regime specificities into the \( A_j(\tau; \phi) \) coefficients.

\(^{29}\)The pricing PDE is an equation which any asset \( U(S,V,t) \) must satisfy and is derived from the standard condition that under the risk neutral measure, the drift of any asset must equal the risk free rate.
in the Heston paper in order to incorporate the varying level of the mean for the volatility and the jumps / regime switches:

\[
\frac{1}{2} V S^2 \frac{\partial^2 U}{\partial S^2} + \rho \sigma V S \frac{\partial^2 U}{\partial S \partial V} + \frac{1}{2} \sigma^2 V \frac{\partial^2 U}{\partial V^2} + r S \frac{\partial U}{\partial V} + \{r(v - V) - \eta V\} \frac{\partial U}{\partial V} - r U + \frac{\partial U}{\partial t} \\
+ \lambda^s E\{U_{\text{post-lojump}} - U_{\text{pre-lojump}}\} + \lambda^h E\{U_{\text{post-hijump}} - U_{\text{pre-hijump}}\} = 0
\]

where the notation \(U_{\text{pre-lojump}}\) represents the value of the asset just before a jump from the low state occurring and \(U_{\text{post-lojump}}\) represents the value of that asset just after the jump from the low state and correspondingly for the jump from the high state.

Now, with the introduction of an additional state variable \(v\) (representing the unconditional volatility mean which has two possible values, \(v^l\) or \(v^h\), depending on the regime), the form of the value of the call option is

\[
\text{Call}(S, V, v, K, \tau) = SP_1(S, V, v, K, \tau) - Ke^{-r\tau}P_2(S, V, v, K, \tau)
\]

where the probabilities are given by

\[
P_j(S, V, v, K, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \exp[-i\phi \log(K)] f_j(S, V, v, K, \tau; \phi) \right] d\phi
\]

The characteristic functions in this generalized case now become

\[
f_j(S, V, v, \tau; \phi) = \exp[A_j(\tau; \phi) + B_j(\tau; \phi)V + C_j(\tau; \phi)v + i\phi \log(S)]
\]

The adjusted PDE’s which the probabilities, and hence the characteristic functions, must satisfy are

\[
\frac{1}{2} V \frac{\partial^2 f_j}{\partial x^2} + \rho \sigma V \frac{\partial^2 f_j}{\partial x \partial v} + \frac{1}{2} \sigma^2 V \frac{\partial^2 f_j}{\partial v^2} + \{r + u_j V\} \frac{\partial f_j}{\partial V} + \{\nu - b_j V\} \frac{\partial f_j}{\partial v} - \frac{\partial f_j}{\partial \tau} \\
+ \lambda^s E\{f_j_{\text{post-lojump}} - f_j_{\text{pre-lojump}}\} + \lambda^h E\{f_j_{\text{post-hijump}} - f_j_{\text{pre-hijump}}\} = 0
\]

again, for \(j = 1, 2\), and where \(u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, b_1 = \kappa + \eta^V - \rho \sigma, b_2 = \kappa + \eta^V\). Given our conjectured form for the characteristic functions, these PDE’s imply the following sets of ODE’s:

\[
-\frac{1}{2} \phi^2 + \rho \sigma i\phi B_j + \frac{1}{2} \sigma^2 B_j^2 + u_j i\phi - b_j B_j - \frac{dB_j}{d\tau} = 0
\]

\[
\kappa B_j - \frac{dC_j}{d\tau} + \lambda^s [\exp(C_j(v^h - v^l)) - 1] + \lambda^h [\exp(C_j(u^l - v^h)) - 1] = 0
\]

\[
ri\phi - \frac{dA_j}{d\tau} + \lambda^s [\exp(C_j(v^h - v^l)) - 1] + \lambda^h [\exp(C_j(v^l - v^h)) - 1] = 0
\]
These complex-valued ODE’s, along with the following boundary conditions \( A_j(0; \phi) = B_j(0; \phi) = C_j(0; \phi) = 0 \), are solved to provide us with the functional forms for the coefficients. We begin by solving the first ODE, which is of the Riccati form, to get a solution for \( B_j \) of

\[
B_j(\tau; \phi) = \frac{b_j - \rho \sigma \phi i + d_j}{\sigma^2} \left[ \frac{1 - \exp(d_j \tau)}{1 - g_j \exp(d_j \tau)} \right]
\]

where

\[
d_j = \sqrt{(\rho \sigma \phi - b_j)^2 - \sigma^2 (2u_j \phi - \phi^2)}
\]

\[
g_j = \frac{b_j - \rho \sigma \phi i + d_j}{b_j - \rho \sigma \phi i - d_j}.
\]

Given this solution, we next solve the second ODE to get a solution for \( C_j \). Unfortunately, this ODE does not result in an analytic solution of \( C_j \) and hence must be solved numerically. As the relevant ode for \( A_j \) involves \( C_j \), it too must be solved numerically. Thus, we are able to calculate the three coefficients which feed into the characteristic functions. These characteristic functions are then inverted to get the probabilities which are then directly used to calculate the option prices.

**B True Long Memory**

**Objective:**

\[
dS_t = \mu S_t dt + \sqrt{V_t} S_t dW_t^{(1)}
\]

\[
V_t = \bar{\nu} + I^{(\alpha)} (\tilde{V}_t - \bar{\nu})
\]

\[
d\tilde{V}_t = \kappa (\bar{\nu} - \tilde{V}_t) dt + \sigma \sqrt{\tilde{V}_t} \left( \rho dW_t^{(1)*} + \sqrt{1 - \rho^2} dW_t^{(2)*} \right)
\]

**Risk Neutral:**

\[
dS_t = r S_t dt + \sqrt{V_t} S_t dW_t^{(1)*}
\]

\[
V_t = \frac{\kappa \bar{\nu}}{\kappa + \eta V_t} + I^{(\alpha)} \left( \tilde{V}_t - \frac{\kappa \bar{\nu}}{\kappa + \eta V_t} \right)
\]

\[
d\tilde{V}_t = [\kappa (\bar{\nu} - \tilde{V}_t) - \eta V_t \tilde{V}_t] dt + \sigma \sqrt{\tilde{V}_t} \left( \rho dW_t^{(1)*} + \sqrt{1 - \rho^2} dW_t^{(2)*} \right)
\]

---

30Actually, this ode is the same as in the simpler Heston specification and thus the solution for \( B_j \) is exactly the same.

31We were not able to find one at least.

32Our numerical approach to solving these ODE’s is to use the fourth order Runge-Kutta method.

33All of the numerical integration in this paper (including that necessary for these inversions) is calculated using an adaptive quadrature scheme based on Gauss-Kronrod formulas.
The true long memory specification, based on that of Comte, Coutin, and Renault (2003), introduces long memory into the volatility process while keeping the affine structure of the model. This specification is inherently different from the first two specifications in that it is non-Markovian. As a result, the option pricing equation will be dependent on the history of the volatility, not only on the current volatility. But, since we do not have a continuous nor infinite record of the data, we have to use a discretized and truncated history of the volatility process.\(^{34}\) I.e., \(\tilde{V}^{\text{hist}}_t = \{\tilde{V}_{t-l}\}\) for \(l = 0, 1, 2, ..., L\) not \(\tilde{V}^{\text{hist}}_t = \{\tilde{V}_{t-l}\}\) for \(0 \leq l \leq \infty\).

Now, the standard partial differential equation which any asset \(U(S, \tilde{V}^{\text{hist}}, t)\) must satisfy becomes:

\[
-rU + \frac{\partial U}{\partial t} + \frac{1}{2} \left[ \frac{\kappa \sigma}{\kappa + \theta} \right] S^2 \frac{\partial^2 U}{\partial S^2} + rS \frac{\partial U}{\partial S} + \sum_{l=0}^{L} \left( \left[ \kappa(\bar{V} - \tilde{V}_{t-l}) - \eta \tilde{V}_{t-l} \right] \frac{\partial U}{\partial \tilde{V}_{t-l}} + \frac{1}{2} \sigma^2 \tilde{V}_{t-l} \frac{\partial^2 U}{\partial \tilde{V}_{t-l}^2} \right) + \rho \sigma \tilde{V}_{t} \sqrt{\tilde{V}_{t}} S \frac{\partial^2 U}{\partial S \partial \tilde{V}_{t}} = 0
\]

At this point, we note that the last term in this PDE contains non-linear terms which prevent us from using the transform based techniques previously used. This term is a result of the correlation between the price and the volatility equations. If there were no correlation, we could proceed without any problem. Unfortunately, with a non-zero correlation, there is non-linearity which cannot be accommodated by the usual technique. Thus, in order to proceed using the transform based approach, we make a simplifying adjustment which implies that our resulting pricing is only approximate and no longer exact. Specifically, we use \(\sqrt{\tilde{V}_{t}}\) in place of \(\sqrt{V_{t}}\) in that last term which now makes the term linear in \(\tilde{V}_{t}\). Justifications for this adjustment include the fact that although in general \(\tilde{V}_{t}\) does not equal \(V_{t}\), their expectations are equal. Furthermore, we are dealing with realized volatility instead of the true volatility in which case this approximation error is not the only approximation error and in fact may be dominated by the error from not knowing the true volatility. Finally, we note that we have checked our option pricing scheme against values generated using Monte-Carlo based pricing and there does not seem to be any systematic bias.

Now, the value of a call option with current stock price \(S\), volatility history \(\tilde{V}^{\text{hist}}\), strike price \(K\), and time to maturity \(\tau\) is given by

\[
\text{Call}(S, \tilde{V}^{\text{hist}}, K, \tau) = SP_1(S, \tilde{V}^{\text{hist}}, K, \tau) - Ke^{-r\tau}P_2(S, \tilde{V}^{\text{hist}}, K, \tau)
\]

\(^{34}\)One may question the validity of discretizing and/or truncating at this point. Unfortunately, without the whole path of volatility, there is no alternative but to replace the integral with a sum. Furthermore, the alternative scheme for option pricing is a Monte Carlo approach which would inherently involve discretization and truncation as well. Lastly, we note that the estimation approach is also based on the discretized data and not the (unobtainable) whole path of volatility.
which when substituted into the above PDE, and taking $X_t = \log(S_t)$, results in the following PDE’s for the probabilities

$$-r P_j - \frac{\partial P_j}{\partial \tau} + \frac{1}{2} \left[ \frac{\kappa \pi}{\kappa + \eta} + I^{(\alpha)} \left( \tilde{V}_t - \frac{\kappa \pi}{\kappa + \eta} \right) \right] \frac{\partial^2 P_j}{\partial \tau^2} + \left( r + u_j \left[ \frac{\kappa \pi}{\kappa + \eta} + I^{(\alpha)} \left( \tilde{V}_t - \frac{\kappa \pi}{\kappa + \eta} \right) \right] \right) \frac{\partial P_j}{\partial \tau} + \sum_{l=0}^{L} \left[ \kappa \pi - (\kappa + \eta^2) \tilde{V}_{t-l} \right] \frac{\partial P_j}{\partial \tilde{V}_{t-l}} + \rho \sigma \tilde{V}_t \frac{\partial^2 P_j}{\partial \tilde{V}_t \partial \tilde{V}_{t-l}} + \left( \frac{1}{2} + u_j \right) \rho \sigma \tilde{V}_t \frac{\partial P_j}{\partial \tilde{V}_t} = 0$$

for $j = 1, 2$, and where $u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$. These probabilities can be seen in the following form

$$P_j(S, \tilde{V}_{hist}, K, \tau) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \frac{\exp[-i \phi \log(K)] f_j(S, \tilde{V}_{hist}, \tau; \phi)}{i \phi} \right] d\phi$$

which represent inversions of the following characteristic functions

$$f_j(S, V, \tau; \phi) = \exp[A_j(\tau; \phi) + \sum_{l=0}^{L} B_{j,l}(\tau; \phi) \tilde{V}_{t-l} + i \phi \log(S)]$$

The above PDE’s imply the following sets of ODE’s:

$$-\frac{1}{2} \phi^2 \pi_l + u_j i \phi \pi_l - (\kappa + \eta^2) B_{j,l} + \frac{1}{2} \sigma^2 B_{j,l}^2 + \rho \sigma i \phi B_{j,l} + \left( \frac{1}{2} + u_j \right) \rho \sigma B_{j,l} = \frac{dB_{j,l}}{d\tau} = 0 \quad \text{for} \quad l = 0$$

$$-\frac{1}{2} \phi^2 \pi_l + u_j i \phi \pi_l - (\kappa + \eta^2) B_{j,l} + \frac{1}{2} \sigma^2 B_{j,l}^2 - \frac{dB_{j,l}}{d\tau} = 0 \quad \text{for} \quad l = 1, 2, ..., L$$

$$r i \phi + \kappa \pi \sum_{l=0}^{L} B_{j,l} = \frac{dA_j}{d\tau} = 0$$

where the $\pi_l$ are defined by replacing the fractional integration operator $I^{(\alpha)}$ with the Grunwald-Letnikov definition\(^{35}\) discretized summation of $\sum_{l=0}^{L} \pi_l B^l$, $\pi_l = \frac{l + \alpha}{\frac{\Gamma(\alpha)}{\Gamma(\alpha + 1)} + 1} \pi_{l-1}$, 0 for $\Delta^\alpha$, $\Delta$ is the time interval between discrete data points (one day in our case), and $B^l$ is the backshift operator such that $B^l z_t = z_{t-l}$. These ODE’s are solved using the following boundary conditions $A_j(0; \phi) = B_{j,0}(0; \phi) = 0$ and the analytic solutions are:

$$A_j(\tau; \phi) = r \phi \pi \tau + \frac{\kappa \pi}{\sigma^2} \left( (\kappa + \eta^2 - (\frac{1}{2} + u_j) \rho \sigma - \rho \sigma \phi i + d_{j,0}) \tau - 2 \log \left[ \frac{1 - g_{j,0} \exp(d_{j,0} \tau)}{1 - g_{j,0}} \right] \right)$$

$$B_{j,0}(\tau; \phi) = \frac{(\kappa + \eta^2 - (\frac{1}{2} + u_j) \rho \sigma - \rho \sigma \phi i + d_{j,0})}{\sigma^2} \left( 1 - \exp(d_{j,0} \tau) \right) \left( 1 - \frac{1 - g_{j,0} \exp(d_{j,0} \tau)}{1 - g_{j,0}} \right)$$

$$B_{j,l}(\tau; \phi) = \left( \frac{(\kappa + \eta^2) + d_{j,l}}{\sigma^2} - \frac{1 - \exp(d_{j,0} \tau)}{1 - g_{j,0} \exp(d_{j,0} \tau)} \right) \text{ for } l = 1, 2, ..., L$$

\(^{35}\)See Oldham and Spanier (1974) for further details of this definition and discretization.
where

\[
\begin{align*}
    d_{j,0} &= \sqrt{(\rho \sigma \phi_i - (\kappa + \eta) V) + \left(\frac{1}{2} + u_j\right) \rho \sigma} - \sigma^2 \pi_0 (2u_j \phi_i - \phi^2) \\
    g_{j,0} &= \frac{(\kappa + \eta) V - (\frac{1}{2} + u_j) \rho \sigma - \rho \sigma \phi_i + d_{j,0}}{(\kappa + \eta) V - (\frac{1}{2} + u_j) \rho \sigma - \rho \sigma \phi_i - d_{j,0}} \\
    d_{j,l} &= \sqrt{(\kappa + \eta V)^2 - \sigma^2 \pi_1 (2u_j \phi_i - \phi^2)} \\
    g_{j,l} &= \frac{(\kappa + \eta V) + d_{j,l}}{(\kappa + \eta V) - d_{j,l}}
\end{align*}
\]

These functional forms for the coefficients feed into the characteristic functions. These characteristic functions are then inverted to get the probabilities which are then directly used to calculate the option prices as in the other cases.

**Appendix B: Moment Conditions for Estimation**

In this appendix, we list specifically the moment conditions used in the GMM estimation. We first begin with the conditions based on the option data, as these conditions are the same for each specification. Then we list the conditions based on the spot data which differ across the specifications.

**A Option-based conditions: \( g_{l}^{o}(\Theta) \)**

We use a panel of nine options per day and we have a moment condition for each of the nine options. We consider the pricing error between the observed (whether in actual or simulated data) option price, \( C^{\text{obs}}(t, \tau, K) \), and the calculated theoretical (given a model and estimated parameter values) option price,
The corresponding moment condition is then that the calculated theoretical price equals the observed option price in expectation (both normalized by strike price). I.e.,

\[ g_\theta^t(\Theta) \equiv \left( \begin{array}{c}
\frac{C_{\text{obs}}(t, \tau_1, K_1) - C_{\text{theor.}}(t, \tau_1, K_1; \Theta)}{K_1} \\
\frac{C_{\text{obs}}(t, \tau_2, K_1) - C_{\text{theor.}}(t, \tau_2, K_1; \Theta)}{K_1} \\
\frac{C_{\text{obs}}(t, \tau_3, K_1) - C_{\text{theor.}}(t, \tau_3, K_1; \Theta)}{K_1} \\
\frac{C_{\text{obs}}(t, \tau_1, K_2) - C_{\text{theor.}}(t, \tau_1, K_2; \Theta)}{K_2} \\
\frac{C_{\text{obs}}(t, \tau_2, K_2) - C_{\text{theor.}}(t, \tau_2, K_2; \Theta)}{K_2} \\
\frac{C_{\text{obs}}(t, \tau_3, K_2) - C_{\text{theor.}}(t, \tau_3, K_2; \Theta)}{K_2} \\
\frac{C_{\text{obs}}(t, \tau_1, K_3) - C_{\text{theor.}}(t, \tau_1, K_3; \Theta)}{K_3} \\
\frac{C_{\text{obs}}(t, \tau_2, K_3) - C_{\text{theor.}}(t, \tau_2, K_3; \Theta)}{K_3} \\
\frac{C_{\text{obs}}(t, \tau_3, K_3) - C_{\text{theor.}}(t, \tau_3, K_3; \Theta)}{K_3}
\end{array} \right) \]

and the condition is \( E[g_\theta^t(\Theta)] = 0 \). We note that the normalization by strike price is included due to the non-stationarity of the price of the underlying asset. And of course, the details of the calculation for the theoretical option price for each specific model are provided in appendix A.

**B Spot-based conditions: \( g_\theta^s(\Theta) \)**

Now we consider the first set of moment conditions, namely those associated with the moments of the state variables. We need these additional moments in order to identify parameters which are specific to the objective measure. There are two such parameters in the first and third specification and three such parameters in the second specification. Thus, this set of conditions based on the moments of the state variables contains three conditions in order to fully identify all of the models. A natural choice for the price state variable is simply the the first and second conditional moments of the return process \( Y_t \equiv \log(S_t/S_{t-1}) = X_t - X_{t-1} \). For the third condition, we use the volatility state variable for which a natural choice is the first conditional moment of the integrated volatility \( Z_t = \int_{t-1}^t V_s ds \). The vector of underlying-based moments thus is,

\[ g_\theta^s(\Theta) \equiv \left( \begin{array}{c}
E_{t-1}[Y_t] - Y_t \\
E_{t-1}[Y_t^2] - Y_t^2 \\
E_{t-1}[Z_t] - Z_t
\end{array} \right) \]

and the condition is \( E[g_\theta^s(\Theta)] = 0 \). Furthermore, we specify the corresponding conditional moments for each specification.
No Long Memory

\[
E_{t-1}[Y_t] = \mu \Delta - \frac{1}{2} \pi \Delta + \frac{1 - \exp(-\kappa \Delta)}{2\kappa} (\overline{\sigma} - V_{t-1})
\]

\[
E_{t-1}[Y^2_t] = \pi \Delta + \frac{1 - \exp(-\kappa \Delta)}{\kappa} (V_{t-1} - \overline{\sigma}) + \left[ \mu \Delta - \frac{1}{2} \pi \Delta + \frac{1 - \exp(-\kappa \Delta)}{2\kappa} (\overline{\sigma} - V_{t-1}) \right]^2
\]

\[
E_{t-1}[Z_t] = \pi \Delta + (V_{t-1} - \overline{\sigma}) \frac{1 - \exp(-\kappa \Delta)}{\kappa}
\]

Spurious Long Memory

\[
E_{t-1}[Y_t] = \mu \Delta - \frac{1}{2} v_{t-1} \Delta + \frac{1 - \exp(-\kappa \Delta)}{2\kappa} (v_{t-1} - V_{t-1})
\]

\[
E_{t-1}[Y^2_t] = v_{t-1} \Delta + \frac{1 - \exp(-\kappa \Delta)}{\kappa} (V_{t-1} - v_{t-1}) + \left[ \mu \Delta - \frac{1}{2} v_{t-1} \Delta + \frac{1 - \exp(-\kappa \Delta)}{2\kappa} (v_{t-1} - V_{t-1}) \right]^2
\]

\[
E_{t-1}[Z_t] = v_{t-1} \Delta + (V_{t-1} - v_{t-1}) \frac{1 - \exp(-\kappa \Delta)}{\kappa}
\]

True Long Memory

\[
E_{t-1}[Y_t] = \mu \Delta - \frac{1}{2} \tilde{v}_{t-1} \Delta + \frac{1 - e^{-\kappa \Delta}}{\kappa} \left( \frac{1 - e^{-\kappa \Delta}}{\kappa} \right) (\tilde{V}_{t-1} - \overline{\sigma})
\]

\[
E_{t-1}[Y^2_t] = \pi \Delta + I^{(\alpha)} \left( \frac{1 - e^{-\kappa \Delta}}{\kappa} (\tilde{V}_{t-1} - \overline{\sigma}) \right) + \left[ \mu \Delta - \frac{1}{2} \pi \Delta + I^{(\alpha)} \left( \frac{1 - e^{-\kappa \Delta}}{\kappa} (\tilde{V}_{t-1} - \overline{\sigma}) \right) \right]^2
\]

\[
E_{t-1}[Z_t] = \pi \Delta + I^{(\alpha)} \left( \frac{1 - e^{-\kappa \Delta}}{\kappa} (\tilde{V}_{t-1} - \overline{\sigma}) \right)
\]

Appendix C: Filtered Probabilities

In this appendix, we specify the procedure for calculating the probability of being in either the low or high mean volatility state of the second specification. As these probabilities are used in calculating the (conditional expectation of the) state variable moments and option prices, we need corresponding probabilities conditional on the available information as of that time. Thus, we want to calculate the filtered probabilities, i.e., \( p(v_t|H_t) \) where \( H_t \) denotes the history of \( V \) up to and including \( V_t \), i.e., \( H_t \equiv \{V_t, V_{t-1}, ..., V_1\} \).

Calculating the filtered probabilities requires calculation of likelihoods conditional on both just the history of \( V \) and the history of \( V \) as well as the past mean volatility state, which are denoted by \( p(V_t|H_{t-1}) \) and

\[36\text{The specific filtering algorithm we use is taken from Hamilton (1990). The interested reader is referred to Appendix B of that paper for more discussion on this algorithm as well as the related procedure of calculating smoothed probabilities.}\]
of \( V \) and the past mean volatility state, the likelihood is given by:

\[
p(V_t|H_{t-1}, v_{t-1}) = ce^{-a-b} \left( \frac{b}{a} \right)^{q/2} I_q(2\sqrt{ab}),
\]

where

\[
c = \frac{2\kappa}{\sigma^2(1 - e^{-\kappa \Delta})}, \\
a = cV_{t-1}e^{-\kappa \Delta}, \\
b = cV_t, \\
q = \frac{2\kappa V_{t-1}}{\sigma^2} - 1
\]

and \( I_q(.) \) is the modified Bessel function of the first kind of order \( q \). This conditional density is actually the non-central \( \chi^2 \) distribution with \( 2q + 2 \) degrees of freedom and non-centrality parameter of \( 2a \). Given this likelihood in combination with the proper filtered probability, the likelihood conditional on only the history of \( V \) can easily be calculated as:

\[
p(V_t|H_{t-1}) = \sum_{v_{t-1}=v^l,v^h} p(V_t|H_{t-1}, v_{t-1})p(v_{t-1}|H_{t-1}).
\]

Now, this equation and the following equation,

\[
p(v_t|H_t) = \frac{\sum_{v_{t-1}=v^l,v^h} p(v_t|v_{t-1})p(V_t|H_{t-1}, v_{t-1})p(v_{t-1}|H_{t-1})}{p(V_t|H_{t-1})},
\]

can be calculated iteratively for \( t = 2, ..., T \) to generate the filtered probabilities. Note that the switching probabilities are determined by the parameter values, i.e., \( p(v_t = v^h|v_{t-1} = v^l) = \lambda^l \Delta \) and \( p(v_t = v^l|v_{t-1} = v^l) = 1 - \lambda^l \Delta \) and the corresponding pair of probabilities for a prior high state. The only remaining issue is the initialization of this iteration, i.e., \( p(v_1|H_1) \). For this we simply set the probability of each state given the first observation equal to the corresponding unconditional probability implied by its stationary distribution, i.e., \( p(v_1 = v^l|H_1) = p(v_2 = v^l) = \frac{\lambda^h}{\lambda^l + \lambda^h} \) and \( p(v_1 = v^h|H_1) = p(v_t = v^h) = \frac{\lambda^l}{\lambda^l + \lambda^h} \).
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Ohanissian, Arek, Jeffrey R. Russell, and Ruey S. Tsay, 2003, Using temporal aggregation to distinguish between true and spurious long memory, working paper.


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Table I

Estimation Results from Real Data

This table presents the estimation results using real data for the three models, no long memory, spurious long memory, and true long memory, in both with risk premia and without risk premia cases. The data used includes 9 years of intra-day returns and 1 year of daily call option prices for the S&P 500 index.

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### Table II
Estimation Results from Simulations: DGP of Spurious Long Memory

This table presents estimation results from 100 simulations where the true DGP is the spurious long memory model in both cases of with and without risk premia. The true values used for the simulations are listed alongside the mean and standard error of the estimated parameter values for all three models.

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Table III
Estimation Results from Simulations: DGP of True Long Memory

This table presents estimation results from 100 simulations where the true DGP is the true long memory model in both cases of with and without risk premia. The true values used for the simulations are listed alongside the mean and standard error of the estimated parameter values for all three models.

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Table IV

Pricing Error Results from Simulations: DGP of Spurious Long Memory w/o Risk Premia

This table presents results from 100 simulations where the true DGP is the spurious long memory model without risk premia. Four measures of pricing errors are reported (one per panel) using both dollar and relative pricing and error and absolute deviation. Both the mean and the standard error are listed. Note, moneyness is defined as StrikePrice/StockPrice.

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### Panel B: Dollar Pricing Absolute Deviation ($)

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### Panel C: Relative Pricing Error (%)

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### Panel D: Relative Pricing Absolute Deviation (%)
Table V
Pricing Error Results from Simulations: DGP of Spurious Long Memory w/ Risk Premia

This table presents results from 100 simulations where the true DGP is the spurious long memory model with risk premia. Four measures of pricing errors are reported (one per panel) using both dollar and relative pricing and error and absolute deviation. Both the mean and the standard error are listed. Note, moneyness is defined as StrikePrice/StockPrice.

### Panel A: Dollar Pricing Error ($)

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<td>7.08</td>
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<td>18.04</td>
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<td>0.48</td>
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### Panel B: Dollar Pricing Absolute Deviation ($)

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### Panel D: Relative Pricing Absolute Deviation (%)

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### Table V

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Notes: moneyness is defined as StrikePrice/StockPrice.
Table VI
Pricing Error Results from Simulations: DGP of True Long Memory w/o Risk Premia

This table presents results from 100 simulations where the true DGP is the true long memory model without risk premia. Four measures of pricing errors are reported (one per panel) using both dollar and relative pricing and error and absolute deviation. Both the mean and the standard error are listed. Note, moneyness is defined as StrikePrice/StockPrice.

### Panel A: Dollar Pricing Error ($)

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### Panel B: Dollar Pricing Absolute Deviation ($)

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### Panel C: Relative Pricing Error (%)

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### Panel D: Relative Pricing Absolute Deviation (%)

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This table presents results from 100 simulations where the true DGP is the true long memory model with risk premia. Four measures of pricing errors are reported (one per panel) using both dollar and relative pricing error and absolute deviation. Both the mean and the standard error are listed. Note, moneyness is defined as StrikePrice/StockPrice.

### Panel A: Dollar Pricing Error ($)

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<td>0.79</td>
<td>0.70</td>
<td>0.64</td>
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### Panel B: Dollar Pricing Deviation (%)

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### Panel D: Relative Pricing Absolute Deviation (%)

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Table VIII
Black-Scholes Implied Volatility Results from Simulations: DGP of Spurious Long Memory

This table presents Black-Scholes implied volatility results from 100 simulations where the true DGP is the spurious long memory model. We list both the mean and standard error of the B-S implied volatilities from the true option prices as well as the ratio of the B-S implied volatilities from the estimated models’ option prices to the true option prices. Note, moneyness is defined as StrikePrice/StockPrice.

### Panel A: Spurious LM w/o Risk Premia

<table>
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<tr>
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<td>19.9%</td>
<td>19.9%</td>
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<tr>
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### Panel B: Spurious LM w/ Risk Premia

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This table presents Black-Scholes implied volatility results from 100 simulations where the true DGP is the true long memory model. We list both the mean and standard error of the B-S implied volatilities from the true option prices as well as the ratio of the B-S implied volatilities from the estimated models’ option prices to the true option prices. Note, moneyness is defined as StrikePrice/StockPrice.

### Panel A: True LM w/o Risk Premia

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### Panel B: True LM w/ Risk Premia

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This figure contains Black-Scholes implied volatility plots as a function of maturity and moneyness for three specific levels of volatility (low, medium, high). The parameter values used for these plots are estimated from real data and listed in Table I. The specification used in this figure is the no long memory model without risk premia.
This figure contains Black-Scholes implied volatility plots as a function of maturity and moneyness for three specific levels of volatility (low, medium, high). The parameter values used for these plots are estimated from real data and listed in Table I. The specification used in this figure is the spurious long memory model without risk premia.
This figure contains Black-Scholes implied volatility plots as a function of maturity and moneyness for three specific levels of volatility (low, medium, high). The parameter values used for these plots are estimated from real data and listed in Table I. The specification used in this figure is the true long memory model without risk premia.

Figure 1. Black-Scholes Implied Volatility Plots: No Risk Premia
(c) True Long Memory
This figure contains Black-Scholes implied volatility plots as a function of maturity and moneyness for three specific levels of volatility (low, medium, high). The parameter values used for these plots are estimated from real data and listed in Table I. The specification used in this figure is the no long memory model with risk premia.
This figure contains Black-Scholes implied volatility plots as a function of maturity and moneyness for three specific levels of volatility (low, medium, high). The parameter values used for these plots are estimated from real data and listed in Table I. The specification used in this figure is the spurious long memory model with risk premia.
This figure contains Black-Scholes implied volatility plots as a function of maturity and moneyness for three specific levels of volatility (low, medium, high). The parameter values used for these plots are estimated from real data and listed in Table I. The specification used in this figure is the true long memory model with risk premia.
Figure 3. Black-Scholes Implied Volatility Plots: All Models. No Risk Premia

This figure contains Black-Scholes implied volatility plots as a function of moneyness for three specific levels of volatility (low, medium, high) for the no risk premia case. The parameter values used for these plots are estimated from real data and listed in Table I. All three specifications are used in this figure.
Figure 4. Black-Scholes Implied Volatility Plots: All Models. Risk Premia

This figure contains Black-Scholes implied volatility plots as a function of moneyness for three specific levels of volatility (low, medium, high) for the risk premia case. The parameter values used for these plots are estimated from real data and listed in Table I. All three specifications are used in this figure.