Effects of non-normality and dependence on the precision of variance estimates using high-frequency financial data *

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Abstract
Volatility is central to financial theory. In practice volatility is not directly observed and must be estimated. Merton’s (1980) seminal work suggests that as the sampling interval approaches zero arbitrarily precise volatility estimates can be obtained. Realistically, however, the limiting case is not attainable since the sampling frequency cannot be any higher than transaction by transaction. We examine the precision of unconditional volatility estimates that use high-frequency data. We derive analytical expression for the precision of volatility estimates as a function of the prominent high frequency data characteristics including leptokurtosis, autocorrelation in the returns, deterministic patterns and volatility clustering in intraday variance. Once these features are accounted for, we find that large amounts of high frequency data do not necessarily translate into very precise estimates. Our results provide a measure of the usefulness of high frequency data in estimating volatility. For two foreign exchange rate series analyzed, we find MSEs are minimized at the sampling intervals of 10 and 20 minutes. The MSEs obtained at the optimal sampling intervals are roughly half of the MSE obtained by using daily data alone.

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1 Introduction

Volatility is central to financial theory and investment decisions. Basic mean variance analysis, for example, requires estimates of the variance for the assets under consideration. Sharpe ratios, often used to characterize risk adjusted returns, also require a variance estimate. Academics and practitioners have long recognized the difficulty in obtaining accurate estimates of both means and variance associated with financial returns.

Estimation of the variance always requires that a choice must be made regarding the frequency of the sampling interval. Historically, the choice of the sampling frequency was driven by the data availability. In the early days of empirical finance weekly data was considered to be high frequency. Later, daily data became available. Now, we have intraday financial data giving us a picture of asset prices perhaps thousands of times within a single day. Despite the availability of intraday data it is still common today to use daily data to estimate the variance of an asset. There is, however, no justification present in the literature. The choice appears to be driven by convention and convenience.

The idea that very high frequency data might be useful in estimating the variance is not new. Working with continuous time geometric Brownian motion, Merton (1980) demonstrated that although precise estimates of the mean return require long spans of data, precise estimates of the variance can be obtained over fixed time intervals provided that the sampling interval approaches zero. If the asset price follows a geometric Brownian motion and is sampled over time period [0, T] at intervals $\Delta T$, straightforward calculations show that the variance of the return variance estimate is proportional to $\Delta T$, or the reciprocal of the number of observations over [0, T]. More recently, Nelson (1992), Foster and Nelson (1996), and Andersen et al. (2002) extend the result to general continuous diffusion process and obtain similar result for conditional instantaneous variance and quadratic variation.

In Merton’s spirit, we might think that with the intraday data now available we should expect that extremely precise estimates of volatility should be at hand. However, there are physical limitations associated with the data collection in financial market. In particular, the highest possible sampling rate is limited by the transaction frequency. Letting the sampling time interval shrink to zero is simply not possible. It is then natural to ask “how precisely can we estimate the volatility using high-frequency financial data?” Not surprisingly, the answer depends on the characteristics
of the return series analyzed.

Financial data, particularly when viewed at a high-frequency, do not conform well with the geometric Brownian motion assumption. First, intraday volatility clustering has been documented and analyzed by numerous studies. Early work includes Engle et al. (1990) for foreign exchange rate returns and Hamao et al. (1990) for equity index returns. Second, asset returns have consistently been documented to be leptokurtic, or fat tailed. The kurtosis tends to increase as the sampling interval shrinks. Andersen and Bollerslev (1998) report increasing kurtosis for the Dollar/Deutschmark exchange rate returns and for the S&P 500 index returns as the sampling frequency increases. For both series the kurtosis exceeds 20 when viewed at 5-minute intervals. Third, Harris (1986), Gerety and Mulherin (1992), and Baillie and Bollerslev (1991) have examined the deterministic intraday behavior of volatility. Among others, these studies suggest that higher volatility occurs near the open and close in the equity markets, or occurs when certain regions of the world are actively transacting in the continuously operated foreign exchange markets. Finally, asset returns tend to be serially correlated. Over time horizons of weeks or months the serial correlation may be quite small, but over short time horizons like daily or intraday, the correlation can be much stronger.

This paper builds on work by Box (1953) and Bai et al. (2001) to construct analytical representations for the precision of variance estimates that use intraday data in the presence of volatility clustering, leptokurtosis, deterministic patterns in the volatility structure, and serial correlation of the returns. We focus on the sum of square estimators often used in empirical studies. The precision of these estimates is then compared to the precision of estimates based on daily data. The ratio of the precision associated with the intraday and daily estimates can then be used to gauge the benefit of using the intraday data. Hence we provide measures of the precision of variance estimates that use intraday data, as well as a measure of the information content or benefit of using the high frequency data relative to the estimates based on the traditional daily data. We demonstrate that the optimal sampling frequencies are achieved at 10 or 20 minutes for the two foreign exchange rate series analyzed. We find that the MSE obtained at the optimal sampling interval is roughly half of the MSE obtained using daily data.

Our work is related to two other branches of literature. Andersen et al. (2001) use intraday data to estimate the variance of a particular day. More recently, Barndorff-Nielsen and Shephard (2002) has considered the precision of realized volatility, but not in the presence of realistic market
micro-structure noise. Bandi and Russell (2003), and Zhang et al. (2003) evaluate MSE of realized volatility using canonical model of market micro-structure noise. Unlike this literature, our approach remains non-parametric in nature and our work is related to realized volatility literature in Section 5.5.

Another related branch is the Drost and Nijman (1993) GARCH aggregation results. They derive the relationship between high and low frequency GARCH parameters. Again, they do not consider the precision of the estimates as we do in this paper. Also, for the sum of squares type estimators considered in this paper we do not impose any model, but rather rely on moments of the return process. Hence, we believe this is the first paper to construct measures for the precision of variance estimates that use high frequency data. Our focus on the unconditional daily variance is purely for exposition. The theoretical results can be applied to estimates of the variance for any time period of interest such as weekly or monthly.

The outline of the paper is as follows. In the following section we introduce notation and discuss the use of high-frequency data in estimating the variance of daily returns. In Section 3, we document several features of high-frequency exchange rate returns. These features are related to findings in earlier empirical work and a set of characteristics of high-frequency data is established. Section 4 proposes a general statistical structure consistent with the established features and discusses two commonly used sum of squares type estimators. In Section 5, theoretical results are presented relating the variance of each estimator to the characteristics of the high-frequency data discussed. Optimal sampling frequency is discussed as well as the relationship of our work and recent realized volatility literature. Finally, Section 6 concludes.

2 Notation

Consider a sequence of prices observed at unit time intervals, and denote the logged sequence of prices by $p_1, p_2, \ldots, p_N$ and the corresponding returns by $r_n = p_n - p_{n-1}$. In the following, we take the unit of time to be one day although the results can be easily carried over to other time periods. We assume that within a day the price is observed on $k$ evenly-spaced time intervals. We denote the $k$ logged intraday prices for day $n$ by $\{p_{nt}, t = 1, \ldots, k\}$ and the $t^{th}$ intraday return by $x_{nt} = p_{nt} - p_{n,t-1}$, where $p_{n0} = p_{n-1,k}$. By transforming the double subscript $(n, t)$ into the single subscript $s(n, t) = (n - 1) \cdot k + t$, the intraday return series $\{x_{nt} : n = 1, 2, \ldots, N; t = 1, 2, \ldots, k\}$
becomes a univariate series \( \{x_s : s = 1, \cdots, N \cdot k \} \). For convenience, both notations \( x_{nt} \) and \( x_s \) will be used interchangeably throughout this paper. To avoid confusion, we use \( x_{s(n,t)} \) to indicate the dependence of the single subscript \( s \) on the double subscript \( (n, t) \) explicitly.

The intraday returns \( \{x_{nt} : t = 1, \cdots, k \} \) on day \( n \) can also identified as a \( k \)-dimensional vector \( \mathbf{x}_n = (x_{n1}, x_{n2}, \cdots, x_{nk})^t \). The daily return \( r_n \) is simply the sum of the intraday returns:

\[
r_n = \sum_{t=1}^{k} x_{nt}
\]

In this paper, we are interested in using the intraday returns to estimate the variance of daily return \( r_n \), which is assumed to follow a covariance stationary process. We denote the mean daily return by \( \mu \) and the variance by \( \sigma^2 \).

Let \( \mathbf{x}_n \) have the joint distribution \( g(\mathbf{x}) \) with mean vector and variance-covariance matrix given by,

\[
\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{bmatrix}, \quad \text{and} \quad \Omega = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{21} & \sigma_2^2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \sigma_{k-1,k} \\ \sigma_{k1} & \cdots & \sigma_{k,k-1} & \sigma_k^2 \end{bmatrix}
\]

(2.2)

If \( \mathbf{x}_n \)'s have the same joint distribution, we have the following identities:

\[
\mu = \mu_1 + \mu_2 + \cdots + \mu_k
\]

(2.3)

\[
\sigma^2 = \sum_{t=1}^{k} \sigma_t^2 + 2 \sum_{t < s} \sigma_{st}
\]

(2.4)

If \( \Omega \) is diagonal, \( \sigma^2 \) simply reduces to the sum of the \( k \) intraday variances \( \sigma_t^2 \). In general, \( \sigma^2 \) is the sum of the \( k \) intraday variances \( \sigma_t^2 \) and terms \( \sigma_{st} \) that account for the correlations among the intraday returns.

The financial data at daily frequency or higher are not collected consecutively in calendar time, either because the data at weekends are not available, or because the data at weekends are very different from the data at weekdays and thus treated differently. We use the same notation \( x_{nt} \) or \( x_{s(n,t)} \) for both consecutive or non-consecutive data which should be clear from the context. If necessary, we will use the subscript \( n^* \) instead of \( n \) to indicate that the data may be collected non-consecutively in calendar time.
3 Features of High-Frequency Data

When viewed over monthly or longer periods, many financial return series are approximately normal white noise. As the sampling period shrinks, however, temporal dependence and departures from normality become strikingly apparent. In fact, at the transaction-by-transaction frequency, the normal approximation is extremely poor since the irregular spacing between transactions, discreteness in transaction prices and dependence introduced by market micro-structure become dominant features of the data\(^1\). We consider financial data sampled at fixed intervals within the day. In particular, our empirical work focuses on 15 minute foreign exchange rate data.

Several features are present in a broad class of intraday financial data and are consistently observed in the literature at various intraday sampling frequencies.

- First, volatility clustering is present in the intraday data. Early work includes Engle, Ito and Lin (1990) for foreign exchange rate returns and Hamao, Masolis and Ng (1990) for equity index returns. Much applied work uses the ARCH type model of Engle (1982) and Bollerslev (1986) or the stochastic volatility models to account for this feature\(^2\).

- Second, intraday volatility also contains a deterministic component. In the foreign exchange rate data, volatility varies as different regions of the world become active in the market. In stock data we tend to find the periods just after the open and just prior to the close are more volatile than the normal trading period. Harris (1986), Gerety and Mulherin (1992), and Baillie and Bollerslev (1991) have examined the deterministic intraday behavior of volatility. More recently, Andersen and Bollerslev (1998) document these patterns in volatility for equity and foreign exchange rate data.

- Third, returns are negatively autocorrelated. This correlation tends to become stronger as the sampling frequency increases.

- Fourth, as the sampling frequency increases, the kurtosis in the data can become exceptionally large. Andersen and Bollerslev (1998), for example, find a kurtosis of 18.5 for the 5-minute Dollar/Deutsch mark returns in their study. Chan and Karolyi (1991) find kurtosis ranging


\(^{2}\)See for example proceedings of the 2nd Olsen and Associates Conference on High-Frequency Data
from 10 to over 25 for 5-minute S&P500 returns over several sub-samples ranging from 1984 to 1989. While it is well understood that time-varying volatility (stochastic or deterministic) increases the overall kurtosis, the common models used, such as the GARCH with conditional normal errors, cannot account for the exceedingly large kurtosis typically observed in the data, e.g., Bollerslev (1987) and Engle and Gonzalez-Rivera (1991). Hence even after accounting for the time-varying volatility, it is still unlikely that the intraday returns have conditional normal distribution.

The data used in our empirical work are foreign exchange rate data obtained from Olsen and Associates. We analyze ten years (1/1987 - 12/1996) of 15-minute returns on the Dollar/Deutsch mark (DD) and Dollar/Japanese Yen (DY). The returns are constructed from irregularly spaced quote data. The equally spaced data are derived from the irregularly spaced quote data by Olsen and Associates. For each point in time marking a 15-minute interval the midpoint of the bid and the ask quotes just prior to, and just after are averaged. Since it is well known that returns over the weekend, when many markets are inactive, are very different from returns during the week we use only the data from 12:00am Tuesday morning to 12:00am Friday morning. This results in a sample of 1560 days and 149,760 15-minute returns. The returns are expressed as percentage. We now examine the 15-minute returns for the two series.

From descriptive statistics given in Table 1, both return series have mean zero, are slightly skewed to the left and are extremely leptokurtic. The statistics presented in Table 1 are affected by the deterministic intraday volatility pattern typically observed in such returns. From the intraday returns series, we construct the sample variance \( s_t^2 \) by time of day. This is given by,

\[
s_t^2 = \frac{1}{N} \sum_{n=1}^{N} x_{nt}^2, \quad t = 1, 2, \cdots, k
\]

Figure 1 shows plots of the sample standard deviations by time of day. The time of day on the horizontal axis is Greenwich Mean Time (GMT).\(^4\) Consistent with previous studies, we find that the most active period is in the afternoon GMT when both U.S. market and European market are

\(^3\)Here, the kurtosis for a random variable \( X \) is defined by \( \gamma_2 = \frac{E[(X - E[X])^4]}{Var(X)^2} - 3 \). The definition given here is sometimes referred to as excess kurtosis.

\(^4\)We first considered the deterministic patterns for the 3 days separately, and found that their patterns are similar. We thus treat them as the same.
Table 1: Descriptive Statistics of Exchange Rate Returns

<table>
<thead>
<tr>
<th>series</th>
<th>mean</th>
<th>std</th>
<th>skewness</th>
<th>kurtosis$^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DD</td>
<td>$-0.904 \times 10^{-4}$</td>
<td>$7.03 \times 10^{-2}$</td>
<td>-0.099</td>
<td>22.34</td>
</tr>
<tr>
<td>DY</td>
<td>$-1.072 \times 10^{-4}$</td>
<td>$7.10 \times 10^{-2}$</td>
<td>-0.0613</td>
<td>18.40</td>
</tr>
</tbody>
</table>

This table provides the descriptive statistics of 15-minute returns for Dollar / Deutsch Mark (DD) and Dollar / Japanese Yen (DY). Each series has 148992 observations since data for 8 out of 1560 days are missing.

active. Clearly, the returns cannot be regarded as homogeneous throughout a day. To partial out the time of day effect, we adjust the returns by dividing the intraday returns by their corresponding intraday sample standard deviations,

$$
\bar{x}_{nt} = \frac{x_{nt}}{\delta_t}
$$

(3.6)

The adjusted series $\bar{x}_{nt}$ are normalized to have unconditional variances close to 1, and should be free of any deterministic volatility pattern. Throughout this paper, this is termed the adjusted return series. The descriptive statistics for the two adjusted return series are given in Table 2.

Table 2: Descriptive Statistics of Adjusted Exchange Rate Returns

<table>
<thead>
<tr>
<th>series</th>
<th>mean</th>
<th>std</th>
<th>skewness</th>
<th>kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{DD}$</td>
<td>$6.488 \times 10^{-4}$</td>
<td>1.000</td>
<td>-0.049</td>
<td>16.49</td>
</tr>
<tr>
<td>$\hat{DY}$</td>
<td>$-11.952 \times 10^{-4}$</td>
<td>1.0000</td>
<td>-0.113</td>
<td>16.83</td>
</tr>
</tbody>
</table>

The adjusted series $\hat{DD}$ and $\hat{DY}$ are obtained from the original return series DD and DY by factoring out their deterministic intra-day volatility pattern, respectively.

As expected, the adjusted return series have smaller overall kurtosis. However, the kurtosis is still quite large.

The sample autocorrelation function (ACF) for the two adjusted return series are presented in Figure 2. For each of the two series only the lag-one sample autocorrelation is significant, which is indicative of a moving average model of order one, MA(1), structure. Negative lag-one
Figure 1: Intra-day Deterministic Pattern of Standard Deviation

Each series contains 148992 15-minute returns from 1552 days.
Figure 2: Sample ACF of Adjusted Return Series

The adjusted series are obtained from the original return series by factoring out the deterministic intra-day volatility pattern.
Figure 3: Sample ACF of Squared MA(1) innovations

For each series of adjusted returns, the innovations are residuals obtained from the fitted normal MA(1) model.
autocorrelation is commonly observed in intraday returns. It is often considered the result of
bid-ask bounce in equity markets and noisy quotes in foreign exchange markets.\textsuperscript{5}

Figure 3 presents the sample ACF of the squared residual series from the fitted normal MA(1)
model for the adjusted return series. The long set of positivity in the ACF plots is the tell tale
mark of volatility clustering typically observed in squared financial returns data.

4 A Statistical Structure of Intraday Returns and Daily Variance
Estimators

In this section we first propose a simple structure for the variance-covariance matrix of intraday
returns and their squares. This structure is consistent with the documented features, and sufficient
to derive analytical results for the precision of volatility estimates. Next, we discuss two intraday
data estimators of the daily variance as well as a typical variance estimator that uses daily returns
data. The precision of these three estimators will be the focus of our theoretical results.

4.1 A general statistical structure

The discussion on the features of high-frequency financial data in Section 3 suggests that the
variance-covariance matrix of $x_n$ need not to be fully saturated as we find evidence of lag-one
autocorrelation only. More specifically, the intraday deterministic volatility pattern in Figure 1
and the lag-one autocorrelation of the adjusted returns in Figure 2 suggest the following stochastic
structure for the intraday returns, in terms of the univariate series $x_{s(n,t)}$

$$
\frac{x_{s(n,t)}}{\delta_t} = \epsilon_{s(n,t)} + \theta \epsilon_{s(n,t)-1}
$$

(4.1)

where $\epsilon_{s(n,t)}$ are uncorrelated with mean 0 and variance 1, and $\{\delta_t, t = 1, \cdots, k\}$ are positive and
deterministically time-varying. Equivalently, the model states that the intraday returns $x_{nt}$ within

\textsuperscript{5}Hasbrouck (1988) discusses how market micro-structure noise can generate this autocorrelation structure.
a day have the following variance-covariance matrix:

$$\Omega = \text{cov}[x_n] = \begin{bmatrix} \delta_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 + \theta^2 & \theta & 0 & \cdot & \cdot \\ \cdot & \theta & 1 + \theta^2 & \theta & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

This model also relates the intraday returns between the days. More explicitly, the first intraday return on the next day is correlated with the last intraday returns on the previous day, and is uncorrelated with other previous intraday returns.

Figure 3 suggests that, while the innovation $\epsilon_{s(n,t)}$, or $\epsilon_{nt}$ in double-subscript, in (4.1) are uncorrelated, the squared innovations $\epsilon^2_{s(n,t)}$ are dynamically related. If we assume that the fourth moment of $\epsilon_{s(n,t)}$ exists and that $\epsilon^2_{s(n,t)}$ is covariance stationary, then the covariance between two squared innovations depends only on the time interval between them. We can therefore represent the auto-covariances of $\epsilon^2_{s(n,t)}$ as:

$$\text{cov}(\epsilon^2_{s(n,t)}, \epsilon^2_{s(n,t)-\ell}) = 2(1 + \frac{\gamma_2}{2}) \cdot \rho_{\ell}$$

where $\gamma_2$ is the kurtosis of $\epsilon_{s(n,t)}$. Since the variance of $\epsilon_{s(n,t)}$ is normalized to 1, it is easy to show that the variance of $\epsilon^2_{s(n,t)}$ is $2(1 + \frac{\gamma_2}{2})$. Note that if $\epsilon_{s(n,t)}$ is normal, then $\gamma_2 = 0$. The sample kurtosis of the adjusted series in Table 2 suggests a positive value of $\gamma_2$. The expression (4.3) shows explicitly the dependence of the variances and covariances of the $\epsilon^2_s$ on the kurtosis $\gamma_2$ of the $\epsilon'_s$.

Autocorrelation of nearby squared innovations are typically positive in financial data, implying that $\rho_{\ell}$s are positive at least for small $\ell$. Furthermore, Figure 3 suggests the strong volatility persistence with positive autocorrelations for squared innovations even farther apart in time. Hence, most $\rho_{\ell}$s should be positive though they may decrease as $\ell$ increases.

The structure given by (4.1) and (4.3) specifies the variance-covariance structure for $x_{s(n,t)}$ and $x^2_{s(n,t)}$. This structure is consistent with the features documented in the previous section: deterministic intraday volatility, serial correlation, fat tails and volatility clustering. With some further technical assumptions, this structure is sufficient to study the impact of documented market micro-structure effects on the precision of sum of squared volatility estimates.
4.2 Daily volatility estimators using intraday observations

In this section we discuss several estimators of $\sigma^2$, the variance of daily returns. The first is the classic estimator based on the sum of squared daily returns. This will be referred to as the daily estimator. Next, we consider two different estimators of the daily variance that use intraday returns data. Both intraday estimators are sum of squared type estimators that have been used in the literature. None of the estimators requires parametric specification of the return process and are therefore considered non-parametric.

First, consider estimating $\sigma^2$ from daily returns. Since for many financial assets the daily returns are essentially uncorrelated, it is common to consider the sum of squares estimator for the daily variance,

$$
\hat{V}_{\text{day}} = \frac{1}{N} \sum_{n=1}^{N} r_n^2
$$

where $N$ is the number of days.

For daily data, if the daily returns are iid normal, this estimator will be efficient. In general, it is unbiased.

Several estimators of low frequency variances that use high-frequency data have been proposed and used in the literature. Perhaps the simplest and most frequently used estimator is the one obtained by summing the squared high frequency returns. This type of estimator has been used in Poterba and Summers (1986), French, Schwert and Stambaugh (1987), and Schwert (1989; 1990). In these studies the ex-post estimate for the variance of a given month was estimated by sampling at daily frequency. Here we focus on estimation of the unconditional variance but this previous research suggests possible estimate that uses intraday returns to consider. More formally we define the simple sum of squares estimator by

$$
\hat{V}_{\text{intra}}^{ss} = \frac{1}{N} \sum_{n=1}^{N} \sum_{t=1}^{k} x_{nt}^2
$$

If the returns are uncorrelated this estimator will be unbiased. Otherwise, this estimator is biased and the magnitude of bias depends on the off-diagonal terms in the variance-covariance matrix $\Omega$ in (2.2). Another simple yet more robust estimator that allows for some serial dependence is given by

$$
\hat{V}_{\text{intra}}^{ssa} = \frac{1}{N} \sum_{n=1}^{N} \left( \sum_{t=1}^{k} x_{nt}^2 + 2 \sum_{t=1}^{k-1} x_{nt} \cdot x_{n,t+1} \right)
$$

13
This estimator is simply $\hat{V}_{\text{intra}}^{\text{ss}}$ plus terms that account for the lag-one autocorrelation, and will be unbiased if at most lag-one autocorrelation exists in the returns. This estimator has been used by French, Schwert and Stambaugh (1987).

It is important to note that for these cited studies, only the daily returns within a given month are used to estimate the volatility of the particular month. That is, $N = 1$. Here we consider using intraday returns across many days to estimate unconditional variance of daily returns.

5 Effects of Kurtosis and Volatility Persistence on the Precision of Variance Estimators

The statistical structure presented in the preceding section is consistent with the established features of the high-frequency data, and thus provides the framework to determine the impacts of various features on the precision of daily variance estimators. In this section, we investigate the impact of the features of the intraday returns data first on the precision of the commonly used daily sum of squares estimator given in (4.4), and then on the intraday sum of squares estimators given in (4.5) and (4.6), using just the moment structure of (4.1) and (4.3). In each case the variance of the estimate is expressed as functions of the intraday data characteristics. When present, the bias of the estimator is also presented. Finally, we discuss some empirical implications of the results.

5.1 The effects of high frequency data characteristics on the sum of squares estimators

To examine the precision of the estimates of $\hat{V}_{\text{day}}$ in (4.4), $\hat{V}_{\text{intra}}^{\text{ss}}$ in (4.5), and $\hat{V}_{\text{intra}}^{\text{ssq}}$ in (4.6) we employ the structure in (4.1) and (4.3) and make the following further assumptions

\( \begin{align*}
(A.1) & \quad E[\epsilon_{ni}^3 \epsilon_{nj}] = 0, \quad E[\epsilon_{ni}^2 \epsilon_{nj} \epsilon_{nk}] = 0, \quad \text{if } i \neq j \neq k \neq l. \\
(A.2) & \quad A_{i,j,l} = O(A), \quad \text{where } A_{i,j,l} = \frac{1}{k} \sum (\delta^i_{j+k} - \bar{\delta}^i)(\delta^j_{l+k} - \bar{\delta}^j), \quad \bar{\delta}^i = \frac{1}{k} \sum \delta^i_j \quad \text{and } A = \frac{1}{k} \sum_{i=1}^{k} (\delta^i - \bar{\delta})^2 \\
(A.3) & \quad \sum_{j=1}^{\infty} \rho_j \text{ is finite.}
\end{align*} \)

Note that together with (4.3) Assumption (A.1) completely specify the fourth moments of $\epsilon_{(n,t)}$. It rules out the leverage effect\(^6\).

\(^6\)The results can be extended to include leverage effects but the derivations and subsequent results are significantly
Next, in Assumption (A.2), $A$ is the squared coefficient of variation for \( \{\delta_1^2, \ldots, \delta_k^2\} \). If the deterministic pattern is absent, i.e., $\delta_1^2 = \delta_2^2 = \cdots = \delta_k^2$, all the terms $A$ and $A_{t,j,t}$ will vanish. This assumption is unnecessary for the result presented below but can greatly simplify the discussion. Finally, Assumption (A.3) simply ensures that the autocorrelations of the squared returns are summable.

Before studying the intraday estimators $\hat{V}_\text{intra}^{ss}$ and $\hat{V}_\text{intra}^{sas}$ in (4.5) and (4.6), respectively, we first consider the estimator $\hat{V}_{\text{day}}$ in (4.4) based on daily returns. This estimator provides a benchmark from which to compare the precision of the intraday estimators. To ensure consistency across theoretical results, the variances of all estimators (including the daily estimator) will be expressed as a function of the high frequency data characteristics. We now present the following proposition concerning the variance of $\hat{V}_{\text{day}}$. The proof is given in the appendix.

**Proposition 5.1** If $x_{s(n,t)}$ satisfies (4.1), (4.3), (A.1), (A.2), and (A.3), $\hat{V}_{\text{day}}$ is an unbiased estimator for $\sigma^2 = \text{var}(r_n)$ with the following variance:

$$\text{var}(\hat{V}_{\text{day}}) = \frac{2(\sigma^2)^2}{N} \left[ 1 + \frac{b_0}{k} + O(N^{-2}, k^{-2}, \theta^2, A^2) \right]$$

where $b_0 = \frac{\gamma_2}{2} + 2(1 + \frac{\gamma_2}{2}) \cdot (\sum_{t=1}^{\infty} \rho_t + 2 \sum_{t=1}^{k} (1 - \frac{1}{k})\rho_t)$.

When the MA(1) coefficient $\theta$, the deterministic intraday volatility pattern $\delta_t$ and the stochastic volatility correlation pattern $\rho_t$ are fixed, the daily return $r_n$ is approximately normal as $k$ gets large by the central limit theorem. Thus, the estimator $\hat{V}_{\text{day}}$ follows approximately a $\chi^2(N)$ distribution with variance $\frac{2(\sigma^2)^2}{N}$. It is interesting to notice that the leptokurtosis measured by $\gamma_2$ and the volatility clustering measured by $\{\rho_t : t = 1, 2, \cdots\}$ affect the asymptotic variance of $\sqrt{N}\hat{V}_{\text{day}}$, that is $N \cdot \text{var}(\hat{V}_{\text{day}})$, at the order of $\frac{1}{k}$ with coefficient $b_0$, while the MA(1) coefficient $\theta$ and the deterministic pattern $\delta_t^2$ measured by $A$ only have effect of order $O(N^{-2}, k^{-2}, \theta^2, A^2)$.

We next examine the simple intraday sum of squares estimator $\hat{V}_\text{intra}^{ss}$ defined by equation (4.5). The following proposition establishes both the bias and the variance as a function of the high frequency data characteristics with the proofs given in the Appendix.

**Proposition 5.2** If $x_{s(n,t)}$ satisfies (4.1), (4.3), (A.1), (A.2), and (A.3), then letting $a_0 = A(1 + \frac{\gamma_2}{2})$, $a_1 = 1 + 2\rho_1(1 + \frac{\gamma_2}{2})$, $a_{11} = (1 + \frac{\gamma_2}{2})(1 + 2 \sum_{t=1}^{\infty} \rho_t)$, we have that

more complicated.
a. \( \text{bias}(\hat{V}_{\text{intra}}^{ss}) = -2\theta \cdot \sigma^2 + O(\theta^2, A^2) \)

b. \( \text{var}(\hat{V}_{\text{intra}}^{ss}) = \frac{2(\sigma^2)^2}{N} \cdot \frac{1}{k} \left[ a_0 + a_{11} + f_1(\theta) + o(N^{-\epsilon}, k^{-\epsilon}, \theta^2, A) \right] \)

where \( \epsilon \) is any positive number, and \( f_1(\theta) = -4a_{11}\theta + (2a_1 + 12a_{11})\theta^2 \)

If \( \theta = 0 \), \( \hat{V}_{\text{intra}}^{ss} \) is unbiased and \( \sqrt{N}\hat{V}_{\text{intra}}^{ss} \) has an asymptotic variance of order \( \frac{1}{k} \). In particular, when the intraday returns are iid normal so that \( A = 0, \theta = 0, \gamma_2 = 0 \) and \( \rho_l = 0, l = 1, 2, ... \), \( \text{var}(\hat{V}_{\text{intra}}^{ss}) = \frac{1}{k} \cdot \text{var}(\hat{V}_{\text{day}}) \). Thus, \( \hat{V}_{\text{intra}}^{ss} \) can dramatically improve the precision in estimating \( \sigma^2 \) in this situation.

In general, however, \( \hat{V}_{\text{intra}}^{ss} \) is biased and inconsistent for \( \theta \neq 0 \). Since the MSE can be expressed as the sum of the squared bias and the variance, the MSE has a lower bound determined by the square magnitude of the bias. Therefore, for large \( N \), \( \hat{V}_{\text{intra}}^{ss} \) would have a larger \( \text{MSE} \) and is less accurate than \( \hat{V}_{\text{day}} \). This suggests, when estimating the aggregate quantities, the use of disaggregated data or high frequency data need not result in more precise estimates than those derived from the use of aggregated data when serial correlation is present in the disaggregated data. Of course, this conclusion hinges on the use of \( \hat{V}_{\text{intra}}^{ss} \) as a choice of the intraday estimates.

In the following, we will concentrate on the adjusted sum of squared estimator \( \hat{V}_{\text{intra}}^{ssa} \), which remains unbiased in the presence of serially correlated returns. The following results for \( \hat{V}_{\text{intra}}^{ssa} \) in (4.6), the sum of squared estimator adjusted for the lag-one autocorrelation, is proved in the Appendix:

**Proposition 5.3** If \( x_{s(n,t)} \) satisfies (4.1), (4.3), (A.1), (A.2), and (A.3), then letting \( a_2 = 1 + 2\rho_2(1 + \frac{\sigma^2}{\theta^2}) \), \( \hat{V}_{\text{intra}}^{ssa} \) is unbiased and it has the following variance
\[
\text{var}(\hat{V}_{\text{intra}}^{ssa}) = \frac{2(\sigma^2)^2}{N} \cdot \frac{1}{k} \left[ a_0 + a_{11} + 2a_1 + f_2(\theta) + o(N^{-\epsilon}, k^{-\epsilon}, \theta^2, A) \right]
\]
where \( \epsilon \) is any positive number, and \( f_2(\theta) = -4a_{11}\theta + (10a_1 + 2\cdot a_2)\theta^2 \)

From the results in propositions 5.1, 5.2, and 5.3 we can make the following observations:

1. If the intraday returns are iid normal, i.e., \( A = 0, \theta = 0, \gamma_2 = 0 \), and \( \rho_l = 0 \) for \( l = 1, 2, \ldots \), then \( a_0 = 0, a_1 = a_2 = a_{11} = 1 \), and \( \text{var}(\hat{V}_{\text{intra}}^{ssa}) = \frac{2(\sigma^2)^2}{N} (1 + 2) \). The ratio \( \frac{\text{var}(\hat{V}_{\text{intra}}^{ssa})}{\text{var}(\hat{V}_{\text{intra}}^{ss})} \) thus quantifies the efficiency loss of the adjusted sum of squares estimator \( \hat{V}_{\text{intra}}^{ssa} \) induced by departures from iid normality, where the departures include the autocorrelations in the returns \( \theta \) and their squares \( \rho_l, l = 1, 2, \ldots, \infty \), the intraday deterministic pattern A, and the leptokurtosis \( \gamma_2 \).
2. We notice that the asymptotic variance of \( \sqrt{N} \hat{\boldsymbol{v}}^{ssa}_{\text{intra}} \) is of order \( \frac{1}{k} \). For large \( k \), \( N \operatorname{var}(\hat{\boldsymbol{v}}^{ssa}_{\text{intra}}) \) can be substantially smaller than the asymptotic variance \( N \operatorname{var}(\hat{\boldsymbol{v}}_{\text{day}}) \). Therefore, \( \hat{\boldsymbol{v}}^{ssa}_{\text{intra}} \) can potentially provide more accurate estimation for \( \sigma^2 \).

We now examine the actual gain of \( \hat{\boldsymbol{v}}^{ssa}_{\text{intra}} \) over \( \hat{\boldsymbol{v}}_{\text{day}} \) in terms of \( \operatorname{var}(\hat{\boldsymbol{v}}^{ssa}_{\text{intra}}) \) and \( \operatorname{var}(\hat{\boldsymbol{v}}_{\text{day}}) \). For simplicity, we focus our discussion on terms not including \( \frac{1}{k} \cdot o(N^{-(1+\epsilon)}, k^{-(1+\epsilon)}, \theta^2, A) \).

First, consider the case where \( \theta = 0 \), that is, when the intraday returns follow covariance stationary white noise. \( \operatorname{Var}(\hat{\boldsymbol{v}}^{ssa}_{\text{intra}}) \) is then given by \( a_0 + a_{11} + 2a_1 \) multiplied by the factor \( \frac{2(a^2)^2}{NK} \).

In the first term \( a_0 \) the factor \( 1 + \frac{\gamma_2}{2} \) quantifies the efficiency loss due to leptokurtosis of the intraday returns. This is potentially important since the sample kurtosis of many high-frequency series exceeds 10, which would result in an efficiency loss by a factor of at least 6. This factor is multiplied by \( A \), which is the squared coefficient of variation of the \( \delta^2 \)'s and quantifies the efficiency loss due to the deterministic intraday volatility pattern. The more heterogeneous the intraday returns are, the larger the value \( A \) is and the greater is the efficiency loss. Thus, \( a_0 \) represents the nonlinear interactive effect of deterministic volatility and leptokurtosis. Similarly, the next term \( a_{11} \) measures the interactive effect between leptokurtosis \( (1 + \frac{\gamma_2}{2}) \) and dynamic volatility \( (1 + \sum_{t=1}^{\infty} \rho_t) \) where \( \sum_{t=1}^{\infty} \rho_t \) is the sum of all the autocorrelations in the squared innovations quantifying the efficiency loss induced by stochastically varying intraday volatility. The stronger the volatility persistence is, the greater \( a_{11} \) is and the less information is contained in the high-frequency data. With strong volatility clustering typically observed in high-frequency data, this term also greatly reduces the benefits of using intraday returns to estimate daily volatility. Overall \( a_0 + a_{11} \) captures the nonlinear effects of leptokurtosis and the changing, deterministic or stochastic, intraday volatility.

It should be noted that under assumptions (4.1), (4.3), and (A.1), the kurtosis \( \gamma_2 \) and the autocorrelations \( (\rho_t, l = 1, 2, \ldots) \) completely characterize the fourth moments of the intraday returns data. In practice, \( \gamma_2 \) is commonly used as a measure of the tail behavior of the data or ”departures from normality” and the \( \rho_t \)'s as measures of the volatility persistence. However, these two types of measures are themselves related since existence of the autocorrelations \( \rho_t \)'s would affect the kurtosis measure \( \gamma_2 \). See Bai, Russell and Tiao (2003) for the GARCH and stochastic volatility models.

The third term \( 2a_1 \) in the variance of \( \hat{\boldsymbol{v}}^{ssa}_{\text{intra}} \) is always positive when \( \gamma_2 > -2 \) and \( \rho_1 > 0 \), as commonly observed in financial time series. Comparing Propositions 5.2 and 5.3, we find that when
\( \theta = 0, \) \( var(\hat{V}_{intra}^{ss}) \) and \( var(\hat{V}_{intra}^{ss}) \) are the same except for this term. This implies that if there is no serial correlation present in the intraday data, \( \hat{V}_{intra}^{ssa} \) is less efficient than \( \hat{V}_{intra}^{ss} \). This term can therefore be interpreted as being induced by estimating the lag-one autocorrelation \( \theta \). Regardless of the features of the intraday returns, the value 2 quantifies the fixed effect of estimating \( \theta \). Hence even if the returns are sampled from a geometric Brownian motion, the benefit of using intraday returns is effectively reduced by a factor of 3 for the estimator \( \hat{V}_{intra}^{ssa} \). In addition, the cost for estimating \( \theta \) is also affected by \( \rho_1 \) and the kurtosis \( \gamma_2 \).

3, we next consider the impact of the autocorrelation of the intraday returns as represented by \( f_2(\theta) \) in \( var(\hat{V}_{intra}^{ss}) \). The parameter \( \theta \), representing the lag-one correlation of intraday returns, measures the deviation of intraday returns from covariance stationary white noise. Notice that the coefficient of \( \theta \) in \( f_2(\theta) \) is \(-4a_1\) which is typically negative. Thus, a negative autocorrelation \( \theta \) results in greater efficiency loss while positive autocorrelation results in less efficiency loss. The second order term, \((10a_1 + 2a_2)\theta^2\) is always positive for non-negative \( \rho_1 \) and \( \rho_2 \) provided \( \gamma_2 > -2 \), thus reducing the efficiency.

5.2 Numerical Comparison of var(\( \hat{V}_{day} \)), var( \( \hat{V}_{intra}^{ss} \)) and var( \( \hat{V}_{intra}^{ssa} \))

Propositions 5.1, 5.2 and 5.3 can be used to study the benefit of using intraday returns to estimate daily volatility when the intraday returns are not iid normal. In the following, we will further illustrate the impacts of autocorrelation and fat-tails using numerical examples.

Table 3 presents the three variances for \( N = 1552, k = 96 \), and when \( A = 0 \) and \( \{ \rho_1 = \rho_2 = \cdots = \sum_{t=1}^{\infty} \rho_t = 0 \} \), hence we focus attention here on the impact of autocorrelation and kurtosis of the intraday return series. Specifically, we consider using ten years of 15-minute exchange returns to estimate the daily volatility when the deterministic pattern and volatility clustering are absent from the intraday return series. To make the numbers in the table comparable, the daily variance \( \sigma^2 \) is normalized to 1.

The \( VARs \) in each column are computed based on the same kurtosis but different values of the autocorrelation \( \theta \). The kurtosis is given in the top row. Reading down the column with \( \gamma_2 = 0 \), for example, corresponds to the case when the intraday returns are normally distributed with each row in that column corresponding to different first order autocorrelations. The last column of the table provides the bias for each value of \( \theta \). The MSE, if desired, is then obtained by summing the
variance terms and the corresponding squared bias term. The following observations can be made along the column direction, and they conform to the discussions in Section 5.1:

- The bias term for $\hat{V}_{\text{intra}}^{ss}$ is given by $-2\theta$ and is large and dominates the variance terms for each of the three estimators whenever $\theta \neq 0$; and this suggests that when estimating aggregated quantities, the bias introduced by neglected autocorrelation can be substantial.

- The variance of $\hat{V}_{\text{day}}$ doesn’t change with $\theta$. This is because we fixed $\sigma^2 = 1$ and $\theta$ affects $\text{var}(\hat{V}_{\text{day}})$ at the order of $O(N^{-2}, k^{-2}, \theta^2, A^2)$.

- $\text{var}(\hat{V}_{\text{intra}}^{ssa})$ and $\text{var}(\hat{V}_{\text{intra}}^{ss})$ decrease as $\theta$ increases.

The impact of kurtosis can be analyzed similarly. Along the row direction, the variance of the estimators correspond to the same lag-one autocorrelation but different kurtosis. We have the following observations:

- Bias only varies with $\theta$.
- All three variances increase with $\gamma_2$. In other words, the estimations become less efficient the greater the kurtosis.
- For $\theta < 0 \text{ var}(\hat{V}_{\text{intra}}^{ss})$ increases faster with $\gamma_2$ than $\text{var}(\hat{V}_{\text{day}})$ and $\text{var}(\hat{V}_{\text{intra}}^{ssa})$, but the situation is reversed for $\theta > 0$
- $\text{var}(\hat{V}_{\text{day}})$ and $\text{var}(\hat{V}_{\text{intra}}^{ssa})$ increase by the same amount with $\gamma_2$ but proportionally $\text{var}(\hat{V}_{\text{intra}}^{ssa})$ increases much faster than $\text{var}(\hat{V}_{\text{day}})$

Based on the above observations from Table 3, we conclude that the benefit of using intraday returns to estimate daily volatility decreases when lag-one autocorrelation becomes negative and/or the leptokurtosis becomes stronger. When $\theta \neq 0$ and $N$ is large, the bias of $\hat{V}_{\text{intra}}^{ss}$ is much bigger than the reduction in $\text{var}(\hat{V}_{\text{intra}}^{ss})$ compared to $\text{var}(\hat{V}_{\text{day}})$, thus $\hat{V}_{\text{intra}}^{ss}$ will not improve the estimation accuracy over $\hat{V}_{\text{day}}$.

Also, we notice that the smallest ratio of the $\text{var}(\hat{V}_{\text{day}})/\text{var}(\hat{V}_{\text{intra}}^{ssa})$ in Table 3 is 10.3 when $\theta = -0.3$ and $\gamma_2 = 9$. Although the ratio of 10.3 is far less than the ideal ratio $\text{var}(\hat{V}_{\text{day}})/\text{var}(\hat{V}_{\text{intra}}^{ssa})=96$ for $\theta = 0$ and $\gamma_2 = 0$, $\hat{V}_{\text{intra}}^{ssa}$ can still substantially improve the estimation efficiency over $\hat{V}_{\text{day}}$ for plausible values of $\theta$ and $\gamma_2$.

Now, we turn to investigate the effects of volatility clustering in addition to the lag-one autocorrelation and leptokurtic property. As in table Table 3, we fix $\theta = -0.3, -0.2, -0.1, 0.0, 0.1$
Table 3: Impacts of Autocorrelation and Kurtosis on Daily Variance Estimates

<table>
<thead>
<tr>
<th>Estimator</th>
<th>$\theta \backslash \gamma_2$</th>
<th>0</th>
<th>3</th>
<th>6</th>
<th>9</th>
<th>Bias</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{V}_{day}$</td>
<td>-0.3</td>
<td>12.89 $\cdot 10^{-4}$</td>
<td>13.09 $\cdot 10^{-4}$</td>
<td>13.29 $\cdot 10^{-4}$</td>
<td>13.49 $\cdot 10^{-4}$</td>
<td>0.0</td>
</tr>
<tr>
<td>$\hat{V}_{ss, intra}$</td>
<td>0.4645 $\cdot 10^{-4}$</td>
<td>1.125 $\cdot 10^{-4}$</td>
<td>1.785 $\cdot 10^{-4}$</td>
<td>2.446 $\cdot 10^{-4}$</td>
<td>1.169</td>
<td></td>
</tr>
<tr>
<td>$\hat{V}_{ssa, intra}$</td>
<td>0.7088 $\cdot 10^{-4}$</td>
<td>0.910 $\cdot 10^{-4}$</td>
<td>1.111 $\cdot 10^{-4}$</td>
<td>1.313 $\cdot 10^{-4}$</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>$\hat{V}_{day}$</td>
<td>-0.2</td>
<td>12.89 $\cdot 10^{-4}$</td>
<td>13.09 $\cdot 10^{-4}$</td>
<td>13.29 $\cdot 10^{-4}$</td>
<td>13.49 $\cdot 10^{-4}$</td>
<td>0.0</td>
</tr>
<tr>
<td>$\hat{V}_{ss, intra}$</td>
<td>0.3168 $\cdot 10^{-4}$</td>
<td>0.7759 $\cdot 10^{-4}$</td>
<td>1.235 $\cdot 10^{-4}$</td>
<td>1.694 $\cdot 10^{-4}$</td>
<td>0.604</td>
<td></td>
</tr>
<tr>
<td>$\hat{V}_{ssa, intra}$</td>
<td>0.5745 $\cdot 10^{-4}$</td>
<td>0.7759 $\cdot 10^{-4}$</td>
<td>0.9772 $\cdot 10^{-4}$</td>
<td>1.179 $\cdot 10^{-4}$</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>$\hat{V}_{day}$</td>
<td>0.0</td>
<td>12.89 $\cdot 10^{-4}$</td>
<td>13.09 $\cdot 10^{-4}$</td>
<td>13.29 $\cdot 10^{-4}$</td>
<td>13.49 $\cdot 10^{-4}$</td>
<td>0.0</td>
</tr>
<tr>
<td>$\hat{V}_{ss, intra}$</td>
<td>0.1342 $\cdot 10^{-4}$</td>
<td>0.3356 $\cdot 10^{-4}$</td>
<td>0.5369 $\cdot 10^{-4}$</td>
<td>0.7383 $\cdot 10^{-4}$</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>$\hat{V}_{ssa, intra}$</td>
<td>0.4027 $\cdot 10^{-4}$</td>
<td>0.6041 $\cdot 10^{-4}$</td>
<td>0.8054 $\cdot 10^{-4}$</td>
<td>1.007 $\cdot 10^{-4}$</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>$\hat{V}_{day}$</td>
<td>0.1</td>
<td>12.89 $\cdot 10^{-4}$</td>
<td>13.09 $\cdot 10^{-4}$</td>
<td>13.29 $\cdot 10^{-4}$</td>
<td>13.49 $\cdot 10^{-4}$</td>
<td>0.0</td>
</tr>
<tr>
<td>$\hat{V}_{ss, intra}$</td>
<td>0.0993 $\cdot 10^{-4}$</td>
<td>0.2443 $\cdot 10^{-4}$</td>
<td>0.3893 $\cdot 10^{-4}$</td>
<td>0.5343 $\cdot 10^{-4}$</td>
<td>-0.162</td>
<td></td>
</tr>
<tr>
<td>$\hat{V}_{ssa, intra}$</td>
<td>0.3651 $\cdot 10^{-4}$</td>
<td>0.5665 $\cdot 10^{-4}$</td>
<td>0.7678 $\cdot 10^{-4}$</td>
<td>0.9692 $\cdot 10^{-4}$</td>
<td>0.0</td>
<td></td>
</tr>
</tbody>
</table>

and $\gamma_2 = 0, 3, 6, 9$ in Table 4. Table 4 relaxes the iid innovation assumption and allows for volatility clustering. Hence $\sum_{t=1}^{\infty} \rho_t$ are not restricted to be zero as in Table 3. Instead, $\sum_{t=1}^{\infty} \rho_t = 26.05$ is derived from the fitted GARCH(1, 1) model for the exchange rate series Deutschmark/Dollar. For detail, see Appendix 2.

For all values of the kurtosis $\gamma_2$ and autocorrelation $\theta$ the variance of the estimates in Table 4 are larger than their counterparts in Table 3. This is particularly true for $\text{var}(\hat{V}_{ss, intra}^{\text{intra}})$ and $\text{var}(\hat{V}_{ssa, intra}^{\text{intra}})$. Both variances are increased by at least a factor of two and become comparable with $\text{var}(\hat{V}_{day})$.

The differences between the two tables represents the impact of volatility clustering on estimating daily volatility. It is apparent that the presence of volatility clustering in the intraday data substantially reduces the benefit of using intraday data to estimate daily volatility. Clearly, the more persistent the volatility is, the less information is gained by sampling more frequently.

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Table 4: Impacts of Autocorrelation and Kurtosis on VARs and bias (continued)

<table>
<thead>
<tr>
<th>Estimator</th>
<th>( \theta \setminus \gamma_2 )</th>
<th>0</th>
<th>3</th>
<th>6</th>
<th>9</th>
<th>Bias</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{V}_{day} )</td>
<td>-0.3</td>
<td>2.709 \cdot 10^{-3}</td>
<td>4.860 \cdot 10^{-3}</td>
<td>7.010 \cdot 10^{-3}</td>
<td>9.161 \cdot 10^{-3}</td>
<td>0.0</td>
</tr>
<tr>
<td>( \hat{V}_{ss}^{ina} )</td>
<td>-0.3</td>
<td>2.343 \cdot 10^{-3}</td>
<td>5.853 \cdot 10^{-3}</td>
<td>9.364 \cdot 10^{-3}</td>
<td>12.870 \cdot 10^{-3}</td>
<td>1.196</td>
</tr>
<tr>
<td>( \hat{V}_{ss}^{ina} )</td>
<td>0.823 \cdot 10^{-3}</td>
<td>1.971 \cdot 10^{-3}</td>
<td>3.119 \cdot 10^{-3}</td>
<td>4.267 \cdot 10^{-3}</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>( \hat{V}_{day} )</td>
<td>-0.2</td>
<td>2.709 \cdot 10^{-3}</td>
<td>4.860 \cdot 10^{-3}</td>
<td>7.010 \cdot 10^{-3}</td>
<td>9.161 \cdot 10^{-3}</td>
<td>0.0</td>
</tr>
<tr>
<td>( \hat{V}_{ss}^{ina} )</td>
<td>-0.2</td>
<td>1.627 \cdot 10^{-3}</td>
<td>4.067 \cdot 10^{-3}</td>
<td>6.506 \cdot 10^{-3}</td>
<td>8.946 \cdot 10^{-3}</td>
<td>0.615</td>
</tr>
<tr>
<td>( \hat{V}_{ss}^{ina} )</td>
<td>0.797 \cdot 10^{-3}</td>
<td>1.927 \cdot 10^{-3}</td>
<td>3.057 \cdot 10^{-3}</td>
<td>4.186 \cdot 10^{-3}</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>( \hat{V}_{day} )</td>
<td>-0.1</td>
<td>2.709 \cdot 10^{-3}</td>
<td>4.860 \cdot 10^{-3}</td>
<td>7.010 \cdot 10^{-3}</td>
<td>9.161 \cdot 10^{-3}</td>
<td>0.0</td>
</tr>
<tr>
<td>( \hat{V}_{ss}^{ina} )</td>
<td>-0.1</td>
<td>1.084 \cdot 10^{-3}</td>
<td>2.710 \cdot 10^{-3}</td>
<td>4.335 \cdot 10^{-3}</td>
<td>5.961 \cdot 10^{-3}</td>
<td>0.247</td>
</tr>
<tr>
<td>( \hat{V}_{ss}^{ina} )</td>
<td>0.778 \cdot 10^{-3}</td>
<td>1.893 \cdot 10^{-3}</td>
<td>3.009 \cdot 10^{-3}</td>
<td>4.125 \cdot 10^{-3}</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>( \hat{V}_{day} )</td>
<td>0.0</td>
<td>2.709 \cdot 10^{-3}</td>
<td>4.860 \cdot 10^{-3}</td>
<td>7.010 \cdot 10^{-3}</td>
<td>9.161 \cdot 10^{-3}</td>
<td>0.0</td>
</tr>
<tr>
<td>( \hat{V}_{ss}^{ina} )</td>
<td>0.0</td>
<td>0.713 \cdot 10^{-3}</td>
<td>1.782 \cdot 10^{-3}</td>
<td>2.851 \cdot 10^{-3}</td>
<td>3.921 \cdot 10^{-3}</td>
<td>0.0</td>
</tr>
<tr>
<td>( \hat{V}_{ss}^{ina} )</td>
<td>0.764 \cdot 10^{-3}</td>
<td>1.870 \cdot 10^{-3}</td>
<td>2.977 \cdot 10^{-3}</td>
<td>4.083 \cdot 10^{-3}</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>( \hat{V}_{day} )</td>
<td>0.1</td>
<td>2.709 \cdot 10^{-3}</td>
<td>4.860 \cdot 10^{-3}</td>
<td>7.010 \cdot 10^{-3}</td>
<td>9.161 \cdot 10^{-3}</td>
<td>0.0</td>
</tr>
<tr>
<td>( \hat{V}_{ss}^{ina} )</td>
<td>0.1</td>
<td>0.514 \cdot 10^{-3}</td>
<td>1.284 \cdot 10^{-3}</td>
<td>2.054 \cdot 10^{-3}</td>
<td>2.824 \cdot 10^{-3}</td>
<td>-0.164</td>
</tr>
<tr>
<td>( \hat{V}_{ss}^{ina} )</td>
<td>0.757 \cdot 10^{-3}</td>
<td>1.858 \cdot 10^{-3}</td>
<td>2.959 \cdot 10^{-3}</td>
<td>4.060 \cdot 10^{-3}</td>
<td>0.0</td>
<td></td>
</tr>
</tbody>
</table>

Results presented in this table are based on GARCH(1, 1) estimates \( \alpha_1 = 0.1443 \) and \( \beta_1 = 0.8381 \) for DM exchange rate series. Consequently, \( \rho_1 = 0.4579, \rho_2 = 0.4498, \cdots \), and \( \sum_{i=1}^{\infty} \rho_i = 26.05 \). The general pattern of impact of the lag-one autocorrelation and kurtosis in Table 4 is similar to the pattern illustrated in Table 3. The differences in the two tables stem from the relatively large value of \( \sum_{i=1}^{\infty} \rho_i \). Table 4 reveals the compounded effect between kurtosis and volatility clustering, which is multiplicative, while Table 3 gives the effect of kurtosis alone.

5.3 Empirical applications to the exchange rate series

Propositions 5.1, 5.2 and 5.3 can be used not only to study the overall benefit of using intraday returns, but also to assess the relative importance of different features on the daily volatility estimators. In order to better understand the impact of each individual feature, we apply the results
in these three propositions to the two 15-minute adjusted exchange rate returns series analyzed in Section 3.

Table 5 presents the sample values of $\sigma^2$, $k$, $N$, $\theta$, $A$, $\gamma_2$, $\rho_1$, $\rho_2$ and $\sum_{t=1}^{\infty} \rho_t$ for both series. These values are inputs to the precision of the variance estimates given in Propositions 5.1 - 5.3 and are used to construct Table 5. To calculate $\sum_{t=1}^{\infty} \rho_t$, we fit both series by GARCH(1, 1) and derive the $\sum_{t=1}^{\infty} \rho_t$ from the estimated parameter. This approach avoids choosing arbitrary lags in calculating $\sum_{i=1}^{\infty} \rho_t$. The estimation results are presented in Table 8 in Appendix 2.

The first row of Tables 6 and 7 gives the variance of the estimator $\hat{V}_{\text{day}}$ and can be treated as a benchmark in assessing the benefit of using intraday data. The first column in both tables gives the bias for the three estimators. Multiplied by the variance 0.42 or 0.43, the bias of 0.05 can significantly reduce the accuracy of $\hat{V}_{\text{intra}}^s$ further. As will be discussed below, $\hat{V}_{\text{intra}}^s$ not only is biased, it also tends to have larger variance than $\hat{V}_{\text{intra}}^s$.

The columns “const”, “$\theta$ term” and “$\theta^2$ term” present the corresponding three terms in Propositions 5.2 and 5.3. Here, “const” refers to the terms in the propositions that do not involve on $\theta$. Thus, apart from the multiplying factor $\frac{2(a_0 + a_{11})}{N}$, $\text{const} = \frac{1}{k}(a_0 + a_{11})$ for $\text{var}(\hat{V}_{\text{intra}}^s)$ and $\text{const} = \frac{1}{k}(a_0 + a_{11} + 2a_1)$ for $\text{var}(\hat{V}_{\text{intra}}^{ssa})$. Likewise, $\theta$ term = $\frac{1}{k} \cdot [-4a_{11}\theta]$ for $\text{var}(\hat{V}_{\text{intra}}^s)$ and $\theta$ term = $\frac{1}{k} \cdot [-4a_1\theta]$ for $\text{var}(\hat{V}_{\text{intra}}^{ssa})$, and similar expression can be obtained for coefficients for $\theta^2$. Consistent with the results of Section 5.1, the constant term of $\text{var}(\hat{V}_{\text{intra}}^{ssa})$ is greater than the corresponding value of $\text{var}(\hat{V}_{\text{intra}}^s)$ in both Tables 6 and 7. On the other hand, the first order and the second order approximations of $\theta$, “$\theta$ term” and “$\theta^2$ term”, of $\hat{V}_{\text{intra}}^{ssa}$ are much smaller than that of $\hat{V}_{\text{intra}}^s$.

The next two columns, “$\text{var}^{(a)}$” and $\text{var}$, present the variances of the three estimators. Here, “$\text{var}^{(a)}$” are the approximations given in Propositions 5.1 - 5.3 and “$\text{var}$” give the exact results by the calculation given in Appendix 1. We notice that the approximation and the exact values are close for all three estimator. The last columns “normalized std” present the standard deviations of three estimators divided by the daily variance $\sigma^2$.

If the intraday returns are iid normal, $\hat{V}_{\text{day}}$, $\hat{V}_{\text{intra}}^s$ and $\hat{V}_{\text{intra}}^{ssa}$ would have normalized standard deviations of $\sqrt{\frac{2}{1552}} = 0.0359$, $\sqrt{\frac{2}{1552-96}} = 0.0037$, and $\sqrt{\frac{2\cdot3}{1552-96}} = 0.0064$. However, the actual corresponding standard deviations are 0.1205, 0.0917 and 0.0781 for Dollar / Deutsch Mark. The series Dollar / Japanese Yen has similar results. Clearly, departures from iid normality in the
### Table 5: Essential Parameters Used in Propositions 5.1 - 5.3

<table>
<thead>
<tr>
<th>Series</th>
<th>$\sigma^2$</th>
<th>$k$</th>
<th>$N$</th>
<th>$\theta$</th>
<th>$A$</th>
<th>$\gamma_2$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$\sum_{t=1}^\infty \rho_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DD</td>
<td>0.4236</td>
<td>96</td>
<td>1552</td>
<td>-0.05654</td>
<td>0.4592</td>
<td>16.49</td>
<td>0.4579</td>
<td>0.4498</td>
<td>26.05</td>
</tr>
<tr>
<td>DY</td>
<td>0.4328</td>
<td>96</td>
<td>1552</td>
<td>-0.05549</td>
<td>0.1828</td>
<td>16.83</td>
<td>0.5013</td>
<td>0.4960</td>
<td>47.07</td>
</tr>
</tbody>
</table>

### Table 6: $\text{var} (\hat{V}_{\text{day}})$, $\text{var} (\hat{V}_{\text{intra}}^{ss})$ and $\text{var} (\hat{V}_{\text{intra}}^{ssa})$ for Dollar / Deutsch Mark

<table>
<thead>
<tr>
<th>Series</th>
<th>bias</th>
<th>const</th>
<th>$\theta$ term</th>
<th>$\theta^2$ term</th>
<th>$\text{var}(a)$</th>
<th>$\text{var}$</th>
<th>normalize std</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{V}_{\text{day}}$</td>
<td>0</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>0.002607</td>
<td>0.002757</td>
<td>0.1205</td>
</tr>
<tr>
<td>$\hat{V}_{\text{intra}}^{ss}$</td>
<td>0.05275</td>
<td>0.001193</td>
<td>0.000270</td>
<td>0.000046</td>
<td>0.001508</td>
<td>0.001531</td>
<td>0.0917</td>
</tr>
<tr>
<td>$\hat{V}_{\text{intra}}^{ssa}$</td>
<td>0</td>
<td>0.001238</td>
<td>0.000007</td>
<td>0.000001</td>
<td>0.001247</td>
<td>0.001281</td>
<td>0.0781</td>
</tr>
</tbody>
</table>

### Table 7: $\text{var} (\hat{V}_{\text{day}})$, $\text{var} (\hat{V}_{\text{intra}}^{ss})$ and $\text{var} (\hat{V}_{\text{intra}}^{ssa})$ for Dollar / Japanese Yen

<table>
<thead>
<tr>
<th>Series</th>
<th>bias</th>
<th>const</th>
<th>$\theta$ term</th>
<th>$\theta^2$ term</th>
<th>$\text{var}(a)$</th>
<th>$\text{var}$</th>
<th>normalize std</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{V}_{\text{day}}$</td>
<td>0</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>0.004148</td>
<td>0.004185</td>
<td>0.1488</td>
</tr>
<tr>
<td>$\hat{V}_{\text{intra}}^{ss}$</td>
<td>0.05254</td>
<td>0.002256</td>
<td>0.000501</td>
<td>0.000083</td>
<td>0.002840</td>
<td>0.002838</td>
<td>0.1231</td>
</tr>
<tr>
<td>$\hat{V}_{\text{intra}}^{ssa}$</td>
<td>0</td>
<td>0.002309</td>
<td>0.000007</td>
<td>0.000001</td>
<td>0.002317</td>
<td>0.002322</td>
<td>0.1112</td>
</tr>
</tbody>
</table>
intraday return significantly reduce the estimation accuracy for all three estimators. More strikingly, the benefit of using intraday data to estimate daily variance is markedly reduced with these departures. When lag-one autocorrelation presents in the series, \( \hat{V}_{intra}^{ss} \) not only is biased, it also has the similar variance as \( \hat{V}_{day} \). These make \( \hat{V}_{intra}^{ss} \) the least desirable estimator among the three. On the other hand, \( \hat{V}_{intra}^{ss} \) can still improve the accuracy over \( \hat{V}_{day} \) though not nearly as substantial as in the iid normal case.

### 5.4 Optimal Sampling Frequency

Section 5.3 examined the precision of variance estimates constructed from 15-minute intraday returns data. This section examines the precision of variance estimates for the dollar/deutchmark and dollar/yen exchange rate series using a sampling intervals ranging from 5 minutes up through half day. In doing so, we summarize how the precision of variance estimates varies with the sampling frequency.

We consider 10 different intraday sampling intervals as well as daily interval, specifically, intraday sampling intervals include 5, 10, \( \cdots \), and 60 minutes, as well as 2, 4, 6, and 12 house. For each of the two intraday foreign exchange rate series we construct the sequence of returns for each of the 10 intraday sampling intervals. Each of the twenty intraday series is then individually deseasonalized using the methods of section 3. The MA coefficients are estimated for each of the deseasonalized series. We again estimate a GARCH(1,1) model for the residuals. The GARCH parameters are used to estimate the sum of the autocorrelations of the squared residuals. Finally, the sample kurtosis is calculated for each series. The parameter estimates, estimated sum of autocorrelations, and the kurtosis are reported in the Tables 9 and 10 in Appendix 3.

For each sampling frequency the statistics are used to evaluate the MSE for \( \hat{V}_{intra}^{ss} \) given in Proposition 5.3. The MSE’s for the dollar/deutchmark are plotted in figure (4.a). The MSE at the one-day sampling frequency is normalized to unity so that the height of the plot can be interpreted relative efficiency of any sampling frequency to that obtained by sampling at the daily frequency. The minimum MSE occurs at 10 minutes for the dollar/deutchmark and the MSE is monotonically increasing for larger or shorter sampling intervals. Intuitively, initially, as the sampling interval is shortened we are using more information and therefore obtaining more accurate estimates. Eventually, however, the market microstructure noise swamps the estimate increasing

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the bias and causing the MSE to rise once the sampling interval is inside 10 minutes. As shown in Table 9, both $\gamma_2$ and $\sum_{t=1}^{\infty} \rho_t$ decrease as sampling intervals increase, and they can at least partially compensate estimation precision lost by few observations. At 10 minutes sampling frequency, at which the minimum achieves, the MSE is about 44% of that obtained from sampling at the daily frequency.

For the dollar/yen we find a similar U-shaped pattern in the relative MSE plot. The trough, however, occurs at a slightly longer sampling interval of 20 minutes. The minimum ratio for this series is about 53%. Hence sampling at the optimal interval provides a benefit in terms of mean squared error of between 44% and 53% for these two series.

5.5 Generalization

A substantial body of research has been developed over the last several years attempting to identify the volatility over a fixed interval of time using very high frequency sampled data. Specifically, if asset prices are generated by a semi-martingale process the daily quadratic variation can be approximated by the sum of intraday squared returns. This literature is summarized in Andersen et al. (2002). See, for example, Andersen et al. (2002) and Barndorff-Neilson and Sheppard (2002) for the asymptotic theory associated with non-parametric inference of the underlying quadratic variation. In this paper we focus on the MSE of the unconditional variance. Nevertheless, our work has immediate implications for inference about the quadratic variation.

We now generalize the analysis of sections 5.1 to allow for different values of $\delta_1, ..., \delta_k$ across different days. We denote these values by $\delta_{n1}, ..., \delta_{nk}$ and now allow for the possibility that these values contain both random and deterministic components. If we make the additional assumption that $\varepsilon^2_{nt}$ is now serially uncorrelated then any variation in the volatility of $\varepsilon_{s(n,t)}$ is captured by variation in $\delta_{n1}, ..., \delta_{nk}$. Equation (4.1) is therefore generalized to

$$\frac{x_{s(n,t)}}{\delta_{tn}} = \varepsilon_{s(n,t)} + \theta \varepsilon_{s(n,t)-1}$$

(5.1)

The variance-covariance matrix associated with day $n$ is given by

Add equation number

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Figure 4: Optimal Sampling Frequency
Given the independece assumption (A.4), the intraday volatility \( \delta_{1n}, \ldots, \delta_n \) has no impact on the distribution of the intraday distribution of \( \varepsilon_{1n} \). We can, therefore, naturally define the day \( n \) volatility conditional on \( \delta_{11}, \ldots, \delta_n \) as

\[
(\text{A.5}) \quad \bar{\sigma}_n^2 = \text{uncorrelated which clearly implies } \sum_{k=1}^\infty \rho_k = 0.
\]

Finally, we assume that the realization of \( \delta_{1n}, \ldots, \delta_n \) on day \( n \) satisfies the following assumption

\[
(\text{A.6}) \quad A_{n, y} y = O(A_n), \text{ where } A_n = \mathbb{E}(\delta_{1n} \ldots \delta_n) = \mathbb{E}(\sum_{k=1}^\infty \delta_k).
\]

Let \( \bar{\sigma}_{n, y}^2 = \sum_{k=1}^n \bar{\sigma}_{kn} = \sum_{k=1}^n \varepsilon_k^2 = \bar{\sigma}^2 + 2 \bar{\sigma} \sum_{k=1}^{n-1} \delta_{1k} \delta_{k+1} \). That is, \( \bar{\sigma}_{n, y}^2 \) is referred to as realized volatility as studied by Andersen et al. (2002) and can be viewed as estimates of \( \sigma_n^2 \). Viewing \( \bar{\sigma}_{n, y}^2 \) and \( \bar{\sigma}_{n, y}^2 \) as

\[
\sum_{k=1}^n \bar{\sigma}_{kn}^2 = \sum_{k=1}^n \bar{\sigma}_{kn}^2 = \sum_{k=1}^\infty \bar{\sigma}_{kn}^2 \approx \sum_{k=1}^\infty \sigma_k^2
\]

by \( \sigma^2 \). The MSE of \( \bar{\sigma}_{n, y}^2 \) and \( \bar{\sigma}_{n, y}^2 \) are then directly obtained from proposition 5.2 and 5.3 under the condition that \( \sum_{k=1}^\infty \rho_k = 0 \) and \( A = \bar{A}_n = \frac{1}{n} \sum_{k=1}^n \delta_k^2 \). These results are summarized in the following corollaries.

Since we assume any variation in the volatility of \( \varepsilon \) is captured by variation in \( \delta_{1n}, \ldots, \delta_n \), it is usual to refer to \( \delta_{1n}, \ldots, \delta_n \) as the intraday volatility of day \( n \). We now summarize the
Corollary 1: Given \( \Omega_n \), (5.1), (A.1), (A.4), (A.5), and (A.6) the (conditional) bias and variance of \( \hat{V}^s_n \) are given by proposition 5.2 with \( N = 1 \), \( a_0 = A_n \left( 1 + \frac{3\epsilon}{2} \right) \), \( a_1 = 1 \), \( a_{11} = \left( 1 + \frac{3\epsilon}{2} \right) \), and \( A = A_n = \frac{1}{n} \sum_{t=1}^{n} (\delta^2_{t+1} - \theta^2)^2 \). The bias and variance are:

\[
\text{bias} \left( \hat{V}^s_n \right) = -2\theta \sigma^2 + O \left( \theta^2, A_n \right)
\]

\[
\text{var} \left( \hat{V}^s_n \right) = 2(\sigma^2_n)^2 \cdot \frac{1}{n} \left[ a_0 + a_{11} + f_1(\theta) + o(k^{-1+\epsilon}, \theta^2, A_n) \right]
\]

where \( \epsilon \) is any positive number and \( f_1(\theta) = -4a_{11}\theta + (2a_1 + 12a_{11})\theta^2 \)

Corollary 2: Given \( \Omega_n \), (5.1), (A.1), (A.4), (A.5), and (A.6) the (conditional) variance of \( \hat{V}^{ssa}_n \) are given by proposition 5.3 with \( a_0 = A_n \left( 1 + \frac{7\epsilon}{2} \right) \), \( a_1 = a_2 = 1 \), \( a_{11} = \left( 1 + \frac{7\epsilon}{2} \right) \), and \( A = A_n = \frac{1}{n} \sum_{t=1}^{n} (\delta^2_{t+1} - \theta^2)^2 \). The variance is

\[
\text{var} \left( \hat{V}^{ssa}_n \right) = 2(\sigma^2_n)^2 \cdot \frac{1}{n} \left[ a_0 + a_{11} + f_2(\theta) + o(k^{-1+\epsilon}, \theta^2, A_n) \right]
\]

where \( \epsilon \) is any positive number and \( f_2(\theta) = -4a_{11}\theta + (10a_1 + 2a_2)\theta^2 \)

From Corollaries 1 and 2 we can draw the following conclusions about realized volatility estimates.

- The simple sum of squares estimate of the day \( n \) volatility is biased in the presence of temporally dependent returns. The magnitude of the bias when first order autocorrelation is present is of the same form as the bias in the unconditional variance estimate. Clearly the bias grows as \( \theta \) grows.

- The variance of both the simple sum of squares as well as the adjusted sum of squares is driven by the kurtosis of \( \epsilon_m \) and the variability of the intraday volatility as captured by \( A_n \) which, unfortunately, is unknown in practice.

It is worth noting that Barndorff Neilson and Shephard (2002) consider using very frequently sampled data to identify the neccessary components of the MSEs in absence of market microstructure effects. In a similar spirit, Bandi and Russell (2003) and Zhang et al. (2003) use a cononical model of returns in the presence of microstructure noise to construct MSEs for \( \hat{V}^{ss}_n \). presence of market microstructure noise.
6 Conclusion

Merton's (1980) seminal work suggested that volatility estimates can be made arbitrarily precise provided that the sampling interval be allowed to shrink to zero. With the availability of high-frequency financial data sets that record asset prices sometimes 1000's of times within a single day it appears that arbitrarily precise estimators may be at hand. While many studies have proceeded along this line, the precision of the estimates has not been considered outside of the unrealistic geometric Brownian motion setting. We identify fat tails, dependence in the returns and in their squared returns, and deterministic patterns in the variance, as departures from this simple continuous time setting. Using a discrete time model that is consistent with the observed features of the high-frequency data, we derive the mean squared error of several sum of squares type estimators.

Our results suggest that, after accounting for these features of the high-frequency data, the precision of the estimators can be greatly reduced. Although the deterministic pattern is a noticeable feature observed in the exchange rate data considered, it has a relatively small impact on the precision of the variance estimates. Most notably, as soon as the returns are autocorrelated, the estimator \( \hat{\sigma}_{\text{intra}}^{\text{as}} \) can be much less efficient than \( \hat{\sigma}_{\text{day}} \) depending the number of days used in the estimation. Also, the efficiency ratio \( ER(\hat{\sigma}_{\text{intra}}^{\text{as}}, \hat{\sigma}_{\text{day}}) \) for \( \hat{\sigma}_{\text{intra}}^{\text{as}} \) triples and may be even larger depending on the magnitude and persistence of the autocorrelation. The kurtosis in the high-frequency data is typically quite large and greatly reduces the precision of the volatility estimate by a factor of half the kurtosis. Furthermore, time persistent varying volatility can also greatly reduce the precision of our volatility estimates. When all the high frequency data characteristics are accounted for, the benefit of using high frequency data to estimate daily volatility is much smaller than the one under ideal situation, namely, the asset price follows the geometric Brownian motion.

An interesting feature of high-frequency data is that often the kurtosis and the autocorrelation increase as the sampling interval decreases. Hence there is a trade off between using higher frequency data to obtain larger sample sizes while exacerbating the influence of the kurtosis and autocorrelation. Our analysis provides a method for selecting optimal sampling interval. For the two foreign exchange rate series analyzed in this paper, we find that optimal sampling intervals are obtained at 10 and 20 minutes.
Appendix 1. Variances for Estimators $\hat{V}_{day}$, $\hat{V}_{intra}^{ss}$ and $\hat{V}_{intra}^{ssa}$

Suppose that that the intra-day returns $x_{nt}$, or $x_{s(n,t)}$, satisfy the structure specified by (4.1) and (4.3), and the assumption (A.1). In this appendix, we will derive the MSEs for the sum of squared estimators $\hat{V}_{day}$, $\hat{V}_{day}^{ss1}$ and $\hat{V}_{day}^{ss2}$ and the ERs for $\hat{V}_{day}^{ss1}$ and $\hat{V}_{day}^{ss2}$. As will be seen, the exact representations can be obtained. To better understand the impacts of different features discussed in Section 3, we also find their expansion at $\theta = 0$.

To simplify the derivation, we ignore the fact the series $\{x_{nt}\}$ doesn’t contain the observations from the weekend. In fact, the omission of weekend observation is easy to incorporate into analysis, see — for detail. In the following, we assume there are $N$ days and each day has $k$ returns observed over equally spaced interval in the discussion.

\[
\hat{V}_{day} = \frac{1}{N} \sum_{n=1}^{N} S_{n}^{2} = \frac{1}{N} \left( \sum_{t=1}^{k} T_{nt} \right)^{2}
\]

\[
= \frac{1}{N} \sum_{n=1}^{N} \left( \sum_{t=1}^{k} \delta_{t} (\epsilon_{nt} + \theta \epsilon_{n,t-1}) \right)^{2}
\]

\[
= \frac{1}{N} \sum_{n=1}^{N} \sum_{t=1}^{k} \left( \theta \delta_{1} \epsilon_{n-1,k} + \delta_{k} \epsilon_{n,k} + \sum_{t=1}^{k-1} (\delta_{t} + \theta \delta_{t+1}) \epsilon_{nt} \right)^{2}
\]

\[
= I_{1} + I_{2}
\]

where

\[
I_{1} = \frac{1}{N} \sum_{n=1}^{N} \sum_{t=1}^{k} u_{t}^{2} \cdot \epsilon_{nt}^{2}, \quad I_{2} = \frac{2}{N} \sum_{n=1}^{N} \sum_{t=0}^{k} \sum_{s=t+1}^{k} v_{(1)t} v_{(1)s} \epsilon_{nt} \epsilon_{ns}
\]

\[
v_{(1)t} = \begin{cases} \theta \delta_{1} & \text{if } t = 0 \\ \delta_{t} + \theta \delta_{t+1} & \text{if } t = 1, \ldots, k - 1 \\ \delta_{k} & \text{if } t = k \end{cases}
\]

\[
u_{(1)t}^{2} = \begin{cases} 0 & \text{if } t = 1, \ldots, k - 1 \\ v_{(1)t}^{2} & \text{if } t = k \end{cases}
\]

We introduce $u_{t}$ here to facilitate later computation.
From (A.1) and the structural equation (4.3), we have

\[
\begin{align*}
\text{var}(I_1) & = \frac{1}{N}[(2 + \gamma_2) \cdot \sum_{t=1}^{k} u_t^4 + 2 \sum_{t<s} u_t^2 u_s^2 \text{cov}(\epsilon_{nt}^2, \epsilon_{ns}^2) + \frac{2}{N} \sum_{n \leq m} \sum_{t,s} u_t^2 u_s^2 \text{cov}(\epsilon_{nt}^2, \epsilon_{ms}^2)] \\
& = \frac{2}{N} \sum_{n \leq m} \sum_{t,s} u_t^2 u_s^2 \text{cov}(\epsilon_{nt}^2, \epsilon_{ms}^2)] \\
& = \frac{2}{N} \sum_{n \leq m} \sum_{t,s} u_t^2 u_s^2 \text{cov}(\epsilon_{nt}^2, \epsilon_{ms}^2)] \\
\end{align*}
\]

\[
\begin{align*}
\text{var}(I_2) & = \frac{4}{N} \left[ \sum_{t=0}^{k} \sum_{s=1}^{1} v_{(1),t}^2 v_{(1),s}^2 \right] + (2 + \gamma_2) \sum_{t=1}^{k-1} \rho_t \sum_{j=0}^{k-1} v_{(1)j}^2 v_{(1)i+j}^2 \\
& = \frac{4}{N} \left[ \sum_{t=0}^{k} \sum_{s=1}^{1} u_t^2 u_s^2 \right] + (2 + \gamma_2) \sum_{t=1}^{k-1} \rho_t \sum_{j=0}^{k-1} u_j^2 u_{i+j}^2 \\
& + \theta^2 (\delta_t^2 \delta_k^2 + (2 + \gamma_2) \sum_{t=1}^{k-1} \rho_t (\delta_t^2 u_t^2 - u_k^2 \delta_t^2) + (2 + \gamma_2) \rho_k \delta_t^2 \delta_k^2) \\
\text{cov}(I_1, I_2) & = 0
\end{align*}
\]

Thus,

\[
\begin{align*}
\text{var}(\tilde{V}_{day}) & = \text{var}(I_1) + \text{var}(I_2) + 2 \cdot \text{cov}(I_1, I_2) \\
& = \frac{1}{N} \left[ 2 \cdot \left( \sum_{t=1}^{k} u_t^2 \right)^2 + 2 \cdot \sum_{t=1}^{k} u_t^4 + 2(2 + \gamma_2) \sum_{t=1}^{k-1} \rho_{N-k+t} \sum_{j=1}^{k-1} u_j^2 u_{i+j}^2 + \sum_{j=k-t+1}^{k} u_j^2 u_{i+j-t-k}^2 \\
& + 2(2 + \gamma_2) \sum_{t=1}^{k-1} \rho_t \sum_{j=0}^{k-1} u_j^2 u_{i+j}^2 + \frac{2}{N} \left( \sum_{n \leq m} \sum_{t=1}^{k-1} u_t^2 u_s^2 \text{cov}(\epsilon_{nt}^2, \epsilon_{ms}^2) \right) + (2 + \gamma_2) \rho_k \delta_t^2 \delta_k^2 \right] \\
& = \frac{2(\sigma_t^2)^2}{N} \left[ 1 + \frac{b_0}{k} + O(N^{-2}, k^{-2}, \theta^2, A^2) \right] \\
\end{align*}
\]

where \( b_0 = \frac{2\theta}{\sigma_t^2} + (2 + \gamma_2) \cdot (\sum_{i=1}^{\infty} \rho_t + 2 \sum_{i=1}^{k} (1 - \frac{1}{k}) \rho_t) \).

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In above derivation, we used the following expression for daily variance:

\[ \sigma^2 = \sum_{t=0}^{k} \nu_{(1)t}^2 = k \delta_0^2 (1 + 2(1 + \sqrt{AA_1,1,1}) \cdot \theta + \theta^2) = k \delta_0^2 (1 + 2\theta + \theta^2 + O(\theta^2, A^2)) \]

and

\[ \sum_{n \neq n'} \text{cov}(I_{1n}, I_{1n'}) = \sum_{i=1}^{N} (N - i) \cdot \text{cov}(I_{1n}, I_{1,1+n}) = N \cdot O(1) \]

when \( \rho_i \) decays exponentially.

For estimators \( \hat{V}_{\text{intra}}^{ss} \) and \( \hat{V}_{\text{intra}}^{ss,ga} \), we have

\[
\hat{V}_{\text{intra}}^{ss} = \frac{1}{N} \sum_{n=1}^{N} \sum_{t=1}^{k} \delta_t^2 (\epsilon_{nt} + \theta \epsilon_{n,t-1})^2
\]

\[
II_1 = \frac{1}{N} \sum_{n=1}^{N} \sum_{t=1}^{k} v_{(2)t} \epsilon_{nt}^2, \quad II_2 = \frac{2\theta}{N} \sum_{n=1}^{N} \sum_{t=1}^{k} \delta_t \epsilon_{nt} \epsilon_{n,t-1}
\]

\[
\hat{V}_{\text{intra}}^{sga} = I_1 + II_2 + III_3
\]

\[
III_2 = \frac{2}{N} \sum_{n=1}^{N} \sum_{t=1}^{k} v_{(3)t} \epsilon_{n,t-1} \epsilon_{nt}, \quad III_3 = \frac{2\theta}{N} \sum_{n=1}^{N} \sum_{t=1}^{k} \delta_t \delta_{t+1} \epsilon_{n,t-1} \epsilon_{n,t+1}
\]

\[
v_{(2)t} = \begin{cases} 
\delta_t^2 + \theta^2 \delta_{t+1}^2 & t = 1, 2, \cdots, k - 1 \\
\delta_t^2 + \theta^2 \delta_1^2 & t = k
\end{cases}
\]

\[
v_{(3)t} = \begin{cases} 
\theta \delta_t^2 + \theta^2 \delta_1 \delta_2 & t = 1 \\
\delta_{t-1} \delta_t + \theta \delta_t^2 + \theta^2 \delta_t \delta_{t+1} & t = 2, \cdots, k - 1 \\
\delta_{k-1} \delta_k + \theta \delta_k^2 & t = k
\end{cases}
\]

Since

\[ E[\hat{V}_{\text{intra}}^{ss}] = E[II_1 + II_2] = \sum_{t=0}^{k} v_{(2)t}^2 \]

we thus have

**Theorem A.1** Suppose that \( \{x_{nt} : t = 1, 2, \cdots, k, n = 1, 2, \cdots, N\} \) satisfy the statistical structure (4.1) and (4.3), then the estimator \( \hat{V}_{\text{intra}}^{ss} \) is biased when \( \theta \neq 0 \):

\[
\text{bias}(\hat{V}_{\text{intra}}^{ss}) = E[\hat{V}_{\text{intra}}^{ss}] - \sigma^2 = 2\theta \sum_{t=1}^{k-1} \delta_t \cdot \delta_{t+1} = 2\theta \cdot \sigma^2 + O(\theta^2, A^2)
\]
Now, we consider the variances of the $\hat{V}_{\text{intra}}^{ss}$ and $\hat{V}_{\text{intra}}^{ss}$

\[
\text{var}(II_1) = \frac{1}{N}[(2 + \gamma_2) \cdot k \sum_{t=1}^{k} v_{(2)t}^4 + 2 \sum_{t<s} v_{(2)t}^2 v_{(2)s}^2 \text{cov}(\epsilon_{nt}, \epsilon_{ns})] + \frac{2}{N} \sum_{n<m} \sum_{t,s} v_{(2)t}^2 v_{(2)s}^2 \text{cov}(\epsilon_{nt}, \epsilon_{ms})
\]

\[
= \frac{(2 + \gamma_2)}{N} \left[ \sum_{t=1}^{k} v_{(2)t}^4 + 2 \sum_{t=1}^{k} \left( \sum_{n=0}^{N-1} \left( 1 - \frac{n}{N} \right) \right) \rho_{n-k+t} \left( \sum_{j=1}^{k-t} v_{(2)t}^2 v_{(2)j+t}^2 \right) \right]
\]

\[
- \frac{2}{N} \sum_{t=1}^{k} \sum_{n=0}^{N-1} \left( \sum_{j=1}^{k-t} v_{(2)t}^2 v_{(2)j+t}^2 \right)
\]

\[
= \frac{(2 + \gamma_2)}{N} \cdot k(\delta^2)^2 \cdot [1 + A + 2\theta^2 + 2(1 + 2\theta^2) \sum_{t=1}^{\infty} \rho_t + o(N^{-1+\epsilon}, k^{-1+\epsilon}, \theta^2, A)]
\]

\[
\text{var}(II_2) = \frac{4\theta^2}{N} \sum_{t=1}^{k} \delta_{t+1}^4 + (2 + \gamma_2) \rho_1 \sum_{t=1}^{k} \delta_{t+1}^4
\]

\[
= \frac{4}{N} \cdot k(\delta^2)^2 \cdot [\theta^2(1 + (2 + \gamma_2)\rho_1) + o(N^{-1+\epsilon}, k^{-1+\epsilon}, \theta^2, A)]
\]

\[
\text{cov}(II_1, II_2) = 0
\]

\[
\text{var}(I_1) = \frac{(2 + \gamma_2)}{N} \cdot k(\delta^2)^2 \cdot [(1 + A + 4\theta + 6\theta^2) + 2(1 + 2\theta^2) \sum_{t=1}^{\infty} \rho_t + o(N^{-1+\epsilon}, k^{-1+\epsilon}, \theta^2, A)]
\]

\[
\text{var}(III_2) = \frac{4\theta^2}{N} (1 + (2 + \gamma_2)\rho_1) \sum_{t=1}^{k} \delta_{t+1}^4
\]

\[
= \frac{4}{N} \cdot k(\delta^2)^2 \cdot [(1 + 2\theta + 3\theta^2)(1 + (2 + \gamma_2)\rho_1) + o(N^{-1+\epsilon}, k^{-1+\epsilon}, \theta^2, A)]
\]

\[
\text{var}(III_3) = \frac{4\theta^2}{N} (1 + (2 + \gamma_2)\rho_2) \sum_{t=1}^{k-1} \delta_{t+1}^2 \delta_{t+1}^2
\]

\[
= \frac{4}{N} \cdot k(\delta^2)^2 \cdot [\theta^2(1 + (2 + \gamma_2)\rho_2) + o(N^{-1+\epsilon}, k^{-1+\epsilon}, \theta^2, A)]
\]

\[
\text{cov}(I_1, III_2) = \text{cov}(I_1, III_3) = \text{cov}(III_2, III_3) = 0
\]
So,

\[
\text{var}(\hat{V}_{\text{intra}}^{ss}) = \text{var}(\hat{V}_{1}) + \text{var}(\hat{V}_{2}) + \text{cov}(\hat{V}_{1}, \hat{V}_{2})
\]

\[
= \frac{2}{N} \cdot k(\bar{x}^2)^2 \cdot [(1 + \frac{\gamma_2}{2})(1 + A + 2\theta^2 + 2(1 + 2\theta^2)^{\sum_{t=1}^{\infty} \rho_t}) + 2\theta^2(1 + (2 + \gamma_2)\rho_1) + o(N^{-1+\epsilon}, k^{-1+\epsilon}, \theta^2, A)]
\]

\[
= \frac{2(\sigma^2)^2}{N} \cdot \frac{1}{k} \cdot [(1 + \frac{\gamma_2}{2})(1 + A + \sum_{t=1}^{\infty} \rho_t) + f_1(\theta) + o(N^{-1+\epsilon}, k^{-1+\epsilon}, \theta^2, A)]
\]

\[
\text{var}(\hat{V}_{\text{intra}}^{s}) = \text{var}(\hat{V}_{1}) + \text{var}(\hat{V}_{2}) + \text{var}(\hat{V}_{3})
\]

\[
= \frac{2}{N} \cdot k(\bar{x}^2)^2 \cdot [(1 + \frac{\gamma_2}{2})(1 + A + 4\theta + 6\theta^2 + 2(1 + 2\theta^2)^{\sum_{t=1}^{\infty} \rho_t}) + 2(1 + 2\theta + 3\theta^2)(1 + (2 + \gamma_2)\rho_1) + 2\theta^2(1 + (2 + \gamma_2)\rho_2) + o(N^{-1+\epsilon}, k^{-1+\epsilon}, \theta^2, A)]
\]

\[
= \frac{2(\sigma^2)^2}{N} \cdot [(1 + \frac{\gamma_2}{2})(1 + A + 2\sum_{t=1}^{\infty} \rho_t) + 2(1 + 2(1 + \frac{\gamma_2}{2})\rho_1) + f_2(\theta) + o(N^{-1+\epsilon}, k^{-1+\epsilon}, \theta^2, A)]
\]

where \( f_1(\theta) = a_{11}\theta + a_{12}\theta^2 \), \( f_2(\theta) = a_{21}\theta + a_{22}\theta^2 \), and

\[
a_{11} = -4(1 + \frac{\gamma_2}{2})(1 + A + \sum_{t=1}^{\infty} \rho_t), \quad a_{12} = 2[1 + (1 + \frac{\gamma_2}{2})6 + 5A + 2\rho_1 + 12\sum_{t=1}^{\infty} \rho_t], \quad a_{21} = -4[1 + (1 + \frac{\gamma_2}{2})(A + 2\rho_1)], \quad a_{22} = 2[6 + (1 + \frac{\gamma_2}{2})(10\rho_1 + 2\rho_2)].
\]

To summarize, we have the following theorem, which gives Proposition (5.2 - 5.3).

**Theorem A.2** Suppose that \( \{x_{it} : t = 1, 2, \cdots, k, n = 1, 2, \cdots, N\} \) satisfy the statistical structure (4.1) and (4.3), and the assumption (A.1), (A.2), then the MSE of the estimators \( \hat{V}_{\text{day}}, \hat{V}_{\text{intra}}^{ss} \) and \( \hat{V}_{\text{intra}}^{s} \) are given by the following expansion at \( \theta = 0, A = 0 \) and large \( k \) :

\[
\text{var}(\hat{V}_{\text{day}}) = \frac{2(\sigma^2)^2}{N} \cdot [1 + \frac{b_0}{k} + o(\theta^2, A^2)]
\]

\[
\text{var}(\hat{V}_{\text{intra}}^{ss}) = \frac{2(\sigma^2)^2}{N} \cdot \frac{1}{k} \cdot [(1 + \frac{\gamma_2}{2})(1 + A + 2\sum_{t=1}^{\infty} \rho_t) + f_1(\theta) + o(N^{-1+\epsilon}, k^{-1+\epsilon}, \theta^2, A)]
\]

\[
\text{var}(\hat{V}_{\text{intra}}^{s}) = \frac{2(\sigma^2)^2}{N} \cdot \frac{1}{k} \cdot [(1 + \frac{\gamma_2}{2})(1 + A + 2\sum_{t=1}^{\infty} \rho_t) + 2(1 + 2(1 + \frac{\gamma_2}{2})\rho_1) + f_2(\theta) + o(N^{-1+\epsilon}, k^{-1+\epsilon}, \theta^2, A)]
\]

where \( f_1(\theta) = a_{11}\theta + a_{12}\theta^2 + o(\theta^2) \), \( f_2(\theta) = a_{21}\theta + a_{22}\theta^2 + o(\theta^2) \) and

\[
a_{11} = -4[(1 + \frac{\gamma_2}{2})(1 + 2\sum_{t=1}^{\infty} \rho_t)]
\]

\[
a_{12} = 2[1 + (1 + \frac{\gamma_2}{2})(6 + 2\rho_1 + 12\sum_{t=1}^{\infty} \rho_t)]
\]

\[
a_{21} = -4[1 + 2\rho_1(1 + \frac{\gamma_2}{2})]
\]

\[
a_{22} = 2[6 + (1 + \frac{\gamma_2}{2})(10\rho_1 + 2\rho_2)]
\]

\[
b_0 = \frac{\gamma_2}{2} + (2 + \gamma_2) \cdot (\sum_{t=1}^{\infty} \rho_t + 2\sum_{t=1}^{k}(1 - \frac{t}{k})\rho_t)
\]
In this theorem, we assumed that only the intra-day data collected in weekdays are used in estimation. If all the intra-day data, including the weekends, satisfy 4.1, 4.3 and assumptions (A.1), (A.2), then the coefficient of $\frac{1}{7}$, 1 for ERs' representation, vanish. Also note that, the exact result can also be obtained as in the proof.

Although the above result is based on the structure (4.1) and (4.3), it can be easily extended to general stochastic volatility given by (4.3) since the expressions on MSE don't depends on (4.3). Similar results for relaxing (4.1) to general ARMA models are also obtained.
Appendix 2. GARCH(1, 1) Estimation

In Appendix 2, we discuss the procedure in GARCH(1, 1) estimation for the two series used in this paper and present the estimation results. In addition, we derive $\rho_1$, $\rho_2$ and $\sum_{t=1}^{\infty} \rho_t$ presented in Table 5 by using the GARCH(1, 1) estimates.

As discussed in Sector 3, the original series are first deseasonalized by the intraday volatility pattern, and then adjusted by the lag one autocorrelation. The derived series are free from deterministic volatility patterns and series autocorrelation, and only have volatility clustering property presented. Since GARCH(1, 1) is widely used to describe volatility clustering property, we use this model to fit out data. The estimates are presented in Table 8, and their standard deviations are presented in the parenthesis:

<table>
<thead>
<tr>
<th>Series</th>
<th>$\omega$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DD</td>
<td>0.030988 (2.0923 $\cdot 10^{-4}$)</td>
<td>0.144314 (6.9073 $\cdot 10^{-4}$)</td>
<td>0.838111 (6.2583 $\cdot 10^{-4}$)</td>
</tr>
<tr>
<td>DY</td>
<td>0.021644 (1.1734 $\cdot 10^{-4}$)</td>
<td>0.127413 (5.8845 $\cdot 10^{-4}$)</td>
<td>0.861935 (4.0396 $\cdot 10^{-4}$)</td>
</tr>
</tbody>
</table>

For GARCH(1, 1) model, the autocorrelation of the squared series is given by $\rho_t = \rho_1 \cdot \phi^{t-1}$, where $\phi = \alpha + \beta$ and $\rho_1 = \alpha \cdot (1 + \alpha \cdot \frac{\phi-\alpha}{1-\phi^{\alpha+1}})$. Thus, series DD has $\rho_1 = 0.4579, \rho_2 = 0.4498$ and $\sum_{t=1}^{\infty} \rho_t = 26.05$, and series DY has $\rho_1 = 0.5013, \rho_2 = 0.49598$ and $\sum_{t=1}^{\infty} \rho_t = 47.06$. These parameters are reported in Table 5 and are used in constructing Table 6 and Table 7.
Appendix 3. Parameters Used in Section 5.4

Table 9: Parameters for Dollar / Japanese Yen series

<table>
<thead>
<tr>
<th>Min .</th>
<th>α</th>
<th>β</th>
<th>$\sigma^2$</th>
<th>$k$</th>
<th>N</th>
<th>θ</th>
<th>$A$</th>
<th>$\gamma_2$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$\sum_{i=1}^{\infty} \rho_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>005</td>
<td>0.1449</td>
<td>0.8506</td>
<td>0.4523</td>
<td>288</td>
<td>1552</td>
<td>-0.0635</td>
<td>0.4347</td>
<td>19.04</td>
<td>0.7431</td>
<td>0.7398</td>
<td>167.3</td>
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<td>010</td>
<td>0.1431</td>
<td>0.8411</td>
<td>0.4365</td>
<td>144</td>
<td>1552</td>
<td>-0.0603</td>
<td>0.4740</td>
<td>17.71</td>
<td>0.4763</td>
<td>0.4688</td>
<td>30.24</td>
</tr>
<tr>
<td>015</td>
<td>0.1443</td>
<td>0.8381</td>
<td>0.4236</td>
<td>96</td>
<td>1552</td>
<td>-0.0565</td>
<td>0.4592</td>
<td>16.49</td>
<td>0.4579</td>
<td>0.4498</td>
<td>26.05</td>
</tr>
<tr>
<td>020</td>
<td>0.1303</td>
<td>0.8463</td>
<td>0.4203</td>
<td>72</td>
<td>1552</td>
<td>-0.0602</td>
<td>0.4506</td>
<td>18.97</td>
<td>0.3580</td>
<td>0.3497</td>
<td>15.34</td>
</tr>
<tr>
<td>030</td>
<td>0.1287</td>
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<td>0.4275</td>
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<td>0.4397</td>
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<tr>
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<td>0.9324</td>
<td>0.4409</td>
<td>24</td>
<td>1552</td>
<td>-0.0111</td>
<td>0.3991</td>
<td>12.00</td>
<td>0.1504</td>
<td>0.1484</td>
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<tr>
<td>120</td>
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<td>0.9338</td>
<td>0.4343</td>
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<td>1552</td>
<td>-0.0197</td>
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<td>0.4200</td>
<td>6</td>
<td>1552</td>
<td>-0.0487</td>
<td>0.3708</td>
<td>4.729</td>
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<td>360</td>
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<td>0.8923</td>
<td>0.4298</td>
<td>4</td>
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<td>-0.0618</td>
<td>0.4518</td>
<td>5.769</td>
<td>0.1292</td>
<td>0.1245</td>
<td>3.519</td>
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<td>720</td>
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<td>0.4560</td>
<td>2</td>
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<td>-0.0021</td>
<td>0.1310</td>
<td>2.323</td>
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<td>0.1186</td>
<td>2.82</td>
</tr>
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</table>

Table 10: Parameters for Dollar / Japanese Yen series

<table>
<thead>
<tr>
<th>Min .</th>
<th>α</th>
<th>β</th>
<th>$\sigma^2$</th>
<th>$k$</th>
<th>N</th>
<th>θ</th>
<th>$A$</th>
<th>$\gamma_2$</th>
<th>$\rho_1$</th>
<th>$\rho_2$</th>
<th>$\sum_{i=1}^{\infty} \rho_t$</th>
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</thead>
<tbody>
<tr>
<td>005</td>
<td>0.1324</td>
<td>0.8669</td>
<td>0.4639</td>
<td>288</td>
<td>1552</td>
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<td>1552</td>
<td>-0.0655</td>
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<td>0.1828</td>
<td>16.83</td>
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<tr>
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References


