Answer Key to Fall 2001 Midterm

1. F

counter example: \( \sigma_{xy} = 0.25, \sigma_x = 0.5, \sigma_y = 0.5 \), then \( \rho = 1 \).

2. F

\[
\begin{align*}
\text{Var}(X+Y) &= \text{Var}(X) + \text{Var}(Y) + 2 \text{Cov}(X,Y) \\
\text{Var}(X-Y) &= \text{Var}(X) + \text{Var}(Y) - 2 \text{Cov}(X,Y)
\end{align*}
\]

3. T

By central limit theorem

4. T

If the covariance is not equal to zero, then they are not independent.

5. F

\[
\begin{align*}
\text{Var(average)} &= \text{Var}\left(\frac{X + Y}{2}\right) = \left(\frac{1}{2}\right)^2 \text{Var}(X + Y) \\
&= 0.25[\text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y)] \\
&= 0.25(25 + 25 + 2 \times 0.21) \\
&= 12.605
\end{align*}
\]

6. T

If independent, joint probability is the product of marginal probabilities.

7. T

Correlation is not affected by changing the units. It is a relative measure. For example,

\[
\begin{align*}
\rho(X, 100Y) &= \frac{\text{Cov}(X, 100Y)}{\text{Std}(X)\text{Std}(100Y)} = \frac{100\text{Cov}(X, Y)}{\text{Std}(X)\times100\text{Std}(Y)} \\
&= \frac{\text{Cov}(X, Y)}{\text{Std}(X)\times\text{Std}(Y)} \\
&= \rho(X, Y)
\end{align*}
\]

8. T
(1-0.8)^1 = 0.0016
9. T

10. T

Let
A: student getting this question correct
B: student scores above 85

Question: what’s P(B|A)?

\[ P(A) = 0.8 \]
\[ P(A, B) = 0.7 \]
\[ P(B | A) = \frac{P(A, B)}{P(A)} = \frac{0.7}{0.8} = 0.875 \]

II. Multiple choice

1. e

\[ X \sim \text{Binomial}(5,.5) \]
\[ Y \sim \text{Binomial}(10,.8) \text{ and independent of } X \]
\[ Z = X + Y \]
\[ E(Z) = E(X) + E(Y) = 2.5 + 8 = 10.5 \]
\[ Var(Z) = Var(X) + Var(Y) = 5 \times 5 + 10 \times 0.8 \times 2 = 2.85 \]

2. a

\[ FC(\text{Fixed Cost}) = 1200 \]
\[ VC(\text{Variable Cost}) = 1.20 \times X \]
\[ R(\text{Revenue}) = 5.50 \times X \]

To breakeven, \[ E(R) = E(VC) + FC \], or \[ E(R) - E(VC) = 1200 \]
\[ (5.50 - 1.20) E(X) = 1200 \]
\[ E(X) = 279 \]
3. b
If we add many iid random variables, then the sum looks more like a normal distribution.

4. c

\[(1 - 0.01)^{50} = 0.605\]

5. c

\[P(R < 0) = P\left(\frac{R - 0.2}{0.1} < \frac{0 - 0.2}{0.1}\right)\]
\[= P(Z < -2)\]
\[= 0.025\]

6. e

\[R = 0.80X + 0.20Y\]
\[E(R) = 0.80E(X) + 0.20E(Y)\]
\[= 0.80 \times 1 + 0.20 \times 0.18\]
\[= 0.116\]
\[Var(R) = 0.80^2 Var(X) + 0.20^2 Var(Y) + 2 \times 0.80 \times 0.20 \times Cov(X, Y)\]
\[= 0.64 \times 1^2 + 0.04 \times 0.31^2 + 2 \times 0.80 \times 0.20 \times (-0.2)\]
\[= -0.016892\]
Negative variance is not possible.

7. a

8. a

Standard deviation of X1 is roughly about 7 and that of X2 is about 5. The correlation between the two variables is positive but it should not be close to 1. The possible correlation among the choices is 0.7. Covariance is the product of correlation and the standard deviations, so it is about \(0.7 \times 7 \times 5\), which is roughly 25.

9. c
$X \sim \text{Binomial}(n, p)$

$Var(X) = np(1 - p)$.

Then

$$Var\left(\frac{X}{n}\right) = \left(\frac{1}{n}\right)^2 Var(X) = \frac{p(1-p)}{n}$$

10. e

95% confidence interval for $\mu$

$$\bar{x} \pm 2\sigma = 132.5 \pm 2 \times \frac{4}{\sqrt{10}}$$

$$= 132.5 \pm 2.53$$

$$= [129.97, 135.03]$$

III

1.

a. .6 (2 pts)

$$P(X_1 = X_2) = P(X_1 = 0, X_2 = 0) + P(X_1 = 1, X_2 = 1)$$

$$= .3 + .3$$

$$= .6$$

b. .1 (2 pts)

$$1 - .3 - .3 - .3 = .1$$

c. Bernoulli(0.4) (2 pts)

Or $X_1 = 0$ with probability 0.6 and $X_1 = 1$ with probability 0.4

d. Bernoulli(0.6) (2 pts)

Or $X_2 = 0$ with probability 0.4 and $X_2 = 1$ with probability 0.6

e. mean = .4 , variance = .24 (2 pts)

$$\text{mean} = p = .4, \text{variance} = p(1-p) = .4 \times .6 = .24$$
f. mean = .6 , variance = .24  (2 pts)

mean = p = .6, variance = p(1-p) = .6 * .4 = .24

g. 0.06  (4 pts)

cov(X1, X2) = \sum p(X1, X2)(X1-\mu_1)(X2-\mu_2)
= .3 \times (0-.4) \times (0-.6) + .3 \times (0-.4) \times (1-.6) + .1 \times (1-.4) \times (0-.6) + .3 \times (1-.4) \times (1-.6)
= 0.06

h. mean = -.2 , variance = .36  (3 pts)

\[
E(X1 - X2) = E(X1) - E(X2) = .4 -.6 = -.2
\]

\[
Var(X1 - X2) = Var(X1) + Var(X2) - 2Cov(X1, X2)
= .24 + .24 - 2 \times .06
= .36
\]

i. Bernoulli(.5)  (3 pts)

\[
P(X1 = 0 | X2 = 1) = \frac{.3}{.6} = .5
\]

\[
P(X1 = 1 | X2 = 1) = \frac{.3}{.6} = .5
\]

j. No  (3 pts)

Any of the following answers is sufficient.

1) Joint probability is not the product of marginal probabilities
2) Covariance is not equal to 0
3) Marginal is not equal to conditional probability.

2.

a. 0.7734  (5 pts)
\[
P(R > 0) = P\left(\frac{R - .15}{.2} > \frac{0 - .15}{.2}\right)
\]
\[
= P(Z > -0.75)
\]
\[
= P(Z < 0.75)
\]
\[
= 0.7734 \quad \text{(from the CDF table)}
\]

b. 0.1915 \quad \text{(5 pts)}

\[
P(0.15 < R < 0.25)
\]
\[
= P\left(\frac{15 - .15}{.2} < \frac{R - .15}{.2} < \frac{25 - .15}{.2}\right)
\]
\[
= P(0 < Z < .5)
\]
\[
= CDF(.5) - CDF(0)
\]
\[
= 0.6915 - 0.5000
\]
\[
= 0.1915
\]

c. 0.2767 \quad \text{(5 pts)}

\[
\text{(answer to a)}^5 = .7734^5 = .2767
\]

d. 0.7734

Since the annual returns are independent, the conditional probability is the same as the marginal probability. Therefore the answer is the same as part a, which is 0.7734.

3.

a. \[P_2 = P_0 + D_1 + D_2 \quad \text{(5 pts)}\]

\[
P_2 = P_1 + D_2
\]
\[
= (P_0 + D_1) + D_2
\]
b. \[ P_n = P_0 + \sum_{i=1}^{n} D_i \]  \hspace{1cm} (5 pts)

\[
P_n = P_{n-1} + D_n = (P_{n-2} + D_{n-1}) + D_n = P_{n-3} + D_{n-2} + D_{n-1} + D_n
\]
... 
\[
= P_0 + D_1 + D_2 + ... + D_{n-1} + D_n = P_0 + \sum_{i=1}^{n} D_i
\]

c. \[ \mu = 0, \sigma^2 = 4n \]  \hspace{1cm} (5 pts)

\[ P_n = P_0 + X_n \hspace{0.5cm} \text{From part b.} \]

\[ X_n = \sum_{i=1}^{n} D_i. \]

\[
E(X_n) = \sum_{i=1}^{n} E(D_i) = 0
\]

\[
Var(X_n) = \sum_{i=1}^{n} Var(D_i) \hspace{0.5cm} (\text{because it's independent})
\]

\[ = \sum_{i=1}^{n} 4 = 4n \]