Financial Econometrics
Midterm Review

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Contents

1 AR(p) models 2

2 MA(q) models 2

3 Forecasting 3
  3.1 In-sample forecasts ........................................... 3
  3.2 Out-of-sample forecasts ........................................ 3

4 Models of volatility 3
  4.1 Historical volatility ........................................... 3
  4.2 Exponential smoothing .......................................... 4
  4.3 ARCH/GARCH .................................................. 4
1 AR(p) models

Recall that an autoregressive model of order $p$ is given by:

\begin{equation}
Y_t = \beta_0 + \beta_1 Y_{t-1} + \ldots + \beta_p Y_{t-p} + \epsilon_t; \quad \epsilon_t \sim \text{i.i.d. } N(0, \sigma^2).
\end{equation}

**Question 1.** What does this model say about the evolution of $Y$ over time?

**Question 2.** Which part of $Y$ depends on the past?

**Question 3.** Which part of $Y$ is not predictable from the past?

**Question 4.** For an AR(1) model, how do we determine if the series is stationary?

**Question 5.** For an AR($p$) model, how do we determine if the series is stationary?

**Question 6.** For an AR(1) model, what happens if $|\beta_1| = 1$?

**Question 7.** For an AR(1) model, what happens if $|\beta_1| > 1$?

**Question 8.** What does the ACF look like for an AR($p$) model? What does the PACF look like for an AR($p$) model?

2 MA(q) models

Recall that a moving average model of order $q$ is given by:

\begin{equation}
Y_t = \beta_0 + \theta_1 \epsilon_{t-1} + \ldots + \theta_q \epsilon_{t-q} + \epsilon_t; \quad \epsilon_t \sim \text{i.i.d. } N(0, \sigma^2).
\end{equation}

**Question 9.** What does this model say about the evolution of $Y$ over time?

**Question 10.** Which part of $Y$ depends on the past?

**Question 11.** Which part of $Y$ is not predictable from the past?

**Question 12.** For an MA(1) model, how do we determine if the series is stationary?

**Question 13.** For an MA($q$) model, how do we determine if the series is stationary?

**Question 14.** What does the ACF look like for an MA($q$) model? What does the PACF look like for an MA($q$) model?
3 Forecasting

3.1 In-sample forecasts

Recall that the models we consider have the form:

\[ y_t = f(y_{t-1}, \ldots, y_0) + \epsilon_t; \quad \epsilon_t \sim N(0, \sigma^2). \]

Thus, the one-step ahead forecast is just given by:

\[ \mathbb{E}[y_t | y_{t-1}, \ldots, y_0] = f(y_{t-1}, \ldots, y_0) \]

Thus, the best one-step-ahead forecast in-sample is just given by \( y_t - \hat{\epsilon}_t \).

3.2 Out-of-sample forecasts

Consider first an AR(1) model of the form:

\[ y_t = \beta_0 + \beta_1 y_{t-1} + \epsilon_t; \quad \epsilon_t \sim N(0, \sigma^2). \]

Recall from your teaching notes that the \( k \)-step ahead forecast is given by:

\[ y_{t+k}^f \equiv \mathbb{E}[y_{t+k} | y_t] = \begin{cases} \frac{\beta_1^k y_t + (1 - \beta_1^k) \mu}{k \beta_0 + y_t}; & |\beta_1| < 1 \\ \frac{1 - \beta_1^k}{1 - \beta_1} \sigma^2; & |\beta_1| = 1 \end{cases} \]

where \( \mu = \beta_0/(1 - \beta_1) \) is the unconditional mean. The forecast error is then given by:

\[ \text{var} \left( \epsilon_{t+k}^f \right) \equiv \text{var} \left( y_{t+k} - y_{t+k}^f | y_t \right) = \begin{cases} \frac{1 - \beta_1^k}{1 - \beta_1} \sigma^2; & |\beta_1| < 1 \\ k \sigma^2; & |\beta_1| = 1 \end{cases} \]

and the 95% confidence interval by:

\[ y_{t+k}^f \pm 2 \sqrt{\text{var} \left( \epsilon_{t+k}^f \right)}. \]

4 Models of volatility

4.1 Historical volatility

Recall that if we have a time series \( r_t \), the \( K \)-period average of historical volatility is given by:

\[ \hat{\sigma}_t = \sqrt{252 \sum_{i=1}^{K} \frac{r_{t-i}^2}{k}} \]

Remember that, in choosing the optimal window for the historical volatility calculations, you are balancing a more accurate estimate (\( k \) high) versus having an estimate that reacts promptly to changes in the time series (\( k \) low).
4.2 Exponential smoothing

For exponential smoothing, the time series of volatility is constructed as:

\[ h_t^2 = \lambda h_{t-1}^2 + (1 - \lambda) r_{t-1}^2. \]

A high value of \( \lambda \) implies that you put more weight on past estimates, while a low value of \( \lambda \) implies that you want to update your data more quickly.

4.3 ARCH/GARCH

An ARCH(p) model is a generalization of the historical volatility model, with the estimation equation given by:

\[ r_t = \sqrt{h_t} z_t, \quad z_t \sim N(0, 1) \]
\[ h_t = \omega + \sum_{i=1}^{p} \alpha_i r_{t-i}^2. \]

A GARCH(p,q) model is a generalization of the exponential smoothing model, with the estimation equation given by:

\[ r_t = \sqrt{h_t} z_t, \quad z_t \sim N(0, 1) \]
\[ h_t = \omega + \sum_{i=1}^{p} \alpha_i r_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j}^2. \]

Consider a GARCH(1,1) model. The \( k \)-step ahead forecast specification is:

\[ h_{t+k|t} = \sigma^2 + (\alpha + \beta)^{k-1}(h_{t+1} - \sigma^2), \]

with the one-step ahead forecast is given by: \( h_{t+1|t} = \omega + \alpha r_t^2 + \beta h_t. \)