VOLATILITY

Finance is risk/return trade-off.

Volatility is risk.

Advance knowledge of risks allows us to avoid them. But what would we have to do to avoid them altogether?? Imagine!

How much should I get paid to take a given risk?

Which risks are not worth taking?
**DEFINE VARIANCE**

- $E(x)$ is the expected value of a random variable $x$
- $x - E(x)$ is the unexpected part of $x$
- $V(x)$ is the variance of $x$
  - $V(x) = E(x - E(x))^2$
  - Standard Deviation is the square root of Variance

**Time Varying Volatility**

- VOLATILITY IS THE STANDARD DEVIATION OF RETURNS - *the unpredictable part of asset prices.*
- The unpredictable part is the surprise part, or the error in the prediction.
- Consider the AR(1) model:
  \[ r_t = \beta_0 + \beta_1 r_{t-1} + \varepsilon_t = \mu_i + \varepsilon_t \]
  - Expected part given the past
  - Unexpected part
- We define the conditional variance as the variance of the surprise part or $\varepsilon_t$. 
More generally for any model we can write it as

\[ r_i = \mu_{\text{expected}} + \epsilon_{\text{unexpected}} \]

The variance is then defined as the variance of the unexpected part or the deviation of the return from its prediction.

\[ \sigma_i^2 = E(\epsilon_i^2 | \text{past}) = E[(r_i - \mu_i)^2 | \text{past}] \]

Before we modeled returns where the variance of the surprise or error was constant. If the errors are Normal then the conditional distribution is given by:

\[ f(r_i | \text{past}) = N(\mu_i, \sigma^2) \]

Now we have acknowledged that the variance may be time varying as well as the mean! \[ f(r_i | \text{past}) = N(\mu_i, \sigma_i^2) \]
So the conditional variance of a return is the variance of the surprise.

- IF the mean $\mu_t=0$ then we have
  \[ r_t = 0 + \varepsilon_t = \varepsilon_t \]
- In other words, the surprise is the return.
- For simplicity, we will assume the mean is zero for now. We do this for convenience.
- IF the mean were not zero we would have to first subtract $\mu_t$ from $r_t$ and then model the variance of the demeaned returns.

**Simplification for now**

- If markets are nearly efficient (in Fama sense) then returns are nearly unpredictable. If the drift is small then the conditional variance is given by:

\[
\sigma^2_t = E\left[ (r_t - \mu_t)^2 \mid F_{t-1} \right] = E(r_t^2 \mid F_{t-1})
\]
**ANNUALIZED VOLATILITY**

- Returns are essentially unpredictable if the efficient market hypothesis is a good approximation.
- Hence the variance over multiple days is the sum of the variances over each of the days (no covariance terms).
- If the variance were the same every day of the year then the annualized variance is given by $n \cdot (\text{daily variance})$ where $n$ is the number of trading days. In the US $n \approx 252$.
- Annualized Volatility is $\sqrt{252} \cdot \text{daily volatility}$ or $\sqrt{12} \cdot \text{monthly volatility}$…

Let’s first take a look at historical volatility. Then we will consider models that capture the time varying features observed in the data.
HISTORY OF THE US EQUITY MARKET: S&P500

S&P500 daily returns 1990 to present
Let’s look more closely at the characteristics of financial returns.

- ALMOST UNPREDICTABLE
  - EFFICIENT MARKET HYPOTHESIS
- SURPRISINGLY LARGE NUMBER OF EXTREMES
  - FAT TAIL DISTRIBUTIONS
- PERIODS OF HIGH AND LOW VOLATILITIES
  - VOLATILITY CLUSTERING
CHECK IT OUT!

- HOW TO CHECK FOR EXCESSIVE EXTREMES
- HOW TO CHECK FOR VOLATILITY CLUSTERING?

HISTORICAL VOLATILITY

- Estimate the standard deviation of a random variable

\[ \hat{\sigma} = \sqrt{\frac{252}{T-K} \sum_{j=T-K}^{T} r_j^2 / K} \]

- What assumptions do we need?
  - Choose K small so that the variance is constant
  - Choose K large to make the estimate as accurate as possible
- Funny boxcars and shadow volatility movements!!
EXPONENTIAL SMOOTHING

- Volatility Estimator used by RISKMETRICS
  \[ \sigma_t^2 = \lambda \sigma_{t-1}^2 + (1 - \lambda) r_{t-1}^2 \]
- Updating
- AN EXAMPLE
- WEAKNESSES
  - How to choose lambda
  - No mean reversion

II ARCH/GARCH MODELS

- GARCH VOLATILITY
- FORECASTING WITH GARCH
- ESTIMATING AND TESTING GARCH
- MANY MODELS
Decisions in Finance often require measures of the volatility

- Risk can be thought of as the variance. Expected returns of an asset are related to their variance. Optimal portfolio selection Markowitz (1952) and Tobin (1958) (Nobel Prize).
- The value of an option

BLACK-SCHOLES AND MERTON

- Options can be used as insurance policies.
- For a fee we can eliminate financial risk for a period.
- What is the right fee?
- Black and Scholes (1972) and Merton (1973) (Nobel Prize) developed an option pricing formula from a dynamic hedging argument.
IMPLEMENTING THESE MODELS

- PRACTITIONERS REQUIRED ESTIMATES OF VARIANCES AND COVARIANCES
- EQUIVALENTLY WE SAY
  - VOLATILITIES
    - (Standard deviation which is the square root of the variance)
  - AND CORRELATIONS
    - (which is the covariance divided by the product of the standard deviations)

ESTIMATES DIFFER FOR DIFFERENT TIME PERIODS

- Volatility is apparently varying over time
- What is the volatility NOW!
- What is it likely to be in the future?
- How can we forecast something we never observe?
Along comes ARCH 2003 Nobel prize to Engle.
WHAT IS ARCH?

- Autoregressive Conditional Heteroskedasticity

THE SIMPLEST PROBLEM – WHAT IS VOLATILITY NOW?

- One answer is the standard deviation over the last 5 years
  - But this will include lots of old information that may not be relevant for short term forecasting
- Another answer is the standard deviation over the last 5 days
  - But this will be highly variable because there is so little information
THE ARCH ANSWER

- Use a weighted average of the volatility over a long period with higher weights on the recent past and small but non-zero weights on the distant past.

- Choose these weights by looking at the past data; what forecasting model would have been best historically? This is a statistical estimation problem.

The ARCH Model

- The ARCH model of Engle(1982) is a family of specifications for the conditional variance.
- The $q^{th}$ order ARCH or ARCH($q$) model is

$$h_t = \omega + \sum_{j=1}^{q} \alpha_j r_{t-j}^2$$

- Where in the ARCH notation

$$h_t = \sigma_t^2 = E(r_t^2 \mid F_{t-1})$$

is the conditional variance.
FROM THE SIMPLE ARCH GREW:

- GENERALIZED ARCH (Bollerslev) a most important extension
- Tomorrow’s variance is predicted to be a weighted average of the
  - Long run average variance
  - Today’s variance forecast
  - The news (today’s squared return)

GARCH

\[ h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1} \]

- Generalization of Exponential Smoothing
- Generalization of ARCH
- Generalization of constant volatility
Suppose the model is:
\[ h_t = .00001 + .05r_{t-1}^2 + .9h_{t-1} \]

And today annualized volatility is 20% and the market return is -3%, what is my estimate of tomorrow’s volatility from this model?
\[ .000198 = .00001 + .05(-.03)^2 + .9(.22/252) \]
Or 22.3% annualized.

REPEAT STARTING AT \( T=1 \)

IF WE KNOW THE PARAMETERS AND SOME STARTING VALUE FOR \( h_1 \), WE CAN CALCULATE THE ENTIRE HISTORY OF VOLATILITY FORECASTS.

OFTEN WE USE A SAMPLE VARIANCE FOR \( h_1 \).
GARCH(p,q)

- The Generalized ARCH model of Bollerslev(1986) is an ARMA version of this model. GARCH(p,q) is

\[ h_t = \omega + \sum_{j=1}^{q} \alpha_j r_{t-j}^2 + \sum_{j=1}^{p} \beta_j h_{t-j} \]

Asymmetric Volatility

- Often negative shocks have a bigger effect on volatility than positive shocks.
- Nelson(1987) introduced the EGARCH model to incorporate this effect.
- I will use a Threshold GARCH or TARCH

\[ h_t = \omega + \sum_{j=1}^{q} \alpha_j r_{t-j}^2 + \sum_{j=1}^{q} \gamma_j r_{t-j}^2 I_{(r_{t-j} < 0)} + \sum_{j=1}^{p} \beta_j h_{t-j}, \]

\[ I_A \]
NEW ARCH MODELS

- GJR-GARCH
- TARCH
- STARCH
- AARCH
- NARCH
- MARCH
- SWARCH
- SNPARCH
- APARCH
- TAYLOR-SCHWERT
- FIGARCH
- FIEGARCH
- Component
- Asymmetric Component
- SQGARCH
- CESGARCH
- Student t
- GED
- SPARCH
- Autoregressive Conditional Density
- Autoregressive Conditional Skewness

ROLLING WINDOW VOLATILITIES

NUMBER OF DAYS=5,260,1300
UNCONDITIONAL, OR LONG RUN, OR AVERAGE VARIANCE

- WHAT IS $E(r^2)$?
  
  - $\sigma^2 \equiv E\left(r^2\right) = E\left(h\right)$
  
  - $E(h_t) = \omega + \alpha E\left(r^2_{t-1}\right) + \beta E(h_{t-1})$
  
  - $\sigma^2 = \omega + (\alpha + \beta)E(h) = \omega + (\alpha + \beta)\sigma^2$
  
  - $\sigma^2 = \frac{\omega}{1 - \alpha - \beta}$

UNCONDITIONAL VARIANCE

- Suppose the model is:
  
  - $h_t = .00001 + .05r_{t-1}^2 + .9h_{t-1}$

- What is the unconditional annualized volatility?
  
  - .0002 or 22.44% annualized.
The GARCH Model Again

\[ r_t = \epsilon_t \]
\[ h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1} \]
\[ = \sigma^2 (1 - \alpha - \beta) + \alpha r_{t-1}^2 + \beta h_{t-1} \]

- The variance of \( r_t \) is a weighted average of three components
  - a constant or unconditional variance
  - yesterday’s forecast
  - yesterday’s news

ERRORS

\[ r_t = \mu_t + \epsilon_t \]

- WE MODEL \( E[(r_t - \mu_t)^2 | F_{t-1}] = E[\epsilon_t^2 | F_{t-1}] \)
- SO THE CONDITIONAL VARIANCE IS THE VARIANCE OF THE \( \epsilon \)s.
- IF \( \mu_t = 0 \) then \( E[r_t^2 | F_{t-1}] = E[\epsilon_t^2 | F_{t-1}] \)
We saw that the errors have fat tails and may be skewed. This is inconsistent with the assumption that errors are normally distributed.

We now introduce a way of thinking about GARCH models where we allow the error to have a different distribution than the normal.

The key is we still need the error to be mean zero and we need $h_t$ to still be the conditional variance...how do we do that?

---

An interesting alternative way to think about the GARCH model

WE CAN WRITE $\varepsilon_i = \sqrt{h_t} z_i$

WHERE $z_i$ is iid (not necessarily normal) with mean zero and variance one.

NOW, $E(\varepsilon_i^2 | F_{t-1}) = E\left[\left(\sqrt{h_t} z_i\right)^2 | past\right] = h_t E(z_i^2) = h_t$

$h_t$ is known given $past$ and $z_i$ is iid with variance 1.
THE ERRORS $z_t$ MUST HAVE VARIANCE 1

THEY COULD BE NORMAL

THEY MIGHT HAVE FATTER TAILS LIKE THE STUDENT –T OR GENERALIZED EXPONENTIAL

DIAGNOSTIC CHECKING

$$r_t = \sqrt{h_t} z_t,$$

SO

$$\frac{r_t}{\sqrt{h_t}} = z_t \quad \text{standardized series}$$

When we divide $r_t$ by its standard
deviation, it should have a variance of
1 for all $t$.

We can check this by creating the
“standardized series” and looking for volatility clustering in them.
Suppose $r_t$ has variance $\sigma^2$. Then $\text{Var} \left( \frac{r_t}{\sigma} \right) = \frac{1}{\sigma^2} \text{Var}(r_t) = \frac{\sigma^2}{\sigma^2} = 1$.

So if we divide a return by its standard deviation, the new return should have variance 1.

Now suppose that returns have different variances. Let $\text{Var}(r_t) = \sigma_t^2$.

So the $\text{Var} \left( \frac{r_t}{\sigma_t} \right) = 1$.

If the GARCH model is the “right” model then the GARCH series should be the “right” variances for the $r_t$’s and and $\sqrt{h_t}$ should be the “right” standard deviation so $\text{Var} \left( \frac{r_t}{\sqrt{h_t}} \right) = 1$. 
This suggests that we can “standardize” each return by dividing by its conditional standard deviation and the resulting series should no longer have time varying volatility. The variance should be constant.

We have a test for time varying volatility so we can check and see if the GARCH model is “right” by testing to see if the standardized returns no longer have time varying volatility.

If we divide by the “wrong” standard deviation then the resulting series will not have constant variance, but rather will have time varying variance.

Time varying volatility is revealed by volatility clustering.

These are measured by the Ljung Box statistic on squared returns.

A failure to reject the null of the standardized series constant volatility suggests the GARCH model is a good one.
Time varying volatility is revealed by volatility clusters.

These are measured by the Ljung Box statistic on squared returns.

The standardized returns $z_t = r_t / \sqrt{\hat{h}_t}$ no longer should show significant volatility clustering.

FORECASTING VOLATILITY
FORECASTING FOR GARCH

- ONE STEP AHEAD FORECAST
- TWO STEP FORECAST

\[ h_{t+2} = \omega + \alpha r_{t+1}^2 + \beta h_{t+1} \]
\[ E(h_{t+2} | F_t) = \omega + (\alpha + \beta) h_{t+1} \]
\[ E(h_{t+2} | F_t) = \sigma^2 + (\alpha + \beta)(h_{t+1} - \sigma^2) \]

MEAN REVERTING VOLATILITY

- Forecasts converge to the same value no matter what the current volatility

\[ E(h_{t+k} | past) = \sigma^2 + (\alpha + \beta)^{k-1}(h_{t+1} - \sigma^2) \]
\[ E(h_{t+k} | past) \to \sigma^2 \text{ if } \alpha + \beta < 1 \]

- LITTLE UPDATING FOR LONG HORIZON VOLATILITY
Monotonic Term Structure of Volatility

\[ \sigma \]

**FORECAST PERIOD**

DOW JONES SINCE 1990

<table>
<thead>
<tr>
<th>Dependent Variable: DJRET</th>
<th>Method: ML - ARCH (Marquardt) - Normal distribution</th>
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<tbody>
<tr>
<td>Date: 01/10/08</td>
<td>Time: 13:42</td>
</tr>
<tr>
<td>Sample: 1/02/1990 1/04/2008</td>
<td>Included observations: 4541</td>
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<tr>
<td>Convergence achieved after 15 iterations</td>
<td>GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*GARCH(-1)</td>
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<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Std. Error</th>
<th>z-Statistic</th>
<th>Prob.</th>
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<tbody>
<tr>
<td>C</td>
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</table>

**Variance Equation**

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<th>Prob.</th>
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<td>GARCH(-1)</td>
<td>0.925645</td>
<td>0.005025</td>
<td>184.2160</td>
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</table>

**Summary Statistics**

R-squared: -0.000371
Mean dependent var: 0.000338
Adjusted R-squared: -0.001032
S.D. dependent var: 0.009795
S.E. of regression: 0.009800
Akaike info criterion: -6.640830
Schwarz criterion: -6.635174
Hannan-Quinn criterion: -6.638838
Durbin-Watson stat: 2.001439
MULTI-STEP FORECASTS

- $E(h_{t+k} | F_t) = E(r_{t+k}^2 | F_t) = \sigma^2 + (\alpha + \beta)(E(h_{t+k-1} | F_t) - \sigma^2) = \sigma^2 + (\alpha + \beta)^{k-1}(h_{t+1} - \sigma^2)$

- Forecast over long horizons is the sum of the forecasts from $t+1$ to $t+K$.

- $V_t(R_k^k) = V_t\left(\sum_{j=1}^{k} r_{t+j}\right) = \sum_{j=1}^{k} E(h_{t+j} | F_t)$
EXOGENOUS VARIABLES IN A GARCH MODEL

- Include predetermined variables into the variance equation
- Easy to estimate and forecast one step
- Multi-step forecasting is difficult
- Timing may not be right

\[
h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1} + \gamma z_{t-1}
\]

EXAMPLES

- Non-linear effects
- Deterministic Effects
- News from other markets
  - Heat waves vs. Meteor Showers
  - Other assets
  - Implied Volatilities
  - Index volatility
- MacroVariables or Events
PARAMETER ESTIMATION

- Historical data reveals when volatilities were large
- Pick parameters to match the historical volatility episodes
- Maximum Likelihood is the estimation procedure used by EVIEWS.

PLAUSIBLE ANSWERS

- WE EXPECT ALL THREE PARAMETERS OF A GARCH(1,1) TO BE POSITIVE.
- WE EXPECT THE SUM OF ALPHA AND BETA TO BE VERY CLOSE TO ONE BUT LESS THAN ONE.
- WE EXPECT THE UNCONDITIONAL VARIANCE TO BE CLOSE TO THE DATA VARIANCE.
DID THE ESTIMATION ALGORITHM CONVERGE?

- Generally the software will reliably find the maximum of the likelihood function and will report it.
- Sometimes it does not. You may get silly values. What then?
  - Check with other starting values
  - Check with other iterations
  - Scale the data so the numbers are not so small
- Often the problem is the data. Look for outliers or peculiar features.
  - Use longer data set

NORMALITY

- THIS ESTIMATION METHOD IS OPTIMAL IF THE ERRORS ARE NORMAL AND IF THE SAMPLE IS LARGE AND THE MODEL IS CORRECT.
- IT IS STILL GOOD WITHOUT NORMALITY
- BUT OTHER ESTIMATORS COULD BE BETTER SUCH AS STUDENT-T.
COMPARE MODELS

- MODELS WHICH ACHIEVE THE HIGHEST VALUE OF THE LOG LIKELIHOOD ARE PREFERRED.
- IF THEY HAVE DIFFERENT NUMBERS OF PARAMETERS – THIS IS NOT A FAIR COMPARISON.
  - USE AIC OR BIC (SCHWARZ) INSTEAD. THE SMALLEST VALUE IS BEST.

WHAT IS THE BEST MODEL?

- The most reliable and robust is GARCH(1,1)
- A student-t error assumption gives better estimates of tails.
- For equities asymmetry is almost always important. See asymmetric volatility section.
- For long term forecasts, a component model is often needed.
- Even better is a model which incorporates economic variables
III VALUE AT RISK ESTIMATION

- VALUE AT RISK
- GARCH
- ASYMMETRIC VOLATILITY
- DOWNSIDE RISK
- BUBBLES AND CRASHES

DOWNSIDE RISK

- THE RISK OF A PORTFOLIO IS THAT ITS VALUE WILL DECLINE, NOT THAT IT WILL INCREASE HENCE DOWNSIDE RISK IS NATURAL.
- MANY THEORIES AND MODELS ASSUME SYMMETRY: c.f. MARKOWITZ, TOBIN, SHARP AND VOLATILITY BASED RISK MANAGEMENT SYSTEMS.
- DO WE MISS ANYTHING IMPORTANT?
MEASURING DOWNSIDE RISK

- Many measures have been proposed. Let $r$ be the one period continuously compounded return with distribution $f(r)$ and mean zero. Let $x$ be a threshold.

$$Skewness = \frac{E(r^3)}{E(r^2)^{3/2}}$$

Probability of loss $= P(r < x)$,

Expected loss $= E(r|r < x)$

$x$ is the $\alpha$ Value at risk if $P(r < -x) = \alpha$

Value at Risk

- For a portfolio the future value is uncertain
- VaR is a number of $\$ that you can be 99% sure, is worse than what will happen.
- It is the 99% of the loss distribution (or the 1% quantile of the gain distribution)
- Simple idea, but how to calculate this?
This single number (a quantile) is used to represent a full distribution. It can be misleading.

- It fails to satisfy some basic principles in rather pathological cases.
- It assumes that you do not change your portfolio over the next day
  - Bad assumption for day traders and proprietary trading desks
  - Does not recognize role of dynamic hedging
  - But it can be calculated on a higher frequency too.
PUZZLER

- $1,000,000 portfolio of SP500.
- Find one day 99% VaR.
- First, just take a guess.

HISTORICAL VaR

- If History repeats, look at worst outcomes in the past
- For example, S&P500 over the last year.
- On a $1,000,000 portfolio, the 99% VaR is ?
HISTOGRAM OF S&P500 GAINS

1% quantile = -0.0445

HISTORICAL S&P500 VaR

- If I use 2 years of data, it is $34,411
- With 3 years, it is $44,594
- And with 30 years it is $30,108
- Which is more accurate?
Dow Jones 99% VaR USING ONE YEAR HISTORICAL QUANTILES

VAR

VAR
HOWEVER

- The extreme events of more than one year ago are not considered
- All outcomes of the last year are considered to be equally likely
- How should you pick the window?
- Look at a picture of VaR over time – it has a boxcar shape implying that risk goes down exactly one year after a big market decline.

VOLATILITY BASED VaR

- With a good volatility forecast, predict the standard deviation of tomorrow's return.
- Assume a Normal Distribution. Then

\[
\text{VaR} = 2.33 \times \sigma_t
\]

- But what do we use for the volatility?
- GARCH forecasts!
- Other volatility estimates?
GARCH MODEL FOR DJ

- Use for example data for 10 years (95-05)
- Forecast out of sample and record the daily standard deviation
- Multiply by 2.33
- We get

RESULTS

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<tr>
<th>GARCH MODEL</th>
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<tbody>
<tr>
<td>C</td>
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<tr>
<td>RESID(-1)^2</td>
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<tr>
<td>GARCH(-1)</td>
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<table>
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<tr>
<th>DATE</th>
<th>RETURN</th>
<th>DAILY SD</th>
<th>VaR</th>
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<td>NA</td>
<td>NA</td>
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<td>2005-01-10</td>
<td>NA</td>
<td>0.006166</td>
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VOLATILITY BASED VaR WITH STUDENT-T ERRORS

- Assume that: \( z_t^* \sim \text{Student} - t, \nu \)
- Because \( V(z^*) = \nu / (\nu - 2), \quad V\left(\frac{z^*}{\sqrt{\nu / (\nu - 2)}}\right) = 1 \)
- Then let \( r_t = \sqrt{h_t} z_t = \sqrt{h_t} z_t^* / \sqrt{\nu / (\nu - 2)} \)
- And estimate volatility and the shape of the error distribution jointly.
- In EViews =@qtdist(.01,\nu)/sqr[\nu/(\nu-2)]

STUDENT-T RESULTS

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<td>.01 QUANTILE OF UNIT STUDENT –T DISTRIBUTION(8.8DF) IS –2.49</td>
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<tr>
<td>DATE</td>
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VOLATILITY BASED VaR
WITHOUT NORMALITY or t

- What is the right multiplier for the true distribution? Maybe neither the normal nor the student t are correct!

- If: \( r_t = \sqrt{h_t} z_t, \quad z_t \sim i.i.d. \)

- Then 1% quantile of the standardized residuals should be used. This is the bootstrap estimator or Hull and White’s volatility adjustment.

HISTOGRAM OF STANDARDIZED RESIDUALS

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Series</td>
<td>GARCHRESID</td>
<td>Sample 1/09/1995 1/20/2005</td>
<td>Observations 2520</td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>-0.040388</td>
<td>Median</td>
<td>-0.021369</td>
<td>Maximum</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.999691</td>
<td>Std. Dev.</td>
<td>0.999691</td>
<td>Skewness</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>4.474130</td>
<td>Kurtosis</td>
<td>4.474130</td>
<td>Jarque-Bera</td>
</tr>
<tr>
<td>Probability</td>
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<td>Probability</td>
<td>0.000000</td>
<td></td>
</tr>
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</table>
BOOTSTRAP VaR

<table>
<thead>
<tr>
<th>DATE</th>
<th>RETURN</th>
<th>DAILY SD</th>
<th>VaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>2005-01-07</td>
<td>-0.001783</td>
<td>NA</td>
<td>NA</td>
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<tr>
<td>2005-01-10</td>
<td>NA</td>
<td>0.006166</td>
<td>15724.38</td>
</tr>
</tbody>
</table>

OVERVIEW AND REVIEW

- HISTORICAL QUANTILES – RESULT IS SENSITIVE TO SAMPLE INCLUDED
- VOLATILITY BASED
  - RESULT IS SENSITIVE TO THE ERROR DISTRIBUTION
  - NORMAL UNDERSTATES EXTREME RISK
  - T AND BOOTSTRAP ARE BETTER.
  - RESULTS ARE NOT SENSITIVE TO THE SAMPLE INCLUDED
ASymmetric Volatility Models

Positive and negative returns might have different weights.

For example:

$$h_t = \omega + \alpha_1 r_{t-1}^2 I_{r_{t-1}>0} + \alpha_2 r_{t-1}^2 I_{r_{t-1}<0} + \beta h_{t-1}$$

$$h_t = \omega + \alpha r_{t-1}^2 + \gamma r_{t-1}^2 I_{r_{t-1}<0} + \beta h_{t-1}$$

We typically find for equities that

$$\alpha_2 > \alpha_1 \ \text{or equivalently} \ \gamma > 0$$
Other Asymmetric Models

**EGARCH: NELSON (1989)**

\[
\log(h_t) = \omega + \beta \log(h_{t-1}) + \alpha \left| \frac{r_{t-1}}{\sqrt{h_{t-1}}} \right| + \gamma \frac{r_{t-1}}{\sqrt{h_{t-1}}}
\]

**NGARCH: ENGLE (1990)**

\[
h_t = \omega + \alpha (r_{t-1} - \gamma)^2 + \beta h_{t-1}
\]
WHERE DOES ASYMMETRIC VOLATILITY COME FROM?

- LEVERAGE - As equity prices fall the leverage of a firm increases so that the next shock has higher volatility on stock prices.
  - This effect is usually too small to explain what we see.

- RISK AVERSION – News of a future volatility event will lead to stock sales and price declines. Subsequently, the volatility event will occur. Since events are clustered, any news event will predict higher volatility in the future.
  - This effect is more plausible on broad market indices since these have systematic risk.
BACK TO VALUE AT RISK

- FIND QUANTILE OF FUTURE RETURNS
  - One day in advance
  - Many days in advance
- REGULATORY STANDARD IS 10 DAY 1% VaR.

MULTI-DAY VaR

- What is the risk over 10 days if you do no more trading? Clearly this is greater than for one day.

- Now we need the distribution of multi-day returns.
10 Day VaR

- If volatility were constant, then the multi-day volatility would simply require multiplying by the square root of the days.
- With normality and constant variance this becomes 7.36 or $\text{sqr(10)} \times 2.33$
- VaR is $7.36 \times \sigma$
  - What is $\sigma$?

MULTI-DAY HORIZONS

- Because volatility is dynamic and asymmetric, the lower tail is more extreme and the VaR should be greater.
TWO PERIOD RETURNS

- Two period return is the sum of two one period continuously compounded returns
- Look at binomial tree version
- Asymmetry gives negative skewness

MULTIPLIER FOR 10 DAYS

- For a 10 day 99% value at risk, conventional practice multiplies the daily standard deviation by 7.36
- For the same multiplier with asymmetric GARCH it is simulated from the example to be 7.88
- Bootstrapping from the residuals the multiplier becomes 8.52
CALCULATION BY SIMULATION

- Evaluate any measure by repeatedly simulating from the one period conditional distribution:
  \[ f_t(r_{t+1}) \]

- Method:
  - Draw \( r_{t+1} \)
  - Update density and draw observation \( t+2 \)
  - Continue until \( T \) returns are computed.
  - Repeat many times
  - Compute measure of downside risk

ESTIMATE TARCH MODEL

<table>
<thead>
<tr>
<th>VARIABLE</th>
<th>COEF</th>
<th>STERR</th>
<th>T-STAT</th>
<th>P-VALUE</th>
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<td>RESID(-1)^2</td>
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<table>
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<td>2005-01-07</td>
<td>0.006835</td>
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<td>2005-01-10</td>
<td>0.006726</td>
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</table>
With Asymmetric Volatility, the multi-period returns are asymmetric with a longer left tail.

For long horizons, the central limit theorem will reduce this effect and returns will be approximately normal.

This is observed in data too.
1 DAY RETURNS ON D.J.

Series: DJRET
Sample 1/03/1995 1/20/2005
Observations 2524

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Value</th>
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<tbody>
<tr>
<td>Mean</td>
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<tr>
<td>Median</td>
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<tr>
<td>Maximum</td>
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<tr>
<td>Minimum</td>
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<td>Std. Dev.</td>
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<tr>
<td>Skewness</td>
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<td>Kurtosis</td>
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<td>Jarque-Bera</td>
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<td>Probability</td>
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10 DAY RETURNS ON D.J.

Series: @MOVSUM(DJRET, 10)
Sample 1/03/1995 1/20/2005
Observations 2524

<table>
<thead>
<tr>
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<tr>
<td>Median</td>
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<td>Maximum</td>
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<td>1117.209</td>
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<td>Probability</td>
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</table>
EVIDENCE FROM DERIVATIVES

- The high price of out-of-the-money equity put options is well documented.
- This implies skewness in the risk neutral distribution.
- Much of this is probably due to skewness in the empirical distribution of returns.
- Data matches evidence that the option skew is only post 1987.
MATCHING THE STYLIZED FACTS

- ESTIMATE DAILY MODEL
- SIMULATE 250 CUMULATIVE RETURNS 10,000 TIMES WITH SEVERAL DATA GENERATING PROCESSES
- CALCULATE SKEWNESS AT EACH HORIZON
IMPLICATIONS

- Multi-period empirical returns are more skewed than one period returns (omitting 1987 crash)
- Asymmetric volatility is needed to explain this.
- Skewness has increased since 1987, particularly for longer horizons.
- Simulated skewness is noisy because higher moments do not exist when the persistence is so close to one. Presumably this is true for the data too.
- Many other asymmetric models could be compared on this basis.

SPIKES AND CRASHES

- RARE BUT EXTREME MOVES DOWN OR UP
- WHAT ARE THE CHARACTERISTICS OF SPIKES AND CRASHES?
- CAN WE PREDICT THEM?
CORRELATIONS

- Find the correlations between the indicator variable for a 3% crash and several independent variables
- 22 day average of down events
- 22 day average of up events
- Up within 3 days
- Exponential smoothed .06
- Exponential smoothed .02
- Garch volatility estimated from 1928-1990 and forecast thereafter
TARCH is estimated from 1928-1990 and forecast to 2006. Downma22 is a moving average of past down days, and similar is Upma22.

**Dependent Variable:** DOWN3
**Method:** ML - Binary Probit (Quadratic hill climbing)
**Sample (adjusted):** 1/02/1990 12/27/2005
**Convergence achieved after 8 iterations**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>z-Statistic</th>
<th>Prob.</th>
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<tbody>
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<td>1.983686</td>
<td>1.299162</td>
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</table>

Mean dependent var 0.006434
S.E. of regression 0.080011
Sum squared resid 25.83771
Rest. log likelihood -157.1161
LR statistic (4 df) 41.41845
Probability(LR stat) 2.20E-08

S.D. dependent var 0.079964
Akaike info crit. 0.069986
Schwarz criterion 0.077786
Hannan-Quinn crit. 0.072750
Avg. log likelihood -0.033756
McFadden R-squared 0.131808
CONCLUSION

- GARCH vols are the best predictors of an impending crash.
- Market declines also have short run predictability.
- Many other variables are also significant in 30’s. But results from 60-now are very similar.

CRASHES AND SPIKES

- The higher volatility, the more likely the market will crash or soar.
- Looking just at the crashes, they are likely preceded by high volatility and market declines.
- Spikes also are more likely with high volatility and market declines.
Markowitz Optimal Portfolios