Problem 1

a. Both the two series can be identified by the pattern of correlations.

series 1. AR(1)  series 2. MA(1).

b. For series 1, the PACF cuts off at lag 1 and ACF decays geometrically, thus it follows an AR(1) model;
For series 2, the ACF cuts off at lag 1 and PACF decays geometrically, thus it follows an MA(1) model.

Problem 2

a. The unconditional mean $\mu = \frac{\beta_0}{1-\beta_1} = \frac{1}{1-0.95} = 20$.

b. The unconditional variance $\sigma^2_y = \frac{\sigma^2}{1-\beta_1^2} = \frac{2}{1-0.95^2} = 20.5128$.

c. For $y_T = 9$, $E(y_{T+1}|y_T) = \beta_0 + \beta_1 y_T = 1 + 0.95 \times 9 = 9.55$.

d. $var(y_{T+1}|y_T = 4.1) = \sigma^2 = 2$.

e. The $k$–step ahead forecast of $y_{t+k}$ given $y_t = 9$ is

$y^k_t = \beta^k_1 y_t + (1 - \beta^k_1)\mu = 0.95^k \times 9 + (1 - 0.95^k)20$

f. The $k$–step ahead forecast error variance of $y_{t+k}$ given $y_t = 9$ is

$var(y^k_t) = \frac{1 - \beta_1^{2k}}{1 - \beta_1^2} \sigma^2 = \frac{1 - 0.95^{2k}}{1 - 0.95^2} \sigma^2$.$\sigma^2$

Problem 3

a. The unconditional mean $\mu = 0.5$.

b. The unconditional variance

$\sigma^2_y = (1 + \theta_1^2 + \theta_2^2)\sigma^2 = (1 + 0.9^2 + 0.2^2) \times 0.2^2 = 0.074$.

c. The 1,2,3 and 4 –step ahead forecast given $\epsilon_t = 0.8$ and $\epsilon_{t-1} = 0.7$ could be written as

$y^1_t = 0.5 + 0.9 \times 0.8 + 0.2 \times 0.7 + E(\epsilon_{t+1}) = 1.36$
\[ y_t^2 = 0.5 + 0.9 \times E(\epsilon_{t+1}) + 0.2 \times 0.8 + E(\epsilon_{t+2}) = 0.66 \]

\[ y_t^3 = 0.5 + 0.9 \times E(\epsilon_{t+2}) + 0.2 \times E(\epsilon_{t+1}) + E(\epsilon_{t+3}) = 0.5 \]

\[ y_t^4 = 0.5 + 0.9 \times E(\epsilon_{t+3}) + 0.2 \times E(\epsilon_{t+2}) + E(\epsilon_{t+4}) = 0.5 \]

d. Since a MA(2) could be expanded to a AR model with very high order lags, we expect lots of lags here.

**Problem 4**

a. \( H_0 : \beta_1 = 1 \).

b. Since the \( p-value = 0.9525 \), we fail to reject the null hypothesis. So the data appears to follow a random walk.

**Problem 5**

for part a and b, we give credits either you use k-month ahead forecast or k-day ahead forecast.

a. The log S&P500 follows a random walk process:

\[ y_t = \beta_0 + y_{t-1} + \epsilon_t, \quad \epsilon_t \sim iidN(0, \sigma^2) \]  \hspace{1cm} (1)

So the return

\[ r_t = y_t - y_{t-1} \sim N(\beta_0, \sigma^2) \]  \hspace{1cm} (2)

From the question we know that \( \beta_0 = 0.008 \) and \( \sigma = 0.0577 \). Since the initial value is \( \log(1331.29) \), the \( k \)-month ahead forecast of the log S&P 500 index level could be written as

\[ y_t^k = k\beta_0 + \log(1331.29) = 0.008k + \log(1331.29) \]

b. Observing the standard deviation of the return as 0.0577, the \( k \)-month ahead forecast error variance could be written as

\[ var(\epsilon_t^k) = k\sigma^2 = (0.0577)^2k \]

c. Assuming there are 30 trading days per month, the return over the next \( k \) days is

\[ r_t^k = y_t^k - y_t \sim N\left(\frac{0.008}{30}k, \frac{0.0577^2}{30}k\right) \]

\[ \sim N(0.000266k, 0.00011098k) \]  \hspace{1cm} (3)

So \( E(r_t) = 0.000266k \).

d. The 95 percent CI for the return over the next \( k \) days is \((0.000266k - 2 \times 0.0105\sqrt{k}, 0.008k + 2 \times 0.0105\sqrt{k})\).
Problem 6

a. From the estimation outputs we know our tarch(1,1) model could be written as:

\[ h_t = 1.8 \times 10^{-6} + 0.0211 r_{t-1}^2 + 0.1039 r_{t-1}^2 (r_{t-1} < 0) + 0.9131 h_{t-1} \]

So given our last observed return is \( r_t = -0.035 \) and \( \sqrt{h_t} = 0.024 \), the one step ahead forecast is

\[ h_{t+1} = 1.8 \times 10^{-6} + 0.0211(-0.035)^2 + 0.1039(-0.035)^2 \times (-0.035 < 0) + 0.9131(0.024^2) = 6.807 \times 10^{-4} \]

And \( \sqrt{h_{t+1}} = 0.0261 \).

b. The one day ahead 1% VaR is given by 2.33 \( \sqrt{h_{t+1}} = 2.33 \times 0.0261 = 0.0608 \).

c. Assuming the GARCH model above used a t–distribution with 5.73 degrees of freedom, The one day ahead 1% VaR is given by

\[ \sqrt{h_{t+1}} \frac{3.36}{\sqrt{5.73/(5.73-2)}} = 0.0261 \frac{3.36}{\sqrt{5.73/(5.73-2)}} = 0.0708 \]

d. -2.8% should be -2.8. But we give credits either you use -2.8 or -2.8%.

The bootstrapped one day ahead 1% VaR is given by 2.8 \( \sqrt{h_{t+1}} = 2.8 \times 0.0261 = 0.0731 \).

e. The bootstrapped one day ahead 1% VaR is larger than the 1% VaR with normal distribution assumption. So if the VaR from normal distribution case is used, the risk is understated.

If -2.8% is used, it is overstated.

Problem 7

a. \[
\text{cov}((r_{1,t}, r_{2,t})|F_{t-1}) = \text{cov}((\sqrt{h_{1,t}} z_{1,t}, \sqrt{h_{2,t}} z_{2,t})|F_{t-1}) = \sqrt{h_{1,t}} \sqrt{h_{2,t}} \text{cov}((z_{1,t}, z_{2,t})|F_{t-1}) = 0.5 \sqrt{h_{1,t}} \sqrt{h_{2,t}}
\]

b. \[
\text{corr}((r_{1,t}, r_{2,t})|F_{t-1}) = \frac{0.5 \sqrt{h_{1,t}} \sqrt{h_{2,t}}}{\sqrt{h_{1,t}} \sqrt{h_{2,t}}} = 0.5
\]

c. \[
E(p_t^2|F_{t-1}) = \text{var}(p_t|F_{t-1}) = 0.5^2 \text{var}(r_{1,t}|F_{t-1}) + 0.5^2 \text{var}(r_{2,t}|F_{t-1}) + 2 \times 0.5 \times 0.5 \text{cov}((r_{1,t}, r_{2,t})|F_{t-1})
\]
\[
= 0.25 h_{1,t} + 0.25 h_{2,t} + 0.25 \sqrt{h_{1,t}} \sqrt{h_{2,t}}
\] (4)

Problem 8

a. Since \( p_t = p_t \xi_t \), it is followed by

\[
\log(p_t^0) = \log(p_t) + \log(\xi_t) = \log(p_t) + \eta_t.
\]

So the observed continuously compounded returns are given by

\[
r_t = \Delta \log(p_t^0) = \epsilon_t + \eta_t - \eta_{t-1} = \epsilon_t + \beta \eta_{t-1} + \gamma_t - \eta_{t-1} = \epsilon_t + (\beta - 1) \eta_{t-1} + \gamma_t
\]
Similarly, the third covariance is given by:

\[
E(r_t r_{t-1}) = E[(\epsilon_t + (\beta - 1)\eta_{t-1} + \gamma_t)(\epsilon_{t-1} + (\beta - 1)\eta_{t-2} + \gamma_{t-1})]
\]

The second covariance is given by:

\[
E(r_t r_{t-2}) = E[(\epsilon_t + (\beta - 1)\eta_{t-1} + \gamma_t)(\epsilon_{t-2} + (\beta - 1)\eta_{t-3} + \gamma_{t-2})]
\]

The first covariance is given by:

\[
E(r_t r_{t-3}) = E[(\epsilon_t + (\beta - 1)\eta_{t-1} + \gamma_t)(\epsilon_{t-3} + (\beta - 1)\eta_{t-4} + \gamma_{t-3})]
\]

Noticing that \(E(\epsilon_t) = 0\), \(E(\gamma_t) = 0\), and \(E(\eta_{t-1}) = 0\), we have \(E(r_t) = 0\).

b. The unconditional variance of the observed continuously compounded returns is given by:

\[
\text{var}(r_t) = \text{var}(\epsilon_t + (\beta - 1)\eta_{t-1} + \gamma_t)
\]

Noticing the fact that there are no correlation between any of these three terms, we have

\[
\text{var}(r_t) = \text{var}(\epsilon_t) + (\beta - 1)^2\text{var}(\eta_{t-1}) + \text{var}(\gamma_t) = \sigma^2 + (\beta - 1)^2 \frac{\sigma^2}{1 - \beta^2} + \sigma^2_\gamma = \sigma^2 + \frac{2}{\beta^2 + 1} \sigma^2_\gamma
\]

c. For the derivations below, we are using the following properties:

\(E(\epsilon_t \epsilon_j) = \sigma^2\), if and only if \(i = j\),

\(E(\gamma_1 \gamma_j) = \sigma^2\), if and only if \(i = j\).

The first covariance is given by:

\[
E(r_t r_{t-1}) = E[(\epsilon_t + (\beta - 1)\eta_{t-1} + \gamma_t)(\epsilon_{t-1} + (\beta - 1)\eta_{t-2} + \gamma_{t-1})]
\]

The second covariance is given by:

\[
E(r_t r_{t-2}) = E[(\epsilon_t + (\beta - 1)\eta_{t-1} + \gamma_t)(\epsilon_{t-2} + (\beta - 1)\eta_{t-3} + \gamma_{t-2})]
\]

The third covariance is given by:

\[
E(r_t r_{t-3}) = E[(\epsilon_t + (\beta - 1)\eta_{t-1} + \gamma_t)(\epsilon_{t-3} + (\beta - 1)\eta_{t-4} + \gamma_{t-3})]
\]
And the fourth covariance is given by:

\[
E(r_t r_{t-4}) = E[(\epsilon_t + (\beta - 1) \eta_{t-1} + \gamma_4)(\epsilon_{t-4} + (\beta - 1) \eta_{t-5} + \gamma_{t-4})]
\]

\[
= E[(\epsilon_t + (\beta - 1)(\beta \eta_{t-2} + \gamma_{t-1}) + \gamma_4)(\epsilon_{t-4} + (\beta - 1) \eta_{t-5} + \gamma_{t-4})]
\]

\[
= E[(\epsilon_t + \beta(\beta - 1)(\beta \eta_{t-3} + \gamma_{t-2}) + (\beta - 1) \gamma_{t-1} + \gamma_4)(\epsilon_{t-4} + (\beta - 1) \eta_{t-5} + \gamma_{t-4})]
\]

\[
= E[(\epsilon_t + \beta^2(\beta - 1)(\beta \eta_{t-4} + \gamma_{t-3}) + (\beta - 1) \gamma_{t-2} + (\beta - 1) \gamma_{t-1} + \gamma_4)(\epsilon_{t-4} + (\beta - 1) \eta_{t-5} + \gamma_{t-4})]
\]

\[
= \beta^4(\beta - 1)^2 \text{var}(\eta_{t-5}) + \beta^3(\beta - 1)\text{var}(\gamma_{t-4})
\]

\[
= \beta^4(\beta - 1)^2 \frac{\sigma_\epsilon^2}{1 - \beta^2} + \beta^3(\beta - 1)\sigma_\gamma^2
\]

So the first 4 autocorrelations are:

\[
corr(r_t, r_{t-1}) = \frac{E(r_t r_{t-1})}{\text{var}(r_t)} = \frac{\beta(\beta - 1)^2}{\sigma^2} \frac{\sigma_\epsilon^2}{1 - \beta^2} + (\beta - 1)\sigma_\gamma^2 + \frac{\beta^2(\beta - 1)}{2} \frac{\sigma_\gamma^2}{1 + \beta} \frac{\sigma_\gamma^2}{\sigma^2}
\]

\[
corr(r_t, r_{t-2}) = \frac{E(r_t r_{t-2})}{\text{var}(r_t)} = \frac{\beta^2(\beta - 1)^2}{\sigma^2} \frac{\sigma_\epsilon^2}{1 - \beta^2} + \beta(\beta - 1)\sigma_\gamma^2 + \frac{\beta(\beta - 1)}{2} \frac{\sigma_\gamma^2}{1 + \beta} \frac{\sigma_\gamma^2}{\sigma^2}
\]

\[
corr(r_t, r_{t-3}) = \frac{E(r_t r_{t-3})}{\text{var}(r_t)} = \frac{\beta^3(\beta - 1)^2}{\sigma^2} \frac{\sigma_\epsilon^2}{1 - \beta^2} + \frac{\beta^2(\beta - 1)}{2} \frac{\sigma_\gamma^2}{1 + \beta} \frac{\sigma_\gamma^2}{\sigma^2}
\]

\[
corr(r_t, r_{t-4}) = \frac{E(r_t r_{t-4})}{\text{var}(r_t)} = \frac{\beta^4(\beta - 1)^2}{\sigma^2} \frac{\sigma_\epsilon^2}{1 - \beta^2} + \frac{\beta^3(\beta - 1)}{2} \frac{\sigma_\gamma^2}{1 + \beta} \frac{\sigma_\gamma^2}{\sigma^2}
\]

d. It is not difficult to guess the form of \(j\)th autocorrelation from the derivations above:

\[
corr(r_t, r_{t-j}) = \frac{E(r_t r_{t-j})}{\text{var}(r_t)} = \frac{\beta^j(\beta - 1)^2}{\sigma^2} \frac{\sigma_\epsilon^2}{1 - \beta^2} + \frac{\beta^{j-1}(\beta - 1)}{2} \frac{\sigma_\gamma^2}{1 + \beta} \frac{\sigma_\gamma^2}{\sigma^2}
\]

So we have

\[
corr(r_t, r_{t-j}) = \begin{cases} 
\frac{(\beta - 1)\sigma_\gamma^2}{(1 + \beta)\sigma^2 + 2\sigma_\gamma^2} & j=1 \\
\beta^{-1} \frac{(\beta - 1)\sigma_\gamma^2}{(1 + \beta)\sigma^2 + 2\sigma_\gamma^2} & j \geq 2 
\end{cases}
\]

The ACF slowly decays in the power of \(\beta\) starting with lag 2. Noticing the fact that ACF for AR(1) model decays exponentially starting with lag 1 while ACF for ARMA(1,1) decays exponentially starting with lag 2, an ARMA(1,1) model would be appropriate.