Integrating Financial and Operational Risks via Multi-Stage Stochastic Programming

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Outline

- Basic Issues
- Single-stage Results
- Multistage Framework
- Rational Tree Structure
- Debt Relationship
- Computational Results
- Conclusions
Basic Issues

- Financing affects production decisions (and v.v.)
- Traditional analyses ignore interactions
- No market valuation for private firms
- Multiples and discounted cash flows ignore contingency (option) value
- Bankruptcy provision necessary for evaluation
- Resulting models are large-scale mixed integer optimization problems
Financing Effects - Single Stage

News vendor with debt \((D)\) and equity \((E)\)

- Convert to risk-neutral equivalent demand distribution \((F)\)
- Suppose debt priced at market rate
- Find optimal mix of debt and equity

Basic single-stage problem:

\[
\begin{align*}
\text{maximize} & \quad p \left( \int_0^x s \, dF(s) + x \int_x^\infty dF(s) \right) - cx(1 + r_f) \\
\text{subject to} & \quad cx \leq E + (1 + r_f)^{-1} \left( \int_0^{s_b} ps \, dF(s) + D(1 + r) \int_{s_b}^\infty dF(s) \right), \\
x & \geq 0,
\end{align*}
\]

where \(k\) represents the initial cash position of the company.
Single Stage Model

No bankruptcy cost or tax shield
- \( D \) and \( E \) do not affect optimal solution
- Modigliani-Miller (MM) result that capital structure is irrelevant
- Problem: Financial distress, interest deductibility, varying tax rates for income, dividends, cap. gains

\[ \Rightarrow \text{Market imperfection} \]

Bankruptcy and tax assumptions
- Proportional bankruptcy cost \( \alpha \)
- Corporate tax rate \( \tau \)
- No loss carry-overs or personal tax effects
Single Stage Results

- Capital structure and production interdependent
- Leverage is a convex function of operating margin $c/p$
- Production more critical than capital structure
- Low-margin companies especially exposed to mis-specifying leverage and to agency costs

![Graphs showing capital structure and production levels for different values of $c/p$, $\sigma$, $\alpha$, and $\tau$.]
Multiple Stage Framework

Key Observations

- Equity and debt investments over time
- Special relevance for valuation of private equity
- Need to include bankruptcy possibility
- Need to include growth and contingencies

Traditional valuation models

- Discount dividend models ($V_E^0$ equity value at 0, $\tilde{d}_t$ dividend, $\tilde{T}V_T$ terminal value, $\rho_e$ discount factor)

$$V_E^0 = \sum_{t=1}^{T} \rho_e^{-t} E_0(\tilde{d}_t) + \rho_e^{-T} E_0(\tilde{T}V_T)$$

- Multiple models ($V_F$ firm value, $S_F$ firm parameter (e.g., sales), $Comp$ - competitor values)

$$V_F = V_{Comp} \times \text{relative size} = V_{Comp} \times \frac{S_F}{S_{Comp}} = \frac{V_{Comp}}{S_{Comp}} \times S_F$$
Issues with Traditional Models

Discount dividend
- Uses only expectations
- No contingent plans
- No bankruptcy costs

Multiple models
- No individual treatment
- Limited number of parameters
- Weights based on accounting data (accuracy, comparability)

$\Rightarrow$ create model with contingencies, bankruptcy, growth, and multiples at horizon
Max \[
\sum_{t=0}^{H} \sum_{s \in \mathcal{S}_t} e^{-r_t} p_r^{s,t} (e_{-}^{s,t} - e_{+}^{s,t}) + e^{-r_T} \sum_{s \in \mathcal{S}_T} p_r^{s,T} \theta(x_{+}^{s,T}, u_{+}^{s,T})
\]

s.t. \[
x_{s,-t-1}^{s,t-1} + y_{s,-t-1}^{s,t-1} \geq x_{s,t}^{s,t} + z_{s,t}^{s,t} \quad \forall \ t \ s
\]
\[
u_{+}^{s,t-1} + e_{+}^{s,t-1} - e_{-}^{s,t-1} - r_{s,-t-1}^{s,t-1} d^{s,t-1} - K I^{s,-t-1} - c y_{s,-t-1}^{s,t} - c y_{s,-t-1}^{s,t-1}
\]
\[
+p z_{s,t}^{s,t-1} - \tau i_{s,t}^{s,t} \geq u_{+}^{s,t} - u_{-}^{s,t} \quad \forall \ t \ s
\]
\[
i_{s,t}^{s,t} - [(p - c)z_{s,t}^{s,t-1} - r_{s,-t-1}^{s,t-1} d^{s,t-1} - K I^{s,-t-1}] \geq 0 \quad \forall \ t \ s
\]
\[
u_{+}^{s,t} + e_{+}^{s,t} - e_{-}^{s,t} + d_{s,t}^{s,t} - c y_{s,t}^{s,t} - K I^{s,t} \geq 0 \quad \forall \ t \ s
\]
\[
M I^{s,t} - \left[ d_{s,t}^{s,t} + e_{-}^{s,t} + e_{+}^{s,t} + x_{s,t}^{s,t} + y_{s,t}^{s,t} + z_{s,t}^{s,t} + i_{s,t}^{s,t} + u_{+}^{s,t} \right] \geq 0 \quad \forall \ t \ s
\]
\[
M (1 - I^{s,t}) - u_{-}^{s,t} \geq 0 \quad \forall \ t \ s
\]
\[
I^{s,-t-1} - I^{s,t} \geq 0 \quad \forall \ t \ s
\]
\[
z_{s,t}^{s,t} \leq q_{s,t}^{s,t} \quad \forall \ t \ s
\]
\[
d^{s,-t-1} (1 + r_{s,-t-1}^{s,t-1}) \int_{q_{b}^{s,t}}^{\infty} dF_t(q) + \alpha p \int_{0}^{q_{b}^{s,t}} q dF_t(q) = d^{s,-t-1} (1 + r_f) \quad \forall \ t \ s
\]
\[
y_{s,t}^{s,t} \geq 0 \quad x_{s,t}^{s,t} \geq 0 \quad U_E \geq e_{+}^{s,t} \geq 0 \quad e_{-}^{s,t} \geq 0, \quad I^{s,t} \in \{0, 1\}, \quad \forall \ t \ s
\]
\[
u_{+}^{s,t} \geq 0 \quad u_{-}^{s,t} \geq 0 \quad i_{s,t}^{s,t} \geq 0 \quad z_{s,t}^{s,t} \geq 0 \quad U_d \geq d_{s,t}^{s,t} \geq 0 \quad \forall \ t \ s.
Integer Issues

Variables $I_{s,t}^t$
- Zero value corresponds to bankruptcy at $t$ under $s$
- All subsequent production must end
- Exponential size growth in stages and scenarios

Optimal solution properties - structural result
- Branches with continued operations at optimality form rational tree
- For Markovian problems where future only depends on current demand state:
  
  Bankruptcy in demand state $s$ at $t$ $\Rightarrow$ bankruptcy in all $s' \leq s$ at $t$
Implications of rational trees

- Only consider variables for rational trees
- At highest demand point $s$ with bankruptcy, sales exactly cover debt face value
- Can solve for nonlinear interest rate function $r_t(D)$ for each rational tree at time $t$

\[
(1 + r_f) D = (1 + r_t(D)) D \int_{q^{b(r_t(D))}}^{\infty} dF_t(q) + \alpha \int_{0}^{q^{b(r_t(D))}} pqdF_t(q)
\]

Methodology

- Pick rational tree
- Solve for given interest rates $r_t(D)$ at time $t$
- Solve linear program
- Update tree (and/or prune)
Three Stage Example

Key Observations

- Varying interest rates for time and scenario
- Allows bankruptcy in one of middle branches (but not all)
- Value increase over two-stage and traditional methods

\[ D: \ 110.9, \ r_0: \ 19\% \]
\[ E: \ 59.6 \]

Terminal Value

- 61.1, 5%  
  - 116.7
  - 185 (Demand)
  - 133
    - Bankrupt 34

- 144.9, 18%
  - 68.64
    - 178
      - 46
      - 0 Bankrupt
**Multistage Results**

Equity value for stochastic program, discount dividend, and multiple methods as function of parameters

- Decreasing in production cost, volatility
- Increasing in bankruptcy recovery, terminal value multiple
- Wider gaps for high margins, large volatility, recovery, terminal value
Multistage Compared to Single Stage

Observations:

- Equity value increases with multiple stages
- Leverage decreases with multiple stages
- Bigger gaps for higher margins (lower costs)
Conclusions

Results:
- Can model production and financing decisions consistently
- Especially useful for private equity
- Use problem structure to reduce complexity of binary decisions on bankruptcy
- Greatest long-term gains for high-margin over single stage (but low-margin loses most by not integrating)

Caveats:
- Assume full disclosure on debt offer
- No agency cost in this model (n.b., high cost for low margins in single stage)
- Only single-period debt (no explicit issuing cost)
- No explicit investment timing
- Structural result for one product only
- No competitor and supply chain interactions (in process)