Problem Set 1
Due Thursday, Week 2

Note: There are a lot of problems, but they’re all easy, and I’m only looking for very short answers.

Part I.
Do problems 1, 4, 8, 9 in Ch 1. Asset Pricing p. 31. In 4, the suggestion is that we can get the Sharpe ratio indicated by the following graph:

\[
\sigma(R) \quad \text{or} \quad \beta
\]

\[
E(R)
\]

In 9, find the mean log return of portfolios \(dP/P = \alpha dR^m + (1 - \alpha) r^I dt\), use reasonable numbers to plot the mean log return as a function of \(\beta\), and find the composition of the growth-optimal portfolio, where \(R^m\) denotes the market return.

Additional problems:

1. Prove that power utility reduces to log if \(\gamma \to 1\). (Hint: You’re always allowed to add constants to utility functions, so write the power function as \(\frac{c^{1-\gamma} - 1}{1-\gamma}\). Hint 2: \(d (c^{1-\gamma})/dc = (1 - \gamma)c^{-\gamma}\) but \(d (c^{1-\gamma})/d\gamma \neq (1 - \gamma)c^{-\gamma}\))

2. Risk aversion coefficients

(a) Suppose you only care about consumption tomorrow, so the utility function is simply \(E[u(c)]\). You have enough income to support consumption \(\bar{c}\) for sure.

i. Suppose you are forced to gamble a small fraction of \(\epsilon\) of future consumption, thus the gamble is \(\pm \epsilon\bar{c}\) with 50% chance of each and your new consumption level will be \(c = \bar{c} + \epsilon\bar{c}\) or \(c = \bar{c} - \epsilon\bar{c}\) with equal probability. How much utility do you lose? Use a second order Taylor approximation to show that the answer is

\[
Eu(c) - u(\bar{c}) = \frac{1}{2}u''(\bar{c})\epsilon^2
\]
ii. Similarly, find out how much better off you are if you are given a gift $\delta c$. (You only need a first order expansion here.)

iii. Now, how much gift $\delta c$ to I have to give you in order to compensate you for taking on the risk $\varepsilon$? Equate your answer in i and ii, solve for $\delta$ in terms of $\varepsilon$. Optional: express your answer in terms of the mean and variance of the risk.

iv. What if you are asked to evaluate a dollar risk $c = \bar{c} + \varepsilon$ or $c = \bar{c} - \varepsilon$ and a dollar bribe $\delta$?

3. An investor has utility function

$$u(c_{t+1}) = \frac{1-\gamma}{1-\gamma}$$

(a) Suppose the investor will consume $50,000 next year. How much would he be willing to give up at $t+1$ to avoid an even bet of gaining or losing $5,50,500,5000$ if $\gamma = 0, 1, 2, 10, 50$?

Hints: To answer this question, equate utility of $50,000$ minus $x$ for sure to the expected utility of winning and losing the bet. It’s prettier if you set the problem up as, what fraction of consumption $x$ would you give up for sure to avoid a bet on a fraction of consumption $y$, i.e. how much would you give up for sure $xc$ to avoid the 50/50 chance of gaining or losing $yc$.

(b) How much would you give up in this situation? Think of $t + 1$ as the rest of your life – right now you expect to consume $50,000 per year for the rest of your life. How much are you willing to give up on average not to take the bet that will raise/lower your consumption by the $5-500$ amounts? What is your risk aversion? (Don’t be surprised if your answers are not consistent across the amounts of money involved.)

4. Suppose an investor has leverage, he must pay back an amount $X$. Equivalently, suppose $X$ represents a backstop level of consumption that the investor is simply not willing to risk no matter what. (“I’d rather die than fly commercial, honey.”)

Now the utility function is

$$u(c) = \frac{(c - X)^{1-\gamma}}{1-\gamma}$$

(a) Plot this utility function. (Use $\gamma = 2$ if you want to use a computer. Freehand sketch is fine too.)

(b) What is the risk aversion coefficient $-\frac{cu''(c)}{u'(c)}$ for this investor? Try to make your formula pretty, showing how $\gamma$ is modified by the ratio $c/X$.

(c) If this investor has a loss, so that it is likely $c$ will be much closer to $X$, does this make him more or less risk averse?
This is an important problem. I think it captures a lot of what happened in the fall of 2008. Many investors have leverage or backstop commitments (mortgages) or even an accustomed level of consumption $X$. As they lose money, they become more risk averse and sell, making markets go down further. There’s nothing “irrational” about it – if you’re leveraged, you have to sell after a loss.

Part II. Some continuous-time problems to get you a bit more familiar with $dz$ and $dt$ and a cool paradox of long term returns.

1. What’s wrong with this:

$$\frac{dx}{x} = \sigma dz$$

$$\int \frac{dx}{x} = \sigma \int dz$$

Look up any integral table and

$$\int_{x_1}^{x_2} \frac{dx}{x} = \ln x_2 - \ln x_1$$

Thus,

$$\ln x_T - \ln x_0 = \sigma (z - z_0).$$

In the notes I came up with an additional $\frac{1}{2}\sigma^2$ term. Where is the mistake?

2. Comparing geometric and arithmetic returns. Consider a stock with zero dividend yield and price process

$$dP = \mu Pdt + \sigma Pdz$$

where $\mu$, $\sigma$, are constants.

(a) Find the instantaneous arithmetic $E(dP/P)$ and geometric $E(d\log P)$ mean stock return. Explain their difference.

(b) Does the difference between arithmetic and geometric mean matter much? Use values for $\sigma$ corresponding to i) the market return which varies about 1% per day ii) a small growth stock which can vary 7% per day. (Transform to annual units. Yes, there are stocks that vary 7% per day, don’t be scared by the annualized volatility.)

3. Suppose that log prices $p$ follow a diffusion with stochastic volatility,

$$dp_t = \mu t + \sigma_t dz_t$$

$$d\sigma_t = v dz_t$$
(Yes, this $\sigma$ can be both positive and negative, but the variance $\sigma^2$ will always be positive. $\sigma_t$ is also unpleasantly nonstationary. A more reasonable specification would have a square root process in the second equation, and a second $dz$, and start with $dp/p$ rather than log prices, but I want to keep this problem simple.) Solve this system — find $p_t$ in terms of $p_0$, $z_t$, $z_0$, $\sigma_0$. Is $p_t$ still normally or lognormally distributed? Hint: First solve $\sigma_s$ in terms of $z_s - z_0$, and plug that back in the $p$ equation. To evaluate $\int_{s=0}^{t} z_s dz_s$ apply Ito’s Lemma to $d(z^2)$. Of course $\int_{s=0}^{t} d(z^2) = z_t^2 - z_0^2$. You will get an answer with $z_t^2 - z_0^2$ as well as $z_t - z_0$ on the right hand side.

4. Solve the continuous-time AR(1),

$$dx_t = -\phi x_t dt + \sigma dz_t$$

to do this, first find

$$d\left(e^{\phi t} x_t\right)$$

from Ito’s lemma. Then you can just integrate. (“solve” means express it with $x_t$ on the left, and some integral against $dz_t$ on the right.) Does the answer remind you of a discrete-time result? In a later problem set we’ll come back to this question and see how could you solve this if you weren’t so clever as to try $d\left(e^{\phi t} x_t\right)$.

5. A useful trick. Prove that if $x$ is normal, $E(e^x) = e^{\mu + \frac{1}{2}\sigma^2}$. a) Write the definition of $E(e^x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{x} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$. Merge the two exponentials, pull a $e^{\mu + \frac{1}{2}\sigma^2}$ out of the integral, leaving an integral of a new normal distribution. The integral of a normal distribution is one, so you’re done.