Problem Set 2
Due Tuesday, October 11

1. Do problem 1, Ch 2 of Asset Pricing

2. Do problem 2 in Ch 2 Asset Pricing Don’t worry about the technicalities needed to get to the present value form of the resource constraint,

\[ k_t + \sum_{j=0}^{\infty} \beta^{j+1} E_t c_{t+j} = \sum_{j=0}^{\infty} \beta^{j+1} E_t c_{t+j}. \]

Yes, a “transversality condition” \( \lim_{T \to \infty} \beta^T k_{t+T} = 0 \) is needed, and yes, there is some danger of the consumer just sitting at his bliss point \( c^*. \) Treat consumption as not freely disposable and otherwise ignore these problems. (To get to 2.7 just take innovations of 2.6 without doing lots of algebra.)

(a) Suppose instead the quadratic utility investor of problem 2 lives in an endowment economy, in which the endowment is given by

\[ c_t = c_{t-1} + \frac{r \beta}{1-\beta \rho} \varepsilon_t \]

where now the shock is simply the shock to the endowment, without any connection to an income stream (which doesn’t exist). How does this change affect your pricing formulas. How bad a mistake is it to “assume an endowment economy” when in fact the true economy generates consumption from production?

3. Repeat problem 2 in continuous time. The problem is

\[ \max -\frac{1}{2} E \int e^{-rt} (c^* - c_t)^2 \text{ s.t.} \]

\[ dk_t = (rk_t + y_t - c_t) \, dt \]

\[ dy_t = -\rho y_t \, dt + \sigma dz_t \]

(I use \( y \) rather than \( e \) for the endowment). You follow the same steps

(a) Express the resource constraint in “present value” form;

(b) Find the first order conditions relating \( c_t \) and \( c_{t+k} \) by a simple variational argument (no need to do value functions here)

(c) Specialize to \( c_t \) and \( c_{t+\Delta} \), and thus express the first-order condition as a restriction on \( dc_t \) (the “random walk”) in continuous time.

(d) Plug from the first order condition into the “present value” resource constraint to obtain the continuous-time analogues to (2.6) and (2.7). (You will find (2.7) from (2.6) by taking its \( d \), i.e. \( dc_t = rdk_t + \ldots \) not by “taking innovations” \( E_{t+1} - E_t \) The operation \( E_{t+\Delta} - E_t \) really doesn’t exist in continuous time.)

(e) Argue that the risk-free rate must be \( r^f_t = r \)

(f) Find the price of the consumption stream. From step d you will have a consumption process that will let you evaluate the required \( E_t c_{t+j}^2 \) terms.

This looks daunting but it is actually much easier in continuous time than discrete. But you have to think straight about how it works!
4. (This is an improved version of problem 3, Ch2). In this problem you will construct a complete simple simulation economy. The economy has either good or bad times; consumption either grows by \( u = 1.06 \) (i.e. 6\%) or goes down by \( d = 0.98 \) (down 2\%). If the economy growing in the \( u \) state, it has probability \( \pi_{u \rightarrow u} = (1 + x) / 2 \) of remaining in the good state, and probability \( \pi_{u \rightarrow d} = (1 - x) / 2 \) of going to the bad state. \( x \) is a parameter which we’ll specify later. If the economy is in the bad state \( d \) it has probability \( \pi_{d \rightarrow u} = (1 - x) / 2 \) of going to the good state, and probability \( \pi_{d \rightarrow d} = (1 + x) / 2 \) of remaining in the bad state. As you can see, the parameter \( x \) controls the persistence of the state of the economy. When \( x = 0 \), consumption growth is independent over time like a coin flip, all probabilities are 1/2. As \( x \) increases, once you are in a good or bad state, you are more and more likely to stay there. Investors have power utility

\[
E \sum_{j=0}^{\infty} \beta^j \frac{c_{t+j}}{1 - \gamma}
\]

with \( \beta = 0.99 \)

I’ll show you how to find the price of a perpetuity, i.e. a bond that pays a coupon of $1 per period forever. You will use the same ideas to find the one period interest rate and the stock price. The value of a perpetuity at any date is

\[
p_t^P = E_t \sum_{j=1}^{\infty} \beta^j \left( \frac{c_{t+j}}{c_t} \right)^{-\gamma} \times 1
\]

A direct attack leads to a mountain of algebra. Instead, think about the one period relation

\[
p_t^P = E_t \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \left( 1 + p_{t+1}^P \right) \right].
\]

All information about the future in this economy is contained in the current value of consumption growth. (Consumption growth is a state variable for this economy.) Thus \( p_t^P \) and \( p_{t+1}^P \) can only depend on consumption growth at \( t \) and \( t + 1 \) respectively. \( p^P \) can only take on two values, \( p^P(u) \) in the good state and \( p^P(d) \) in the bad state. We can find these prices as follows

\[
\begin{align*}
\left\{ \begin{array}{l}
p^P(u) = \beta \pi_{u \rightarrow u} u^{-\gamma} [1 + p^P(u)] + \beta \pi_{u \rightarrow d} d^{-\gamma} [1 + p^P(d)] \\
p^P(d) = \beta \pi_{d \rightarrow u} u^{-\gamma} [1 + p^P(u)] + \beta \pi_{d \rightarrow d} d^{-\gamma} [1 + p^P(d)]
\end{array} \right.
\end{align*}
\]

This is a system of equations in the two unknowns \( p^P(u) \) and \( p^P(d) \). One fun way to solve this is to start with \( p^P(u) = 0 \) and \( p^P(d) = 0 \) on the right hand side, plug through to find new values on the left hand side, use those new values on the right hand side, and so on. This will converge, because you’re finding at each step \( j \) the value of a series of coupons that lasts \( j \) periods. This approach is useful in more complicated and nonlinear model. Here though, the system is linear and it’s easy enough to just solve for \( p^P(u) \) and \( p^P(d) \).

\[
\begin{bmatrix}
1 - \beta \pi_{u \rightarrow u} u^{-\gamma} & -\beta \pi_{u \rightarrow d} d^{-\gamma} \\
-\beta \pi_{d \rightarrow u} u^{-\gamma} & 1 - \beta \pi_{d \rightarrow d} d^{-\gamma}
\end{bmatrix}
\begin{bmatrix}
p^P(u) \\
p^P(d)
\end{bmatrix}
= \begin{bmatrix}
\beta \pi_{u \rightarrow u} u^{-\gamma} + \beta \pi_{u \rightarrow d} d^{-\gamma} \\
\beta \pi_{d \rightarrow u} u^{-\gamma} + \beta \pi_{d \rightarrow d} d^{-\gamma}
\end{bmatrix}
\]

\[
\begin{bmatrix}
p^P(u) \\
p^P(d)
\end{bmatrix}
= \begin{bmatrix}
1 - \beta \pi_{u \rightarrow u} u^{-\gamma} & -\beta \pi_{u \rightarrow d} d^{-\gamma} \\
-\beta \pi_{d \rightarrow u} u^{-\gamma} & 1 - \beta \pi_{d \rightarrow d} d^{-\gamma}
\end{bmatrix}^{-1}
\begin{bmatrix}
\beta \pi_{u \rightarrow u} u^{-\gamma} + \beta \pi_{u \rightarrow d} d^{-\gamma} \\
\beta \pi_{d \rightarrow u} u^{-\gamma} + \beta \pi_{d \rightarrow d} d^{-\gamma}
\end{bmatrix}.
\]

I couldn’t make it look pretty by expanding the right hand side, but this is good enough to plug in numbers and find the price in the two states.
Your objective is to write a little program that will find the numbers, such as \( p^u(d) \) and \( p^d(d) \) for given values of the parameters. You do not have to find pretty algebraic expressions. Start your program with \( \gamma = 1 \) and \( x = 0 \) (but leave these parameters as variables so we can change them later.)

(a) Now, it’s your turn. Find the one period interest rate in this economy. Like everything else, the interest rate can take on only two values, one value in the good state and a different value in the bad state. In each state, the interest rate is

\[
R_t^f = \frac{1}{E_t} \left[ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \right]
\]

Thus, your job is to find the two values for the interest rate \( R^f(u), R^f(d) \).

(b) To model a stock, we can think of an asset that pays consumption as its dividend. The value of the whole U.S. economy is the value of a claim to the consumption it can provide us. Now, the stock price

\[
p_t = E_t \sum_{j=1}^{\infty} \beta^j \left( \frac{c_{t+j}}{c_t} \right)^{-\gamma} c_{t+j}
\]

depends on the level of consumption, not just the growth rate at each date. However, we can apply the same trick as for the perpetuity to the ratio of price to consumption, i.e. the price/dividend ratio of the stock,

\[
\frac{p_t}{c_t} = E_t \left[ m_{t+1} \left( \frac{p_{t+1} + c_{t+1}}{c_{t+1}} \right) \right]
\]

\[
\frac{p_t}{c_t} = E_t \left[ \beta \left( \frac{p_{t+1}}{c_{t+1}} + 1 \right) \left( \frac{c_{t+1}}{c_t} \right)^{1-\gamma} \right]
\]

\( \frac{p_t}{c_t} \) can take on only two values, \( \frac{p_t}{c_t}(u) \) and \( \frac{p_t}{c_t}(d) \). Proceed as with the perpetuity to find those two values.

(c) Find stock returns and excess returns (return-riskfree rate), and the long term bond returns and excess returns. For each one, you have to find four values, the return from each state u and d to states u and d. Hint: for the stock return:

\[
R_{t+1} = \frac{p_{t+1} + c_{t+1}}{p_t} = \left( \frac{p_{t+1}}{c_{t+1}} + 1 \right) \left( \frac{c_{t+1}}{c_t} \right)
\]

(d) Find the expected stock and perpetuity returns in the two states. If something takes two values \( x(u), x(d) \), then its expected value if today is \( u \) is \( E_t(x) = \pi_{u \rightarrow u} x(u) + \pi_{ud} x(d) \).

(e) Find these quantities for the following parameter configurations:

i. \( \gamma = 1, x = 0 \)

ii. \( \gamma = 1, x = 0.3 \). Motivation: everything looks the same across states in i.

iii. \( \gamma = 5, x = 0.3 \). Motivation: maybe raising the risk aversion coefficient will give us a more reasonable expected stock excess returns.

iv. \( \gamma = 5, x = -0.3 \). Motivation: iii gives you a negative equity premium, maybe reversing \( x \) will fix that.

For each set of parameters, make a table with the bond price, the stock price/consumption ratio, the riskfree rate, the expected excess stock and bond returns, and the actual stock and
bond returns to the two following states. For example, here is a table of my results for the first question. (I report returns as percentages, \( r = (R - 1) \times 100 \).) Explain the behavior of the variables. Why are \( p \) or \( p/c \) higher or lower in one state than the other? Why is the equity or bond expected excess return positive or negative? Use the economics of the class and readings, for example starting with the intuition behind \( R^f \approx 1 + \delta + \gamma E_i (\Delta c_{t+1}) \) to explain interest rate behavior. (You may find it helpful to print out extra variables, like consumption growth, expected consumption growth, returns rather than just excess returns, and so forth. You may also find it useful to simulate some data and plot the variables over time. However, this is not required.) What we want here is to find parameters that generate prices and returns like the ones we see in the data – a decent equity premium, slow-moving price/dividend ratios that forecast returns, and so forth. We won’t get there, but we’ll learn how hard it is!

Results for gamma x

\[
\begin{array}{c c c c c}
\text{Bond} & \text{p} & \text{p/c} & \text{Rf} & \text{ER-Rf} & \text{ERp-Rf} \\
good state & u & 34.82 & 99.00 & 2.87 & 0.16 & 0.00 \\
bad state & d & 34.82 & 99.00 & 2.87 & 0.16 & 0.00 \\
\end{array}
\]

stock excess bond excess
To state: u d u d
From u: 4.20 -3.88 0.00 0.00
From d: 4.20 -3.88 0.00 0.00