Problem Set 3  
Due Tuesday, Oct. 18  

1. Problem 1 in Ch 4 Asset Pricing. Note: Some other authors define absence of arbitrage to include the LOOP in which case AA—LOOP is trivial. I didn’t, so for this problem the definition of AA only involves the price of nonnegative payoffs. The answer is that neither NA nor LOOP imply the other. So, your job is to construct examples (i.e. describe a \(\{x, p(x)\}\) with the required properties.)

(a) An example in which LOOP holds but NA doesn’t.
(b) An example in which NA holds but LOOP doesn’t.

Hint: There are examples in which all assets are traded, but it’s easier and more realistic to generate examples with a short-sale constraint or fixed cost of transacting.

2.  
(a) Problem 2 in Ch 4 Asset Pricing.
(b) Also answer, “if the law of one price or absence of arbitrage do not hold in population, will you see that failure in a finite sample drawn from that population.”

The issue here is that in a sample you don’t see every “state” envisioned by a population. Hint: how well did risk management based on empirical probabilities work during the 2008 financial crisis? As usual, if the answer is “yes” give a proof, if the answer is “no” provide a counterexample. (If you observe \(x+y=z\) in a sample, must you see \(p(x+y) = p(z)\)?)

3. Instead of Problem 3 in Ch 4 Asset Pricing, do this restated and improved version of the problem. It explores a third source of inspiration for discount factors, \(m_{t+1} = R_{t+1}^{-1}\)

(a) Suppose you have a single return \(R\). Find \(x^*\) Is \(x^* > 0\)? (Hint. Sometimes yes, sometimes no, think about which cases apply when.)
(b) \(m = 1/R\) is a discount factor since \(1 = E_t(R_{t+1}^{-1}R_{t+1})\). How does this relate to the discount factors you know: Is \(R_{t+1}^{-1} = x^*?\) Is \(R_{t+1}^{-1} \in \mathbb{X}\)? Is \(R_{t+1}^{-1} = m \geq 0\) as promised by the no-arbitrage theorem? Are there kinds of return for which \(R_{t+1}^{-1}\) won’t work as a discount factor?
(c) Generalizing, show that if you have a single asset with \(R_{t+1} > 0\), then you can always discount with its ex-post return,

\[
P_t = \sum_{j=1}^{\infty} \left( \Pi_{k=1}^{j} \frac{1}{R_{t+k}} \right) D_{t+j}
\]

Show that this expression holds ex-post as well as with \(E_t\) in front of the summation. Does \(E_t\) need to be the true probability?
(d) Now we need to generalize to multiple assets. Let \(R\) now denote an \(N \times 1\) vector of asset returns. Show that the inverse of the “growth optimal portfolio” return \(\alpha^* R\) that solves

\[
\max_{\{\alpha\}} E[\ln(\alpha' R)] \quad \text{ s.t. } \alpha' 1 = 1
\]

is a discount factor which prices all the returns. (Note a typo in the book, it should be “the inverse of the return on the portfolio... ” as it is here.) You do not have to solve for the weights \(\alpha^*\), just show that \(m = (\alpha^* R)^{-1}\) is a discount factor.
(e) A more beautiful way to set up the problem: Consider any discount factor \(m_{t+1}\) that prices a large number of assets. Show that the random variable \(R_{t+1} = m_{t+1}^{-1}\)
i. is a rate of return (i.e. has price 1).
ii. \( R_{t+1}^{-1} \) is a discount factor.
iii. \( R_{t+1} \) is the “growth optimal” return among all returns priced by this discount factor, i.e. it solves
\[
\max E [\ln(R)] \text{ s.t. } 1 = E(mR).
\]
specifically, this means we pick \( R \) state by state
\[
\max_{(R(s))} \sum_s \pi(s) \ln(R(s)) \text{ s.t. } 1 = \sum_s \pi(s)m(s)R(s)
\]
(look carefully at how this maximization is different from the last one. It’s not the same problem!)

(f) Though the “growth optimal” portfolio has a mystical place in finance, there really isn’t much special going on here. Show that the marginal utility of any optimal portfolio calculation is a discount factor. If we maximize
\[
\max_{\{\alpha\}} E [u'(\alpha'R)] \text{ s.t. } \alpha'1 = 1,
\]
show that \( u'(c'R) \) is a discount factor. Thus,

i. Show that we can define a whole family of discount factors for \( u'(c) = c^{-\gamma} \).
ii. But what if markets are complete? How can there be more than one discount factor? (no big algebra needed, just explain the puzzle.)

(g) Use the approach in part e to find a generalization for continuous time with many assets. Start with a discount factor \( \Lambda_t \) that prices a large set (more than one) asset return.

i. Show that \( V_t/V_0 = \Lambda_0/\Lambda_t \) is a value process. (This is like e part i)
ii. Show that \( (V_t/V_0)^{-1} \) is a discount factor for any security priced by \( \Lambda \).
iii. Show that \( V_t/V_0 = \Lambda_0/\Lambda_t \) is the “growth optimal” value process, that solves
\[
\max E \left[ \ln \left( \frac{V_T}{V_0} \right) \right] \text{ s.t. } V_0\Lambda_0 = E_t [V_T\Lambda_T].
\]

(h) In this case, we can easily answer the puzzling questions from discrete time. (These are important, take your time):

i. Is the discount factor \( (V_t/V_0)^{-1} \) positive?
ii. Is the discount factor \( (V_t/V_0)^{-1} \) traded – is it in the continuous time equivalent of \( x^* \)?
iii. We know that \( V \) is a value process, but is it traded? If we started with a set of securities that are less than contingent claims, does this construction produce an value process spanned by those securities? Or does it produce some fictitious process priced by the discount factor but not achievable from the original securities? If “not necessarily” (hint, that’s the right answer,) can we choose a \( \Lambda \) so that the value process \( V_t \) is traded? I.e. if we had an original incomplete set of assets, can we make sure that this \( V \) is the growth-optimal portfolio in that set of assets?

4. Make a state preference diagram with consumption in state 1 and consumption in state 2 on the axes and consider an investor maximizing \( E[u(c)] \).

(a) Show analytically and graphically that a risk-averse investor will always buy full insurance if it is actuarially fair (\( \pi_i = p_i \) for all states \( i \)).
(b) What happens if \( p_1 > \pi_1 \)?
(c) What happens if there is one risk-neutral investor for whom \( u(c) = kc \)?
(d) What happens if there are two investors with identical risk averse utility, but they disagree on probabilities?