Problem Set 4

1. Do chapter 1, problem 2, 6

2) This problem conceptually follows \( p = E(mx) = E(m \times (\text{proj}(x|m) + \varepsilon)) = E(m \times (\text{proj}(x|m))) \) so “only the systematic component of risk generates prices.” Translating that idea to returns and expected returns takes a few extra lines. Hint: Consider the regression \( R^t = a + bm + \varepsilon \), then take \( E() \) and \( E(m \cdot) \) of both sides to show that \( a + bm \) is the “efficient” return to the left of \( R^t \) on the frontier. In ch1, you are allowed to assume complete markets.

6) Assume the investor does want a mean-variance portfolio, to rule out the easy answer “inter-temporal investors don’t want mean-variance portfolios. There’s not a lot of math here, just a few lines to explain the puzzle, why do people hold assets like B in their portfolios?

2. Do problems 1, 2; 5, 6 in Chapter 5, Asset Pricing, p. 97. Obviously, the list of properties of \( R^* \) and \( R^{e*} \) in section 5.5 will help. (3, 4 are in the list of properties, but you should know how to do them like all the other properties.) Hints:

1. A clearer statement of the problem. In Figure 5.2 I drew \( R^{e*} \) perpendicular to planes of constant mean in \( \overline{R^c} \) Provide equations to justify this picture. (Hint: Remember that \( R^* \) is perpendicular to planes of constant price. This means that \( 0 = E(m(R - R^*)) \) not \( 0 = E(mR) \!)

2) Hint. The nonstochastic case is a one-line argument. What’s \( x^* \) in a nonstochastic economy? There are lots of ways to do the stochastic case. I derived first \( \|x^*\| \|R^*\| = 1 \) (remember, we define \( \|x\|^2 = E(x^2) = \langle x|x \rangle \) and then thought about how big (second moment) \( R^* \) is. If you make a mean-standard deviation picture, you will see that higher Sharpe ratios argue for smaller \( R^* \), while larger risk free rates argue for larger \( R^* \). Use the \( R^f = 1.01 \) and market Sharpe ratio= 0.5 to back out \( E(R^{e*}) \). The book has a typo, \( R^f = 1.01 \) not 0.01.

2d) Show that even if \( x^* \) lies above the set of returns, \( \|x^*\| > \|R^*\| \), \( \text{proj}(x^*|1) \) most likely lies below the set of returns, generalizing the riskfree case.
5) The question should read just “What happens to the mean-variance frontier if investors are risk neutral.” Explain what happens to the \( R^* + wR^* \) expression of the frontier. Make drawings both in state space and \( E(R) \) vs. \( \sigma(R) \) space.

6) The point here is, just what do we mean by “\( R^* \) is the discount-factor mimicking return?” We use “mimicking portfolios” a lot, so it’s an important question. (Hint: ‘Project’ by using formulas like \( \text{proj}(y|\mathcal{X}) = E(x|x)^{-1}E(xy)x \) always includes the possibility zero. \( 0 \notin R^* \)

6b) Also, looking forward to the usual “no, but” answer, what is the \( \text{right} \) relationship between \( R^* \) and \( \mu \)? I.e. how would you do the analogue of drawing a line from something (\( m \)) to the plane \( \mathbb{R}^2 \)?

3. Do problems 2, 3, in Chapter 6, *Asset Pricing*, p. 120

3) In addition to the problem, use our standard numbers on \( E(R^m)\sigma(R^m) \) and \( R^f \) to plot \( R^m, R^f \), and \( R^* \) on a sketch of a mean-variance frontier. A point of this problem is again to relate the market return to \( R^* \) and further dispel the idea that they are the same.

You will have to find \( a \) and \( b \) in terms of \( R^m, R^f \) along the way. You will find a prettier representation for \( m \) of the form \( m = a + b(R^m - E(R^m)) \).

Work hard to get a pretty representation for \( R^* \). Use \( S = (E(R^m) - R^f) / \sigma(R^m) \) to denote the market Sharpe ratio. Express \( R^* \) as a portfolio, \( R^* = (\mu)R^f + (\sigma)R^m \) or \( R^* = (\mu)R^f + (\sigma)(R^m - R^f) \). That will help you to locate \( R^* \) on the frontier.

4) Here’s the issue restated a bit. We know that the factor-mimicking portfolio returns span the mean-variance frontier. The question is, as we go around the frontier, do we change weight in the factor mimicking portfolios stay the same?

New Problems

1. We showed that given a mean-variance efficient return \( R^{\text{mv}} \) we can find a discount factor of the form \( m = a - bR^{\text{mv}} \). It seems we have a problem – this discount factor need not be positive. For example, if \( R^{\text{mv}} \) is normally distributed, \( m \) can be negative. Even if \( R^{\text{mv}} \) is never negative (for example, unleveraged stock returns are not negative), \( m = a - bR^{\text{mv}} \) will typically be able to take on negative values. In incomplete markets we can just answer “this is an \( m \) not the \( m \).” But suppose markets are complete, so there is only one \( m \). Do we have to add something about arbitrage, or does this tell us something about frontier returns? (Hint: work backwards from \( m > 0 \). Can \( R^* \) be less than zero? Can \( R^* \) be normally distributed? Can any frontier return be normally distributed? Assume there is a risk free rate.)

2. The traditional approach to the Roll theorem: Prove directly that if \( R^{\text{mv}} \) is mean-variance efficient, then every other asset obeys \( E(R^i) = \gamma + \beta_{i,\text{mv}}(E(R^{\text{mv}}) - \gamma) \). If there is a risk free rate, \( \gamma = R^f \). If not, \( \gamma \) is the same number for every \( R^i \) but different depending on which \( R^{\text{mv}} \) you choose. Outline: Start with \( R^{\text{mv}} \), and consider portfolios \( R^p = R^{\text{mv}} + \varepsilon(R^p - R^{\text{mv}}) \). Find the mean and variance of such portfolios and take the limit as \( \varepsilon \to 0 \). You’ll use the facts that \( \varepsilon^2 << \varepsilon \) for small \( \varepsilon \), and that \( dE(R^p)/d\sigma^2(R^p)|_{\varepsilon=0} \) is the same for every variation \( R^p \) if and only if \( R^{\text{mv}} \) is on the frontier. Why must that be the case? Make a picture of how \( E(R^p) \) and \( \sigma(R^p) \) must vary as we vary \( \varepsilon \) and pass through \( R^{\text{mv}} \), and argue that all portfolio lines must pass through \( (\sigma(R^{\text{mv}}), E(R^{\text{mv}})) \) tangent to the frontier.

3. Obtain Fama French 25 portfolio returns from Ken French’s website
   https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html. You want the “25 Portfolios Formed on Size and Book-to-Market (5 x 5)” Start in 1932 after the last of the -99 returns. Pay attention to units to get these data to \( R_{t+1} \). Also get the Fama French factors. The factors are excess returns, so add back the risk free rate to make them returns.
(a) Plot the mean vs. standard deviation of each portfolio, along with the mean-variance frontier, in units of percent return. Calculate the mean-variance frontier in two ways and make sure you get the same answer:

i. Use the formulas for the Lagrangian approach on p.85-86.

ii. Construct a time-series of $R^*$ and $R^*$. Then calculate the mean and variance of $R^* + wR^*$ for a bunch of $w$. Include the mean and variance of $R^*$ on your plot. (You may need two plots at different scales)

iii. Construct $x^*$ (i.e. construct the time series $x_1^*$) and verify that $1 = E(x^* R)$ for the 25 portfolios. Verify that $E(R^2) = E(R^* R)$ for the 25 portfolios

(b) Choose an arbitrary return $R_{mv}$ on the frontier (but not the minimum variance return). Calculate the beta of each asset against that return, and plot $E(R^i)$ vs. $\beta_{i, mv}$.

i. Why do Fama and French think they need a 3 factor model? Here is a one-factor representation that is working very well in their data!

ii. Come to think of it, this plot was made using sample moments. Where is the sample variation we should see in any sample estimate?

(c) Find and interpret the portfolio weights of $R^*$, those of a return on the upper portion of the mean-variance frontier (I used the return with 2% mean), and those of the minimum variance portfolio. (You can put these in 5 x 5 boxes, small to big up and down and growth to value side to size, like Fama and French. If the weights seem a bit wild, that’s the point.)

(d) Add the mean and standard deviation of the market index to your plot. Is the market mean-variance efficient, or close to it? If not, is this a puzzle? Does the frontier look like a good investment opportunity? Do you think you could achieve this mean and variance in reality? If not what’s wrong?

(e) Now, repeat your calculations for part b and using data only through 1980. Use the moments from the pre-1980 data to calculate time series for $x_1^*, R_1^*$ etc. for the remainder of the sample. Plot the $E(R^i)$ vs. $\beta_{i, mv}$ in the post-1980 sample, and present the results of $1 = E(x^* R)$ and $E(R^2) = E(R^* R)$ calculations in the post 1980 samples.

Note: please include your code along with your solutions.