When some consumers are uncertain about a product’s quality, product availability conveys information about the propensity of other, better-informed, consumers to purchase. The willingness-to-pay of less-informed consumers to buy may therefore increase after observing a stock-out. We develop a model in which the firm can profit from the increased willingness-to-pay after a stock-out via expensive, in-season replenishment. Consumers with better knowledge about the quality (“savvy” consumers) and consumers less knowledge about the quality, but with a low out-side option (“eager, novice” consumers) buy early in the season. Consumers with less knowledge about the quality, but, a relatively high out-side option (“less eager, novice” consumers) buy later in the season, only after an early-season stock-out. We show that stock-outs are informative only when the in-season replenishment is expensive enough, because in this case consumers know the firm has an incentive to purchase a large initial inventory and hence a stock-out provides a credible signal of high quality. We further find that the incentive to reduce inventory to facilitate such signalling is the highest for intermediate levels of consumer heterogeneity in expertise (savvy vs. novice consumers). However, equilibrium profits are then the lowest in this case as well.

Key words: Strategic consumer behavior, Bayes’ rule, newsvendor models

1. Introduction and Motivation

In 1994, Mighty Morphin Power Rangers were hard to find, leading parents to a frenzied search. Many even camped outside stores in order to buy the toys as soon as they came in (Collins 1994). Other toys and innovative products have experienced similar phenomena: Cabbage Patch Kids in 1983, Beanie Babies in the 1990s, Tickle Me Elmo in 1998, Pokémon in 1999, PlayStation 2 in 2000, Nike Airforce1 in 2002, iPod mini and Nintendo DS in 2004, iPod nano in 2005 (Wingfield and Guth 2005). Why do we observe so many stock-outs for these products?

Classical newsvendor logic provides one explanation. Demand for innovative products is difficult to forecast and production processes may be inflexible, suppliers and subcontractors may have to be lined up in advance, lead times can be long, and so on. Hence, firms may have to commit to a production decision long before they can observe demand. If production exceeds demand, the firm
incurs overstock costs; if the reverse, it incurs the cost of lost sales. The optimal production quantity will trade off these two costs. Sometimes firms lose the “bet” on the upside and demand is greater than supply, which can account for the availability problems reported above. This explanation no doubt accounts for some observed shortages. Still, finding such extreme unavailability repeatedly over many generations and types of products is surprising given the often high margins that are lost. Such high-profile shortages often raise suspicion about company motives in the popular business press. Could something more subtle than an unlucky production decision be at work?

We think so. Common characteristics of the products cited above are that they are new, innovative, difficult to evaluate, and have little (or no) market history, creating a great deal of uncertainty among potential consumers about the utility (quality) of the product. As a result, consumers may try to acquire information about product quality. An important source of information is the purchasing decisions of fellow consumers. The fact that droves of other consumers are “voting” their approval with their wallets—is in many ways the truest indication of product quality. Hence, the stock-outs generated by such droves of consumers are a signal of high quality. Shouldn’t a rational firm leverage this?

Of course there are other explanations for stock-outs of new, innovative products. For one, firms may try to spread out demand when production costs decrease because of learning, which can lead to more stock-outs for products that are early in their lifecycle (see, e.g., Holloway et al. 2006). Also a limited inventory may increase customer store visits and lead to sales of other products while the customer is in the store. While all plausible reasons for observed shortages, our focus here is on the signal value stock-outs provide about product quality and how firms might rationally exploit this effect.

Consumers learn about the purchasing decisions of others in many ways. Simple word of mouth helps spread information about product quality: Consumers may consult “expert” opinions, product reviews, the advice of friends and colleagues, and so on. Also shopping web sites frequently rank products by popularity and post ratings and narrative reviews from prior consumers. But with new or highly experiential products, reviews can at best convey only a general sense of product quality. Hence, despite these other information sources, simple availability (or lack thereof) remains a strong signal of which products are most popular. Just as the prospect of a sell-out concert or sporting event creates buzz and stimulates interest among potential fans, backlogs and stock-outs create a sense that a product is “hot” and widely in demand. Such information can confirm positive (but uncertain) beliefs about a product and create a sense of affirmation that stimulates

\[^1\] See, for example, *Business Week*, Nov. 21, 2005, “Moore Addresses Xbox 360 Shortage “Conspiracy,”” in which a Microsoft executive addressed criticism that the company created an artificial scarcity of its popular game console to whip up holiday hype.
new consumers to buy.

The positive “buzz” generated by stock-outs appears real and has not gone unnoticed. Yet, it remains an open question to many industry observers and consumers whether firms deliberately create stock-outs. Leveraging the buzz from stock-outs leads to an interesting paradox: Can it ever be optimal for a firm to limit its inventory investment in order to trigger stock-outs and create more sales? We address this paradox by introducing the option for the firm to replenish—at some extra cost—demand during the (short) selling season. This in-season replenishment option leads to an interesting dilemma for consumers: Purchase early in the season, or, wait until later in the season, after learning about the initial success of the product via reported stock-outs (or absence of them)? We analyze the consumers incentives to wait for information via stock-outs and of the firm to create stock-outs via low inventory investment. Our work allows us to characterize the environments in which a firm has the greatest incentives to “create” stock-outs.

2. Related Literature

Different research streams have explored the link between product availability and product quality. In one behavioral experiment (Verhallen 1982), experimenters showed subjects three recipe books that differed in availability (available, unavailable, and unavailable that changed to available). When the market explanation for unavailability was given, subjects rated the unavailable books higher; That is, consumers inferred from the limited availability that the product must have high demand and therefore be of high quality. The author explains this reaction using commodity theory, which is rooted in psychology and predicts that scarcity enhances the value (or desirability) of anything that can be possessed, is useful to its possessor, and is transferable from one person to another (Lynn 1991).

In the economics literature, a recent stream of research has studied consumer inference from other consumers’ actions. Banerjee (1992) and Bikchandani et al. (1992) analyze the equilibrium outcome when a sequence of individuals makes decisions with incomplete information about the value of an asset. The asset can either be of negative or positive value. Each individual has private but inaccurate information about the asset value and observes the outcome of the decisions (to buy the asset or not) of his predecessor. Agents do not observe the predecessor’s private information. The authors demonstrate that the influence of the predecessors’ observed decisions could be so strong that individuals completely ignore their own information and follow their predecessors’ decisions. This effect is called “herding.” Herding can be socially inefficient as consumers can make the wrong decision, buying an asset with negative value or not buying a highly positive valued asset. In a our context, stock-outs act as an imperfect signal of previous consumers’ purchasing deci-
Psychology and herding literature, typically, focus on explaining the decisions of individual consumers or subjects. They do not study how a firm can influence these decisions.

Our work has some connections to prior operations literature as well. Traditionally, availability levels are considered a consequence of exogenous consumer demand and the firm’s inventory policy. The examples above suggest consumer purchasing behavior and availability may be determined simultaneously—they may be endogenously determined in equilibrium (Gaur and Park 2007, Cachon and Kok 2007). Especially when consumers do not have accurate information about a product but observe public product-availability information, consumer purchasing behavior and availability need to be determined in an equilibrium. Consumers may then complement their own private information with availability information in order to make a purchasing decision. One literature stream focuses on the management of a category of products distinguished by some attribute (van Ryzin and Mahajan 1999, Gans 2002, Gaur and Park 2007). van Ryzin and Mahajan study how to optimally select which variants firms need to offer in the category and how much inventory each store should stock, considering consumer characteristics and the cost of supply. They model a trend-following population as an exogenously given probability that all demand for the category will be for one particular variant (as in herding). Gans (2002) studies customer search behavior. He studies customer loyalty to a certain vendor when the quality experience is noisy. Customers may sample different vendors and accumulate their experiences before settling on one supplier. During each visit, consumers update their prior about the quality of the vendor. Stock and Balachander (2005) analyze when “scarcity strategies” signal product quality to uninformed consumers and may yield higher profits for the seller.

A stream of papers explores how inventory levels impact demand. Balakrishnan et al. (2004) analyze optimal lot-sizing when stocking large quantities stimulates demand. Another stream focuses on the problem of how scarcity impacts consumers’ waiting behavior. Liu and van Ryzin (2005) find that the threat of a shortage creates an incentive for consumers to purchase early at high prices. Several papers look at mechanisms to alleviate the impact the strategic consumer behavior; Su and Zhang (2008) study quantity and price commitment, Lai et al. (2007) study posterior price-matching policies, and Cachon and Swinney (2007) study quick, in-season replenishment. These authors find these mechanisms can increase the seller’s profit. Tereyagloglu and Veeraraghavan (forthcoming) study production and pricing strategies for “exclusive” goods that are used by conspicuous consumers to display their social status. They find that creating scarcity may improve

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2 Web sites may also list, for example, rankings of recent sales of books or CDs (the New York Times) or may announce publicly when a product has reached a certain threshold sales (e.g., a CD has earned gold or platinum).
Firm invests in $Q$ at cost $c$ 

Nature draws $\hat{\nu}$ 

\begin{align*} 
\text{time} & \quad t=0 & t=1 & \quad \text{time} \\
\text{Informed consumers, observe } (\hat{\nu}, \theta) & \quad \text{All consumers that decided Wait in } t=0 & \quad \text{observe } m \\
\text{Decide: Buy/Wait} & \quad \text{Decide: Buy/Quit} \\
\text{Uninformed consumers observe } \hat{\theta} & \\
\text{Decide: Buy/Wait} & \\
\text{If total demand}>Q, then } m=1, \text{ o/w } m=0 & \text{Firm satisfies demand at cost } c+\varepsilon \\
\text{Firm satisfies excess demand at cost } c+\varepsilon & 
\end{align*}

Figure 1  Model time line.

profits of the firm, but not always. Finally, Debo et al. (2012) study strategic queue-joining behavior when the queue is generated for a service of unknown value. They show that some consumers may not join the queue in equilibrium unless it is long enough. Veeraraghavan and Debo (2009, 2011) study the selection of a queue when the relative value of the services is unknown. They show that some consumers may join the longer queue in equilibrium, depending on the waiting costs, the queue buffer size and the heterogeneity with respect to prior service-value information.

None of these papers studies the question of how a firm can potentially benefit from inducing herd behavior through stock-outs.

3. Model Setup

In Figure 1, we present the time line of the game. In the first stage of our game, the firm determines the investment in initial inventory, after which Nature determines the product quality. These events take place before the start of the selling season. Next, at the beginning of the selling season (referred to as $t = 0$) when the initial inventory becomes available, all consumers decide whether to purchase the product or wait based on their information about the product. At the end of the selling season (referred to as $t = 1$), consumers that did not yet purchase, can decide whether to purchase or quit, based on their information, which now includes whether the initial inventory stocked out or not at $t = 0$.

Let $V$ be the value of the product, which is a random variable with a density (distribution) $f(v)$ ($F(v)$) over $[0, 1]$ with mean $\mu_V$. We assume that $f(v)$ is strictly positive over $(0, 1)$ and absolutely continuous and common knowledge to all consumers and the firm. The potential market size is normalized to 1. There is a continuum of consumers of type $\theta$, which is uniformly distributed over $[0, 1]$ and is the value of an outside option. One can think of $1 - \theta$ as representing a consumer’s willingness-to-pay for the product. A fraction, $\alpha$, of the consumers, the “informed” consumers, knows the realization of $V$, denoted $\hat{v}$. The remaining fraction, the “uninformed” consumers, does not know the realization of $V$. We assume that the value of the product to consumers is discounted
over time; when \( \tilde{v} \) is the value at \( t = 0 \), then \( \delta \tilde{v} \) is its value at \( t = 1 \), where \( \delta \in (0, 1) \) is the discount factor. Again, \( \alpha \), \( \delta \) and the distribution of \( \theta \) are common knowledge; the realized value of \( \theta \) for each customer, however, is private information.

The firm does not know the realization of \( \tilde{v} \) when it commits to cheap production of initial inventory, \( Q \), at a cost \( c \) per unit, that will become available at \( t = 0 \). We refer to \( c \) as the “preseason” inventory replenishment cost. Consumers do not observe the initial inventory. The firm observes the quality and the initial demand at \( t = 0 \), after which the firm can produce additional inventory at a cost \( c + \varepsilon \) per unit to satisfy the potential excess demand of \( t = 0 \). We refer to \( \varepsilon \) as the “in-season” inventory replenishment premium. The sales price is \( r (> c + \varepsilon) \). If the initial inventory is insufficient (sufficient) to cover the demand at \( t = 0 \), a (no) stock-out signal becomes observable to all remaining consumers at \( t = 1 \) (but still the initial inventory remains unobserved). As noted, at \( t = 1 \), consumers who did not buy yet decide whether to purchase or quit. All demand at \( t = 1 \) is then satisfied at a cost of \( c + \varepsilon \).

The informed consumers’ purchasing strategy is trivial: as the value of the product only decreases, they purchase the product at \( t = 0 \) when their type, \( \theta \), is less than the realized quality, \( \tilde{v} \). The uninformed consumers’ purchasing strategy depends on the posterior expected quality, which is a function of the stock-out information and the uninformed consumer’s belief about the firm’s initial inventory, \( Q_c \). Let \( \bar{\theta} \) denote the highest type that buys at \( t = 0 \) and \( \bar{\theta}_m (\geq \theta) \) be the highest type that buys at \( t = 1 \) when there is a (no) stock-out: \( m = 1 \) (0). Hence, the consumers’ purchasing strategies are characterized by \((Q_c, \theta)\), where \( \theta = (\bar{\theta}, \bar{\theta}_0, \bar{\theta}_1) \). The firm’s production strategy is determined by the inventory at time \( t = 0 \), \( Q_f \). After the \( t = 0 \) sales are realized, the firm’s decision is trivial: The firm produces just enough products to satisfy possible leftover demand from \( t = 0 \) and the \( t = 1 \) demand.\(^3\)

We assume that all players are rational Bayesian decision makers. We first impose consumer rationality conditions: given their information at each point in time and their beliefs about the other consumers’ and the firm’s strategy, consumers make purchasing decisions that maximize their expected utility. Being rational, the uninformed consumers’ posterior expected quality associated with a belief about the inventory investment needs to be consistent with Bayes’ rule. We also impose the firm’s rationality conditions; the firm’s inventory investment must maximize the firm’s expected profits, given the firm’s belief about the consumer’s purchasing behavior.

Let \( U_W(Q_c, \theta, \theta) \)\(^4\) be the \( \theta \)-consumer’s ex ante (at \( t = 0 \)) expected utility increase with respect to \( \theta \) when delaying the purchase to \( t = 1 \), given the belief \( Q_c \) about the firm’s inventory and that

\(^3\)Because the firm has the same information as the uninformed consumers, it can perfectly forecast the uninformed consumer’s utility and, hence, demand.

\(^4\)Detailed expressions for \( U_W \) and other functions in the Model Setup section will be provided in the Analysis section.
all other consumers’ purchasing strategy is characterized by $\theta$. Similarly, $U(m, Q_c, \theta, \theta)$ is the ex post (at $t = 1$) expected utility increase (with respect to $\theta$), after stock-out realization $m$ has been observed, given $Q_c$ and $\theta$. The uninformed consumers’ purchasing strategy given belief $Q_c$, $\theta^*(Q_c)$, must be rational in every period:

$$
\mu_V - \theta \geq U_W(Q_c, \theta, \theta) \quad \text{for } \theta \leq \theta \quad \text{and} \quad U_W(Q_c, \theta, \theta) < \mu_V - \theta \quad \text{for } \theta \leq \theta
$$

and

$$
U(m, Q_c, \theta, \theta) \geq 0 \quad \text{for } \theta \leq \theta \leq \theta_m \quad \text{and} \quad U(m, Q_c, \theta, \theta) < 0 \quad \text{for } \theta_m \leq \theta \leq 1 \quad \text{and} \quad m \in \{0, 1\}.
$$

In words, the $\theta$-consumer buys now when the expected utility increase from buying the product at $t = 0$, $\mu_V - \theta$, is greater than the expected increase in utility when delaying the purchase, $U_W$. Otherwise, the consumer delays the purchase. For these types, conditional on the stock-out information observed at $t = 0$, if the increase in utility is positive at $t = 1$, the consumer purchases the product; otherwise, the consumer quits.

Given this rational behavior on the part of consumers, the firm makes inventory investments to maximize its profits. The firm’s profits depend on both the consumer’s strategy, $\theta$, and the firm’s inventory, $Q_f$:

$$
Q_f^*(\theta) \in \arg \max_{Q_f} \Pi(Q_f, \theta).
$$

$Q^*$ is an equilibrium if $Q_f^*(\theta^*(Q^*)) = Q^*$. That is, when the optimal initial inventory investment is equal to the consumers’ beliefs about the initial inventory, $Q^*$ (on which the purchasing strategy, $\theta^*(Q^*)$, is determined) we obtain an equilibrium.

We assume that the conditional expected quality tends to one (zero) when the quality tends to the lower (upper) bound:

$$
\lim_{\theta \to 0^+} \frac{\int_{-\infty}^{\theta} v dF(v)}{F(v)} = 0 \quad \text{and} \quad \lim_{\theta \to 1^-} \frac{\int_{\theta}^{1} v dF(v)}{F(v)} = 1.
$$

A few comments on the model are in order. In reality of course there may be many reasons for consumers might delay purchasing, such as an anticipated price decrease or information about the quality that may become available via sources other than stock-outs (e.g., new information may be revealed via consumer reviews). For sake of clarity, however, we exclude such motivations for consumers to wait from our model and focus only on waiting motivated by learning from stock-outs. Also, for simplicity, we exclude other sources of uncertainty, such as on the total market size or on the relative size of the informed vs. uninformed consumers. Classical inventory-management theory mainly studies such sources of uncertainty while ignoring potential information content in
stock-outs. Hence, our focus is on the main source of uncertainty that determines the information content in stock-outs; quality uncertainty.

As for the prior $f(v)$, one can interpret this as being due to brand reputation or a history of successful introductions of new products. For example, consumers arguably had a high prior on iPhones due to Apple’s success with iPods; “iPod killers” from other manufacturers (e.g., Sirius or Microsoft) arguably started with lower priors. In contrast, although Fisher Price toys may have had a stronger prior than the less well-known Bandai toys, the latter had an enormous unanticipated success with Mighty Morphin Power Rangers in the mid-90s. Game boxes provide another example that fits well; the success of previous product launches sets expectations for a new product. Industry hype about a new product introduction may also affect the market prior. In our model, $f(v)$ is exogenous and the same for the firm and the consumers. While the firm could have a different prior than consumers via, for example, market research, to keep matters simple we assume that the prior about the quality is the same for all players.

As in Miklós-Thal and Zhang (2011), the fraction of informed consumers, $\alpha$, can be thought of as the size of the “enthusiast” market segment, those who have ample previous experience with similar products and a superior ability to evaluate the product accurately before purchasing. While every consumer who considers the product conducts a private inspection, we assume that the enthusiasts have better ability to evaluate the product than the “novices”. In the absence of any other information about the quality, the novice segment mainly bases its purchasing decisions on its prior belief about quality, such as the “reputation” of the firm. Obviously, the assumption that there are only two classes of consumers, one class that is perfectly informed and one class that is uninformed, is a stylized way to capture heterogeneity in the consumers’ ability to assess a new product on the market.

The in-season replenishment option, at a cost of $c + \varepsilon$ can be considered as the “reactive capacity” of the firm, where the premium is due to lead times are significantly shorter than for the preseason inventory (e.g., domestic production or fast transportation). Hence, the unit cost is significantly higher. Furthermore, as such capacity is deployed during the selling season, much more accurate forecasts are available, so no newsvendor-like problem needs to be solved by the firm.

Finally, there are two relevant margins, $r - c$ and $r - c - \varepsilon$. For new and innovative products $r - c$ is typically high (see, e.g., Fisher 1997); firms that create an innovative product can command high prices. However, due the expensive in-season replenishment, the margin on deliveries during the season may be substantially reduced to $r - c - \varepsilon$. We refer to this margin as the “in-season margin.”
4. Analysis

In this section, we analyze the consumer equilibrium first. For a given belief about the firm’s initial inventory, we determine the consumers’ purchasing strategy; Wait or Buy at $t = 0$, and Quit or Buy at $t = 1$ for those that waited at $t = 0$. We examine how the belief about the initial inventory impacts a consumer’s purchasing strategy. Given this consumer behavior, we then analyze the firm’s optimal inventory investment. Next, we determine conditions for equilibrium inventory investment and consumer purchasing strategies. In the subsection following, we provide comparative statics for the special case where the prior quality is uniformly distributed and the discount factor is large. In the last subsection, we perform numerical experiments for more general settings.

4.1. General Results

4.1.1. The consumers’ purchasing strategy. Recall that $Q_c$ is the consumer’s belief about the firm’s inventory. To simplify notation, we drop the index $c$ on $Q$ in this subsection. Recall also that $\theta$ is the highest type of the uninformed consumers that buy in the beginning of the season ($t = 0$). A stock-out will occur at the end of $t = 0$ when $\alpha \tilde{v} + (1 - \alpha) \theta > Q$, or $\tilde{v} > v(Q, \theta)$, where

$$v(Q, \theta) = \min \left\{ 1, \left\{ \frac{Q - (1 - \alpha) \theta}{\alpha} \right\}^+ \right\}.$$  

Any inference about the quality depends only on $\theta$, not on $(\theta_0, \theta_1)$. With a slight abuse of notation, let the posterior utility at $t = 1$ after observing a stock-out ($m = 1$) or not ($m = 0$) at $t = 0$ be $U(m, Q, \theta, \theta)$. With the assumption in Equation (4), the conditional expectations when $v = 0$ or 1 are well defined. The increase in expected utility (with respect to $\theta$) after observing the stock-out situation is outlined in Lemma 1.

**Lemma 1.** (i) The ex post increase in expected utility at $t = 1$ is

$$U(0, Q, \theta, \bar{\theta}) = \delta \frac{\int_{0}^{v(Q, \bar{\theta})} vdF(v)}{F(v(Q, \theta))} - \theta$$

and $U(1, Q, \theta, \bar{\theta}) = \delta \frac{\int_{v(Q, \theta)}^{1} vdF(v)}{F(v(Q, \theta))} - \theta$

(where $\bar{F} = 1 - F$) and satisfies $U(0, Q, \theta, \bar{\theta}) < \delta \mu_v - \theta < U(1, Q, \theta, \bar{\theta})$.

(ii) $\frac{\partial}{\partial Q} U(m, Q, \theta, \bar{\theta}) \geq 0$ for all $m \in \{0, 1\}$, $Q \in [0, 1]$ and $\theta \in [0, 1]$.

As stock-outs occur only for high-quality realizations, it is easy to see that the posterior expected quality when observing a stock-out is greater than when not observing a stock-out; $U(0, Q, \theta, \bar{\theta}) < U(1, Q, \theta, \bar{\theta})$ (Lemma 1(i)). As a consequence, the posterior utility after observing a stock-out is higher than after observing no stock-out. Furthermore, the posterior quality assessment increases in the uninformed consumers’ belief about the initial inventory (Lemma 1(ii)). There are two causes to a stock-out—the product quality is high or the initial inventory is low. When the belief about the initial inventory increases, the uninformed consumers put more weight on the explanation that
the stock-out is caused by high quality. For notational convenience, we introduce $V(m, Q, \theta)$—the posterior quality after observing a stock-out situation $m \in \{0, 1\}$, given the belief about the initial inventory, $Q$, and the consumer purchasing behavior characterized by $\theta$. $V(0, Q, \theta) = \int_{\mathbb{Q}(Q, \theta)} v d\mathbb{F}(v) / \mathbb{F}(\mathbb{Q}(Q, \theta))$ and $V(1, Q, \theta) = \int_{\mathbb{Q}(Q, \theta)}^1 v d\mathbb{F}(v) / \mathbb{F}(\mathbb{Q}(Q, \theta))$.

For any consumer of type $\theta$, the expected utility from waiting is $U_W(Q, \theta, \theta) = \delta \mu V(0, Q, \theta) + \bar{\mu} V(1, Q, \theta)$ or

$$U_W(Q, \theta, \theta) = \begin{cases} \delta \mu V(0, Q, \theta) & \theta < \delta V(0, Q, \theta) \\ \delta \int_{\mathbb{Q}(Q, \theta)}^1 v d\mathbb{F}(v) - \theta \bar{\mu} V(1, Q, \theta) & \delta V(0, Q, \theta) < \theta < \delta V(1, Q, \theta) \\ \delta V(1, Q, \theta) & \theta > \delta V(1, Q, \theta). \end{cases}$$

Uninformed consumers with a very low outside option, as in the first case, would purchase at $t = 1$, irrespective of the stock-out status. Due to the martingale property, the future expected utility (at $t = 1$) is equal to the current expected utility (at $t = 0$). Hence, the expected utility when delaying the purchase is $\delta \mu V - \theta$. From Lemma 1(i), the uninformed consumer only buys in the second case when there is a stock-out, in which case the increase in utility is strictly positive. When there is no stock-out, the increase in utility is zero as the best option for consumer is to retain the outside option. Finally, in the third case, for high values of the outside option, irrespective of the signal realization, the consumer does not purchase at $t = 1$. Hence, the increase in utility is exactly equal to zero. With an expression for $U_W$, we can now characterize the equilibrium purchasing strategy as a function of the consumers’ belief about the inventory, $\theta^*(Q)$:

**Lemma 2.** (i) $0 < \theta^*(Q) \leq \mu V$.

(ii) $\theta^*(Q)$ is a root of $\theta$ in $[0, \mu V]$ of

$$\delta \int_{\mathbb{Q}(Q, \theta)}^1 v d\mathbb{F}(v) - \theta \bar{\mu} V(1, Q, \theta) - (\mu V - \theta) = 0 \quad (5)$$

if such root exists, otherwise: $\theta^*(Q) = \mu V$.

(iii) When there is no stock-out in the first period, uninformed consumers do not purchase in the second period: $\theta^*_0(Q) = \theta^*(Q)$.

(iv) When there is a stock-out in the first period, uninformed consumers of type $\theta$ purchase in the second period when $\theta^*(Q) \leq \theta \leq \theta^*_1(Q)$, where $\theta^*_1(Q) = \delta V(1, Q, \theta^*(Q))$.

It is obvious that the increase in utility from delaying purchasing cannot be negative (as the consumer types do not change over time). Hence, the highest type that buys at $t = 0$ will be lower.

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5 If the firm could credibly communicate the initial inventory level, there would be a trade-off between the incentive to create a stock-out (low enough inventory) and the posterior quality after observing a stock-out (high enough inventory).

6 This assumption can be interpreted as follows: At every point in time, the consumers’ belief about the utility of alternative products remains the same. The utility of the focal product, however, declines over time.
Figure 2 Illustration of the first- and second-period expected utility as a function of the consumer type for the numerical example. In the left panel, the solid line indicates $\max\{\mu_V - \theta, U_W(Q, \theta, \bar{\theta})\}$. In the right panel, the solid line for values greater than $\bar{\theta}$ indicates $\max\{U(1, Q, \bar{\theta}, \theta), 0\}$ and $\max\{U(0, Q, \bar{\theta}, \theta), 0\}$.

than the prior expected quality (Lemma 2(i)). From Equation (5) in Lemma 2(ii), it follows that the highest consumer type that purchases at $t = 0$ is indifferent between purchasing at $t = 0$ and delaying and purchasing in the future only when a stock-out has been reported at $t = 0$; that is, consumers only wait for stock-out information when they benefit from acting on it. If not, the future expected quality is the same as the current expected quality (via the martingale property of future beliefs), only reduced due to the value loss over time. When the expected utility increase in the future is greater than the expected utility increase from purchasing now, the consumer waits. Equation (5) says these increases are equal for the marginal consumer type $\bar{\theta}$ (Lemma 2(ii)). Consumers with higher types (higher outside options) will delay their purchase and quit when there is no stock-out (Lemma 2(iii)). When there is a stock-out, the highest consumer type that buys will have a posterior value (including the value loss due to discounting) that is equal to her type (Lemma 2(iv)). As uninformed consumers only purchase when there is a stock-out, we drop the index 1 from $\theta_1$.

Numerical example. Consider the following parameters: $\delta = 0.9$, $\alpha = 0.5$, $c = 0.2$, $r = 1$, $\varepsilon = 0.4$ and $f(v)$ is uniformly distributed over $[0, 1]$. For $Q = 0.6$, we compute that $\bar{\theta}^* = 0.416$ and $\bar{\theta} = 0.802$. Figure 2 shows a plot of the expected prior utility as a function of $\theta$ and the ex post utility. Notice that in equilibrium, consumers with $\theta < \bar{\theta}^*$ prefer buying at $t = 0$, and consumers with $\bar{\theta}^* < \theta \leq 1$ prefer waiting until $t = 1$. Those with $\bar{\theta} < \theta < \bar{\theta}^*$ buy at $t = 1$ only after a stock-out has been reported at $t = 0$ and quit otherwise; consumers with $\bar{\theta} < \theta$ quit, irrespective of the stock-out realization at $t = 0$. □

Define the quality realization $v(\delta)$ as the realization that makes the conditional future quality (including the discount factor, $\delta$) equal to the prior quality: $v(\delta)$ is the solution to $\delta \int_{v(\delta)}^{1} u dF(u) = \mu_V$. For the remainder of the paper, we assume that
\[ \delta > \mu_V. \]  

Hence, \( v(\delta) \) is always defined and less than 1 (note: \( v(1) = 0 \) and \( v(\mu_V) = 1 \)).

We are now ready to analyze how the purchasing strategy depends on the belief about the inventory. We show in the Lemma below that for beliefs below \( \bar{Q}(\delta) \equiv \alpha \mu_V(\delta) + (1 - \alpha) \mu_V \), consumers never wait. Above \( \bar{Q}(\delta) \), consumers do wait:

**Lemma 3.**

(i) If \( 0 < Q < \bar{Q}(\delta) \), then \( \theta^*(Q) = \bar{\theta}^*(Q) = \mu_V \).

(ii) If \( \bar{Q}(\delta) < Q < \alpha + \mu_V(1 - \alpha) \), \( \theta^*(Q) < \mu_V \). That is, Equation (5) has at least one root in \((0, \mu_V)\) and \( \bar{\theta}^*(Q) = \delta V(1, Q, \mu_V) \).

(iii) If \( \alpha + \mu_V(1 - \alpha) < Q \), then \( \theta^*(Q) = \mu_V \) and \( \bar{\theta}^*(Q) = \delta \).

When consumers believe the initial inventory is low (less than \( \bar{Q}(\delta) \)), they are not willing to wait (Lemma 3(i)). Hence, all uninformed consumers for whom \( \theta \leq \mu_V \) purchase at \( t = 0 \). Irrespective of the stock-out signal at \( t = 1 \), remaining consumers with \( \mu_V < \theta \) wait and quit in the next period (due to the value decrease, even after a stock-out). When consumers believe the initial inventory is neither very high nor very low, they are willing to wait because a stock-out will be informative (Lemma 3(ii)). In Lemma 3(iii), consumers believe that the initial inventory is so high that all informed consumers and uninformed consumers with an outside option less that the prior quality can be satisfied at \( t = 0 \). Of course, in this case they do not expect a stock-out. But if a stock-out occurs (which happens off the equilibrium path), they know for sure that the quality is high. Hence, consumer types higher than the prior quality wait until \( t = 1 \) and buy when a stock-out occurs.

**Numerical example (continued).** We continue the numerical example about with \( Q = 0.60 \) and demonstrate the consumers’ purchasing strategy as a function of their initial-inventory beliefs. (See Figure 3.) Notice that for some low inventory beliefs (less than \( \bar{Q}(\delta) \)), three consumer equilibria exist, one with \( \theta^*(Q) = \bar{\theta}^*(Q) = \mu_V \) (see Lemma 3(i)) and two with \( \theta^*(Q) < \mu_V \) (corresponding with the inner and outer branches in Figure 3). For these beliefs about the inventory, Equation (5) has two roots in \((0, \mu_V)\). Uniqueness of the equilibrium of this game is thus not guaranteed. Overall, as the initial inventory belief increases, the volume of consumers buying the product with strictly positive probability (either at \( t = 0 \), or at \( t = 1 \), after a stock-out at \( t = 0 \)) increases roughly in the conjectured inventory. This is consistent with Lemma 1; when the initial inventory belief is higher, uninformed consumers interpret an early-season stock-out as due to high product quality detected by informed consumers who purchase at the beginning of the season. Therefore, with a higher initial inventory belief, more uninformed consumers are willing to delay their purchase and buy only if an early-season stock-out has been reported. \( \Box \)
Figure 3 Illustration of the strategic consumer purchasing behavior as a function of the consumers’ conjectured inventory. For this example, $\tilde{Q}(\delta) = 0.305$ (which corresponds with the vertical solid line)—below which $\theta^* = (\mu_V, \mu_V)$ is an equilibrium—and $\alpha + (1 - \alpha)\mu_V = 0.75$—above which $\theta^* = (\mu_V, \delta)$ is an equilibrium.

4.1.2. The firm’s inventory investment. Per our discussion above, we focus on purchasing strategies with $\tilde{\theta}_0 = \theta$ and $\tilde{\theta}_1 \geq \theta$ as these are the only ones that can occur in equilibrium (Lemma 2). Given $\theta = (\tilde{\theta}, \overline{\theta})$, the demand depends on the realization of the quality, $\tilde{v}$, and the inventory $Q$. $\alpha \tilde{v} + (1 - \alpha)\overline{\theta}$ is the demand at $t = 0$; the informed consumers (fraction $\alpha$) purchase when their outside option is less than $\tilde{v}$. At $t = 1$, the demand depends on the stock-out realization. When $m = 0$ (no stock-out), the demand is 0 (see Lemma 2). When $m = 1$ (a stock-out), the demand is $(1 - \alpha)(\overline{\theta} - \theta)$. No expensive replenishment is required when there is no stock-out at $t = 0$. When there is a stock-out, due to the expensive in-season replenishment, the margin on the demand at $t = 0$ and $t = 1$ that cannot be satisfied from initial inventory, $\alpha \tilde{v} + (1 - \alpha)\overline{\theta} - Q$, is only $r - (c + \varepsilon)$.

The firm’s optimal initial inventory is determined by the marginal expected profits, $\frac{d}{dQ} \int_0^1 \Pi(v, Q, \theta) dF(v)$. The optimal firm inventory investment is $Q_f(\theta) = \alpha F^{-1}(v^*(\theta)) + (1 - \alpha)\bar{\theta}$, where $v^*(\theta)$, when it is interior (in $(0, 1)$), satisfies the first-order condition (FOC),

$$F(v) = \frac{\varepsilon}{c + \varepsilon} - \left( \frac{r}{c + \varepsilon} - 1 \right) \frac{1 - \alpha}{\alpha} (\overline{\theta} - \theta) f(v),$$

and the second-order condition (SOC),
\[-f(v) - \left( \frac{r}{c+\varepsilon} - 1 \right) \frac{1-\alpha}{\alpha} (\overline{\theta} - \theta) f'(v) < 0. \]  

(8)

$v$ is the threshold quality level above which a stock-out occurs, and hence, at $t = 1$, an extra demand volume of $(1 - \alpha)(\overline{\theta} - \theta)$ uninformed consumers is generated upon a stock-out. The firm’s inventory investment has a one-to-one relationship with the threshold quality level. Hence, it is convenient to analyze the optimality conditions for the inventory in terms of $v$. The inventory can be decomposed in two parts: (a) inventory for the informed consumers, $\alpha F^{-1}(v^*(\theta))$, and (b) inventory for the uninformed consumers, $(1 - \alpha)\theta$. Inventory for the uninformed consumers is equal to their “deterministic” demand; their demand does not depend on the quality realization because uninformed consumers do not know the quality and purchase based on the prior (reputation) of the firm.\footnote{Recall that, for sake of clarity, we excluded other sources of uncertainty, see Section 3.}

As the informed consumers’ demand depends on the true quality, which is random for the firm at the moment of committing to initial inventory, the firm trades off cheap preseason inventory (at a cost of $c$) with expensive, in-season replenishment (with additional cost $\varepsilon$). We recognize, thus, the service level and the critical fractile, $F(v) = \varepsilon/(c + \varepsilon)$, in the firm’s FOC of Equation (7).

Besides the critical fractile, the second term in (7) is due to the discontinuous increase in sales after a stock-out realization (demand $(1 - \alpha)(\overline{\theta} - \theta)$ is added). This term, which is negative, reduces the critical fractile and hence can be viewed as the reason to (rationally) “create stock-outs” by increasing the stock-out probability (above the critical fractile obtained ignoring strategic consumer reactions). When $\overline{\theta} = \theta$ or $\alpha = 1$, no additional uninformed consumers purchase at $t = 1$ after a stock-out at $t = 0$, so the term is zero and the optimal newsvendor investment is determined by the classical critical fractile. Similarly, when the in-season margin is zero ($r = c + \varepsilon$), no extra revenues can be made, even when $\overline{\theta} > \theta$ and $\alpha < 1$. Hence, the term is also zero. A small increase in the threshold quality from $v$ to $v + dv$ level leads to an increase in probability of a stock-out of $f(v)dv$. Hence, the additional term is proportional to the density. From the FOC and SOC of Equations (7) and (8), we can now analyze how the consumer behavior, characterized by $(\overline{\theta}, \theta)$, determines the optimal inventory investment.

**Lemma 4.** The firm’s inventory depends on the consumer behavior as follows: (i) $\frac{\partial Q^*_I(\theta)}{\partial \theta} < 0$ and (ii) $\frac{\partial Q^*_I(\theta)}{\partial \theta} > 1 - \alpha$.

When the demand *after* the stock-out increases (i.e., $\overline{\theta}$ increases), keeping the demand before the stock-out constant ($\theta$ is fixed) in Lemma 4(i), the firm’s incentive is to invest in less initial inventory to profit from subsequent increased sales triggered by the stock-out. When the demand *before* the stock-out increases ($\overline{\theta}$ increases), keeping the total demand the same ($\overline{\theta}$ fixed), the firm
invests in more inventory, \( \frac{d}{d\theta} Q^*_f(\theta) > 1 - \alpha \), for two reasons: First, there is more known demand before the stock-out, hence, the firm invests in more initial inventory (at a rate of \( 1 - \alpha \)). Second, the demand after a stock-out decreases; hence, the firm’s incentive to reduce inventory to harvest a stock-out is lower, via a similar, but, opposite reasoning as for Lemma 4(i).

The “myopic consumer” case. For the remainder of this paper, we will refer to the “myopic consumer” case as the case in which the uninformed consumers ignore stock-out information. That is, they do not update their quality expectations after observing or not observing a stock-out early in the season. Such a situation could exist when, for example, stock-out information is not available to consumers. The uninformed consumers’ purchasing strategy is simply \( \theta^o = \theta^o = \mu_V \); as the product loses value over time and there are no potential gains of waiting. Also, the uninformed consumers purchase the product as early as possible. The firm trades off cheap preseason inventory for the informed consumers with expensive, in-season inventory. Hence, \( F(v_o) = \frac{\epsilon}{c + \epsilon} \) or \( Q_o = \alpha F^{-1}(\epsilon/(c + \epsilon)) + (1 - \alpha)\mu_V \), where \( v^o = v(Q^o, \mu_V) \). The stock-out probability with myopic consumers is then \( c/(c + \epsilon) \), and the sales season starts and finishes at \( t = 0 \). We will compare the equilibrium results of our model with this myopic consumer case. To make the comparison easier, we refer to all the consumers in our model as “strategic” consumers and the consumers in the myopic case as “myopic” consumers.

4.1.3. Equilibrium. For notational convenience, we define \( \gamma = \mu_V \left( \frac{\epsilon}{c + \epsilon} - 1 \right) \frac{1 - \alpha}{\alpha} \) and \( \epsilon = \frac{\epsilon}{c + \epsilon} \); \( \epsilon \) is the classical critical fractile. A necessary condition for equilibrium with an interior \( v^* \) is the solution of

\[
F(v) = \frac{\epsilon}{c + \epsilon} - \gamma \left( \frac{\delta}{\nu_V} \int_v^{v^*} \frac{u}{\mu_V} dF(u) F(v) - \frac{1 - \delta + \delta \int_0^{v^*} \frac{u}{\mu_V} dF(u) F(v)}{F(v)} \right) f(v). \tag{9}
\]

At \( v^* \), the SOC (see Equation (7)), keeping \( \theta^* \) fixed must be satisfied:

\[
-f(v^*) - \gamma \left( \frac{\delta}{\nu_V} \int_v^{v^*} \frac{u}{\mu_V} dF(v) F(v^*) - \frac{1 - \delta + \delta \int_0^{v^*} \frac{u}{\mu_V} dF(v) F(v^*)}{F(v^*)} \right) f'(v^*) < 0. \tag{10}
\]

For the remainder of the paper, we assume that Equations (9) and (10) have a unique solution.

**Proposition 1.** (i) When \( F(v(\delta)) > \epsilon \), an equilibrium with inventory investment \( Q^* = Q^o \) and consumer strategy \( \bar{\theta} = \theta^* = \mu_V \) exists. Stock-outs are uninformative, no consumer waits, and the sales season ends at \( t = 0 \).

(ii) When, \( F(v(\delta)) < \epsilon \), an equilibrium with inventory investment \( Q^* < Q^o \) and consumer strategy \( \bar{\theta} < \theta^* \) exists and \( F(v^*) < \epsilon \). Stock-outs are informative, and some consumers wait to purchase.

When there is no stock-out at \( t = 0 \), the sales season ends at \( t = 0 \); otherwise, it ends at \( t = 1 \).
Note the condition of Proposition 1 can be written as $v(\delta) > F^{-1}(\epsilon)$. Recall that $v(1) = 0$ and $v(\mu V) = 1$. Hence, the proposition identifies a threshold on the discount factor $\delta \in (\mu V, 1)$ such that if the value declines steeply ($\delta$ is below the threshold $\delta$), stock-outs are uninformative, and when the value loss is small ($\delta$ is above $\delta$), stock-outs are informative. This threshold depends on the critical fractile, $\epsilon$. When in-season replenishment is cheap ($\epsilon$ is low) or preseason replenishment is expensive ($c$ is high), the critical fractile, $\epsilon$, is low and, hence, the firm has a weak incentive to invest in high initial inventory levels. Consumers in turn, will anticipate this and so expect early-season stock-outs, irrespective of the quality realization. Hence, these stock-outs will be uninformative. As a consequence, no consumers will be willing to wait (see Lemma 3(i) with $\theta = \bar{\theta}$), which justifies the low inventory investment based on the critical fractile only (see Lemma 4 with $\theta = \bar{\theta}$). This result has interesting consequences. While a firm may think of “creating” early-season stock-outs to trigger increased willingness to pay and then profit from those customers who wait to buy by reducing its in-season replenishment cost, Proposition 1 demonstrates that such strategy will not work in equilibrium; consumers, knowing in-season margins are high, will attach no value to stock-out information. In effect the firm has to “commit” to low in-season margins (or face them naturally) to make any signal of quality due to stock-out credible.

The equilibrium also has interesting consequences for the variability and length of the selling season. The equilibrium stock-out probability is $1 - F(\v^*) > 1 - \epsilon$, where recall the right hand side is the stock-out probability with myopic consumers. Early-season stock-outs only occur due to high quality realizations spotted by early informed consumers. When the uninformed consumers know they will learn about the stock-outs, some will delay their purchase until later in the season. In response, the firm invests in less inventory, increasing the probability of an early-season stock-out. This implies that the selling season with strategic consumers is longer and more variable than with myopic consumers (for whom the selling season ends at $t = 0$).

Numerical example (continued). We continue the numerical example that we evaluated at $Q = 0.60$. Recall in equilibrium the firm’s inventory investment, given its beliefs about the consumer strategy, must be optimal and the consumer’s strategy must be optimal, given their beliefs about the firm’s inventory investment; that is, $Q^*_f(\theta^*(Q^*)) = Q^*$. We illustrate the firm’s best reaction to all the consumers’ equilibrium purchasing strategies for a given believed inventory in Figure 4. Recall from Figure 3 that for some conjectures about the initial inventory, three equilibrium purchasing strategies exists. We plot the firm’s best reaction to each of these three strategies (to the left of the vertical solid line). We find the equilibrium inventory investment and show graphically that for a given consumer strategy, $Q^* = 0.377$ is optimal and for this given $Q^*$, the consumer decision is optimal too. For this $Q^*$, we compute $\theta^*$: $\theta^* = 0.312$ and $\bar{\theta}^* = 0.648$. □
Figure 4 Illustration of the equilibrium inventory investment. The left panel presents the strategic consumer purchasing behavior as a function of the consumers’ conjectured strategy. The right panel presents the firm’s optimal inventory investment for a purchasing behavior based on a given consumer’s conjecture about the inventory.

In the following subsections, we analyze the properties of the equilibrium inventory investment.

4.2. Results for Uniformly Distributed Quality

In this subsection, we analyze the case in which the quality is uniformly distributed over $[0, 1]$; the expected quality, $\mu_V$, is thus $1/2$. With myopic consumers, the optimal initial inventory investment is $Q^o = \alpha v^o + (1 - \alpha)\frac{1}{2}$, where $v^o = \epsilon$. With strategic consumers, we have:

**Proposition 2.** When $F(v) = v$:

(i) Let $\delta = \frac{1}{1 + \epsilon} < \delta$. Then for all $\delta > \delta$, in equilibrium, $v^* \in (1/\delta - 1, \epsilon)$ is the solution to the quadratic equation

$$v = \epsilon - \frac{\gamma (1 + v) \delta - 1}{\mu_V v}$$

from which: $\theta^* = \frac{1}{2} \left( \frac{1 - \delta}{v^*} + \delta u^* \right)$, $Q^* = \alpha v^* + (1 - \alpha)\theta^*$, and $\theta^* = \delta \frac{1}{2} (1 + v^*)$ with $\theta^* < \theta^*$ for $\alpha \in (0, 1)$.

(ii) When $\delta < \delta$, $\theta^* = \theta^* = \frac{1}{2}$ and $Q^* = Q^o$ for $\alpha \in (0, 1)$.

(iii) When $\delta \to 1^-$, $\lim_{\delta \to 1^-} v^* = \epsilon - \frac{\gamma}{\gamma - \epsilon}$ when $\epsilon > \gamma$, and $\lim_{\delta \to 1^-} \frac{v^*}{1 - \delta} = \frac{\gamma}{\gamma - \epsilon}$ when $\epsilon < \gamma$.

Consistent with Proposition 1, only when the discount factor is small enough (Proposition 2(i)), do consumers strategically wait for informative stock-out information. Otherwise, (Proposition 2(ii)), they do not. When the discount factor is large (Proposition 2(iii)), the equilibrium quality threshold is the critical fractile reduced by $\gamma$ when it is positive, $\epsilon - \gamma$. When $\epsilon - \gamma < 0$, the threshold...
will be close to zero (but strictly positive) as \(\lim_{\delta \to 1^-} v^* = 0\). The inequality \(\epsilon < \gamma\) can be written as a function of the fraction of informed consumers, \(\alpha < \hat{\alpha}\), where \(\hat{\alpha} = \frac{r - c}{r + c} (\leq 1)\). This is interesting since for the uniform distribution the stock-out probability is \(1 - v^*\). Hence, \(\gamma\) captures the increase in stock-out probability when the discount factor is large. The incentive to “create” stock-outs is thus driven by \(\gamma = \frac{1}{2} \left( \frac{r}{r + c} - 1 \right) \frac{1 - \alpha}{\alpha}\), which involves the in-season margin (after expenses for in-season replenishment are taken into account) and the market composition. With a lower fraction of informed consumers, the incentive to create a stock-out is greater. When the incentive is so strong that \(\gamma\) exceeds the critical fractile, the quality threshold is so low that a stock-out is almost surely generated.

Next, we analyze how the equilibrium depends on the parameters of the environment, namely the in-season and preseason replenishment costs and the fraction of informed consumers. To do so, we analyze the equilibrium quality threshold above which a stock-out first occurs. With myopic consumers, the equilibrium quality threshold, \(v^o\), is \(\epsilon\), from which \(\frac{\partial v^o}{\partial \epsilon} > 0\), \(\frac{\partial v^o}{\partial c} < 0\) and \(\frac{\partial v^o}{\partial \alpha} = 0\). With strategic consumers, we have:

**Proposition 3.** When \(F(v) = v\), in the limit, and \(\delta \to 1^-:\)

(i) The equilibrium quality threshold level is nondecreasing in the in-season replenishment costs: \(\alpha > \hat{\alpha} : \frac{\partial v^*}{\partial \epsilon} > 0\) and \(\alpha < \hat{\alpha} : \frac{\partial v^*}{\partial \epsilon} = 0\).

(ii) The equilibrium service level is nonmonotonic in the preseason replenishment costs: \(\alpha > \hat{\alpha} : \frac{\partial v^*}{\partial c} > 0\) \(\Leftrightarrow\) \(\alpha < \frac{r}{r + 2c}\) and \(\alpha < \hat{\alpha} : \frac{\partial v^*}{\partial c} = 0\). Only when \(\alpha > \frac{1}{3}\) is \(\frac{\partial v^*}{\partial c} > 0\) possible.

(iii) The equilibrium service level is nondecreasing in the fraction of informed consumers: \(\alpha > \hat{\alpha} : \frac{\partial v^*}{\partial \alpha} > 0\) and \(\alpha < \hat{\alpha} : \frac{\partial v^*}{\partial \alpha} = 0\).

(iv) The higher the quality threshold, the lower (higher) the consumer threshold type that buys at \(t = 0\) (at \(t = 1\) after a stock-out) \(\frac{\partial \theta^*}{\partial v} < 0\) and \(\frac{\partial \theta^*}{\partial v} > 0\).

The comparative statics for the quality threshold follow immediately from Proposition 2(iii). When the fraction of informed consumers is low, the quality threshold tends to zero as the discount factor tends to one. Hence, the threshold does not change as a function of the cost or market structure. When the fraction of informed consumers is high, \(v^* \approx \epsilon - \gamma = \frac{r}{c + \varepsilon} - \frac{1}{2} \left( \frac{r}{c + \varepsilon} - 1 \right) \frac{1 - \alpha}{\alpha}\). It is now easy to see that the threshold increases in the fraction of informed consumers, \(\alpha\), and in the in-season replenishment cost, \(\varepsilon\). These comparative statics are intuitive: The informed consumers reveal quality information to the uninformed ones via stock-outs. With more informed consumers the increase in demand after a stock-out (stemming from uninformed consumers) decreases, hence, the incentive to create stock-outs decreases. Furthermore, a more expensive in-season replenishment cost, \(\varepsilon\), leads to a higher quality threshold (via the newsvendor logic as initial inventory
is a cheaper substitute for in-season replenishment), but also to a lower in-season margin on the
stocked-out items and hence a lower incentive to create stock-outs. These incentives go in the same
direction: Higher in-season replenishment costs yield a higher quality threshold. Interestingly, the
equilibrium quality threshold is ambiguous in the preseason replenishment costs, $c$. A higher pre-
season replenishment cost would, via the newsvendor logic, reduce the quality threshold (and create
more stock-outs). However, because of the reduced margin after a stock-out, a higher preseason
replenishment cost reduces the incentive to create a stock-out, increasing the quality threshold.

Finally, as the quality threshold increases, the stock-out probability increases and hence more
uninformed consumers wait at $t = 0$ and, if the stock-out occurs, more uninformed consumers
purchase at $t = 1$ (Proposition 3(iv)).

Recall from the myopic case that the inventory investment is determined by
$$Q^* = \alpha \varepsilon + \frac{1}{2} (1 - \alpha),$$
where $\varepsilon^* = \varepsilon$. The comparative statics are intuitive: $\frac{\partial Q^*}{\partial \varepsilon} > 0$ and $\frac{\partial Q^*}{\partial c} < 0$. If $\varepsilon > 1/2$, which holds
if and only if $\varepsilon > c$, the inventory is always monotonically increasing in the fraction of informed
consumers, $\frac{\partial Q^*}{\partial \alpha} > 0$. Otherwise, $\frac{\partial Q^*}{\partial \alpha} < 0$.

With strategic consumers, we have:

**Proposition 4.** When $F(v) = v$, in the limit, and $\delta \to 1^{-}$:

(i) The equilibrium inventory is nonmonotonic in the in-season replenishment costs: $\alpha > \hat{\alpha}$: $\frac{\partial Q^*}{\partial \varepsilon} > 0$ and $\alpha < \hat{\alpha}$: $\frac{\partial Q^*}{\partial \varepsilon} < 0$.

(ii) The equilibrium inventory is nonmonotonic in the preseason replenishment costs: $\alpha > \hat{\alpha}$: $\frac{\partial Q^*}{\partial c} > 0$, $\alpha < \hat{\alpha}$: $\frac{\partial Q^*}{\partial c} < 0$. Only when $\alpha > \frac{1}{3}$ is $\frac{\partial Q^*}{\partial c} > 0$ possible.

(iii) The equilibrium inventory reaches a minimum in the fraction of informed consumers at $\alpha = \hat{\alpha}$:
$$\alpha > \hat{\alpha} : \frac{\partial Q^*}{\partial \alpha} > 0$$
and $\alpha < \hat{\alpha}$: $\frac{\partial Q^*}{\partial \alpha} < 0$.

The comparative statics with respect to the replenishment costs are the reverse of the myopic
consumer case. When the fraction of informed consumers is low, their contribution to the change
in inventory will be low (Proposition 2(iii)): $\alpha \varepsilon^* \approx 0$ for $\delta \to 1^{-}$. The main change in inventory is
determined by the uninformed consumers who buy in the beginning of the season: $Q^* \approx (1 - \alpha)\varepsilon^*$. From Proposition 2(ii) and (iii), the inventory is thus $Q^* \approx (1 - \alpha)\frac{1}{2} \left( \frac{1 - \delta}{\delta} + \delta \varepsilon^* \right)$. Even though $\varepsilon^* \approx 0$ when the value loss is small, $\frac{1 - \delta}{\delta} \approx \frac{\gamma}{\gamma + \delta}$. Therefore, the inventory is approximately $\frac{1}{2} (1 - \alpha)(1 - \xi)$. As $\xi = \frac{2\varepsilon}{\gamma - \varepsilon - 1 - \alpha} > Q^* \approx \frac{1}{2} \left( (1 - \alpha) - \alpha \frac{2\varepsilon}{\gamma - \varepsilon - 1 - \alpha} \right)$. Now, it is clear that when the in-season replenishment
cost, $\varepsilon$, increases, the inventory decreases (when $\alpha < \hat{\alpha} = \frac{\varepsilon + \gamma}{\gamma - \varepsilon - \gamma}$). The opposite is true for the
preseason replenishment cost, $c$.

In the case where the fraction of informed consumers is high, the quality threshold is determined
by $\varepsilon^* \approx \varepsilon - \gamma > 0$, which determines the inventory investment for informed consumers, $\alpha(\varepsilon - \gamma)$, and
the inventory for the uninformed consumers who buy in the beginning of the season \( \frac{1}{2}(1-\alpha)(\epsilon-\gamma) \).

In total, the inventory is approximately \( Q^* \approx \frac{1}{2}(1+\alpha)(\epsilon-\gamma) \). The comparative static is similar to the comparative static of the quality threshold (Proposition 3). Now, it is clear that the inventory increases in the fraction of informed consumers and in the in-season replenishment costs. The inventory is ambiguous in the preseason inventory costs.

Interestingly and contrary to the case with myopic consumers, the inventory is not monotonic in the fraction of informed consumers. In fact, it reaches a minimum at \( \hat{\alpha} \) (Proposition 4(iii)). At this minimum inventory level, the incentive to create stock-outs is thus the highest. Recall that \( \hat{\alpha} = \frac{r-c+\epsilon}{r-c+\epsilon+\epsilon} \). Hence, the market heterogeneity level at which the inventory is the lowest is determined by the cost structure. When the in-season margin \( (r-c-\epsilon) \) is low, the lowest inventory occurs for markets with few informed consumers.

Finally, we assess how the cost structure and market composition impacts the firm’s equilibrium profit. We define (for the uniform distribution) the myopic newsvendor profit as a function of the quality threshold, \( v \):

\[
\pi^o(v) = \int_0^1 \Pi(v, \alpha v + \frac{1-\alpha}{2}, (\frac{1}{2}, \frac{1}{2})) dv = \alpha \left\{ \frac{r}{2} - \frac{(c+\epsilon)(1-v)^2}{2} - \epsilon v \right\} + \frac{(1-\alpha)(r-c)}{2},
\]

(12)

Recall that the discount factor does not play a role in the firm’s profit with myopic consumers as no consumer delays purchasing. Hence, \( \pi^o(v) \) does not depend on \( \delta \). With myopic consumers, the firm earns the highest margin on uninformed consumers: \( r-c \). The volume of uninformed consumers depends on the prior expectation about quality and their market share: \( \frac{1}{2}(1-\alpha) \). Due to the quality uncertainty at the time of committing to preseason production, the firm may run out of stock due to the (fickle) informed consumers. The firm’s ex-ante profits on that segment are equal to the expected revenues, \( \frac{1}{2} \alpha r \), based again on the informed consumers’ market share and the prior about the quality, reduced by the minimized newsvendor losses: \( \frac{1}{2}(c+\epsilon)(1-v^o)^2 + c\epsilon^o \).

Let \( \pi^s \) indicate \( \pi^s(v^o) \), the optimal profits with myopic consumers. The comparative statics of \( \pi^o \) are clear: from the envelope theorem, the newsvendor losses increase in both the preseason and in-season cost, even after the firm optimally adjusts its inventory investment in reaction to cost changes: \( \frac{\partial \pi^o}{\partial c} < 0 \) and \( \frac{\partial \pi^o}{\partial \epsilon} < 0 \). It is easy to show that the profits decrease when the market becomes more informed as the firm is exposed to more fickleness due to the informed consumers who only buy when the quality realization is high enough, which is unknown to the firm when committing to the preseason inventory: \( \frac{\partial \pi^o}{\partial \alpha} < 0 \).

Now we can compare the equilibrium profit with strategic consumers to the myopic profit, \( \pi^o(v^o) \):
Lemma 5. When \( F(v) = v \) and \( \delta > \frac{1}{1 + \epsilon} \), the firm’s expected profits are 
\[
\Pi^* = \pi^*(v^*) - \frac{1}{2}(1 - \alpha)\epsilon(1 - v^*)\delta(1 - \frac{1}{\delta - 1}).
\]

In the literature on strategic consumer behavior with forward looking consumers, the firm typically looses profit due to strategic behavior (see, e.g., Lai et al. 2010). Lemma 5 allows us to address this issue. With myopic consumers, the newsvendor solution maximizes the firm’s profit, so we have that \( \pi^o(v^o) \geq \pi^o(v^*) \). Furthermore, due to equilibrium condition (11) in Proposition 2: 
\[
\epsilon(1 - v^*) \frac{(1 + v^*)\delta - 1}{\epsilon} > 0 \text{ as } v^* > 1/\delta - 1.
\]
Therefore from the decomposition of \( \Pi^* \) in Lemma 5, we see that the firm can never make more profits with strategic consumers than with myopic consumers when the market is heterogeneously informed. In addition, we have:

Proposition 5. When \( F(v) = v \), in the limit, and \( \delta \to 1^- \):

(i) The equilibrium profits are nonmonotonic in the in-season replenishment costs: \( \alpha > \hat{\alpha} : \frac{\partial \Pi^*}{\partial c} \leq 0 \) and \( \alpha < \hat{\alpha} : \frac{\partial \Pi^*}{\partial c} < 0 \).

(ii) The equilibrium profits are nonmonotonic in the preseason replenishment costs: \( \alpha > \hat{\alpha} : \frac{\partial \Pi^*}{\partial c} \leq 0 \) and \( \alpha < \hat{\alpha} : \frac{\partial \Pi^*}{\partial c} < 0 \). Only when \( \alpha < \sqrt{2} - 1 = 0.41421 \) is \( \frac{\partial \Pi^*}{\partial c} > 0 \) possible.

(iii) The equilibrium profits reach a minimum in the fraction of informed consumers at \( \alpha = \hat{\alpha} \): \( \alpha > \hat{\alpha} : \frac{\partial \Pi^*}{\partial \alpha} > 0 \) and \( \alpha < \hat{\alpha} : \frac{\partial \Pi^*}{\partial \alpha} < 0 \).

We have seen that, paradoxically, an increase in in-season production cost, \( \epsilon \), can lead to increased profits when \( \alpha > \hat{\alpha} \) (Proposition 5(i)). This result never holds under classical newsvendor reasoning with myopic consumers. One of the main drivers of the profit increase with strategic consumers is the increased satisfied demand. The total satisfied demand is \( \frac{1}{2}\alpha + (1 - \alpha)\theta^* \). Recall that \( \frac{\partial \theta^*}{\partial v} > 0 \) (Proposition 3(iv)) and \( \frac{\partial v^*}{\partial \epsilon} > 0 \) (Proposition 3(i)). Combining these two effects, we see that the total expected demand increases in the in-season replenishment costs. Even though the margin decreases when the in-season costs increase, the increase in total expected demand makes the expected profits increase overall. This comparative static result is also in line with Proposition 1; when the in-season replenishment costs are too low, stock-outs may be uninformative. Therefore, with strategic consumers, it may be in a firm’s interest to keep expensive in-season replenishment systems to support the signaling value about quality contained in early-season stock-outs. Without such systems, strategic consumers believe that the stock-outs are due to low initial inventory levels and hence have little to do with high quality. It is also interesting to note that the profits can increase in the preseason costs when \( \alpha > \hat{\alpha} \) (Proposition 5(ii)). This may happen when the fraction of informed consumers is not too small. Recall that an increase in preseason procurement costs can lead to a higher initial inventory (see Proposition 4(ii)), hence, increased satisfied demand. The resulting increase in revenues can more than compensate for the increased cost.
Interestingly, the profit is not monotonically decreasing in the fraction of informed consumers. As with inventory, the profits reach a minimum at $\hat{\alpha}$ (Proposition 5(iii)). Therefore, when the incentive to create stock-outs via low inventory investment is the highest, the profits are also the lowest. This implies firms facing a market with a fraction of $\hat{\alpha}$ informed consumers can earn more profits by either reducing or increasing the fraction of informed consumers (provided they have the ability to influence $\alpha$). Both strategies will entail an increase in inventory.

In the next section, we check the robustness of our findings with respect to the assumption of uniformly distributed quality. We also investigate general value losses.

4.3. Results from Numerical Experiments
In this subsection, we numerically demonstrate the robustness of our findings for (1) For a generic discount factor, but large enough such that consumers have an incentive to wait (as determined in Proposition 1) and (2) a Beta-distributed prior density about the quality. We assess the change in profit, inventory and total satisfied demand for $\delta = 0.9$, $c = 0.2$, $r = 1$, $\varepsilon = 0.4$ and $\alpha = 1/2$ as our base case. The margin, $r - c = 0.8$, is relatively high, as is common for new and innovative products. The in-season shipment cost is twice as expensive as the preseason shipment cost. The in-season margin is $r - c - \varepsilon = 0.4$, which is half of $r - c$. We use the Beta distribution with generic parameters $(\nu, \omega)$. For our base case, we consider $\nu = \omega = 2$, which yields a mean, $\mu_V$, of $1/2$ and a standard deviation $(2\sqrt{5})^{-1}$, which is less than the standard deviation of the uniform distribution. The results are shown in Figures 5 and 6, where the solid lines are equilibrium results (i.e., with strategic consumers) and the dashed lines are results for myopic consumers, for which $\theta^o = \bar{\theta}^o = \mu_V = 1/2$ and $Q^o (= (1 - \alpha)\mu_V + \alpha F^{-1}(\varepsilon))$ is the inventory investment. For our base case, as the critical fractile, $\varepsilon$, is $2/3$ and $F^{-1}(2/3) > 1/2$, from which it follows that the myopic inventory is increasing in the fraction of informed consumers.

![Figure 5](image_url)

**Figure 5** Profits, inventory and total demand as a function of the fraction of uninformed consumers. The solid lines are the strategic consumers and the dashed lines are the myopic consumers.
The impact on the firm’s profit and inventory of consumers strategically waiting for quality information is greatest for heterogeneous markets, i.e. intermediate values of $\alpha$. (See Figure 5.) Both profit and inventory decline due to informed consumers. This is intuitive as Propositions 4(iii) and 5(iii) establish a critical fraction of informed consumers, $\hat{\alpha} = \frac{0.4}{1.2} = 0.333...$ above (below) which the inventory and profits increase (decrease). For homogenous markets ($\alpha$ either very high or very low), strategic consumers do not matter much.

The strategic consumer reaction can lead to a 7% increase in total demand volume for heterogeneously informed markets. (See Figure 5.) Therefore, interestingly, the stock-out information may lead to more consumers buying the product. However, the firm cannot translate this increase into higher profits.

The equilibrium profits can increase in the in-season inventory replenishment cost (Proposition 5(i)), as seen in Figure 6. Intuitively, with higher costs of replenishing after a stock-out, the stock-out information is a more credible signal of high quality rather than low initial inventory. Hence, consumers are more willing to delay their purchase until later in the season and more will buy after stock-outs are observed early in the season. Similarly, the equilibrium inventory may also decrease in the in-season replenishment costs (Proposition 4(i)), as shown in Figure 6. This is in contrast with the newsvendor logic that holds with myopic consumers.

The counterintuitive effects that inventory and profits may increase in the preseason procurement cost (Proposition 4(ii) and 5(ii)) are not very robust. These effects do not hold for $\delta = 0.9$. We tested the comparative statics with $\delta = 0.9999$ and did clearly observe the limiting comparative statics. However, for more realistic discount factors, in our numerical example the equilibrium inventory and profits always decrease in the preseason replenishment cost. We do not report these experiments.

Next, we examine some numerical results on the impact of an increase in the prior on quality
(the firm’s “quality reputation”), keeping the coefficient of variation constant. The mean prior is \( \nu/(\nu+\omega) \). We consider different parameterizations of the demand distribution, \((\nu, \omega)\). We increase the mean prior quality, \(\mu_V\), keeping the squared coefficient of variation equal to 1/10. Then, the parameter values are \(\nu = 10 - 11\mu_V\) and \(\omega = \nu(1 - \mu_V)/\mu_V\). In order to retain a unimodal density function with \(f(0) = f(1) = 0\), we need that \(\omega > 1\) and \(\nu > 1\). Hence, we limit the range of \(\mu_V\) to \(\{1/11, 2/11, 3/11, \ldots, 7/11\}\). With these parameter values, \(\omega\) and \(\nu\) are integers and the distribution function is hypergeometric, which allows for efficient computation of the equilibrium from Equation (9). For \(\mu_V = 1/11\), the parameters are \(\omega = 90\) and \(\nu = 9\). The high value of \(\omega\) yields numerically instable evaluations of the hypergeometric function, so we left out this parameter value. Obviously, as the mean prior increases, profits (and inventory) increase. To analyze the relative impact of the mean prior, we plot the ratio of the equilibrium over myopic profits and inventory as a function of the fraction of informed consumers. Observe from Figure 7 that the shape qualitatively corresponds with our theoretical results for the uniform distribution: The (relative) difference between the equilibrium profits and inventory is the largest for heterogeneously informed markets. This suggests our findings above are robust with respect to the mean of the density function, as long as the uncertainty, characterized by the squared coefficient of variation, remains constant. Interestingly, for low fractions of informed consumers, the profit loss compared to the myopic case is the highest for firms with a “strong” reputation for quality. Only when the fraction of informed consumers is high are the losses the lowest for such firms. However, the difference relative to firms with a weak reputation is small and is almost not discernable in the figure.

Figure 7 Relative profits (left panel) and inventory (right panel) as a function of the fraction of uninformed consumers. The dotted line corresponds with \(\mu_V = 2/11\), the dashed line corresponds with \(\mu_V = 7/11\). For all experiments, the coefficient of variation is \(1/\sqrt{10}\).
5. Conclusions

Although there is speculation in the business and popular press about the reasons for frequent stock-outs of new innovative products, to the best of our knowledge no information-based theory provides a rationale for how stock-outs actually generate the positive “buzz” for a rational firm with rational consumers. Our model sheds new light on this issue. It explains how firms can generate sales through stock-outs.

In our model, unknown quality is the key source of uncertainty for the main decision makers. From the firm’s perspective, we decompose essentially the market demand into two distributions: A continuous distribution\(^8\) for the “savvy” market segment and a bivalued distribution for the “novice” segment.\(^9\) A “high” realization of the latter corresponds with the positive “buzz”. Bivalued demand distributions have been used in the operations management literature to capture herd behavior (Lippman and McCardle, 1997, example 4). A novelty in our model is that the demand distributions are endogenized via strategic behavior of differently informed consumers. From the consumer’s perspective, in essence, we find that the savvy consumers always buy early as there are no gains from waiting. The novice, but “eager” consumers (with a low value for their outside option) do not care about waiting and also buy early in the season. The novice and less eager consumers delay their purchase decision and only purchase the product when early-season stock-outs have been reported. Stock-outs are informative only when the in-season replenishment costs are high enough. In essence, a high cost of in-season replenishment serves as a commitment device; consumers will believe that any stock-outs observed when replenishment is costly are due to high quality. In this case, the firm increases the total expected sales from the novice segment whose demand becomes more uncertain (contingent on an stock-out) and spread over the whole season. Moreover, we find that inventory and equilibrium profits are the lowest for markets with intermediate levels of heterogeneity.

Are these findings consistent with stylized facts? To the best of our knowledge, no empirical study rigorously investigates the potential positive information that can be contained in early-season stock-outs (except for some experimental research, see, e.g., Verhallen 1982). As noted, some recent high-profile stock-outs have been extensively reported in the media. For example, Apple’s 3G iPhone shortage in 2005. At that time Apple’s Chief Operating Officer, Tim Cook, acknowledged “a number of stock-outs” of Apple 3G iPhones on July 18, 2005, attributing the stock-outs to “overwhelming demand.” Apple’s supply chain extends into Asia. FedEx in-season replenishment from the Shenzhen factory (operating at full capacity) reduces the margin; moreover,

\(^8\) The Beta distribution with parameters \(\omega > 1\) and \(\nu > 1\) resembles a normal distribution.

\(^9\) Either \((1 - \alpha)\mathbf{g}^*\) with probability \(F(\mathbf{g}^*)\) or \((1 - \alpha)\bar{\mathbf{g}}^*\) with probability \(\bar{F}(\mathbf{g}^*)\).
capacity constraints can be interpreted as equivalent to infinite marginal costs of production beyond capacity. Hence, it is credible that any shortages are due to high demand. If Apple’s supply chains were fully domestic, making in-season replenishment cheap, early-season shortages may have been less credible signals of quality. In contrast, lack of initial shortages may reduce the perception of a new product. For example, the availability of the iPad2 4S in Apple stores makes some question the appeal of this new version of the iPad.\footnote{See for example: “... there doesn’t appear to be as much as of a frenzy as we expected over the new iPad. We walk into any Apple store and get one today, easily, and that may be a problem.” Savitz, 2012.}

We hope that our paper provides an empirically testable hypotheses about consumer reactions to stock-outs. Our theory does not assume any status-like utility increases due to scarcity (like a “snob” effect, see Tereyagolu and Veeraraghavan, forthcoming). On the contrary, in our model, if consumers knew the exact quality realization, they would all value the product in exactly the same way, independently of each other. Our theory is mainly based on heterogeneity in the consumer’s ability to quickly assess the true utility of a product. We only posit that some consumers can do this well and purchase early, generating valuable information via early-season stock-outs for other consumers who delay their purchase until later in the season.

6. References


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7. Appendix: Proofs

**Proof of Lemma 1:** This proof is straightforward and hence omitted. □

**Proof of Lemma 2:** (i) Is obvious because $U'_W(Q, \theta, \theta) \geq 0$.

(ii) Is the root in $[0, \mu_v]$ of $H(\theta, Q) = 0$, where $H(\theta, Q) = \delta \int_{|\|Q, \theta\|}^1 \nu dF(v) - \theta \bar{F}(\nu(Q, \theta)) - (\mu_v - \theta)$. If no such root exists, it is easy to see that $H(\theta, Q) < 0$ for all $\theta \in [0, \mu_v]$ (because $H(0, Q) = \delta \int_{|\|Q, \theta\|}^1 \nu dF(v) - \mu_v < (\delta - 1)\mu_v < 0$). Then, no consumer can obtain positive utility from delaying their purchase.

(iii) Consumers that delay purchasing do not want to buy at $t = 1$ after observing no stock-out at $t = 0$, or: $\delta \frac{\int_{0}^{\nu(Q, \theta)} \nu dF(v)}{\frac{\nu(Q, \theta)}{F(\nu(Q, \theta))}} < \theta$. We consider two cases:

(iii-a): When $\theta \in [0, \mu_v]$, then $\delta \int_{0}^{\nu(Q, \theta)} \nu dF(v) - \theta \bar{F}(\nu(Q, \theta)) = \mu_v - \theta$. Let $S$ be such that: $\int_{0}^{\nu(Q, \theta)} \nu dF(v) - \theta \bar{F}(\nu(Q, \theta)) + S = 0$. Then adding this equality to the equality that defines $S$, we obtain $\int_{0}^{1} \nu dF(u) - S = \mu_v - \theta > 0$ or $S = (1 - \delta)\mu_v > 0$. Therefore, $\delta \int_{0}^{\nu(Q, \theta)} \nu dF(v) - \theta \bar{F}(\nu(Q, \theta)) < 0$, or, it is optimal for consumers types above $\theta$ not to purchase the product when there is no stock-out.

(iii-b): When $\theta = \mu_v$, then $\int_{0}^{\nu(Q, \mu_v)} \nu dF(v) - \mu_v \bar{F}(\nu(Q, \mu_v)) < 0 \equiv \delta \frac{\int_{0}^{\nu(Q, \mu_v)} \nu dF(v)}{\frac{\nu(Q, \mu_v)}{F(\nu(Q, \mu_v))}} < \mu_v$. As the conditional expectation, $\int_{0}^{\nu(Q, \mu_v)} \nu dF(v)$, is always less than $\mu_v$ (as $\frac{d}{d\nu} \frac{\int_{0}^{\nu} \nu dF(v)}{\frac{\nu}{F(\nu)}} > 0$, $\delta \frac{\int_{0}^{\nu(Q, \mu_v)} \nu dF(v)}{\frac{\nu(Q, \mu_v)}{F(\nu(Q, \mu_v))}}$ is also less than $\mu_v$.

(iv) Is obvious and hence, its proof is omitted.

**Proof of Lemma 3:** We drop $Q$ from $H(\theta, Q)$ (defined in Lemma 2). When we find a root of $H(\theta) = 0$ that is less than $\mu_v$, an equilibrium exists with some consumers buying at $t = 0$ and some waiting to buy until $t = 1$ and a stock-out occurs at $t = 0$. Otherwise, if $H(\theta) < 0$ for all $\theta \in [0, \mu_v]$, all consumers buy at $t = 0$ and none buy at $t = 1$, irrespective of the stock-out at $t = 0$. As we consider $Q < 1$ and $0 < \alpha < 1$, $\frac{Q - (1 - \alpha)}{\alpha} < 1 \Rightarrow Q < 1$, which is thus always satisfied. With $\nu(Q, \theta)$, we can write $H(\theta)$ as follows:
\[
\begin{array}{c|c|c|c}
\alpha < 1/2 & 0 < Q < \alpha < 1 - \alpha & 0, \frac{\alpha}{1-\alpha} : H_{II}(\theta) & \frac{\alpha}{1-\alpha}, 1 : H_{III}(\theta) \\
0 < \alpha < Q < 1 - \alpha & 0, \frac{\alpha}{1-\alpha} : H_{I}(\theta) & \frac{\alpha}{1-\alpha}, \frac{\alpha}{1-\alpha} : H_{II}(\theta) & \frac{\alpha}{1-\alpha}, 1 : H_{III}(\theta) \\
0 < \alpha < 1 - \alpha < Q & 0, \frac{\alpha}{1-\alpha} : H_{I}(\theta) & \frac{\alpha}{1-\alpha}, \frac{\alpha}{1-\alpha} : H_{II}(\theta) & \frac{\alpha}{1-\alpha}, 1 : H_{III}(\theta) \\
\alpha < 1/2 & 0 < Q < 1 - \alpha < \alpha & 0, \frac{\alpha}{1-\alpha} : H_{I}(\theta) & \frac{\alpha}{1-\alpha}, \frac{\alpha}{1-\alpha} : H_{II}(\theta) \\
0 < 1 - \alpha < Q < \alpha & 0, \frac{\alpha}{1-\alpha} : H_{I}(\theta) & \frac{\alpha}{1-\alpha}, \frac{\alpha}{1-\alpha} : H_{II}(\theta) & \frac{\alpha}{1-\alpha}, 1 : H_{III}(\theta) \\
0 < 1 - \alpha < \alpha < Q & 0, \frac{\alpha}{1-\alpha} : H_{I}(\theta) & \frac{\alpha}{1-\alpha}, \frac{\alpha}{1-\alpha} : H_{II}(\theta) & \frac{\alpha}{1-\alpha}, 1 : H_{III}(\theta)
\end{array}
\]

where \( H_1(\theta) = -\mu_V + \theta, H_{II}(\theta) = \delta \int_0^{Q/(\alpha (1-\alpha))} v dF(u) - \theta F(Q/(\alpha (1-\alpha))) - \mu_V + \theta \) and \( H_{III}(\theta) = -(1 - \delta) \mu_V \). It follows that there can never be an equilibrium in \( H_{III}(\theta) \). There can only be an equilibrium in \( H_1(\theta) \), \( \theta = \mu_V \) when \( \mu_V < \frac{Q}{\alpha} \iff \alpha + \mu_V (1 - \alpha) < Q \). However, in that case: \( \frac{(Q-(1-\alpha)\mu_V)^+}{\alpha} > 1 \), hence, no stock-out will occur at \( t = 0 \). If no stock-out occurs, none of the remaining uninformed buyers will buy because the posterior value will be \( \delta \mu_V \). Otherwise, when \( \alpha + \mu_V (1 - \alpha) > Q \) the equilibrium needs to be in \( H_{II}(\theta) \). Now, assume that \( \alpha + \mu_V (1 - \alpha) > Q \). A sufficient condition is to find a \( Q \) less than \( \alpha + \mu_V (1 - \alpha) \) such that \( H_{II}(\mu_V) > 0 \) or:

\[
\delta \int_0^{1} \frac{(Q-(1-\alpha)\mu_V)}{\alpha} udF(u) - \mu_V F(Q/(\alpha (1-\alpha))) - \mu_V + \mu_V > 0.
\]

From the definition of \( \bar{v}(\delta) \), it follows that \( \delta \int_0^1 udF(u) - \mu_V F(u) > 0 \) when \( v > \bar{v}(\delta) \) with \( \bar{v}(1) = 0 \).

Hence, when \( \delta < 1 \), there will exist some \( \bar{v}(\delta) \in (0, 1) \) for which the above inequality holds. Therefore, if we select \( Q = \bar{Q}(\delta) \) such that \( \bar{v}(\delta) = \frac{Q-(1-\alpha)\mu_V}{\alpha} \); \( \bar{Q}(\delta) = \alpha \bar{v}(\delta) + (1 - \alpha)\mu_V \), then, we know that \( \alpha + \mu_V (1 - \alpha) < \bar{Q}(\delta) \) and hence, for all \( Q < \bar{Q}(\delta), H_{II}(\mu_V) < 0 \), which is thus an equilibrium. For all \( \bar{Q}(\delta) < Q < \alpha + \mu_V (1 - \alpha) \), at least, there exist one \( \theta \in [0, \mu_V] \) such that \( H_{II}(\theta) = 0 \). □

**Proof of Lemma 4:** The proof is straightforward and hence omitted. Note that the second order condition (SOC) is: \(-f'(\bar{u}^*) - (\frac{\varepsilon}{\bar{c}^* - \bar{c}^*}) \frac{\bar{c}^* - \bar{c}^*}{\bar{c}^* - \bar{c}^*} f'(\bar{u}^*) < 0 \), which, we assume holds. The comparative statics are obtained from implicit derivation of \( \bar{v}^* = F(Q/(\alpha (1-\alpha))) \) and of the equilibrium condition Equation (9). We obtain:

\[
\frac{dQ_I^*}{d\bar{c}} = \alpha \frac{d\bar{c}^*}{f(F^{-1}(\bar{c}^*))} + (1 - \alpha) \quad \text{and} \quad \frac{dQ_I^*}{d\bar{c}} = \alpha \frac{d\bar{c}^*}{f(F^{-1}(\bar{c}^*))} + (1 - \alpha)
\]

which gives after substituting \( \frac{\varepsilon}{\bar{c}^* - \bar{c}^*} \frac{\bar{c}^* - \bar{c}^*}{\bar{c}^* - \bar{c}^*} = (\frac{\varepsilon}{\bar{c}^* - \bar{c}^*}) \frac{\bar{c}^* - \bar{c}^*}{\bar{c}^* - \bar{c}^*} \),

\[
\frac{dQ_I^*}{d\bar{c}} = \alpha \frac{d\bar{c}^*}{f(F^{-1}(\bar{c}^*))} + (1 - \alpha)
\]

and \( \frac{dQ_I^*}{d\bar{c}} = \alpha \frac{d\bar{c}^*}{f(F^{-1}(\bar{c}^*))} + (1 - \alpha) \)

Hence:

\[
\frac{dQ_I^*}{d\bar{c}} + \frac{dQ_I^*}{d\bar{c}} = 1 - \alpha \quad \text{and} \quad \frac{dQ_I^*}{d\bar{c}} < 0 \iff \frac{(F(\bar{c}^*))^2}{\bar{c}^*} < f'(\bar{c}^*)\]

where \( F(\bar{c}^*) < \frac{\varepsilon}{\bar{c}^* - \bar{c}^*} \). Given the second order condition (SOC) holds (by assumption), after substituting \( \frac{\varepsilon}{\bar{c}^* - \bar{c}^*} \frac{\bar{c}^* - \bar{c}^*}{\bar{c}^* - \bar{c}^*} \) by \( F(\bar{c}^*) - \frac{\varepsilon}{\bar{c}^* - \bar{c}^*} \)
(which are equal in equilibrium), we obtain \( f'(\bar{w}) > \frac{[f'(\bar{w})]^2}{F'(\bar{w}) - \bar{w}} \), from which follows that \( \frac{dQ^*_F}{dw} < 0 \) and \( \frac{dQ^*_F}{dw} = 1 - \alpha - \frac{d\bar{Q}}{d\bar{w}} > 1 - \alpha \). \( \square \)

**Proof of Proposition 1:** When \( Q \in [0, \bar{Q}(\delta)] \), an equilibrium purchasing strategy is \( \theta^*(Q) = \bar{\theta}^*(Q) = \mu_F \) (Lemma 3). The optimal inventory investment with such purchasing strategy is \( Q^o \). Therefore, when the best reaction of the firm is \( Q^o \) for conjectured inventory over \([0, \bar{Q}(\delta)]\) and \( Q^o \in (0, \bar{Q}(\delta)) \), the best reaction intersects the 45 degree line and is thus an equilibrium. From the definitions, it follows that: \( Q^o < \bar{Q}(\delta) \Leftrightarrow \epsilon < F(\bar{Q}(\delta)) \). Now, consider the case \( Q^o > \bar{Q}(\delta) \), or \( F(\bar{Q}(\delta)) < \epsilon \). Notice that by definition of \( \bar{Q}(\delta) \), \( \delta \int_{\bar{Q}(\delta)}^1 \frac{w}{\mu_F} dF(u) = F(\bar{Q}(\delta)) \), it follows that \( 1 - \delta + \delta \int_{\bar{Q}(\delta)}^1 \frac{w}{\mu_F} dF(u) > 1 - \delta + \int_{\bar{Q}(\delta)} \frac{w}{\mu_F} dF(u) \) (we omit the proof for brevity). Hence, as \( f(v) \geq 0 \), the second term is nonnegative over \((\bar{Q}(\delta), 1)\). By continuity, there must be at least one intersection point, \( v^* \), in \((\bar{Q}(\delta), 1)\) that satisfies Equation (9). By assumption, it satisfies Equation (10) and determines \( \bar{\theta}^* = \delta \int_{v^*} \frac{w}{\mu_F} dF(u) \), \( \theta^* = \frac{(1-\delta)\mu_F}{\mu_F} + \theta \int_{\bar{Q}(\delta)} v^* \frac{w}{\mu_F} dF(u) \) (with \( \bar{\theta}^* > \theta^* \)) and \( Q^o = \alpha v^* + (1 - \alpha) \theta^* \). \( \square \)

**Proof of Proposition 2:** (i) When \( F(v) = v \), \((\bar{w}^*, \theta^*, \bar{\theta}^*, Q^*)\) satisfy the following four conditions:

\[
\delta \int_0^1 v d\omega - \theta(1 - \bar{\omega}) = \frac{1}{2} - \theta, \quad \bar{\theta} = \delta \int_0^1 v d\omega - \frac{1}{2} = \frac{Q - (1 - \alpha)\theta}{\alpha} \quad \text{and} \quad v = \frac{\epsilon}{c + \epsilon} - \frac{r}{c + \epsilon} - 1 \frac{1 - \alpha}{\alpha} (\bar{\theta} - \theta)
\]

and hence, we can first solve for \((\bar{w}^*, \theta^*, \bar{\theta}^*)\), from which \( Q^o \) follows directly. We first solve for \( \theta \) and \( \bar{\theta} \) as a function of \( v \): \( \bar{\theta} = \frac{1}{2} \frac{1 - \delta(v^2 - 1)}{v^2}, \quad \bar{\theta} = \delta \bar{\theta}^* (1 + v^2) \). Now, we substitute \( \bar{\theta} - \bar{\theta}^* \) in the firm’s equilibrium condition, which gives a quadratic equation in \( v^* \): \( v^* = \frac{\epsilon - \gamma \delta}{2} \left( 1 + \frac{\gamma (1 - \delta)}{(c - \gamma \delta)^2} \right) \). The root is:

\[
v^* = \left\{ \begin{array}{ll}
\frac{\epsilon - \gamma \delta}{2} \left( 1 + \frac{\gamma (1 - \delta)}{(c - \gamma \delta)^2} \right) & \frac{\epsilon - \gamma \delta}{2} > 0 \\
\frac{\epsilon - \gamma \delta}{2} \left( 1 - \frac{\gamma (1 - \delta)}{(c - \gamma \delta)^2} \right) & \frac{\epsilon - \gamma \delta}{2} < 0
\end{array} \right.
\]

Hence \( v^* \in (0, 1) \) and \( \bar{\theta}^* - \theta^* > 0 \) when: \( v^* \in \left( \frac{1}{2}, 1 \right) \Leftrightarrow \bar{\theta}^* > \theta^* \) and \( 1 + \frac{\gamma (1 - \delta)}{(c - \gamma \delta)^2} < \frac{2}{c - \gamma \delta} - 1 \). Notice that:

\[
\frac{dv^*}{d\gamma} = \left\{ \begin{array}{ll}
\frac{1}{2} - \frac{\gamma + 2}{2} \frac{1 + \frac{\gamma (1 - \delta)}{(c - \gamma \delta)^2}}{1 + \frac{\gamma (1 - \delta)}{(c - \gamma \delta)^2}} & \frac{\epsilon - \gamma \delta}{2} > 0 \\
- \frac{1}{2} - \frac{\gamma + 2}{2} \frac{1 + \frac{\gamma (1 - \delta)}{(c - \gamma \delta)^2}}{1 + \frac{\gamma (1 - \delta)}{(c - \gamma \delta)^2}} & \frac{\epsilon - \gamma \delta}{2} < 0.
\end{array} \right.
\]

Via straightforward algebra, which we omit for brevity, we find that when \( \frac{1}{2} - 1 < \epsilon, \frac{dv^*}{d\gamma} < 0 \). Notice also that \( \lim_{\gamma \to +\infty} v^* = \lim_{\gamma \to +\infty} \frac{\gamma^2}{2} \frac{1 - \delta}{3 - \gamma \delta} \) (with \( 1 - \sqrt{1 + 2x} \approx -\frac{1}{2}x \)), hence: \( \lim_{\gamma \to +\infty} v^* = \frac{1}{2} - 1 \). Also: \( \lim_{\gamma \to +\infty} v^* = \epsilon \). As \( v^* \) decreases in \( \gamma \), we obtained \( v^* \in \left[ \frac{1}{2}, 1 - \epsilon \right] \). When \( \epsilon = \frac{1}{2} \), then, it is easy to see that \( \bar{\theta}^* = \theta^* = \frac{1}{2} \) because \( v^* = \frac{1}{2} - 1, \theta^* = \frac{1}{2} - \frac{\gamma (1 - \delta)}{(c - \gamma \delta)^2} \) and \( \bar{\theta}^* = \delta \bar{\theta}^* (1 + \frac{1}{2} - 1) = \frac{1}{2} \).
Therefore, for $\frac{1}{\delta} - 1 < \epsilon$, there will be no waiting uninformed consumers.

(ii) Follows immediately from Proposition 1.

(iii) We can rewrite the condition $\nu = \epsilon - \gamma \frac{(1+\nu)\delta - 1}{\nu}$ as follows: $\Psi(\nu) = 0$, where $\Psi(\nu) = \frac{(\epsilon - \nu)}{2(\epsilon - \nu - 1)} \nu - \frac{1-\alpha}{\alpha} ((1+\nu)\delta - 1)$. Note that when $\delta \to 1$, the solution of $\Psi(\nu) = 0$ becomes:

$$\frac{1}{2} (\epsilon - \gamma + |\gamma - \epsilon|) + (1-\delta) \frac{1}{2} \left( \gamma + 2\gamma - \gamma^2 + \gamma \epsilon \right) \text{ and } \frac{1}{2} (\epsilon - \gamma - |\gamma - \epsilon|) + (1-\delta) \frac{1}{2} \left( \gamma - 2\gamma - \gamma^2 + \gamma \epsilon \right).$$

When $\alpha > \hat{\alpha}$, the roots are: $\epsilon - \gamma + (1-\delta)\gamma \frac{2 - \epsilon}{1 - \epsilon}$ and $(1-\delta) \frac{2}{1 - \epsilon}$. When $\alpha < \hat{\alpha}$, the roots are: $(1-\delta) \frac{2}{1 - \epsilon}$ and $\epsilon - \gamma + (1-\delta)\gamma \frac{2 - \epsilon}{1 - \epsilon}$. It is easy to see that one root can be discarded and we obtain:

$$\nu^* \approx \begin{cases} \epsilon - \gamma, & \alpha > \hat{\alpha} \\ (1-\delta) \frac{2}{1 - \epsilon}, & \alpha < \hat{\alpha} \end{cases}$$

**Proof of Proposition 3:** We compute $\frac{\partial \nu^*}{\partial x} = -\frac{\partial \nu}{\partial x}$, $\frac{\partial \nu^*}{\partial \nu} = -\frac{\partial \nu}{\partial \nu}$ and $\frac{\partial \nu^*}{\partial \alpha} = -\frac{\partial \nu}{\partial \alpha}$.

For every $\delta \in (\frac{1}{1+x},1)$, we can identify one $x$ that satisfies $\gamma \frac{(1+x)\delta - 1}{\delta - 1} = \epsilon - x$. Therefore, we consider $\delta$ as an independent parameter and substitute $\delta = \frac{1+x}{1+\frac{c\alpha}{\alpha C}}$ in the above expressions. We focus on high values of $\delta$ as for low value of $\delta$, the system behaves similar to a myopic one.

(i-i-ii-iii) It is easy to see that $\alpha < \hat{\alpha}$: $\frac{\partial \nu^*}{\partial x} = \frac{\partial \nu^*}{\partial \nu} = \frac{\partial \nu^*}{\partial \alpha} = 0$ and for $\alpha > \hat{\alpha}$: $\frac{\partial \nu^*}{\partial x} > 0$, $\frac{\partial \nu^*}{\partial \nu} > 0 \iff -2\epsilon \alpha + (1-\alpha)\hat{\nu} > 0$ and $\frac{\partial \nu^*}{\partial \alpha} > 0$.

(iv-v) From which the signs are derived. Recall that $\nu^* \in [\frac{1}{2} - 1, \epsilon]$ and $\epsilon < 1$, hence, $(\nu^*)^2 < \frac{1}{2} - 1$.

Now, we obtain: $\frac{\partial \nu}{\partial x} = \frac{1}{x} \frac{\partial \nu}{\partial x} \frac{\nu^2 - (\frac{1}{2} - 1)}{\nu^2} < 0$ and $\frac{\partial \nu}{\partial \nu} = \frac{\nu^2 - (\frac{1}{2} - 1)}{\nu^2} > 0$.

**Proof of Proposition 4:** Now, we can analyze $Q^* = Q^*(\nu^*)$, where $Q^*(\nu) = \alpha \nu + (1 - \alpha) \frac{1}{\nu} \frac{1+\nu}{\nu} \frac{(1-x)^2}{\nu}$. We obtain: $\frac{\partial Q^*}{\partial \nu} = -\frac{\partial Q}{\partial \nu} + \frac{\partial Q}{\partial \nu} + \frac{\partial Q}{\partial \nu}$ and $\frac{\partial Q^*}{\partial \alpha} = -\frac{\partial Q}{\partial \alpha} + \frac{\partial Q}{\partial \alpha} + \frac{\partial Q}{\partial \alpha}$. Then (A) assuming that $\frac{x}{c} - \gamma > 0$, we let: $\nu^* = \frac{x}{c} - \gamma$ and obtain straightforward algebra that for $\alpha < \hat{\alpha}$:

$$\frac{\partial Q^*}{\partial \nu} < 0, \frac{\partial Q^*}{\partial \nu} < 0 \text{ and } \frac{\partial Q^*}{\partial \alpha} < 0$$

and (B) assuming that $\frac{x}{c} - \gamma < 0$, we have that $r - c - \epsilon + \alpha - r \alpha - \alpha \epsilon > 0$ with $\delta$ close to one, with $\epsilon = \frac{x}{c} - \gamma$, we let: $\nu^* = (1-\delta) \frac{x}{c} - \gamma$ and obtain straightforward algebra that for $\alpha > \hat{\alpha}$:

$$\frac{\partial Q^*}{\partial \nu} > 0, \frac{\partial Q^*}{\partial \nu} > 0 \iff -2\epsilon \alpha + (1-\alpha)\hat{\nu} > 0 \text{ and } \frac{\partial Q^*}{\partial \alpha} > 0 \iff \alpha^2 (r + \epsilon - c) + r - \epsilon - c > 0.$$
**Proof of Proposition 5:** Now, we compute the comparative static of the equilibrium profit:

\[ \Pi^* = \Pi^* (\nu^*), \]

where

\[
\Pi^* (\nu) = \alpha \left( \frac{1}{2} r - \left\{ (c + \varepsilon) \frac{1}{2} (1 - v)^2 + cv \right\} - (1 - v) (\varepsilon - v) \frac{\varepsilon}{c + \varepsilon - 1} \right) + (1 - \alpha) \frac{1}{2} (r - c)
\]

We obtain: \( \frac{\partial \Pi^*}{\partial \varepsilon} = - \frac{\partial \nu^*}{\partial \varepsilon} + \frac{\partial \Pi^*}{\partial \varepsilon} = - \frac{\partial \nu^*}{\partial \varepsilon} + \frac{\partial \Pi^*}{\partial \varepsilon} \) and \( \frac{\partial \Pi^*}{\partial \varepsilon} = - \frac{\partial \nu^*}{\partial \varepsilon} + \frac{\partial \Pi^*}{\partial \varepsilon} \). Then (A) assuming that \( \frac{\delta}{c + \varepsilon} - \gamma > 0 \), we let: \( v^* = \frac{\delta}{c + \varepsilon} - \gamma \) and obtain via straightforward algebra that for \( \alpha < \hat{\alpha} \):

\[
\frac{\partial \Pi^*}{\partial \varepsilon} < 0 \iff - (r - c)^2 - \varepsilon (r - c + r - c - \varepsilon) < 0,
\]

\[
\frac{\partial \Pi^*}{\partial \varepsilon} < 0 \iff - (r - c)^2 + \varepsilon (2r - 2c - \varepsilon) - 2\varepsilon^2 \alpha < 0
\]

and (B) assuming that \( \frac{\delta}{c + \varepsilon} - \gamma < 0 \), we have that \( r - c - \varepsilon + \alpha \alpha - r \alpha - \alpha \varepsilon > 0 \) with \( \delta \) close to one, with \( \varepsilon = \frac{\delta}{c + \varepsilon} \), we let: \( v^* = (1 - \delta) \frac{\delta}{c + \varepsilon} \) and obtain via straightforward algebra that for \( \alpha > \hat{\alpha} \):

\[
\frac{\partial \Pi^*}{\partial \varepsilon} \Leftrightarrow \left( (r - c)^2 + 2c \varepsilon + \varepsilon^2 \right) (\alpha^2 + 1) - 2 \left( (c + \varepsilon)^2 + (c - r)^2 + \varepsilon \right) \alpha > 0
\]

\[
\frac{\partial \Pi^*}{\partial c} \Leftrightarrow \left( r^2 + 2r \varepsilon - (c + \varepsilon)^2 \right) (\alpha^2 + 1) - 2 \left( (c + \varepsilon)^2 + (r + \varepsilon)^2 + \varepsilon \right) \alpha > 0 \text{ and } \frac{\partial \Pi^*}{\partial \varepsilon} > 0.
\]

The inequality \( \frac{\partial \Pi^*}{\partial v} \leq 0 \) is because:

\[
- (r - c)^2 + \varepsilon (2r - 2c - \varepsilon) - 2\varepsilon^2 \alpha < - (r - c)^2 + \varepsilon (2r - 2c - \varepsilon) < - (r - c)^2 + (r - c) (2r - 2c - (r - c)) = 0,
\]

where in the second inequality, we substitute \( \varepsilon (2r - 2c - \varepsilon) \) by its upper bound, which is determined by: \( \varepsilon^* = r - c \). We find conditions when \( \frac{\partial \Pi^*}{\partial c} > 0 \). We characterize the intersection between \( \frac{\partial \Pi^*}{\partial c} = 0 \) and \( \alpha > \hat{\alpha} \)

\[
\begin{align*}
\left\{ &r - c - \varepsilon - \alpha (r + \varepsilon - c) = 0 \\
&\left( (c + \varepsilon)^2 - \varepsilon^2 + (r + \varepsilon)^2 \right) (\alpha^2 + 1) - 2 \alpha \left( (c + \varepsilon)^2 + (r + \varepsilon)^2 + \varepsilon \right) = 0
\end{align*}
\]

which has as solution: \( e^o = r \frac{(1 - \alpha)(1 - 2\alpha - \alpha^2)}{1 - \alpha + 3\alpha^2 + \alpha^3}, e^o = 2r \frac{\alpha(1 - \alpha)}{1 - \alpha + 3\alpha^2 + \alpha^3} \) (the other one can be discarded for sure), which is positive when \( 1 - 2\alpha - \alpha^2 > 0 \) and \( 1 - \alpha + 3\alpha^2 + \alpha^3 > 0 \) which is true only if \( \alpha < \sqrt{2} - 1 \). □