Technical Appendix to Accompany
Was There a Nasdaq Bubble in the Late 1990s?

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Appendix

(A) The Stochastic Discount Factor

The properties of the SDF are described in detail in Pástor and Veronesi (2005; PV). This appendix contains a brief summary. The process in equation (10) implies a normal unconditional distribution for \( y_t \) with mean \( \psi \) and variance \( \sigma_y^2/2k_y \). Let \( y_D = \psi - 4\sigma_y/\sqrt{2k_y} \) and \( y_U = \psi + 4\sigma_y/\sqrt{2k_y} \) be the boundaries between which \( y_t \) lies 99.9% of the time. To ensure that \( s_t \) (log surplus) conforms to the economic intuition of a habit formation model, PV impose the following parametric restrictions: \( a_2 < 0, a_1 > -2a_2y_U \) and \( a_0 < 1/4 (a_1/a_2) \). The resulting process for the stochastic discount factor \( \pi_t = e^{-\eta t - \gamma (\varepsilon t + s_t)} \) is given by

\[
d\pi_t = -r_{f,t}\pi_t dt - \pi_t \sigma_{\pi,t} dW_{0,t},
\]

where

\[
r_{f,t} = R_0 + R_1 y_t + R_2 y_t^2,
\]

with

\[
R_0 = \eta + \gamma \mu_\varepsilon + \gamma a_1 k_y \psi - \frac{1}{2} \gamma^2 \sigma_\varepsilon^2 = (\gamma a_2 - \frac{1}{2} \gamma^2 a_1^2) \sigma_y^2 - \gamma^2 a_1 \sigma_\varepsilon \sigma_y \n\]

\[
R_1 = \gamma (2a_2 k_y \overline{y} - a_1 k_y - 2a_2 \gamma (\sigma_\varepsilon \sigma_y + a_1 \sigma_y^2)) \n\]

\[
R_2 = 2a_2 \gamma (\overline{k}_y - \gamma a_2 \sigma_y^2) \n\]

and

\[
\sigma_{\pi,t} = \gamma (\sigma_\varepsilon + (a_1 + 2a_2 y_t) \sigma_y). \quad (A1)
\]

The parameter restrictions imposed earlier imply that \( \sigma_{\pi,t} \) decreases as \( y_t \) increases. As a result, expected return and return volatility are low when \( y_t \) is high. See PV for more details.

(B) Proofs

Lemma 1: Let \( \bar{b}_t \) follow the process

\[
d\bar{b}_t = (\zeta_0 \bar{v}_t + \zeta_1 \rho_i^t - \zeta_2) dt,
\]

where \( \rho_i^t \) and \( \bar{v}_t \) follow the processes in equations (2) and (4), and \( \zeta_i \) are constants. Define \( Y_t = \left( \bar{v}_t - \gamma \varepsilon_t, y_t, \bar{v}_t, \rho_i^t, \psi_i^t \right) \) and \( g(Y_T) = e^{Y_{1,T} - \gamma a_1 Y_{2,T} - \gamma a_2 Y_{2,T}^2} \), where \( \nu \) is a constant, \( Y_{i,t} \) denotes the \( i \)-th element of \( Y_t \), and \( \gamma, a_1, \) and \( a_2 \) are taken from equations (8) and (9). Then

\[
E_t \left[ e^{-\eta(T-t)} g(Y_T) \right] = H(Y_t, t; T) = e^{K_0(t; T) + K(t; T)'} \left( Y_{1,t} + K_0(t; T)Y_{2,t} \right), \quad (A2)
\]

where \( K_0(t; T), \left( K(t; T) = (K_1(t; T), .., K_5(t; T))' \right) \), and \( K_0(t; T) \) satisfy a system of ordinary differential equations (ODE)

\[
\frac{dK_0(t; T)}{dt} = \eta - K(t; T)'. A_Y - \frac{1}{2} K(t; T)'. \Sigma_Y \Sigma_Y' K(t; T) - K_0(t; T) \sigma_y^2 \quad (A5)
\]

\[
\frac{dK(t; T)}{dt} = -K(t; T)'. B_Y + 2K_0(t; T) \left[ \Sigma_Y \Sigma_Y' \right]_2 e_2 - 2K_0(t; T) k_y \bar{v}_t \quad (A4)
\]

\[
\frac{dK_0(t; T)}{dt} = -K(t; T)'. B_Y - 2K(t; T) \left[ \Sigma_Y \Sigma_Y' \right]_2 e_2 - 2K_0(t; T) k_y \bar{v}_t \quad (A3)
\]
subject to the final condition \( K_0 (T ; T) = -\gamma a_2 \), \( K (T ; T) = (1, -\gamma a_1, 0, 0) \), and \( K_0 (T ; T) = 0 \). In the above, \( \epsilon_2 = (0, 1, 0, \ldots, 0) \), and

\[
A_Y = \begin{pmatrix}
-\gamma \mu_e - v \zeta_2 \\
\kappa y & \phi \\
\kappa L \phi \\
0 & 0
\end{pmatrix} ;
B_Y = \begin{pmatrix}
0 & 0 & v \zeta_0 & v \zeta_1 \\
0 & -k \gamma & 0 & 0 \\
0 & 0 & -k_L & 0 \\
0 & 0 & -\phi & -\phi
\end{pmatrix} ;
\Sigma_Y = \begin{pmatrix}
-\gamma \sigma_e & 0 & 0 \\
\sigma_y & 0 & 0 \\
\sigma_{L,0} & \sigma_{L,L} & 0 \\
\sigma_{i,0} & 0 & \sigma_i
\end{pmatrix} .
\]

**Proof of Lemma 1:** From the definition of the vector \( Y_t \), we have

\[
dY_t = (A_Y + B_Y Y_t) dt + \Sigma_Y dW_t,
\]

The Feynman-Kac theorem implies that \( H (Y_t, t) \) from (A2) solves the partial differential equation

\[
\frac{\partial H}{\partial t} + \sum_{i=1}^{5} \left( \frac{\partial H}{\partial Y_i} \right) [A_Y + B_Y Y_t]_i + \frac{1}{2} \sum_{i=1}^{5} \sum_{j=1}^{5} \frac{\partial^2 H}{\partial Y_i \partial Y_j} [\Sigma_Y \Sigma_Y^t]_{ij} = \eta H
\]

subject to the boundary condition

\[
H (Y_T, T) = g (Y_T).
\]

It can be easily verified that the exponential quadratic function (A2) indeed satisfies (A6) subject to (A7), as long as \( K_0 (t ; T) \), \( K (t ; T) \), and \( K_0 (t ; T) \) are the solutions to the system of ODEs in (A3) through (A5) under the final conditions presented in the claim of the Lemma. \( \blacksquare \)

**Lemma 2.** If average excess profitability \( \bar{\psi}_t \) is observable and \( T_i \) is known, the firm’s ratio of market value to book value of equity is given by

\[
\frac{M_t}{B_t} = e^{c_t} \int_0^{T_i-t} \tilde{Z}_i \left( y_t, \bar{\rho}_t, \bar{\psi}_t, s \right) ds + \tilde{Z}_i \left( y_t, \bar{\rho}_t, \bar{\psi}_t, T_i - t \right),
\]

where

\[
\tilde{Z}_i \left( y_t, \bar{\rho}_t, \bar{\psi}_t, s \right) = e^{\tilde{Q}_0 (s) + Q(s)' N_t + Q_5 (s) \eta^2},
\]

and where \( N_t = \left( y_t, \bar{\rho}_t, \bar{\psi}_t \right) \) is the vector of state variables characterizing firm \( i \), \( \tilde{Q}_0 (s) = K_0 (0; s) \), \( Q_i (s) = K_{i+1} (0; s) \) for \( i = 2, 3, 4 \), \( Q_1 (s) = K_2 (0; s) + \gamma a_1 \), and \( Q_5 (s) = K_6 (0; s) + \gamma a_2 \), where \( K_i (.; s) \) are in Lemma 1, all for the parameterization \( \zeta_0 = 0 \), \( \zeta_1 = v = 1 \), and \( \zeta_2 = c^i \).

**Proof of Lemma 2:** Let \( t = 0 \), for notational simplicity. For given \( T_i \), the pricing formula is

\[
M_t = E_0 \left[ \int_0^{T_i} \frac{\pi_s}{\pi_0} D^i_s ds \right] + E_0 \left[ \frac{\pi_{T_i}}{\pi_0} B_{T_i}^i \right] = e^{c_t} \int_0^{T_i} E_0 \left[ \frac{\pi_s}{\pi_0} B^i_s \right] ds + E_0 \left[ \frac{\pi_{T_i}}{\pi_0} B_{T_i} \right] .
\]

We need to compute the following expectation:

\[
E_0 \left[ \frac{\pi_s}{\pi_0} B^i_s \right] = e^{\gamma t_0 + \gamma a_1 y_0 + \gamma a_2 y^2} e^{-\gamma t_0 - \gamma a_1 y_0 - \gamma a_2 y^2} = e^{\gamma t_0 + \gamma a_1 y_0 + \gamma a_2 y^2} H (Y_T, 0; s),
\]

2
where the $H$ function is given in equation (A2). Since $B_Y$ has only zeros in its first column, we have $[K(t; T_i) \cdot B_Y]_1 = 0$ in equation (A4). This implies $\frac{dK_i(t; T_i)}{dt} = 0$ and thus $K_1(t; T_i) = 1$ for $t \leq T_i$. By substituting in $H(Y, 0; s)$, we obtain

$$E_0 \left[ \frac{\pi_s}{\pi_0} B_s^i \right] = e^{\gamma_{y_0} + \gamma_{y_2}y_2^2} \times H(Y, 0; s) = B_0^i \times e^{\gamma_{y_1}y_0 + \gamma_{y_2}y_2^2} \times e^{K_0(0; s) + \sum_{i=2}^{5} K_i(0; s) Y_i, 0 + K_6(0; s) Y_2^2}.$$

This expression leads immediately to the claim upon redefinition of the variables. ■

Proof of Proposition 1:

The density of the exponential distribution is $h(s, p) = pe^{-ps}$. We assume throughout that parameters are chosen such that $Q_0(s) = -ps + \tilde{Q}_0(s) \rightarrow -\infty$, and all $Q_i(s)$ for $i \neq 0$ converge to finite numbers, where $\tilde{Q}_0(s)$ and $Q_i(s)$ are defined in Lemma 2. Such parameters exist, because $B_Y$ in Lemma 1 has negative eigenvalues, and thus the convergence conditions are met for instance if $\Sigma_Y = 0$. We now prove Proposition 1 under these conditions.

For given $T$, the expected discounted value of the future cash flow is given in Lemma 2:

$$E_t \left[ \int_t^T \frac{\pi_s}{\pi_t} D_s^i d\tau \right] + E_t \left[ \frac{\pi_s}{\pi_t} B_s^i | T \right] = B_t^i c^i \int_t^T \tilde{Z}^i \left( y_t, \bar{p}_t, \bar{\rho}_t, \bar{\psi}_t, s - t \right) ds + B_t^i \tilde{Z} \left( y_t, \bar{p}_t, \bar{\rho}_t, \bar{\psi}_t, T - t \right).$$

Integrating over all possible $T$'s, the value of the stock today is given by

$$M_t^i = B_t^i c^i \int_t^\infty \left( pe^{-p(T-t)} \right) T^i \tilde{Z} \left( y_t, \bar{p}_t, \bar{\rho}_t, \bar{\psi}_t, s - t \right) ds dT + B_t^i \int_t^\infty pe^{-p(T-t)} \tilde{Z} \left( y_t, \bar{p}_t, \bar{\rho}_t, \bar{\psi}_t, T - t \right) dT.$$

Using integration by parts and recalling that $\int pe^{-p(T-t)} dT = \frac{p}{p} e^{-p(T-t)} = -e^{-p(T-t)}$, we find

$$\int_t^\infty \left( pe^{-p(T-t)} \right) T^i \tilde{Z} \left( y_t, \bar{p}_t, \bar{\rho}_t, \bar{\psi}_t, s - t \right) ds dT = \left[ -e^{-p(T-t)} \int_t^T \tilde{Z} \left( y_t, \bar{p}_t, \bar{\rho}_t, \bar{\psi}_t, s - t \right) ds \right]_{T=t}^{T=\infty} - \int_t^\infty -e^{-p(T-t)} \tilde{Z} \left( y_t, \bar{p}_t, \bar{\rho}_t, \bar{\psi}_t, T - t \right) dT.$$

Under the assumption stated earlier ($Q_0(s) \rightarrow -\infty$ and $Q_i(s)$'s converge to finite numbers), we have $e^{-p(T-t)} \int_t^T \tilde{Z} \left( y_t, \bar{p}_t, \bar{\rho}_t, \bar{\psi}_t, s - t \right) ds \rightarrow 0$ as $T \rightarrow \infty$. From equation (A5), the leading term in $\tilde{Q}_0(s)$ is linear in $s$, while the other terms converge to finite numbers. Thus, the properties of the integral $\int_t^T \tilde{Z} \left( y_t, \bar{p}_t, \bar{\rho}_t, \bar{\psi}_t, s - t \right) ds$ as $T \rightarrow \infty$ are determined by a term of the form $\int_t^T e^{m(s-t)} ds$ for some constant $m$ determined as part of the solution of (A5). Under the assumptions stated earlier, $e^{-p(T-t)} \int_t^T e^{m(s-t)} ds = 1/m (e^{-p+m}(T-t) - e^{-p(T-t)}) \rightarrow 0$ as $T \rightarrow \infty$. Thus,

$$\int_t^\infty \left( pe^{-p(T-t)} \right) T^i \tilde{Z} \left( y_t, \bar{p}_t, \bar{\rho}_t, \bar{\psi}_t, s - t \right) ds dT = \int_t^\infty e^{-p(T-t)} \tilde{Z} \left( y_t, \bar{p}_t, \bar{\rho}_t, \bar{\psi}_t, T - t \right) dT.$$

Substituting this back into equation (A8), we find the relation (13) in Proposition 1. ■

Proof of Proposition 2: By the law of iterated expectations, the pricing function is

$$M_t^i = E_t \left[ \int_t^T \frac{\pi_s}{\pi_t} D_s^i ds + \frac{\pi_s}{\pi_t} B_s^i | T \right] = E_t \left[ \int_t^T \frac{\pi_s}{\pi_t} D_s^i ds + \frac{\pi_s}{\pi_t} B_T^i | \bar{\psi}_t \right].$$
The inner expectation is computed in Proposition 1. Thus
\[ M^i_t = B^i_t \left( c^i + p \right) \times E_t \left[ \int_0^\infty e^{Q_0(s) + Q_1(s) \cdot N_t + Q_2(s) \cdot \Sigma^2_t} ds \right]. \]

Under the assumptions stated in the proof of Proposition 1, the integral exists. The only variable in \( N_t \) that is not known at \( t \) is \( \bar{\psi}_i \). The claim of Proposition 2 then follows from the rules of the lognormal distribution, as
\[ E_t \left[ e^{Q_4(s) \cdot \bar{\psi}_i} \right] = e^{E \left[ Q_4(s) \cdot \bar{\psi}_i \right] + \frac{1}{2} \text{Var} \left[ Q_4(s) \cdot \bar{\psi}_i \right]} = e^{Q_4(s) \cdot \bar{\psi}_i + \frac{1}{2} Q_4(s) \cdot \bar{\sigma}^2_{i,t}}. \]

**Lemma 3 (Learning).** Let \( \bar{\psi}_i \) follow equation (5), \( Z_t = (\rho^i_t, \bar{\sigma}^2_{i,t}, \bar{y}_t)' \), and the prior distribution of \( \bar{\psi}_i \) at \( t = 0 \) be normal, \( N(\bar{\psi}_0, \bar{\sigma}^2_{i,0}) \). The posterior of \( \bar{\psi}_i \) conditional on \( F_t = \{ Z_r : 0 < r \leq t \} \) is also normal, and the posterior moments \( \hat{\psi}_i^t = E_t \left[ \bar{\psi}_i^t \right] \) and \( \bar{\sigma}^2_{i,t} = E_t \left[ \left( \hat{\psi}_i^t - \hat{\psi}_i^t \right)^2 \right] \) at \( t > 0 \) follow
\[
\begin{align*}
    d\hat{\psi}_i^t &= -k_{\psi} \hat{\psi}_i^t dt + \bar{\sigma}^2_{i,t} \left( \frac{\phi^i_t}{\sigma_{ii}} \right) d\hat{W}_i^t, \\
    \frac{d\bar{\sigma}^2_{i,t}}{dt} &= -2k_{\psi} \bar{\sigma}^2_{i,t} - \left( \bar{\sigma}^2_{i,t} \right)^2 \left( \frac{\phi^i_t}{\sigma_{ii}} \right)^2.
\end{align*}
\]

Above, \( \hat{W}_i^t \) is the third entry in the vector of expectation errors, \( \hat{W}_t = [\hat{W}_{0,t}, \hat{W}_{L,t}, \hat{W}_{i,t}] \), which follows \( d\hat{W}_t = \Sigma_{-1} [dZ_t - E_t(dZ_t)] \). To obtain the dynamics of \( Z_t \), we can define matrices \( A_Z, B_Z, C_Z \) and \( \Sigma_Z \) such that equations (2), (4), and (10) can be combined into one as
\[
dZ_t = (A_Z + B_Z Z_t + C_Z \bar{\psi}_t^t) dt + \Sigma_Z dW_t,
\]
where \( W_t = [W_{0,t}, W_{L,t}, W_{i,t}] \). Proof of Lemma 3 follows from Liptser and Shiryaev (1977).

**Expected Return and Volatility.** Let \( M_i^i/B_i^i = \Phi^i \left( \rho^i_t, \bar{\sigma}^2_{i,t}, \bar{y}_t, \hat{\psi}_i^t, \bar{\sigma}^2_{i,t} \right) \), following Proposition 2. Ito’s Lemma implies that firm \( i \)'s return volatility is given by \( \sqrt{\sigma_{R,1}^i \sigma_{R,1}^i} \), where
\[
\sigma_{R}^i = \left( \frac{\partial \Phi^i}{\partial \rho^i_t} \sigma_y + \frac{\partial \Phi^i}{\partial \bar{\rho}_t} \sigma_L + \frac{\partial \Phi^i}{\partial \bar{\rho}_t} \sigma_i + \frac{\partial \Phi^i}{\partial \bar{\psi}_t^t} \sigma_{\bar{\psi},t} \right),
\]
\[
\sigma_y = (\sigma_y, 0, 0), \quad \sigma_L = (\sigma_{L,0}, \sigma_{L,L,0}, 0), \quad \sigma_i = (\sigma_{i,0}, 0, \sigma_{i,i}), \quad \text{and} \quad \sigma_{\bar{\psi},t} = \left( 0, 0, \frac{\phi^i}{\sigma_{ii}} \bar{\sigma}^2_{i,t} \right).
\]

We also have
\[
E \left[ dR_t^i \right] = \sigma_{R,1}^i \sigma_{i,t},
\]
for expected excess return, where \( \sigma_{R,1}^i \) is the first element in \( \sigma_{R}^i \) and \( \sigma_{i,t} \) is given in equation (A1).

**Proposition 3:** The M/B value of the old economy is given by
\[
M^O_t/B^O_t = \Phi \left( \bar{\rho}_t, \bar{y}_t \right) = c^O \int_0^\infty Z (y_t, \bar{\rho}_t, s) ds,
\]
where
\[
Z (y_t, \bar{\rho}_t, s) = e^{Q_0^O(s) + Q_1^O(s)y_t + Q_2^O(s)\bar{\rho}_t + Q_3^O(s)y^2}.
\]
and \( Q^O_0(s) = K_0(0; s) \), \( Q^O_1(s) = K_2(0; s) + \gamma a_1 \), \( Q^O_2(s) = K_3(0; s) \) and \( Q^O_3(s) = K_6(0; s) + \gamma a_2 \), where \( K_i(.; s) \) are in Lemma 1, all for the parametrization \( \zeta_0 = \zeta_2 = v = 1 \), and \( \zeta_1 = 0 \).

**Proof of Proposition 3:** The claim follows from the same argument as in Proposition 1, but for the parameterization \( \zeta_0 = \zeta_2 = v = 1 \), and \( \zeta_1 = 0 \) in Lemma 1. The functions of time \( Q^O_j(s) \), \( j = 0, \ldots, 3 \), are computed as in Proposition 1. ■

The return volatility of the old economy is \( \sqrt{\sigma^O_R \sigma^O_R} \), where \( \sigma^O_R = \Phi \left( \frac{\partial \Phi}{\partial y} \bar{\sigma}_y + \frac{\partial \Phi}{\partial \rho} \sigma_L \right) \), \( \bar{\sigma}_y = (\sigma_y, 0) \), and \( \sigma_L = (\sigma_{L,0}, \sigma_{L,L}) \). The old economy’s expected excess return is \( E \left[ dR^O_t \right] = \sigma^O_{R,1} \sigma_{\pi,t} \), where \( \sigma^O_{R,1} \) is the first element in \( \sigma^O_R \).

**Lemma 4.** (e.g., Duffie, 1996). For any linear vector process \( z_t \) that satisfies
\[
\frac{dz_t}{dt} = (A_z + B_z z_t) dt + \Sigma_z dW_t, \tag{A13}
\]
we have
\[
z_{t+\tau} | z_t \sim N \left( \mu_z(z_t, \tau), S_z(\tau) \right),
\]
where \( \mu_z \) and \( S_z \) are given by
\[
\mu_z(z_t, \tau) = \Psi(\tau) z_t + \int_0^\tau \Psi(\tau - s) A_z ds
\]
\[
S_z(\tau) = \int_0^\tau \Psi(\tau - s) \Sigma_z \Sigma_z' \Psi(\tau - s) ds
\]
and \( \Psi(\tau) = U e^{\Lambda \tau} U^{-1} \), where \( \Lambda \) is the diagonal matrix with the eigenvalues of \( B_z \) on its principal diagonal, \( U \) is the matrix collecting the respective eigenvectors on each column, and \( e^{\Lambda \tau} \) is the diagonal matrix with \( e^{\lambda_{ii} \tau} \) on its principal diagonal.

\( \text{(C) The Gordon growth model with an uncertain growth rate.} \)

This section formalizes the discussion in the third paragraph of the introduction. Let \( \pi_t \) denote the dividend rate. The Gordon model assumes that the drift rate \( g \) of dividends is constant:
\[
\frac{dD_t}{D_t} = g dt + \sigma_D dW_D. \tag{A14}
\]
We consider two different specifications of the stochastic discount factor.

**C.1. The Stochastic Discount Factor Independent of the Dividend Process.**

Suppose that the SDF is governed by the following process with constant drift and volatility:
\[
\frac{d\pi_t}{\pi_t} = -r_f dt - \sigma_{\pi} dW_{\pi}.
\]
The price of the asset is then given by
\[
P_t = E_t \left[ \int_t^\infty \frac{\pi_s}{\pi_t} D_s ds \right] = D_t E_t \left[ \int_t^\infty \frac{\pi_s D_s}{\pi_t D_t} ds \right], \tag{A15}
\]
assuming that the expectation exists. Let $x_t = \log(\pi_tD_t)$. Ito’s lemma implies that

$$dx_t = \left(-r_f + g - \sigma_x \sigma_D \rho_{D,\pi} - \frac{1}{2} \left(\sigma_{\pi}^2 + \sigma_D^2 - 2\sigma_x \sigma_D \rho_{D,\pi}\right)\right)dt - \sigma_D dW_D,$$

where $\rho_{D,\pi}$ is the correlation between $dW_D$ and $dW_\pi$. Using the properties of the lognormal distribution,

$$E_t \left[\frac{\pi_tD_s}{\pi_tD_t}\right] = E_t \left[e^{(x_s - x_t)}\right] = e^{-\left(r_f + \sigma_x \sigma_D \rho_{D,\pi} - g\right)(s-t)}.$$

The price of the asset is then

$$P_t = D_t \int_t^\infty E_t \left[\frac{\pi_tD_s}{\pi_tD_t}\right] ds = D_t \int_t^\infty e^{-\left(r_f + \sigma_x \sigma_D \rho_{D,\pi} - g\right)(s-t)} ds = \frac{D_t}{r_f + \sigma_x \sigma_D \rho_{D,\pi} - g},$$

where $r = r_f + \sigma_x \sigma_D \rho_{D,\pi}$ is the sum of the risk-free rate and the risk premium. This is the well-known Gordon growth formula in a continuous-time framework.

When $g$ is unknown, it follows from the law of iterated expectations that

$$P_t = E_t \left[\int_t^\infty \frac{\pi_tD_s}{\pi_tD_t} ds\right] = E_t \left[E_t \left[\int_t^\infty \frac{\pi_tD_s}{\pi_tD_t} ds \mid g\right]\right] = D_t E_t \left[\frac{1}{r - g}\right].$$

That is, the P/D ratio is equal to the expectation of the P/D ratio in the case where $g$ is known. (This expectation exists only under the assumption that the distribution of $g$ assigns positive likelihood only to the values of $g$ that satisfy a transversality condition.) Note that the risk premium is unchanged compared to the case of known $g$. By explicitly modeling the learning process, this fact can be proven directly by adapting the results in Veronesi (2000).


Following Campbell (1986) and Abel (1999), we assume the existence of a representative agent with a CRRA utility over aggregate consumption, which is given by

$$C_t = D_t^\lambda.$$

In a dynamic economy, the SDF is given by the marginal utility of consumption

$$\pi_t = e^{-\eta t} C_t^{-\gamma},$$

where $\gamma$ is the coefficient of risk aversion. The SDF then follows the process

$$\frac{d\pi_t}{\pi_t} = -\left(\eta + \lambda \gamma g - \frac{1}{2} \lambda \gamma (1 + \lambda \gamma) \sigma_D^2\right) dt - \lambda \gamma \sigma_D dW_D \quad (A16)$$

$$= -\left(\eta + \lambda \gamma E_t[g] - \frac{1}{2} \lambda \gamma (1 + \lambda \gamma) \sigma_D^2\right) dt - \lambda \gamma \sigma_D d\tilde{W}_D. \quad (A17)$$

The process (A16) is written with respect to the true Brownian motion $W_D$ from equation (A14), whereas the (equivalent) process (A17) is written with respect to the Brownian motion $\tilde{W}_D$ perceived by the agent with incomplete information about $g$. The equality between the processes in
equations (A16) and (A17) follows from Girsanov’s theorem. Importantly, equation (A17) shows that uncertainty about \( g \) has no impact on the volatility of the SDF.

The price of the asset is given by

\[
P_t = E_t \left[ \int_t^\infty \frac{\pi_s}{\pi_t} D_s ds \right] = D_t E_t \left[ \int_t^\infty e^{-\eta(s-t)} \left( \frac{D_s}{D_t} \right)^{1-\lambda\gamma} ds \right], \tag{A18}
\]

assuming that the expectation exists.

If \( g \) is observable, the same calculation as in Section C.1 shows that

\[
\frac{P_t}{D_t} = \int_t^\infty e^{-\eta - (1-\lambda\gamma)g + \frac{1}{2}\lambda\gamma(1-\lambda\gamma)\sigma_D^2}(s-t)ds = \frac{1}{\eta - (1-\lambda\gamma)g + \frac{1}{2}\lambda\gamma(1-\lambda\gamma)\sigma_D^2}. \tag{A19}
\]

Note that as long as \( \lambda\gamma \neq 1 \), the \( P/D \) ratio is convex in \( g \).

If \( g \) is unobservable, the law of iterated expectations implies that

\[
\frac{P_t}{D_t} = E_t \left[ \frac{1}{\eta - (1-\lambda\gamma)g + \frac{1}{2}\lambda\gamma(1-\lambda\gamma)\sigma_D^2} \right].
\]

Due to the previously mentioned convexity, an increase in uncertainty about \( g \) (i.e., a mean-preserving spread on the density of \( g \)) leads to an increase in the \( P/D \) ratio.

By following the approach in Veronesi (2000), it is also possible to show that an increase in uncertainty decreases (increases) expected excess return if and only if \( \gamma > 1/\lambda \) (\( \gamma < 1/\lambda \)).

REFERENCES


