Special Notes:
1. This is a closed-book exam. You may use an 8 × 11 piece of paper for the formulas.
2. Throughout this paper, \( N(\mu, \sigma^2) \) will denote a normal distribution with mean \( \mu \) and variance \( \sigma^2 \).
3. This is a 2 hour exam.

**Problem A. True or False:** Please Explain your answers in detail. Partial credit will be given (50 points)

1. The probability of triplets is 0.001. Then the probability of exactly one set of triplets among 700 births is greater than 0.3.

   *True. If we use a Poisson approximation with mean 0.7 we have \( P(X = 1) = 0.347 > 0.3 \).*

2. The function \( f(x) = \exp (-x - e^{-x}) \) is a valid probability density function.

   *True. The density is non-negative and integrates to one.*

3. The probability that exactly one of two events \( A \) and \( B \) occurring is \( P(A) + P(B) - 2P(A \cap B) \),

   *True. The probability is \( P(A \cup B) - P(A \cap B) = P(A) + P(B) - 2P(A \cap B) \).*
4. A gambler can either double or half his wealth on a fair toss of a coin. At time $T$ his expected wealth is $(5/4)^T$ times his original wealth.

\[ \text{True. We need to calculate } E \left( 2 \sum X_i 0.5^{T-X_i} \right) \text{ where } X_i \text{ is } 1 \text{ or } -1 \text{ with probability } 0.5. \text{ Now } E \left( 2^{X_i 0.5^{T-X_i}} \right) = \frac{5}{4}. \text{ Hence the desired result.} \]

5. Let $B_t$ be a standard Brownian motion then the sequence of random variables $X_T = e^{B_t-t}$ is a martingale.

\[ \text{False. } E(X_t|X_s) = E \left( e^{B_t-t} | B_s \right) = e^{-(t-s)/2} e^{B_t-(t-s)} = e^{-(t-s)/2} X_s. \text{ Hence we have a super-martingale.} \]

6. A box contains three cards. One card is red on both sides, one card is green on both sides and one card is red and green. A card is selected at random and you observe the color on one side is green. The probability that the other side is green is 0.5.

\[ \text{False. There are three possibilities either we have the green/red card or we have the green/green card in either order. In } 2 \text{ out of } 3 \text{ cases the other side is green, hence } P(\text{green}) = 2/3. \]

7. Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on $(0, 1)$. Then the expected value of the minimum of the sample is $1/(n + 1)$.

\[ \text{True. The cdf of the minimum } X_{(1)} \text{ is given by } 1 - F_{X_{(1)}} = (1 - F_X(x))^n. \text{ For a uniform } F_X(x) = x \text{ and we get } E \left( X_{(1)} \right) = \frac{1}{n+1}. \]
8. The moment generating function of a Poisson distribution with mean $\lambda$ is $\exp(\lambda(e^t - 1))$.

True. The mgf is defined by $M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!} = \exp(\lambda(e^t - 1))$.

9. The stochastic process $tB_{\frac{1}{t}}$ is a standard Brownian motion.

True. Let $X_t = tB_{\frac{1}{t}}$. The increment $X_t - X_s$ has distribution

$$X_t - X_s = tB_{\frac{1}{t}} - sB_{\frac{1}{t}} = tN\left(0, \frac{1}{t}\right) - sN\left(0, \frac{1}{s}\right) \sim N(0, t - s)$$

Hence $X_t$ is a standard Brownian motion.

10. Let $C$ be a standard Cauchy distribution $C(0, 1)$, then $E\left(\frac{1}{C}\right) < \infty$.

False. If $C \sim C(0, 1)$ then $\frac{1}{C} \sim C(0, 1)$ and so $E\left(\frac{1}{C}\right) = \infty$.

11. Let $U = \max(X, Y)$ where $X$ and $Y$ are independent with distribution function $F(x)$. Then the probability density for $U$ is $f_U(x) = f(x)F(x)$.

False. The cdf for $U$ is given by $P(U \leq x) = F(x)^2$ and so the density is $f_U(x) = 2f(x)F(x)$.

12. Suppose that $u(x)$ is a concave function and $X$ a random variable. Then $E\left(u(X)\right) \geq u(E(X))$.

False. Jensen's inequality says that if $u(x)$ is convex then $E\left(u(X)\right) \geq u(E(X))$.
Problem B. (20 points)
Let $X$ and $Y$ be independent exponential random variables with means $\lambda$ and $\mu$, respectively.

1. Find the joint distribution of $U = X + Y$ and $V = X/(X + Y)$.
2. Find the marginal distributions for $U$ and $V$.

First, the joint distribution of $X$ and $Y$ is given by

$$f_{X,Y}(x, y) = \frac{1}{\lambda \mu} \exp \left( -\frac{x}{\lambda} - \frac{y}{\mu} \right)$$

The transformation $u = x + y$ and $v = x/(x + y)$ has inverse given by $x = uv$, $y = u(1 - v)$. Hence, the Jacobian is $u$.

The joint distribution of $(U, V)$ is then

$$f_{U,V}(u, v) = \frac{u}{\lambda \mu} \exp \left( -\frac{uv}{\lambda} - \frac{u(1 - v)}{\mu} \right)$$

The marginal distribution of $U$ is given by

$$f_U(u) = \int_0^1 f_{U,V}(u, v)dv$$

$$= \frac{1}{\lambda - \mu} \left( \exp \left( -\frac{u}{\lambda} \right) - \exp \left( -\frac{u}{\mu} \right) \right)$$

The marginal distribution of $V$ is given by

$$f_V(v) = \int_0^\infty f_{U,V}(u, v)du$$

Hence

$$f_V(v) = \frac{\lambda \mu}{(\mu v + \lambda(1 - v))^2}$$
Problem C. (20 points).
You are designing a toll road that carries trucks and cars. Each week you see an average of 19,000 vehicles pass by. The current toll for cars is 50 cents and you wish to set the toll for trucks so that the revenue reaches $11,500 per week. You observe the following data: three of every four trucks on the road are followed by a car, while only one of every five cars is followed by a truck.

1. What is the equilibrium distribution of trucks and cars on the road?

2. What should you charge trucks so as to reach your goal of $11,500 in revenues per week?

The transition matrix for the number of trucks and cars is given by

\[
P = \begin{pmatrix}
\frac{1}{4} & \frac{3}{4} \\
\frac{1}{5} & \frac{4}{5}
\end{pmatrix}
\]

The equilibrium distribution \( \pi = (\pi_1, \pi_2) \) is given by \( \pi P = \pi \). This has solution given by

\[
\frac{1}{4}\pi_1 + \frac{1}{5}\pi_2 = \pi_1 \quad \text{and} \quad \frac{3}{4}\pi_1 + \frac{4}{5}\pi_2 = \pi_2
\]

with \( \pi_1 = \frac{4}{19} \) and \( \pi_2 = \frac{15}{19} \).

With cars at a toll of 50 cents we see that we need to charge trucks $1.00 to generate revenues of $11,500 per week.