Special Notes:

1. This is a closed-book exam. You may use an 8 × 11 piece of paper for the formulas.
2. Throughout this paper, \( N(\mu, \sigma^2) \) will denote a normal distribution with mean \( \mu \) and variance \( \sigma^2 \).
3. This is a 2 hr exam.

Honor Code: By signing my name below, I pledge my honor that I have not violated the Booth Honor Code during this examination.

Signature:

Problem A. True or False: Please Explain your answers in detail. Partial credit will be given (60 points)

1. Let \( X \) be a random variable with cdf \( F_X(x) \). Then \( F_X(X) \) is uniformly distributed.

   True. \( P(F_X(X) \leq x) = P\left(X \leq F_X^{-1}(x)\right) = F_X\left(F_X^{-1}(x)\right) = x.\)

2. The Binomial distribution converges to a Poisson distribution with rate \( \lambda \) as the number of trials tends to infinity with proportion given by \( p = \lambda/n.\)

   True. The moment generating functions converge and so we also have convergence in distribution.
3. You toss a fair coin until either the sequence \( THT \) or \( HTT \) appears. The waiting time for \( THT \) is longer than that of \( HTT \).

\( \text{True.} \) The expected waiting time of \( THT \) is 10 and \( HTT \) only 8.

4. For a non-negative random variable with finite expectation, \( E(\log X) \geq \log E(X) \).

\( \text{False.} \) From Jensen’s inequality, as \( \log(\cdot) \) is a concave function, we must have that:

\[
E(\log X) \leq \log E(X)
\]

5. The Gamma distribution is a special case of the \( \chi^2 \)-distribution.

\( \text{False.} \) It’s the opposite.

6. The Cauchy distribution is self-reciprocal, i.e. \( 1/C \) is also Cauchy.

\( \text{True.} \) The density of a Cauchy is \( f_C(x) = 1/\pi(1 + x^2) \) which is self-reciproal.

7. Let \( U \sim U(0, 1) \). Then \( 1/U \) has a Beta distribution.

\( \text{False.} \) Let \( Y = g(U) = \frac{1}{U} \). Then, the transformation formula gives \( g^{-1}(y) = y^{-1} \)

\[
f_y(y) = f_U(g^{-1}(y)) \left| \frac{\partial g^{-1}(y)}{\partial y} \right| = y^{-2} \quad \text{for} \ 1 \leq y < \infty
\]
which is a Pareto distribution.

8. Consider the hierarchical model specified by the conditional and marginal distributions 
   \( X|\theta \sim N(2\theta, 1) \) and \( \theta \sim N(0, 1) \). Then \( E(X) = 2 \) and \( Var(X) = 1 \).

   \textit{False} Using Law of Iterated Expectation, we have that:
   \[
   E[X] = E\left[E[X|\theta]\right] = E[2\theta] = 2 \times 0 = 0
   \]
   \[
   E[X^2] = E\left[E[X^2|\theta]\right] = E\left[Var(X|\theta) + E[X|\theta]^2\right]
   \]
   \[
   = E\left[1 + 4\theta^2\right] = 1 + 4 = 5
   \]

9. Let \( B_t \) denote a standard Brownian motion. Then \( E(B_t^2) = 1 \).

   \textit{False} Given \( B_0 = 0 \) and \( B_t \sim N(0, t) \), we have \( E[B_t^2] = t \)

10. A chest has three drawers; one contains two gold coins, one contains two silver coins, 
    and one contains one gold and one silver coin. Assume that one drawer is selected 
    randomly and that a randomly selected coin from that drawer turns out to be gold. 
    Then the probability that the chosen drawer contains two gold coins is 50%.

    \textit{False}. Three drawers \( A, B, C \). Prior \( P(G) = \frac{1}{2} \).
    Posterior \( P(A|G) = P(A \cap G)/P(G) = (1/3)(1/2) = 2/3 \).
Problem B. (20 points)

Let $X \sim \text{Exp}(\lambda_1)$ and $Y \sim \text{Exp}(\lambda_2)$ be independent exponential random variables where the rates $\lambda_1, \lambda_2 > 0$. Calculate the following distributions:

1. If $\lambda_1 = \lambda_2$, find the distribution of $Z = X + Y$.

2. If $\lambda_1 > \lambda_2$, show that the distribution of $Z = X + Y$ is a weighted sum of exponentials. Identify the weights and components.

Answers

1. We have that:

$$f_{x+y}(z) = \int_0^z f_x(x)f_y(z-x)\,dx$$

$$= \int_0^z \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2(z-x)}\,dx$$

$$= \lambda_1 \lambda_2 \int_0^z e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 x} \,dx$$

$$= \lambda_1 \lambda_2 \int_0^z e^{-(\lambda_1-\lambda_2)x} \,dx$$

Thus, if we have that $\lambda_1 = \lambda_2 = \lambda$, the density of $X + Y$ is then equal to:

$$f_{x+y}(z) = \lambda_1 \lambda_2 e^{-\lambda z} \int_0^z \,dx$$

$$= \lambda_1 \lambda_2 z e^{-\lambda z}$$

$$= \lambda^2 z e^{-\lambda z}$$

Note that this is Erlang distribution with parameter 2 and $\lambda$.

2. If however, we have that $\lambda_1 > \lambda_2$, we then have that:

$$f_{x+y}(z) = \lambda_1 \lambda_2 e^{-\lambda_2 z} \int_0^z e^{(\lambda_2-\lambda_1)x} \,dx$$

$$= \lambda_1 \lambda_2 e^{-\lambda_2 z} \left. \frac{e^{(\lambda_2-\lambda_1)x}}{\lambda_2 - \lambda_1} \right|_0^z$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} e^{-\lambda_2 z} (e^{(\lambda_2-\lambda_1)z} - 1)$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} (e^{-\lambda_1 z} - e^{-\lambda_2 z})$$
Problem C. (20 points)

Elvis Presley had a twin brother (Jesse Garon Presley) who died at birth.

1. What is the probability that Elvis was an identical twin?

**Background Information:** Twins are estimated to be approximately 1.9% if the world population. Mono-zygotic (“identical”) twins making up 0.2% of the total world population and 8% of all twins. You can also assume that fraternal twins are equally likely to be of opposite sex.

- Explain clearly any laws of probability that you use.

The hypotheses are:
A: Elvis’s birth event was an identical birth event
B: Elvis’s birth event was a fraternal twin event
If identical twins are 8% of all twins, then identical birth events are 8% of all twin birth events, so the priors are

\[ P(A) = 0.08 \quad \text{and} \quad P(B) = 0.92 \]

The evidence is \( E \): Elvis’s twin was male
The likelihoods are

\[ P(E|A) = 1 \quad \text{and} \quad P(E|B) = 1/2 \]

Because identical twins are necessarily the same sex, but fraternal twins are equally likely to be opposite sex by assumption. Hence

\[ P(A|E) = \frac{8}{54} = 0.15. \]

The tricky part of this one is realizing that the sex of the twin provides relevant information!

**Full Solution:** Using Bayes’ rule, we have that:

\[
Pr(A|E) = \frac{Pr(E|A)Pr(A)}{Pr(E)} = \frac{Pr(E|A)Pr(A)}{Pr(E|A)Pr(A) + Pr(E|B)Pr(B)}
\]

\[
= \frac{(1)(.08)}{(1)(.08) + (.5)(.92)}
\]

\[
= \frac{.08}{.54} = \frac{4}{27} = 0.148148...
\]