1. (Stein’s Identity). If $X \sim N(\mu, \sigma^2)$ show that

$$E((X - \mu)g(X)) = \sigma^2 E(g'(X))$$

when both sides exist.

2. (Inequalities). Is it ever true that the identity $E\left(\frac{1}{X^2}\right) = \frac{1}{E(X^2)}$ holds for a non-trivial random variable $X$?

3. (Mutual Information Inequality). Suppose that $p(x)$ and $q(x)$ are two different probability distributions. Show that the Kullback-Leibler information divergence, $KL$, between the two densities is positive, that is

$$KL(p, q) = \int p(x) \log \left(\frac{p(x)}{q(x)}\right) dx > 0.$$ 

What can you say about them case $KL(p, q) = 0$?

4. (Jensen). Suppose that $X > 0$ with $E(X) = 1$. Show that $E(X \log X) > 0$.

5. (Pareto). The Pareto distribution with parameters $\alpha$ and $\beta$ has probability density

$$f_x(x|\alpha, \beta) = \frac{\alpha \beta^\alpha}{x^{\alpha+1}}$$

where $\beta < x < \infty$ and $\alpha, \beta > 0$.

Find the mean and variance.

Show that the variance does not exist if $\alpha \leq 2$.

6. (Cauchy). Let $C$ have a standard Cauchy distribution.

Show that $1/C$ has the same distribution.

7. (Poisson). Suppose that $X \sim \text{Poi}(\mu)$ and independently $Y \sim \text{Poi}(\lambda)$.

Show that $X + Y \sim \text{Poi}(\mu + \lambda)$.

Suppose that $X|Y \sim \text{Poi}(Y)$ and $Y \sim \text{Poi}(\lambda)$.

Find the moment generating function of $X + Y$. 