Topics in Dynamic Asset Pricing

Teaching Notes 1 (Addendum 2).

An Example of Portfolio Selection
with time-varying opportunity set
in Complete Markets

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The Portfolio Problem

- Consider the following portfolio problem with time-varying expected returns.
- Let \((\beta, S)\) be processes of the form
  \[
  d\beta_t = r_t\beta_t dt \quad \text{with} \quad \beta_0 > 0
  \]
  \[
  dS_t^i = \mu_t^i S_t^i dt + S_t^i \sigma_t^i dB_t \quad \text{with} \quad S_0^i > 0
  \]
- where \(r_t, \mu_t = (\mu_1^t, ..., \mu_d^t)\) and \(\sigma^i\) are bounded, adapted processes.
- For concreteness, assume that \(\mu_t = (\mu_1^t, ..., \mu_d^t)\) follow a continuous time, VAR process
  \[
  d\mu_t = (A_0 + A_1 \mu_t) dt + \Sigma dB_t
  \]
- Notice that the set of Brownian motions moving \(\mu_t\) is the same as the ones moving \(S_t\).
- Instead, assume \(r_t = r\) is constant and that \(\sigma_t^i = \sigma\) is also constant, for every \(i\).
- Assume again that \(\{\sigma\}\) is invertible.
The dynamics of the market price of risk

- Define by $\lambda_t = \mu_t - r1_d$ the excess return process.
- Let the $d \times 1$ market price of risk process
  \[ \nu_t = \sigma^{-1} \lambda_t \]
- Clearly, also $\nu_t$ follows a VAR process
  \[ d
\nu_t = \sigma^{-1} d\mu_t \]
  \[ = \left( \sigma^{-1} A_0 + \sigma^{-1} A_1 \mu_t \right) dt + \sigma^{-1} \Sigma dB_t \]
  \[ = \left( \tilde{A}_0 + \tilde{A}_1 \nu_t \right) dt + \tilde{\Sigma} dB_t \]
- with $\tilde{A}_0 = \sigma^{-1} (A_0 + A_1 1_d r)$, $\tilde{A}_1 = \sigma^{-1} A_1 \sigma$ and $\tilde{\Sigma} = \sigma^{-1} \Sigma$. 
**Result:** Given an initial condition \( \nu_0 = \hat{\nu} \), then for \( \tau > 0 \)

\[
\nu_\tau \sim N(\alpha(\nu_0, \tau), S(\tau))
\]

where

\[
\alpha(\nu_0, \tau) = \Psi(\tau) \nu_0 + \zeta(\tau)
\]

\[
S(\tau) = \int_0^\tau \Psi(\tau - s) \tilde{\Sigma} \tilde{\Sigma}' \Psi(\tau - s)' \, ds
\]

\[
\zeta(\tau) = \int_0^\tau \Psi(\tau - s) \tilde{A}_0 \, ds
\]

and \( \Psi(\tau) \) solves the system of differential equation

\[
\frac{d\Psi(t)}{dt} = \tilde{A}_1 \Psi(t)
\]

with initial condition \( \Psi(0) = I \).

If \( \tilde{B} \) has distinct and real eigenvalues, then the solution is

\[
\Psi(\tau) = U \exp(\Lambda \cdot \tau) U^{-1}
\]

where, \( \Lambda \) is the diagonal matrix with \( \tilde{A}_1 \) eigenvalues on the principal diagonal, \( U \) is the matrix of the associated eigenvectors, and \( \exp(\Lambda \cdot T) \) is the diagonal matrix with \( e^{\lambda_i T} \) in its \( ii \)-th position.

Clearly, we need \( \lambda_i \leq 0 \) to ensure that the solution does not explode.
• In this case, we have that the Novikov’s condition is satisfied:

\[ E \left[ \exp \left( \frac{1}{2} \int_0^T \nu'_t \nu_t dt \right) \right] < \infty \]

• Thus,

\[ \xi_t = \exp \left( -\frac{1}{2} \int_0^t \nu'_u \nu_u du - \int_0^t \nu'_u dB_u \right) \]

defines a P-martingale.
The Optimal Consumption Plan

- Define the *state-price deflator*

\[ \pi_t = e^{-rt} \xi_t = \exp \left( - \left( \int_0^t r + \frac{1}{2} \nu'u' \nu_u \, du \right) - \int_0^t \nu'_u \, dB_u \right) \] (3)

- We then have that the optimal consumption is given by

\[ C^*_t = \mathcal{I}_u (\lambda \pi_t, t) \] (4)

- where \( \mathcal{I}_u \) is the inverse of the utility functions.

- In addition, by defining again

\[ \hat{w} (\lambda) = E \left( \int_0^T \pi_t \mathcal{I}_u (\lambda \pi_t, t) \, dt \right) \] (5)

- the solution to \( \lambda^* \) is given by the equality \( \hat{w} (\lambda) = w \).

- Assume for instance a power utility

\[ u (C, t) = e^{-\rho t} \frac{C^{1-\gamma}_t}{1-\gamma} \]
• This implies

\[ u_c = e^{-\rho t} C_t^{-\gamma} \quad \Rightarrow \quad I_u(x, t) = e^{-\frac{\rho}{\gamma} t} x^{-\frac{1}{\gamma}} \]

• Hence

\[ C_t^* = e^{-\frac{\rho}{\gamma} t} (\lambda \pi_t)^{-\frac{1}{\gamma}} \]

\[ = e^{-\frac{\rho}{\gamma} t} \left( \lambda \exp \left( - \int_0^t r + \frac{1}{2} \nu'_u \nu_u du - \int_0^t \nu'_u dB_u \right) \right)^{-\frac{1}{\gamma}} \]

\[ = \lambda^{-\frac{1}{\gamma}} \exp \left( \int_0^t \frac{1}{\gamma} (r - \rho) + \frac{1}{2} \nu'_u \nu_u du + \frac{1}{\gamma} \int_0^t \nu'_u dB_u \right) \]

• where \( \lambda \) is a constant determined by the budget constraint.

• The process for log consumption \( c_t = \log (C_t) \) is then given by

\[ c_t = -\frac{1}{\gamma} \log (\lambda) + \left( \int_0^t \frac{1}{\gamma} (r - \rho) + \frac{1}{2} \nu'_u \nu_u du + \frac{1}{\gamma} \int_0^t \nu'_u dB_u \right) \]

• implying

\[ dc_t = \left( \frac{1}{\gamma} (r - \rho) + \frac{1}{2} \nu'_t \nu_t \right) dt + \frac{1}{\gamma} \nu'_t dB_t \]
The Optimal Portfolio Weights

• Consider now all the processes under $Q$: Define the new Brownian motion

$$d\hat{B}_t = dB_t + \nu_t dt$$

• so that, under $Q$, we have

$$dc_t = \left( \frac{1}{\gamma} (r - \rho) - \frac{1}{2\gamma} \nu_t \nu_t \right) dt + \frac{1}{\gamma} \nu_t d\hat{B}_t$$  \hspace{1cm} (6)$$

$$d\nu_t = \left( \tilde{\Lambda}_0 + \hat{\Lambda}_1 \nu_t \right) dt + \tilde{\Sigma} d\hat{B}_t$$

• with $\hat{\Lambda}_1 = \tilde{\Lambda}_1 - \tilde{\Sigma}$.

• We now find the portfolio weights that support $C^*_t$.

• First, define the $Q$-martingale

$$M_t = E^Q \left[ \int_0^T \beta_u^{-1} C_u du \right]$$  \hspace{1cm} (7)$$

• We know that $M_0 = w$ = wealth at time 0.

• From the martingale representation theorem, we know that there exists $\hat{\eta}_t$ such that

$$dM_t = \hat{\eta}_t d\hat{B}_t$$
• The (discounted) wealth is given by

\[ \hat{W}_t = \beta_t^{-1}W_t = E_t^Q \left[ \int_t^T \beta_u^{-1}C_u du \right] = M_t - J_t \]  

(8)

• where

\[ J_t = \int_0^t \beta_u^{-1}C_u du \]

• Thus, the process for the discounted wealth is

\[ d\hat{W}_t = -\beta_t^{-1}C_t dt + \hat{\eta}_t d\hat{B}_t \]  

(9)

• We also have that the wealth is always equal to the total amount invested in stocks and bonds, which must satisfy the budget constraint

\[ \hat{W}_t = \theta_t^0 + \theta_t \hat{S}_t = \int_0^t \theta_u d\hat{S}_t - \int_0^t \beta_u^{-1}C_u du \]

• where \( \hat{S}_t = \beta_t^{-1}S_t \) is a martingale under Q

\[ d\hat{S}_t = \hat{S}_t \sigma d\hat{B}_t \]

• Thus, the process for the discounted wealth under Q is

\[ d\hat{W}_t = -\beta_t^{-1}C_t + \theta_t \hat{S}_t \sigma d\hat{B}_t \]

• which, comparing with (9), yields immediately

\[ \theta_t \hat{S}_t \sigma = \hat{\eta}_t \]
How do we get \( \hat{\eta}_t \) practically?

- Notice that we can rewrite (8) as

\[
\hat{W}_t = E_t^Q \left[ \int_t^T \beta_u^{-1} C_u du \right]
\]

\[
= \beta_t^{-1} C_t E_t^Q \left[ \int_t^T \frac{\beta_u^{-1} C_u}{\beta_t^{-1} C_t} du \right]
\]

\[
= \beta_t^{-1} C_t E_t^Q \left[ \int_t^T \frac{\beta_u^{-1} e^{c_u - c_t}}{\beta_t^{-1}} du \right]
\]

- From the process for optimal consumption (6), the conditional expectation \( E_t^Q \left[ \int_t^T \frac{\beta_u^{-1} e^{c_u - c_t}}{\beta_t^{-1}} du \right] \) depends only on \( \nu_t \).

- In other words, we can define the function

\[
F (\nu_t, t, T) = E_t^Q \left[ \int_t^T \frac{\beta_u^{-1} e^{c_u - c_t}}{\beta_t^{-1}} du \right]
\]

(10)

- and therefore the process

\[
\hat{W}_t = \beta_t^{-1} C_t F (\nu_t, t; T)
\]
• Thus, using Itô’s lemma, the diffusion part of the discounted wealth process $d\hat{W}_t$ must be given by

$$\tilde{\sigma}'_W = \hat{W}_t \left( \frac{1}{\gamma} \nu' + \frac{1}{F} \sum_{i=1}^{n} \frac{\partial F}{\partial \nu^i} \tilde{\sigma}^i \right)$$

• with $\tilde{\sigma}^i = \left[ \tilde{\Sigma} \right]_i$, the $i$–th row of $\tilde{\Sigma}$.

• In other words, from (9) we must have $\hat{\eta}_t = \hat{\sigma}'_W$.

• This yields

$$\theta_t I_S \hat{\sigma} = \hat{W}_t \left( \frac{1}{\gamma} \nu' + \frac{1}{F} \sum_{i=1}^{n} \frac{\partial F}{\partial \nu^i} \tilde{\sigma}^i \right)$$

• or

$$\theta_t I_S = \hat{W}_t \left( \frac{1}{\gamma} \nu' + \frac{1}{F} \sum_{i=1}^{n} \frac{\partial F}{\partial \nu^i} \tilde{\sigma}^i \right) \sigma^{-1}$$
Myopic and Hedging Demand

• In terms of fraction of wealth, and recalling \( \nu_t = \sigma^{-1}(\mu_t - r1_n) \)

\[
\vartheta_t = \frac{\theta_t I_S}{W_t} = \frac{\theta_t I S}{W_t} = \frac{(\mu_t - r1_n)'(\sigma\sigma')^{-1}}{\gamma} + \sum_{i=1}^{n} \frac{1}{F} \frac{\partial F}{\partial \nu_i} \tilde{\sigma}^i \sigma^{-1}
\]

(11)

• The first term on the RHS is the usual “myopic term”: Higher excess return increase the portfolio holding, while higher risk and higher risk aversion decreases it.

• The second term is the hedging demand component. Notice first we can write

\[
\tilde{\sigma}^i \sigma^{-1} = \left(\tilde{\sigma}^i \sigma\right)(\sigma\sigma')^{-1}
\]

• In addition, it turns out (see later) that

\[
\frac{1}{F} \frac{\partial F}{\partial \nu_i} = -\frac{J_{W\nu_i}}{W J_{WW}}
\]

• where \( J(W, t; T) = E_t \left[ e^{-\rho(\tau-t)\frac{C_1^{1-\gamma}}{1-\gamma}} \right] \) is the indirect utility function.

• Thus, the hedging demand is given by

\[
\text{Hedging Demand} = -\sum_{i=1}^{n} \frac{J_{W\nu_i}}{W J_{WW}} \left(\tilde{\sigma}^i \sigma\right)(\sigma\sigma')^{-1}
\]

• I give an intuition of this term below in the context of a specific example.
How do we compute $F(\nu_t, t; T)$ in (10) ?

• From contingent claim valuation: Consider a security that pays out $C_t$ over time as dividend.
• Under Q, its value is

$V(C_t, \nu_t, t; T) = E_t^Q \left[ \int_t^T e^{-r(\tau-t)} C_\tau d\tau \right] = C_t F(\nu_t, t; T)$

• The total expected return (under Q) on this security must equal the risk free rate, so that

$E_t^Q [dV] + C dt = V r dt$ (12)

• From Ito’s Lemma

$E_t^Q [dV] = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial C} C_t \mu_C(\nu_t) + \sum_{i=1}^d \frac{\partial V}{\partial \nu_i} \mu_i(\nu_t)$

$+ \sum_{i=1}^d \frac{\partial^2 V}{\partial C \partial \nu_i} C_t \frac{\nu_t}{\gamma} \tilde{\sigma}_i + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \frac{\partial^2 V}{\partial \nu_i \partial \nu_j} \tilde{\sigma}_i \tilde{\sigma}_j'$

• where

$\mu_C(\nu_t) = \frac{1}{\gamma} (r - \rho) + \frac{\nu_t'}{2\gamma} \left( \frac{1}{\gamma} - 1 \right)$ (13)

$\mu_i(\nu_t) = \tilde{A}_{0,i} + \tilde{A}_{1,i} \nu_t$ (14)
Finally, since
\[ \frac{\partial V}{\partial t} = C \frac{\partial F}{\partial t}; \quad \frac{\partial V}{\partial C} = F; \quad \frac{\partial V}{\partial \nu^i} = C \frac{\partial F}{\partial \nu^i}; \quad \frac{\partial^2 V}{\partial C \partial \nu^i} = \frac{\partial F}{\partial \nu^i}; \quad \frac{\partial^2 V}{\partial \nu^i \partial \nu^j} = \frac{\partial^2 F}{\partial \nu^i \partial \nu^j} C \]

• substituting everything into (12), we find
\[
F(r - \mu_C(\nu_t)) = 1 + \frac{\partial F}{\partial t} + \sum_{i=1}^{d} \frac{\partial F}{\partial \nu^i} \left( \tilde{A}_{0,i} + \tilde{A}_{1,i} + \frac{\tilde{\sigma}_i}{\gamma} \right) \nu_t
\]
\[
+ \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 F}{\partial \nu^i \partial \nu^j} \tilde{\sigma}_i \tilde{\sigma}_j'
\]
\[ (15) \]

• with final condition
\[
F(\nu_T, T; T) = 0.
\]

• If a solution is not available, the PDE can often be computed numerically.

• From \( F \), one can then obtain \( \eta_t \) and, in addition, also consumption. In fact, recall that
\[
\widehat{W}_t = \beta_t^{-1} C_t F(\nu_t, t; T)
\]

• which gives
\[
C_t = W_t F(\nu_t, t; T)^{-1}
\]
The Hedging Demand Again

- We can finally show that
  \[
  \frac{1}{F} \frac{\partial F}{\partial \nu^i} = - \frac{J_{W \nu^i}}{W J_{WW}}
  \]  
  (16)

- From the Bellman equation (see TN 1), we have that the first order conditions with respect to consumption \( C \) is
  \[ J_W = u_c (C^*_t) \]

- where \( C^*_t = C(W_t, \nu_t) = W_t F(\nu_t, t; T)^{-1} \) is the optimal policy function.

- Differentiating both sides with respect to \( W \) and, say, \( \nu^i \) yields the equalities
  \[
  J_{WW} = u_{cc} \frac{\partial C^*_t}{\partial W} = F(\nu_t, t; T)^{-1}
  \]
  \[
  J_{W \nu^i} = u_{cc} \frac{\partial C^*_t}{\partial \nu^i} = -WF(\nu_t, t; T)^{-2} \frac{\partial F}{\partial \nu^i}
  \]

- It is immediate to verify that the RHS and LHS of (16) coincide.
The case $\gamma = 1$

- Notice that if $\gamma = 1$, then $F(\nu_t, t; T) = F(t; T)$ is the solution to the PDE.

- In fact, in this case, we obtain
  \[
  F(t; T)\rho = 1 + F'(t; T)
  \]
  which yields
  \[
  F(t; T) = He^{\rho t} + \frac{1}{\rho}
  \]

- The final condition
  \[
  F(T; T) = He^{\rho T} + \frac{1}{\rho} = 0
  \]
  yields
  \[
  H = \frac{e^{-\rho T}}{\rho}
  \]

- Thus
  \[
  F(t; T) = \frac{1}{\rho} \left( 1 - e^{-\rho(T-t)} \right)
  \]
• This implies that the consumption to wealth ratio is deterministic and given by

\[ C_t = \frac{W_t \rho}{(1 - e^{-\rho(T-t)})} \]

• In addition, the optimal portfolio choice is

\[ \vartheta_t = \frac{(\mu_t - r \mathbf{1}_n)'}{\gamma} (\sigma \sigma')^{-1} \]
Consider the univariate case \((d = 1)\).

In this case, the portfolio holding of the market is given by

\[
\varrho_t = \frac{\mu_t - r}{\gamma \sigma^2} + \left( \frac{1}{F} \frac{\partial F}{\partial \nu} \right) \tilde{\sigma}
\]

where \(F\) has to satisfy the PDE (15) becomes

\[
F(r - \mu_C(\nu_t)) = 1 + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial \nu} \left( \tilde{A}_0 + \left( \tilde{A}_1 + \frac{\tilde{\sigma}}{\gamma} \right) \nu_t \right) + \frac{1}{2} \frac{\partial^2 F}{\partial \nu^2} \tilde{\sigma}^2
\]

Unfortunately, it does not have a closed form solution.

Wachter (2002) however finds a nice way out: Rather than considering a claim to the whole sequence of consumption plan \(\{C_{\tau}\}_{t}^{T}\), that is, the security

\[
V(C, \nu, t; T) = E^Q \left[ \int_{0}^{T} e^{-r(\tau-t)} C_{\tau} d\tau \right]
\]

she first considers a set of claims, each to the exact consumption “coupon” \(C_{\tau}\) paid at that particular \(\tau\), for \(\tau \in [0, T]\).
• By no arbitrage, the value of the security paying the process \( \{C_{\tau}\} \) will be the sum (i.e. the integral) of all these individual claims.

• Let the value of each claim to the “coupon” \( C_{\tau} \) be given by

\[
v(C, \nu, t; \tau) = E^Q \left[ e^{-r(\tau-t)}C_{\tau} \right].
\]

• The homogeneity discussed earlier entails that \( v(C, \nu, t; \tau) = Cf(\nu, t; \tau) \) for some \( f(.) \).

• Under \( Q \) also this claim must earn the risk free rate

\[
E^Q [dv] = rvdt
\]

• Ito’s Lemma then gives

\[
rv = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial C} C_t \mu_C(\nu_t) + \frac{\partial v}{\partial \nu} \left( \hat{A}_0 + \hat{A}_1 \nu_t \right) + \frac{1}{2} \frac{\partial^2 v}{\partial \nu^2} \hat{\sigma}^2 + \frac{\partial^2 v}{\partial C \partial \nu} \frac{\nu_t}{\gamma}
\]

• with final condition \( v(C_{\tau}, \nu_{\tau}, \tau; \tau) = C_{\tau} \)

• Recall also that under \( Q \)

\[
\mu_C(\nu_t) = \frac{1}{\gamma} (r - \rho) + \frac{\nu_t^2}{2\gamma} \left( \frac{1}{\gamma} - 1 \right)
\]

• As before, we can substitute the following quantities

\[
\frac{\partial v}{\partial t} = C \frac{\partial f}{\partial t}; \quad \frac{\partial v}{\partial C} = f; \quad \frac{\partial v}{\partial \nu} = C \frac{\partial f}{\partial \nu}; \quad \frac{\partial^2 v}{\partial \nu^2} = C \frac{\partial^2 f}{\partial \nu^2}; \quad \frac{\partial^2 v}{\partial C \partial \nu} = \frac{\partial f}{\partial \nu}
\]
• Substituting also $v = Cf$, $\mu_C(\nu)$ and deleting $C'$ on both sides yields

$$rf = \frac{\partial f}{\partial t} + f\left(\frac{1}{\gamma}(r - \rho) + \frac{\nu^2_t}{2\gamma}\left(\frac{1}{\gamma} - 1\right)\right) + \frac{\partial f}{\partial \nu}\left(\hat{A}_0 + \hat{A}_1\nu_t\right) + \frac{1}{2}\frac{\partial^2 f}{\partial \nu^2}\hat{\sigma}^2 + \frac{\partial f}{\partial \nu}\nu_t\hat{\sigma}$$

• with final condition $f(\nu_\tau, \tau; \tau) = 1$

• Fortunately, a “solution” to this PDE instead exists.

• How can we find it constructively?

• Use the method of undetermined coefficients
  — This methodology is extensively used to obtain the prices of Fixed Income Securities.

• Conjecture (this comes with experience)

$$f(\nu, t; \tau) = \exp\left(a_0(t; \tau) + a_1(t; \tau)\nu + a_2(t; \tau)\nu^2\right)$$

• Then we obtain

$$\frac{\partial f}{\partial t} = (a'_0 + a'_1\nu_t + a'_2\nu^2_t)f$$
$$\frac{\partial f}{\partial \nu} = (a_1 + 2a_2\nu_t)f$$
$$\frac{\partial^2 f}{\partial \nu^2} = (2a_2 + (a_1 + 2a_2\nu_t)^2)f$$
• Substitute and delete $f$ on both sides, to find

$$r = a_0' + a_1'\nu + a_2'\nu^2 + \left(\frac{1}{\gamma}(r - \rho) + \frac{\nu_t^2}{2\gamma} \left(\frac{1}{\gamma} - 1\right)\right) + (a_1 + 2a_2\nu) \left(\hat{A}_0 + \hat{A}_1\nu + \frac{\nu_t\hat{\sigma}}{\gamma}\right)$$

$$+ \frac{1}{2}(2a_2 + (a_1 + 2a_2\nu)^2)\hat{\sigma}^2$$

• Finally, bunch up together terms in $\nu$, $\nu^2$ etc.

• One obtains:

$$0 = a_0' - r + \left(\frac{1}{\gamma}(r - \rho)\right) + a_1\hat{A}_0 + \frac{1}{2} \left(2a_2 + a_1^2\right)\hat{\sigma}^2 +$$

$$+ \left(a_1' + a_1 \left(\frac{\hat{A}_1 + \hat{\sigma}}{\gamma}\right) + 2a_2\hat{A}_0 + 2a_1a_2\hat{\sigma}^2\right)\nu_t$$

$$+ \left(a_2' + \frac{1}{2\gamma} \left(\frac{1}{\gamma} - 1\right) + 2a_2 \left(\hat{A}_1 + \frac{\hat{\sigma}}{\gamma}\right) + 2a_2^2\hat{\sigma}^2\right)\nu_t^2$$

• This equation is satisfied if the following system of ODEs is satisfied

$$a_2' + 2a_2 \left(\frac{\hat{A}_1 + \hat{\sigma}}{\gamma}\right) + 2a_2^2\hat{\sigma}^2 + \frac{1}{2\gamma} \left(\frac{1}{\gamma} - 1\right) = 0$$

$$a_1' + a_1 \left(\frac{\hat{A}_1 + \hat{\sigma}}{\gamma}\right) + 2a_2\hat{A}_0 + 2a_1a_2\hat{\sigma}^2 = 0$$

$$a_0' - r + \frac{1}{\gamma}(r - \rho) + a_1\hat{A}_0 + \frac{1}{2} \left(2a_2 + a_1^2\right)\hat{\sigma}^2 = 0$$
• Note that the system can be easily solved recursively: Solve the first equation (for \(a_2\)), then plug in the solution into the second equation (for \(a_1\)) and then finally, obtain the solution of the last equation (for \(a_0\)).

• It is possible to find exact closed formulas for \(a_2\) and \(a_1\). However, this is as easy to obtain numerically.

• ODEs are infinitely simpler than PDE, as you can solve them backward: Start with the final condition \(a_0(\tau) = a_1(\tau) = a_2(\tau) = 0\) and then simply move backwards. Below are the details.

• Once we have the solution for \(a_0(t; \tau), a_1(t; \tau)\) and \(a_2(t; \tau)\) for every \(\tau \in [0, T]\), the function \(F(\nu, t; T)\) can be obtained easily.

In fact, recall that

\[
V(C, \nu, t; T) = CF(\nu, t; T) = E_t^Q \left[ \int_t^T e^{-r(\tau-t)} C_\tau d\tau \right]
\]

\[
= \int_t^T E_t^Q \left[ e^{-r(\tau-t)} C_\tau \right] d\tau = \int_t^T \nu(C, \nu, t; \tau) d\tau
\]

\[
= C \int_t^T f(\nu, t; \tau) d\tau
\]

• This implies that simply

\[
F(\nu, t; T) = \int_t^T f(\nu, t; \tau) d\tau = \int_t^T \exp \left( a_0(t; \tau) + a_1(t; \tau)\nu_t + a_2(t; \tau)\nu_t^2 \right) d\tau
\]
• The portfolio holdings require the computation of
\[
\frac{\partial F}{\partial \nu} = \int_t^T (a_1(t; \tau) + a_2(t; \tau) \nu_t) \exp \left( a_0(t; \tau) + a_1(t; \tau) \nu_t + a_2(t; \tau) \nu_t^2 \right) d\tau
\]

• To conclude, we have the following portfolio rule

\[ \vartheta_t = \text{Miopic Demand} + \text{Hedging Demand} \]

with

\[
\text{Myopic Demand} = \frac{\mu_t - r}{\gamma \sigma^2}
\]

\[
\text{Hedging Demand} = \left( \frac{1}{F} \frac{\partial F}{\partial \nu} \right) \frac{\tilde{\sigma}}{\sigma} \int_t^T (a_1(t; \tau) + a_2(t; \tau) \nu_t) f(\nu_t, t; \tau) d\tau
\]

\[
= \frac{\tilde{\sigma}}{\sigma} \int_t^T f(\nu_t, t; \tau) d\tau
\]

• Also note that the optimal consumption was given by \( C_t = W_t F(\nu, t; T)^{-1} \) implying that the \( C/W \) ratio is given by

\[
\frac{C}{W} = \frac{1}{\int_t^T f(\nu_t, t; \tau) d\tau}
\]
• An obvious application of the setting above is the one where returns $\mu_t$ are predictable from the dividend price ratio.

• We can think then of $\mu_t$ to be just

$$\mu_t = \alpha + \beta \log \left( \frac{D_t}{P_t} \right)$$

where $\alpha$ and $\beta$ are the regression coefficients of some sort of predictive regression

• Note that if $\log \left( \frac{D_t}{P_t} \right)$ follows a mean reverting process, so does $\mu_t$ and so we are back to the case discussed in this section.

• Also, there is a natural negative correlation between returns and $D/P$ ratio: a negative return implies that $P_t$ decreased. Since dividends do not move much, this implies that $D/P$ went up.

• How do we impose a negative correlation in the model? Just assume that $\hat{\sigma} < 0$

• The following parameters have been used by many, including Barberis (JF 2000), and Wachter (JFQA 2002)

<table>
<thead>
<tr>
<th>Parameter Choice</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rate of time preference $\rho$</td>
<td>0.0624</td>
</tr>
<tr>
<td>Risk free rate $r$</td>
<td>0.0168</td>
</tr>
<tr>
<td>Volatility of stock prices $\sigma$</td>
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</tr>
<tr>
<td>Volatility of $\nu_t$</td>
<td>-0.0655</td>
</tr>
<tr>
<td>Mean Reversion $A_1$</td>
<td>-0.0226</td>
</tr>
<tr>
<td>Drift $A_0$</td>
<td>0.0062</td>
</tr>
</tbody>
</table>
• The following figures show the $C/W$ ratio, Hedging Demand and Total Demand as a function of expected return $\mu_t$.

**Figure 1: $C/W$ ratio**

![Graph showing $C/W$ ratio](image)

- Note that the $C/W$ ratio declines with $\gamma$, as intuition would have it: Higher risk aversion implies a higher desire to save and thus consume less.
Figure 2: Hedging Demand

- The hedging demand is positive. The intuition is simple:
  - If we have a bad shock to returns, we have that $\mu_t$ increases (intuitively, the $D/P$ increases, implying higher expected return).
  - But a higher $\mu_t$ implies that investor now want to buy more of the stock.
  - Anticipating this correlation, the investor buys more of the stock today, compared to the case where the hedging demand is zero.

- This finding is bad news for the portfolio holding puzzle: We already showed that the agent would hold too much of the stock even with simple myopic demand (no time varying investment opportunity set).

- The total demand now of the stock is even higher, deepening the puzzle.
• We will see other channels that would decrease the holding of stocks later on.
The predictability, however, helps a little to generate asset holding that depend on life cycle.

Using the same parameters, with $\gamma = 5$ but for three different maturities $T$ we obtain the following.

Figure 4: Total Demand

As it can be see, the shorter the time to “death” the lower the share in stocks, especially if current expected return is high.

In this case, mean reversion kicks in and the investor is wary about the negative consequences of a decrease in expected returns.