Conditional Heteroscedastic Models

What is stock volatility?
Answer: conditional standard deviation of stock returns

Why is volatility important?
Has many important applications

• Option (derivative) pricing, e.g., Black-Scholes formula
• Risk management, e.g. value at risk (VaR)
• Asset allocation, e.g., minimum-variance portfolio; see pages 184-185 of Campbell, Lo and MacKinlay (1997).
• Interval forecasts

A key characteristic: Not directly observable!!

How to calculate volatility?

1. Use high-frequency data: French, Schwert & Stambaugh (1987); see Section 3.15.
   • Realized volatility of daily returns in recent literature.
   • Use daily high, low, and closing prices.

2. Implied volatility of options data, e.g, VIX of CBOE. See Figure 1.

3. Econometric modeling

We focus on the econometric modeling first. Use of high frequency data will be discussed later.

Basic idea of econometric modeling
Shocks of asset returns are NOT serially correlated, but dependent.
See ACF of squared and absolute returns of some stocks

Basic structure

\[ r_t = \mu_t + a_t, \quad \mu_t = \phi_0 + \sum_{i=1}^{p} \phi_i r_{t-i} - \sum_{i=1}^{q} \theta_i a_{t-i}, \]

Volatility models are concerned with time-evolution of

\[ \sigma_t^2 = \text{Var}(r_t | F_{t-1}) = \text{Var}(a_t | F_{t-1}). \]

the conditional variance of a return.

Two general categories
Figure 1: Time plot of the daily closing value of VIX of the COBE: January 2, 2004 to April 13, 2006.

- “Fixed function” and
- Stochastic function

of the available information.

**Univariate volatility models**

1. Autoregressive conditional heteroscedastic (ARCH) model of Engle (1982),
2. Generalized ARCH (GARCH) model of Bollerslev (1986),
3. GARCH-M models
4. IGARCH models
5. Exponential GARCH (EGARCH) model of Nelson (1991),
7. Conditional heteroscedastic ARMA (CHARMA) model of Tsay (1987),

**ARCH model**

\[ a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2, \]

where \( \{\epsilon_t\} \) is a sequence of iid r.v. with mean 0 and variance 1, \( \alpha_0 > 0 \) and \( \alpha_i \geq 0 \) for \( i > 0 \). Distribution of \( \epsilon_t \): Standard normal, standardized Student-t, generalized error dist (GED), or skewed Student-t.

**Properties of ARCH models**

Consider an ARCH(1) model

\[ a_t = \sigma_t \epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2, \]

where \( \alpha_0 > 0 \) and \( \alpha_1 \geq 0 \).

1. \( \text{E}(a_t) = 0 \)
2. \( \text{Var}(a_t) = \alpha_0/(1 - \alpha_1) \) if \( 0 < \alpha_1 < 1 \)
3. Under normality,

\[ m_4 = \frac{3\alpha_0^2(1 + \alpha_1)}{(1 - \alpha_1)(1 - 3\alpha_1^2)}, \]

provided \( 0 < \alpha_1^2 < 1/3 \).

The 3rd property implies heavy tails.

**Advantages**

- Simplicity
- Generates volatility clustering
- Heavy tails (high kurtosis)

**Weaknesses**

- Symmetric btw positive & negative prior returns
- Restrictive
- Provides no explanation
- Not sufficiently adaptive in prediction

**Building an ARCH Model**

1. Modeling the mean effect and testing
   
   Use Q-statistics of squared residuals; McLeod and Li (1983) & Engle (1982)
2. Order determination
   Use PACF of the squared residuals

3. Estimation: Conditional MLE

   Skewness & Kurtosis of standardized residuals.

5. Software: Many available. We use R and S-Plus in class.

**Estimation**: Conditional MLE or Quasi MLE  
**Example**: Monthly log returns of Intel stock

**R demonstration:**
**Special Note**: R uses “OX” package with “garchOxFit” command to estimate GARCH models. The GARCH order in OX is different from that of the textbook and S-Plus. A GARCH($r$, $m$) model in the textbook and S-Plus is called a GARCH($m$, $r$) model in R. In other words, ARCH order is the 2nd argument in R.

```r
> library(fSeries)
> source("garchoxfit_R.txt") %Modification needed to make the program works.
> da=read.table("m-intc7303.txt")
> intc=ts(log(da[,2]+1),frequency=12,start=c(1973,1)) % log returns
> acf(intc)
> acf(intc^2)
> pacf(intc)
> pacf(intc^2)
> Box.test((intc)^2,lag=10,type='Ljung') % Test for ARCH effect.
  Box-Ljung test

  data: (intc)^2
  X-squared = 59.7216, df = 10, p-value = 4.091e-09

> m1=garchOxFit(formula.mean=~arma(0,0),formula.var=~garch(0,3),series=intc)
(* Output edited to simplify the handout *)

***************
** SPECIFICATIONS **
***************
Dependent variable : X
Mean Equation : ARMA (0, 0) model.
No regressor in the mean
Variance Equation : GARCH (0, 3) model.
  No regressor in the variance
The distribution is a Gauss distribution.
Strong convergence using numerical derivatives
Log-likelihood = 233.614
Please wait: Computing the Std Errors...

Maximum Likelihood Estimation (Std. Errors based on Second derivatives)

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Std.Error</th>
<th>t-value</th>
<th>t-prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cst(M)</td>
<td>0.016390</td>
<td>0.006413</td>
<td>2.555</td>
</tr>
<tr>
<td>Cst(V)</td>
<td>0.011999</td>
<td>0.001572</td>
<td>7.630</td>
</tr>
<tr>
<td>ARCH(Alpha1)</td>
<td>0.215677</td>
<td>0.13180</td>
<td>1.636</td>
</tr>
<tr>
<td>ARCH(Alpha2)</td>
<td>0.071882</td>
<td>0.048948</td>
<td>1.469</td>
</tr>
<tr>
<td>ARCH(Alpha3)</td>
<td>0.049396</td>
<td>0.049304</td>
<td>1.002</td>
</tr>
</tbody>
</table>

No. Observations: 372  No. Parameters: 5
Mean (Y): 0.01799
Variance (Y): 0.01784
Skewness (Y): -0.60142
Kurtosis (Y): 5.92148
Log Likelihood: 233.614

Estimated Parameters Vector:
0.016390; 0.011999; 0.215677; 0.071882; 0.049396

***************
** FORECASTS **
***************
Number of Forecasts: 15

<table>
<thead>
<tr>
<th>Horizon</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.01639</td>
<td>0.01414</td>
</tr>
<tr>
<td>2</td>
<td>0.01639</td>
<td>0.01532</td>
</tr>
<tr>
<td>...</td>
<td>(edited)</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.01639</td>
<td>0.01809</td>
</tr>
</tbody>
</table>

************
** TESTS **
************

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Statistic</th>
<th>t-Test</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Skewness</td>
<td>-0.71214</td>
<td>5.6299</td>
<td>1.8027e-008</td>
</tr>
<tr>
<td>Excess Kurtosis</td>
<td>2.9629</td>
<td>11.743</td>
<td>7.6940e-032</td>
</tr>
<tr>
<td>Jarque-Bera</td>
<td>167.52</td>
<td>.NaN</td>
<td>4.2100e-037</td>
</tr>
</tbody>
</table>

Information Criterium (to be minimized)
Akaike: -1.229107
Shibata: -1.229462
Schwarz: -1.176434
Hannan-Quinn: -1.208189

-------------
Q-Statistics on Standardized Residuals
Q(10) = 11.0752 [0.3516885]
Q(15) = 19.5637 [0.1893145]
Q(20) = 20.8711 [0.4047564]

HO : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

Q-Statistics on Squared Standardized Residuals
--> P-values adjusted by 3 degree(s) of freedom
Q(10) = 5.55174 [0.5929504]
Q(15) = 22.8241 [0.0292570]
Q(20) = 23.8158 [0.1245260]

HO : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

ARCH 1-2 test: F(2,364) = 0.32557 [0.7223]
ARCH 1-5 test: F(5,358) = 0.32365 [0.8987]
ARCH 1-10 test: F(10,348) = 0.57556 [0.8339]

Residual-Based Diagnostic for Conditional Heteroskedasticity of Tse (2001)

RBD(10) = 3.62592 [0.9626488]
RBD(15) = 16.5646 [0.3455497]
RBD(20) = 8.58065 [0.9872751]

P-values in brackets

Adjusted Pearson Chi-square Goodness-of-fit test

<table>
<thead>
<tr>
<th># Cells(g)</th>
<th>Statistic</th>
<th>P-Value(g-1)</th>
<th>P-Value(g-k-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>48.8602</td>
<td>0.133868</td>
<td>0.047509</td>
</tr>
<tr>
<td>50</td>
<td>61.8710</td>
<td>0.102550</td>
<td>0.038865</td>
</tr>
<tr>
<td>60</td>
<td>74.7742</td>
<td>0.080766</td>
<td>0.032113</td>
</tr>
</tbody>
</table>

Rem.: k = 5 = # estimated parameters

> m1=garchOxFit(formula.mean="arma(0,0),formula.var="garch(0,1),series=intc)
(* Output edited *)

********************
** SPECIFICATIONS **
********************
Dependent variable : X
Mean Equation : ARMA (0, 0) model.
No regressor in the mean
Variance Equation : GARCH (0, 1) model.
No regressor in the variance
The distribution is a Gauss distribution.

Strong convergence using numerical derivatives
Log-likelihood = 230.454
Maximum Likelihood Estimation (Std. Errors based on Second derivatives)

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<thead>
<tr>
<th>Coefficient</th>
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<th>t-value</th>
<th>t-prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cst(M)</td>
<td>0.016548</td>
<td>2.696</td>
<td>0.0073</td>
</tr>
<tr>
<td>Cst(V)</td>
<td>0.012391</td>
<td>8.020</td>
<td>0.0000</td>
</tr>
<tr>
<td>ARCH(Alpha1)</td>
<td>0.373638</td>
<td>2.794</td>
<td>0.0055</td>
</tr>
</tbody>
</table>

Mean (Y) : 0.01799 Variance (Y) : 0.01784
Skewness (Y) : -0.60142 Kurtosis (Y) : 5.92148
Log Likelihood : 230.454 Alpha[1]+Beta[1]: 0.37344

***************
** FORECASTS **
***************
Number of Forecasts: 15

<table>
<thead>
<tr>
<th>Horizon</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.01655</td>
<td>0.01383</td>
</tr>
<tr>
<td>2</td>
<td>0.01655</td>
<td>0.01755</td>
</tr>
<tr>
<td>.... (edited)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.01655</td>
<td>0.01978</td>
</tr>
</tbody>
</table>

***************
** TESTS **
***************

<table>
<thead>
<tr>
<th>Statistic</th>
<th>t-Test</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Skewness</td>
<td>-0.67997</td>
<td>5.3756</td>
</tr>
<tr>
<td>Excess Kurtosis</td>
<td>2.4472</td>
<td>9.6989</td>
</tr>
<tr>
<td>Jarque-Bera</td>
<td>121.49</td>
<td>NaN</td>
</tr>
</tbody>
</table>

Information Criterium (to be minimized)
Akaike -1.222870 Shibata -1.222999
Schwarz -1.191266 Hannan-Quinn -1.210319

Q-Statistics on Standardized Residuals
Q( 10) = 13.6710 [0.1885356]
Q( 15) = 22.2135 [0.1023272]
Q( 20) = 23.7844 [0.2519364]
H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

Q-Statistics on Squared Standardized Residuals
--> P-values adjusted by 1 degree(s) of freedom
Q( 10) = 12.2592 [0.1990864]
\[ Q(15) = 29.6965 \pm 0.0084008 \]
\[ Q(20) = 31.0595 \pm 0.0397697 \]

Homogeneity of serial correlation: Accept \( H_0 \) when probability is high \([Q < \text{Chisq}(\text{lag})]\)

\begin{align*}
\text{ARCH 1-2 test: } & F(2,366) = 2.5957 [0.0760] \\
\text{ARCH 1-5 test: } & F(5,360) = 1.4194 [0.2164] \\
\text{ARCH 1-10 test: } & F(10,350) = 1.1103 [0.3533]
\end{align*}

\begin{verbatim}
> names(m1)
[1] "x" "csts" "cond.dist" "arma.orders" "arfima"[6] "garch.orders" "arch.in.mean" "model" "ox" "call"
[11] "residuals" "condvars" "coef" "title" "description"

> par(mfcol=c(2,1))  % To plot the returns and volatilities on the sample page
> plot(intc,type='l')
> plot(sqrt(m1$condvars),type='l')
> par(mfcol=c(1,1))
> sresi=m1$residuals/sqrt(m1$condvars)  % Compute standardized residuals
> qqnorm(sresi)  % Obtain normal probability plot (ideal case: a straight line)
> qqline(sresi)  % Impose a straight line on the QQ-plot.

S-Plus demonstration:

> da=read.table("m-intc7303.txt")
> dim(da)
> intc=log(da[,2]+1) % Compute log returns.
> autocorTest(intc,lag.n=12)  % Test serial correlation using Q(12).
Test for Autocorrelation: Ljung-Box

Null Hypothesis: no autocorrelation

Test Statistics:
Test Stat 18.5664
  p.value 0.0995

Dist. under Null: chi-square with 12 degrees of freedom
  Total Observ.: 372

> archTest(intc,lag.n=12)  % Test ARCH effect using 12 lags.
Test for ARCH Effects: LM Test

Null Hypothesis: no ARCH effects

Test Statistics:
Test Stat 43.5041
  p.value 0.0000
\end{verbatim}
Dist. under Null: chi-square with 12 degrees of freedom

> acf(intc^2,lag.max=12,type='partial')  # Compute PACF of squared series.
Call: acf(x = intc^2, lag.max = 12, type = "partial")

Partial Correlation matrix:

<table>
<thead>
<tr>
<th>lag</th>
<th>intc</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1402</td>
</tr>
<tr>
<td>2</td>
<td>0.1703</td>
</tr>
<tr>
<td>3</td>
<td>0.1557</td>
</tr>
<tr>
<td>4</td>
<td>0.0633</td>
</tr>
<tr>
<td>5</td>
<td>0.0407</td>
</tr>
<tr>
<td>6</td>
<td>0.0143</td>
</tr>
<tr>
<td>7</td>
<td>-0.0399</td>
</tr>
<tr>
<td>8</td>
<td>0.1614</td>
</tr>
</tbody>
</table>

% First 3 lags are relatively large.

> arch3=garch(intc~1,~garch(3,0))  # Fit a Gaussian ARCH(3) model
> summary(arch3)

Call:
garch(formula.mean = intc ~ 1, formula.var = ~ garch(3, 0))

Mean Equation: intc ~ 1

Conditional Variance Equation:  ~ garch(3, 0)

Conditional Distribution: gaussian

--------------------------------------------------------------
Estimated Coefficients:

| Value | Std.Error | t value | Pr(>|t|) |
|-------|-----------|---------|---------|
| C     | 0.01713   | 0.006626| 2.5860  | 0.01009 |
| A     | 0.01199   | 0.001107| 10.8325 | 0.00000 |
| ARCH(1)| 0.17874   | 0.080294| 2.2260  | 0.02662 |
| ARCH(2)| 0.07720   | 0.050552| 1.5271  | 0.12760 |
| ARCH(3)| 0.05722   | 0.076928| 0.7438  | 0.45749 |

AIC(5) = -456.5791
BIC(5) = -436.9846

Normality Test:

<table>
<thead>
<tr>
<th>Jarque-Bera P-value</th>
<th>Shapiro-Wilk P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>173.1</td>
<td>0.9696</td>
</tr>
<tr>
<td>0.002337</td>
<td></td>
</tr>
</tbody>
</table>
Ljung-Box test for standardized residuals:
---------------------------------------------------------------
<table>
<thead>
<tr>
<th>Statistic</th>
<th>P-value</th>
<th>Chi^2-d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.79</td>
<td>0.3848</td>
<td>12</td>
</tr>
</tbody>
</table>

Ljung-Box test for squared standardized residuals:
---------------------------------------------------------------
<table>
<thead>
<tr>
<th>Statistic</th>
<th>P-value</th>
<th>Chi^2-d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>21.42</td>
<td>0.04453</td>
<td>12</td>
</tr>
</tbody>
</table>

Lagrange multiplier test:
---------------------------------------------------------------

<table>
<thead>
<tr>
<th>Lag</th>
<th>Lag</th>
<th>Lag</th>
<th>Lag</th>
<th>Lag</th>
<th>Lag</th>
<th>Lag</th>
<th>Lag</th>
<th>Lag</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1646</td>
<td>-0.05844</td>
<td>0.1577</td>
<td>0.2978</td>
<td>0.4671</td>
<td>0.8066</td>
<td>1.037</td>
<td>1.449</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Lag</th>
<th>Lag</th>
<th>Lag</th>
<th>Lag</th>
<th>Lag</th>
<th>C</th>
<th>TR^2</th>
<th>P-value</th>
<th>F-stat</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.02206</td>
<td>-0.8262</td>
<td>3.857</td>
<td>-0.3651</td>
<td>-0.5014</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

> arch1=garch(intc~1,~garch(1,0)) % Simplify to an ARCH(1) model.
> summary(arch1)

Call:
garch(formula.mean = intc ~ 1, formula.var = ~ garch(1, 0))

Mean Equation: intc ~ 1

Conditional Variance Equation: ~ garch(1, 0)

Conditional Distribution: gaussian

---------------------------------------------------------------
Estimated Coefficients:
---------------------------------------------------------------
| Value | Std.Error | t value | Pr(>|t|) |
|-------|-----------|---------|---------|
| C     | 0.01741   | 0.006231| 2.794   | 5.475e-03|
| A     | 0.01258   | 0.001246| 10.091  | 0.000e+00|
| ARCH(1)| 0.35258  | 0.088515| 3.983   | 8.189e-05|

AIC(3) = -454.4589
BIC(3) = -442.7022

Normality Test:
---------------------------------------------------------------
Jarque-Bera P-value Shapiro-Wilk P-value
120.9 0 0.9713 0.0008877

Ljung-Box test for standardized residuals:
---------------------------------------------------------------
Statistic P-value Chi^2-d.f.
15.37 0.2221 12

Ljung-Box test for squared standardized residuals:
---------------------------------------------------------------
Statistic P-value Chi^2-d.f.
26.28 0.009796 12

....(edited)
TR^2 P-value F-stat P-value
22.18 0.03552 2.149 0.0761

> names(arch1)
[1] "residuals" "sigma.t" "df.residual" "coef" "model"
[6] "cond.dist" "likelihood" "opt.index" "cov" "prediction"
[11] "call" "asymp.sd" "series"
> sresi=arch1$residuals/arch1$sigma.t  %Compute standardized residuals
> autocorTest(sresi,lag.n=12)  % Repeat the default output.
Test for Autocorrelation: Ljung-Box
Null Hypothesis: no autocorrelation

Test Statistics:

Test Stat 15.3651
p.value 0.2221

> autocorTest(sresi^2,lag.n=12)
Test for Autocorrelation: Ljung-Box

Null Hypothesis: no autocorrelation

Test Statistics:

Test Stat 26.2798
p.value 0.0098  % Same as the default output.

> autocorTest(sresi^2,lag.n=10)
Test for Autocorrelation: Ljung-Box

Null Hypothesis: no autocorrelation
Test Statistics:

Test Stat 12.7024
p.value 0.2408

> predict(arch1,5) % Prediction 1-step to 5-step ahead forecasts.
$series.pred:
[1] 0.01741175 0.01741175 0.01741175 0.01741175 0.01741175 %return

$sigma.pred:
[1] 0.1181940 0.1322976 0.1369244 0.1385189 0.1390767 % volatility

$asymp.sd: % Unconditional variance of a(t).
[1] 0.1393796

> qqnorm(sresi) % Normal probability plot to check normal assumption
> qqline(sresi) % add line to help read the plot.

From the R output, we obtain that, under normality,

$$r_t = 0.0165 + a_t, \quad \sigma_t^2 = 0.012 + 0.374a_{t-1}^2.$$ 

Model checking:

Standardized shocks \( \{\tilde{a}_t\} \)

\(Q(10) = 13.67(0.19)\)

For \( \{\tilde{a}_t^2\} \)

\(Q(10) = 12.26(0.20), \text{ but } Q(15) = 29.70(.01)\)

Implications

- Expected monthly log return is about 1.7%.
- \( \hat{\alpha}_1^2 = 0.374^2 < 1/3 \) so that 4th moment exists.

From the S-Plus output, we obtain that, under normality,

$$r_t = 0.0174 + a_t, \quad \sigma_t^2 = 0.013 + 0.353a_{t-1}^2.$$ 

Model checking:

Standardized shocks \( \{\tilde{a}_t\} \)

\(Q(12) = 15.37(0.22)\)

For \( \{\tilde{a}_t^2\} \)

\(Q(12) = 21.28(.01).\)

The two programs give similar, but not identical, results.

Next, consider Student-t innovations.

R demonstration
** SPECIFICATIONS **

Dependent variable : X
Mean Equation : ARMA (0, 0) model.
No regressor in the mean
Variance Equation : GARCH (0, 1) model.
No regressor in the variance
The distribution is a Student distribution, with 5.99816 degrees of freedom.

Strong convergence using numerical derivatives
Log-likelihood = 243.116
Please wait : Computing the Std Errors ...

Maximum Likelihood Estimation (Std.Errors based on Second derivatives)

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Std.Error</th>
<th>t-value</th>
<th>t-prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cst(M)</td>
<td>0.021513</td>
<td>0.0060427</td>
<td>3.560</td>
</tr>
<tr>
<td>Cst(V)</td>
<td>0.013332</td>
<td>0.0019620</td>
<td>6.795</td>
</tr>
<tr>
<td>ARCH(Alpha1)</td>
<td>0.268092</td>
<td>0.12224</td>
<td>2.193</td>
</tr>
<tr>
<td>Student(DF)</td>
<td>5.998160</td>
<td>1.6661</td>
<td>3.600</td>
</tr>
</tbody>
</table>

No. Observations : 372 No. Parameters : 4
Mean (Y) : 0.01799 Variance (Y) : 0.01784
Skewness (Y) : -0.60142 Kurtosis (Y) : 5.92148
Log Likelihood : 243.116 Alpha[1]+Beta[1]: 0.26809

***************
** FORECASTS **
***************

Number of Forecasts: 15

<table>
<thead>
<tr>
<th>Horizon</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.02151</td>
<td>0.01453</td>
</tr>
<tr>
<td>2</td>
<td>0.02151</td>
<td>0.01723</td>
</tr>
<tr>
<td>.... (edited)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.02151</td>
<td>0.01821</td>
</tr>
</tbody>
</table>

*************
** TESTS **
*************

<table>
<thead>
<tr>
<th>Statistic</th>
<th>t-Test</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Skewness</td>
<td>-0.68834</td>
<td>5.4417</td>
</tr>
<tr>
<td>Excess Kurtosis</td>
<td>2.5502</td>
<td>10.107</td>
</tr>
<tr>
<td>Jarque-Bera</td>
<td>130.18</td>
<td>.NaN</td>
</tr>
</tbody>
</table>

---------------
Information Criterion (to be minimized)
Akaike       -1.285572  Shibata     -1.285800
Schwarz      -1.243433  Hannan-Quinn -1.268837

Q-Statistics on Standardized Residuals
 Q( 10) = 14.2606  [0.1614340]
 Q( 15) = 23.2423  [0.0791288]
 Q( 20) = 24.7769  [0.2101018]
H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

Q-Statistics on Squared Standardized Residuals
---> P-values adjusted by 1 degree(s) of freedom
 Q( 10) = 15.0259  [0.0902279]
 Q( 15) = 33.5172  [0.0024250]
 Q( 20) = 35.0263  [0.0138654]
H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

ARCH 1-2 test:  F(2,366) = 3.1174  [0.0454]*
ARCH 1-5 test:  F(5,360) = 1.8061  [0.1108]
ARCH 1-10 test: F(10,350)= 1.2727  [0.2443]
P-values in brackets

> sresi=m1$residuals/m1$condvars^.5
> pacf(sresi^2)

> m2=garch0xFit(formula.mean=~arma(0,0),formula.var=~garch(0,2),series=intc,cond.dist="t")

************************************************************
** SPECIFICATIONS **
************************************************************
Dependent variable : X
Mean Equation : ARMA (0, 0) model.
No regressor in the mean
Variance Equation : GARCH (0, 2) model.
No regressor in the variance
The distribution is a Student distribution, with 6.09662 degrees of freedom.

Strong convergence using numerical derivatives
Log-likelihood = 245.913
Please wait : Computing the Std Errors ...

Maximum Likelihood Estimation (Std.Errors based on Second derivatives)

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Std.Error</th>
<th>t-value</th>
<th>t-prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cst(M)</td>
<td>0.022100</td>
<td>0.0060294</td>
<td>3.665</td>
</tr>
<tr>
<td>Cst(V)</td>
<td>0.012420</td>
<td>0.0018284</td>
<td>6.793</td>
</tr>
</tbody>
</table>
ARCH(Alpha1)  0.184359  0.10805  1.706  0.0888
ARCH(Alpha2)  0.110735  0.068533  1.616  0.1070  <= insigh. at 5%
Student(DF)  6.096618  1.6919  3.603  0.0004

No. Observations :  372  No. Parameters :  5
Mean (Y)        :  0.01799  Variance (Y) :  0.01784
Skewness (Y)    :  -0.60142  Kurtosis (Y) :  5.92148
Log Likelihood  :  245.913  Alpha[1]+Beta[1]:  0.29509

> qqplot(rt(10000,6.1),sresi)  % qq-plot for student-t dist.
> qline(sresi)

S-Plus demonstration

> m1=garch(intc~1,garch(1,0),cond.dist="t")  %Use student-t innovations
> summary(m1)

Call:
garch(formula.mean=intc ~ 1,formula.var= ~ garch(1, 0),cond.dist="t")

Mean Equation: intc ~ 1

Conditional Variance Equation:  ~ garch(1, 0)

Conditional Distribution:  t
    with estimated parameter 6.159751 and standard error 1.647094

Estimated Coefficients:

| Value  | Std.Error | t value | Pr(>|t|) |
|--------|-----------|---------|---------|
| C      | 0.02213   | 0.006010| 3.681   | 2.666e-04|
| A      | 0.01338   | 0.001965| 6.809   | 4.002e-11|
| ARCH(1)| 0.24916   | 0.115574| 2.156   | 3.174e-02|

AIC(4) = -477.9073
BIC(4) = -462.2317

Normality Test:

<table>
<thead>
<tr>
<th>Jarque-Bera P-value</th>
<th>Shapiro-Wilk P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>128.9</td>
<td>0.9707</td>
</tr>
</tbody>
</table>

Ljung-Box test for standardized residuals:

<table>
<thead>
<tr>
<th>Statistic</th>
<th>P-value</th>
<th>Chi^2-d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>16.1</td>
<td>0.1868</td>
<td>12</td>
</tr>
</tbody>
</table>
Ljung-Box test for squared standardized residuals:
--------------------------------------------------------------
<table>
<thead>
<tr>
<th>Statistic</th>
<th>P-value</th>
<th>Chi^2-d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>29.91</td>
<td>0.002882</td>
<td>12</td>
</tr>
</tbody>
</table>

> tresi = m1$residuals / m1$sigma.t
> autocorTest(tresi, lag.n=12)
Test for Autocorrelation: Ljung-Box
Null Hypothesis: no autocorrelation

Test Statistics:

Test Stat 16.0974
  p.value 0.1868

> autocorTest(tresi^2, lag.n=12)
Test for Autocorrelation: Ljung-Box
Null Hypothesis: no autocorrelation

Test Statistics:

Test Stat 29.9089
  p.value 0.0029

> autocorTest(tresi^2, lag.n=10) # use 10 lags
Test for Autocorrelation: Ljung-Box

Null Hypothesis: no autocorrelation

Test Statistics:

Test Stat 15.6545
  p.value 0.1100  # The result confirms that lag-12 is significant.
                  # See below PACF of squared residuals.
> predict(m1, 5)  # Prediction
$series.pred:
[1] 0.02212715 0.02212715 0.02212715 0.02212715 0.02212715

$sigma.pred:
[1] 0.1204767 0.1303599 0.1327079 0.1332865 0.1334302

$asymp.sd:
[1] 0.1334779
attr(, "class"): [1] "predict.garch"

> acf(tresi^2,type='partial',lag.max=12)
Call: acf(x = tresi^2, lag.max = 12, type = "partial")

Partial Correlation matrix:
   lag tresi
1  1  0.0352
2  2  0.1273
3  3  0.0643

.....
11 11 -0.0634
12 12  0.1639

> m2=garch(intc~1,~garch(2,0),cond.dist="t") % Increase the order
> summary(m2)
Mean Equation: intc ~ 1

Conditional Variance Equation:  ~ garch(2, 0)

Conditional Distribution:  t
with estimated parameter 6.02561 and standard error 1.565027

--------------------------------------------------------------
Estimated Coefficients:
--------------------------------------------------------------
    Value Std.Error  t value  Pr(>|t|)
C   0.02264 0.006005 3.770 1.899e-04
A   0.01247 0.001918 6.500 2.627e-10
ARCH(1) 0.17457 0.105950 1.648 1.003e-01
ARCH(2) 0.12109 0.092888 1.304 1.932e-01 % Insignificant

--------------------------------------------------------------
AIC(5) = -481.6892
BIC(5) = -462.0947

Question: What is the fitted ARCH(1) model in R?

\[ r_t = 0.022 + a_t, \quad \sigma_t^2 = 0.013 + 0.268a_{t-1}^2, \]

and the t-distribution has 6.00 d.f.

Question: What is the fitted ARCH(1) model in S-Plus?

\[ r_t = 0.022 + a_t, \quad \sigma_t^2 = 0.013 + 0.249a_{t-1}^2, \]

and the t-distribution has 6.16 d.f.

Comparison with normal models:

- Using a heavy-tailed dist for \( \epsilon_t \) reduces the ARCH effect
The difference between the models is small for this particular instance. You may try the generalized error distribution.

**GARCH Model**

\[ a_t = \sigma_t \epsilon_t, \]
\[ \sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2 \]

where \( \{\epsilon_t\} \) is defined as before, \( \alpha_0 > 0, \alpha_i \geq 0, \beta_j \geq 0, \) and \( \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) < 1. \)

Re-parameterization:

Let \( \eta_t = a_t^2 - \sigma_t^2. \) \( \{\eta_t\} \) un-correlated series.

The GARCH model becomes

\[ a_t^2 = \alpha_0 + \sum_{i=1}^{\max(m,s)} (\alpha_i + \beta_i) a_{t-i}^2 + \eta_t - \sum_{j=1}^{s} \beta_j \eta_{t-j}. \]

This is an ARMA form for the squared series \( a_t^2. \)

Use it to understand properties of GARCH models, e.g. moment equations, forecasting, etc.

Focus on a GARCH(1,1) model

\[ \sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \]

- **Weak stationarity:** \( 0 \leq \alpha_1, \beta_1 \leq 1, (\alpha_1 + \beta_1) < 1. \)
- **Volatility clusters**
- **Heavy tails:** if \( 1 - 2\alpha_1^2 - (\alpha_1 + \beta_1)^2 > 0, \) then
  \[
  \frac{E(a_t^4)}{[E(a_t^2)]^2} = \frac{3[1 - (\alpha_1 + \beta_1)^2]}{1 - (\alpha_1 + \beta_1)^2 - 2\alpha_1^2} > 3. \]

- For 1-step ahead forecast,
  \[ \sigma_h^2(1) = \alpha_0 + \alpha_1 a_h^2 + \beta_1 \sigma_h^2. \]

For multi-step ahead forecasts, use \( a_t^2 = \sigma_t^2 \epsilon_t^2 \) and rewrite the model as

\[ \sigma_{t+1}^2 = \alpha_0 + (\alpha_1 + \beta_1) \sigma_t^2 + \alpha_1 \sigma_t^2 (\epsilon_t^2 - 1). \]

2-step ahead volatility forecast

\[ \sigma_h^2(2) = \alpha_0 + (\alpha_1 + \beta_1) \sigma_h^2(1). \]

In general, we have

\[ \sigma_h^2(\ell) = \alpha_0 + (\alpha_1 + \beta_1) \sigma_h^2(\ell - 1), \quad \ell > 1. \]

This result is exactly the same as that of an ARMA(1,1) model with AR polynomial

\[ 1 - (\alpha_1 + \beta_1)B. \]
**Example:** Monthly excess returns of S&P 500 index starting from 1926 for 792 observations. The fitted of a Gaussian AR(3) model

\[ r_t = 0.088r_{t-1} - 0.023r_{t-2} - 0.123r_{t-3} + 0.007 + a_t, \]

\[ \hat{\sigma}_a^2 = 0.00333. \]

For the GARCH effects, use a GARCH(1,1) model (R output).

A joint estimation:

\[ r_t = 0.032r_{t-1} - 0.030r_{t-2} - 0.010r_{t-3} + 0.0076 + a_t \]
\[ \sigma_t^2 = 0.00008 + 0.853\sigma_{t-1}^2 + 0.125a_{t-1}^2. \]

Implied unconditional variance of \( a_t \) is

\[ \frac{0.0000794}{1 - 0.85298 - 0.1247} = 0.00356 \]

close to the expected value.

A simplified model:

\[ r_t = 0.0074 + a_t, \sigma_t^2 = 0.00008 + 0.854\sigma_{t-1}^2 + 0.122a_{t-1}^2. \]

Model checking:

For \( \hat{a}_t \): \( Q(10) = 11.22(0.34) \) and \( Q(20) = 24.29(0.23) \).

For \( \hat{a}_t^2 \): \( Q(10) = 9.92(0.27) \) and \( Q(20) = 16.74(0.54) \).

Forecast: 1-step ahead forecast:

\[ \sigma_h^2(1) = 0.00008 + 0.854\sigma_{h-1}^2 + 0.122a_{h}^2 \]

<table>
<thead>
<tr>
<th>Horizon</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>∞</th>
</tr>
</thead>
<tbody>
<tr>
<td>Return</td>
<td>.0074</td>
<td>.0074</td>
<td>.0074</td>
<td>.0074</td>
<td>.0074</td>
<td>.0074</td>
</tr>
<tr>
<td>Volatility</td>
<td>.053</td>
<td>.052</td>
<td>.052</td>
<td>.051</td>
<td>.051</td>
<td>.050</td>
</tr>
</tbody>
</table>

**R demonstration:**

```r
> library("fSeries")
> source("garch0xFit_R.txt")
> sp5=scan(file="sp500.dat")
> plot(sp5,type='l')
> m1=arima(sp5,order=c(3,0,0))
> m1
Call:
arima(x = sp5, order = c(3, 0, 0))

Coefficients:
ar1  ar2  ar3  intercept
0.0890  -0.0238  -0.1229  0.0062
s.e.  0.0353   0.0355  0.0353  0.0019
```
sigma^2 estimated as 0.00333: log likelihood = 1135.25, aic = -2260.5

> x=ts(sp5)
> m2=garchOxFit(formula.mean="arma(3,0)",formula.var="garch(1,1)",series=x)

********************
** SPECIFICATIONS **
********************
Dependent variable : X
Mean Equation : ARMA (3, 0) model.
No regressor in the mean
Variance Equation : GARCH (1, 1) model.
No regressor in the variance
The distribution is a Gauss distribution.

Strong convergence using numerical derivatives
Log-likelihood = 1272.18

Maximum Likelihood Estimation (Std.Errors based on Second derivatives)

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Std.Error</th>
<th>t-value</th>
<th>t-prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cst(M)</td>
<td>0.007639</td>
<td>0.0015225</td>
<td>5.017</td>
</tr>
<tr>
<td>AR(1)</td>
<td>0.031987</td>
<td>0.038368</td>
<td>0.8337</td>
</tr>
<tr>
<td>AR(2)</td>
<td>-0.030276</td>
<td>0.038407</td>
<td>-0.7883</td>
</tr>
<tr>
<td>AR(3)</td>
<td>-0.010637</td>
<td>0.037558</td>
<td>-0.2832</td>
</tr>
<tr>
<td>Cst(V)</td>
<td>0.793989</td>
<td>0.28174</td>
<td>2.818</td>
</tr>
<tr>
<td>ARCH(Alpha1)</td>
<td>0.124710</td>
<td>0.022615</td>
<td>5.514</td>
</tr>
<tr>
<td>GARCH(Beta1)</td>
<td>0.852981</td>
<td>0.021891</td>
<td>38.97</td>
</tr>
</tbody>
</table>

No. Observations : 792 No. Parameters : 7
Mean (Y) : 0.00614 Variance (Y) : 0.00341
Skewness (Y) : 0.41134 Kurtosis (Y) : 12.30025
Log Likelihood : 1272.183 Alpha[1]+Beta[1]: 0.97749

Warning : To avoid numerical problems, the estimated parameter Cst(V), and its std.Error have been multiplied by 10^-4.

***************
** FORECASTS **
***************
Number of Forecasts: 15

<table>
<thead>
<tr>
<th>Horizon</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.01248</td>
<td>0.002889</td>
</tr>
<tr>
<td>2</td>
<td>0.005195</td>
<td>0.002824</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>15</td>
<td>0.007639</td>
<td>0.002101</td>
</tr>
</tbody>
</table>
** TESTS **

<table>
<thead>
<tr>
<th>Statistic</th>
<th>t-Test</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Skewness</td>
<td>-0.38960</td>
<td>4.4847</td>
</tr>
<tr>
<td>Excess Kurtosis</td>
<td>1.2654</td>
<td>7.2920</td>
</tr>
<tr>
<td>Jarque-Bera</td>
<td>72.877</td>
<td>.NaN</td>
</tr>
</tbody>
</table>

Information Criterion (to be minimized)

Akaike    = -3.194906
Shibata   = -3.195060
Schwarz   = -3.153590
Hannan-Quinn = -3.179027

Q-Statistics on Standardized Residuals

--> P-values adjusted by 3 degree(s) of freedom

Q( 10) = 11.5651 [0.1157988]
Q( 15) = 17.7852 [0.1223710]
Q( 20) = 24.1110 [0.1164494]

H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

Q-Statistics on Squared Standardized Residuals

--> P-values adjusted by 2 degree(s) of freedom

Q( 10) = 10.3145 [0.2436438]
Q( 15) = 14.2072 [0.3594178]
Q( 20) = 16.7703 [0.5389502]

H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

ARCH 1-2 test: F(2,785) = 0.69609 [0.4988]
ARCH 1-5 test: F(5,779) = 0.62957 [0.6772]
ARCH 1-10 test: F(10,769) = 1.0519 [0.3976]

S-Plus demonstration:

> x=scan(file='sp500.dat')
> spfit=garch(x~ar(3),~garch(1,1)) % Fit an AR(3) + GARCH(1,1) model.
> summary(spfit)

Mean Equation: x ~ ar(3)
Conditional Variance Equation: ~ garch(1, 1)
Conditional Distribution: gaussian

Estimated Coefficients:

| Value Std.Error  | t value | Pr(>|t|) |
--------------------|---------|----------|

C 7.751e-03 1.603e-03 4.8359 1.595e-06
AR(1) 3.267e-02 3.849e-02 0.8488 3.903e-01 % AR coefs are insign. at 5%
AR(2) -2.884e-02 3.823e-02 -0.7543 4.509e-01
AR(3) -8.407e-03 3.550e-02 -0.2368 8.129e-01
A 8.374e-05 2.436e-05 3.4382 6.164e-01
ARCH(1) 1.213e-01 2.030e-02 5.9774 3.439e-09
GARCH(1) 8.523e-01 1.969e-02 43.2803 0.000e+00

AIC(7) = -2526.239, BIC(7) = -2493.517

Normality Test:

<table>
<thead>
<tr>
<th>Jarque-Bera</th>
<th>P-value</th>
<th>Shapiro-Wilk</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>72.25</td>
<td>2.22e-16</td>
<td>0.9817</td>
<td>0.04185</td>
</tr>
</tbody>
</table>

Ljung-Box test for standardized residuals:

<table>
<thead>
<tr>
<th>Statistic</th>
<th>P-value</th>
<th>Chi^2-d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.78</td>
<td>0.4636</td>
<td>12</td>
</tr>
</tbody>
</table>

Ljung-Box test for squared standardized residuals:

<table>
<thead>
<tr>
<th>Statistic</th>
<th>P-value</th>
<th>Chi^2-d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>13.44</td>
<td>0.338</td>
<td>12</td>
</tr>
</tbody>
</table>

> spfit=garch(x~1,~garch(1,1)) % A refined model
> summary(spfit)

Call: garch(formula.mean = x ~ 1, formula.var = ~ garch(1, 1))

Mean Equation: x ~ 1
Conditional Variance Equation: ~ garch(1, 1)
Conditional Distribution: gaussian

Estimated Coefficients:

| Value Std.Error t value Pr(>|t|) |
|--------------------------------|
| C 7.647e-03 1.545e-03 4.950 9.096e-07 |
| A 8.561e-05 2.412e-05 3.549 4.097e-04 |
| ARCH(1) 1.216e-01 1.974e-02 6.159 1.165e-09 |
| GARCH(1) 8.511e-01 1.899e-02 43.280 0.000e+00 |

AIC(4) = -2530.821, BIC(4) = -2512.122

Normality Test:

<table>
<thead>
<tr>
<th>Jarque-Bera</th>
<th>P-value</th>
<th>Shapiro-Wilk</th>
<th>P-value</th>
</tr>
</thead>
</table>

22
Ljung-Box test for standardized residuals:

<table>
<thead>
<tr>
<th>Statistic</th>
<th>P-value</th>
<th>Chi^2-d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.99</td>
<td>0.4468</td>
<td>12</td>
</tr>
</tbody>
</table>

Ljung-Box test for squared standardized residuals:

<table>
<thead>
<tr>
<th>Statistic</th>
<th>P-value</th>
<th>Chi^2-d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>13.11</td>
<td>0.3609</td>
<td>12</td>
</tr>
</tbody>
</table>

Lagrange multiplier test:

<table>
<thead>
<tr>
<th>Lag 1</th>
<th>Lag 2</th>
<th>Lag 3</th>
<th>Lag 4</th>
<th>Lag 5</th>
<th>Lag 6</th>
<th>Lag 7</th>
<th>Lag 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.9755</td>
<td>0.5875</td>
<td>-0.4926</td>
<td>-0.8138</td>
<td>-0.1367</td>
<td>-1.018</td>
<td>1.497</td>
<td>1.859</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Lag 9</th>
<th>Lag 10</th>
<th>Lag 11</th>
<th>Lag 12</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5532</td>
<td>1.758</td>
<td>0.2104</td>
<td>0.1441</td>
<td>-0.947</td>
</tr>
</tbody>
</table>

\[ \text{TR}^2 \text{ P-value F-stat P-value} \]

\[ 13.15 \ 0.3583 \ 1.216 \ 0.3824 \]

```r
> predict(spfit,5)
$series.pred:
[1] 0.007647292 0.007647292 0.007647292 0.007647292 0.007647292

$sigma.pred:
[1] 0.05358398 0.05365175 0.05371758 0.05378154 0.05384369
```

> mean(x)
[1] 0.006143056  % Point forecasts are higher than sample mean!

Compare the Splus result with that of R!

Turn to Student-t innovation.

Estimation of degrees of freedom:

\[ r_t = 0.0085 + a_t, \]
\[ \sigma^2_t = .00012 + .113a^2_{t-1} + .842\sigma^2_{t-1}, \]

where the estimated degrees of freedom is 6.99.

**Forecasting evaluation**

Not easy to do; see Andersen and Bollerslev (1998).

**IGARCH model**

An IGARCH(1,1) model:

\[ a_t = \sigma_t \epsilon_t, \sigma^2_t = \alpha_0 + \beta_1 \sigma^2_{t-1} + (1 - \beta_1)a^2_{t-1}. \]
For the monthly excess returns of the S&P 500 index, we have

\[ r_t = 0.007 + a_t, \sigma_t^2 = 0.0001 + 0.806\sigma_{t-1}^2 + 0.194a_{t-1}^2 \]

For an IGARCH(1,1) model,

\[ \sigma_h^2(\ell) = \sigma_h^2(1) + (\ell - 1)\alpha_0, \quad \ell \geq 1, \]

where \( h \) is the forecast origin.

Effect of \( \sigma_h^2(1) \) on future volatilities is persistent, and the volatility forecasts form a straight line with slope \( \alpha_0 \). See Nelson (1990) for more info.

Special case: \( \alpha_0 = 0 \).

used in RiskMetrics to VaR calculation.

**Example:** An IGARCH(1,1) model for the monthly excess returns of S&P500 index from 1926 to 1991 is given below via R.

\[ r_t = 0.0074 + a_t, \quad a_t = \sigma_t \epsilon_t \]

\[ \sigma_t^2 = 0.00005 + 0.143a_{t-1}^2 + 0.857\sigma_{t-1}^2. \]

**R demonstration**

```r
> m2=garchOxFit(formula.mean=~arma(0,0),formula.var=~igarch(1,1),series=sp5)
```

```
********************
** SPECIFICATIONS **
********************
Dependent variable : X
Mean Equation : ARMA (0, 0) model.
No regressor in the mean
Variance Equation : IGARCH (1, 1) model.
No regressor in the variance
The distribution is a Gauss distribution.

Strong convergence using numerical derivatives
Log-likelihood = 1268.24

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Std.Error</th>
<th>t-value</th>
<th>t-prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cst(M)</td>
<td>0.007416</td>
<td>0.0015254</td>
<td>4.861</td>
</tr>
<tr>
<td>Cst(V)</td>
<td>0.512441</td>
<td>0.17527</td>
<td>2.924</td>
</tr>
<tr>
<td>ARCH(Alpha1)</td>
<td>0.142948</td>
<td>0.021444</td>
<td>6.666</td>
</tr>
<tr>
<td>GARCH(Beta1)</td>
<td>0.857252</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

| No. Observations : | 792 | No. Parameters : | 4 |
| Mean (Y) :          | 0.00614 | Variance (Y) :   | 0.00341 |
| Skewness (Y) :      | 0.41134 | Kurtosis (Y) :    | 12.30025 |
| Log Likelihood :    | 1268.238 |              |        |
Warning : To avoid numerical problems, the estimated parameter 
Cst(V), and its std.Error have been multiplied by 10^-4.

***************
** FORECASTS **
***************
Number of Forecasts: 15

<table>
<thead>
<tr>
<th>Horizon</th>
<th>Mean</th>
<th>Variance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.007416</td>
<td>0.003079</td>
</tr>
<tr>
<td>2</td>
<td>0.007416</td>
<td>0.003079</td>
</tr>
<tr>
<td>.......</td>
<td>.......</td>
<td>.......</td>
</tr>
<tr>
<td>15</td>
<td>0.007416</td>
<td>0.003079</td>
</tr>
</tbody>
</table>

---------------

***********
** TESTS **
***********

<table>
<thead>
<tr>
<th></th>
<th>Statistic</th>
<th>t-Test</th>
<th>P-Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Skewness</td>
<td>-0.40577</td>
<td>4.6708</td>
<td>3.0001e-006</td>
</tr>
<tr>
<td>Excess Kurtosis</td>
<td>1.2112</td>
<td>6.9797</td>
<td>2.9578e-012</td>
</tr>
<tr>
<td>Jarque-Bera</td>
<td>70.146</td>
<td>.NaN</td>
<td>5.8618e-016</td>
</tr>
</tbody>
</table>

Information Criterium (to be minimized)

<table>
<thead>
<tr>
<th></th>
<th>Akaike</th>
<th>Shibata</th>
<th>-3.195073</th>
</tr>
</thead>
<tbody>
<tr>
<td>Schwarz</td>
<td>-3.177338</td>
<td>Hannan-Quinn</td>
<td>-3.188239</td>
</tr>
</tbody>
</table>

Q-Statistics on Standardized Residuals

Q( 10) = 10.8290 [0.3709943]
Q( 15) = 17.6387 [0.2821344]
Q( 20) = 23.6418 [0.2583909]

H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

Q-Statistics on Squared Standardized Residuals

--> P-values adjusted by 2 degree(s) of freedom

Q( 10) = 10.0513 [0.2614457]
Q( 15) = 13.4740 [0.4119030]
Q( 20) = 15.9382 [0.5968584]

H0 : No serial correlation ==> Accept H0 when prob. is High [Q < Chisq(lag)]

ARCH 1-2 test:  F(2,785) = 1.1501 [0.3171]
ARCH 1-5 test:  F(5,779) = 0.79599 [0.5527]
ARCH 1-10 test: F(10,769)= 1.0004 [0.4413]

The GARCH-M model

\[ r_t = \mu + c\sigma_t^2 + a_t, \quad a_t = \sigma_t\epsilon_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1a_{t-1}^2 + \beta_1\sigma_{t-1}^2 \]
where \( c \) is referred to as risk premium, which is expected to be positive.

**Example**: A GARCH(1,1)-M model for the monthly excess returns of S&P 500 index from January 1926 to December 1991. The fitted model is

\[
 r_t = 0.0054 + 1.01\sigma_t^2 + a_t, \sigma_t^2 = .00008 + .123a_{t-1}^2 + .852\sigma_{t-1}^2. 
\]

Std err of risk premium is 0.818.

**R demonstration**

```r
> m3=garch0xFit(formula.mean=~arma(0,0),formula.var=~garch(1,1),series=sp5,arch.in.mean=T)
```

**SPECIFICATIONS**

Dependent variable : X
Mean Equation : ARMA (0, 0) model.
No regressor in the mean
Variance Equation : GARCH (1, 1) model.
No regressor in the variance
The distribution is a Gauss distribution.

Strong convergence using numerical derivatives
Log-likelihood = 1270.1

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Std.Error</th>
<th>t-value</th>
<th>t-prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cst(M)</td>
<td>0.005420</td>
<td>0.0023602</td>
<td>2.297</td>
</tr>
<tr>
<td>Cst(V)</td>
<td>0.829654</td>
<td>0.29321</td>
<td>2.830</td>
</tr>
<tr>
<td>ARCH(Alpha1)</td>
<td>0.123127</td>
<td>0.022286</td>
<td>5.525</td>
</tr>
<tr>
<td>GARCH(Beta1)</td>
<td>0.852256</td>
<td>0.022400</td>
<td>38.05</td>
</tr>
<tr>
<td>ARCH-in-mean(var)</td>
<td>1.008013</td>
<td>0.88853</td>
<td>1.134</td>
</tr>
</tbody>
</table>

Warning : To avoid numerical problems, the estimated parameter Cst(V), and its std.Error have been multiplied by 10^-4.

**S-Plus demonstration**

```r
> spfit=garch(x~1+var.in.mean,~garch(1,1))
> summary(spfit)
```

garch(formula.mean = x ~ 1 + var.in.mean, formula.var = ~ garch(1, 1))

Mean Equation: x ~ 1 + var.in.mean
Conditional Variance Equation:  ~ garch(1, 1)
Conditional Distribution:  gaussian

Estimated Coefficients:  

| Value Std.Error t value Pr(>|t|) |  |
|---------------------------------|--|
C 5.487e-03 2.262e-03 2.426 7.747e-03
ARCH-IN-MEAN 1.088e+00 8.182e-01 1.330 9.203e-02
A 8.764e-05 2.507e-05 3.496 2.494e-04
ARCH(1) 1.227e-01 2.047e-02 5.993 1.571e-09
GARCH(1) 8.494e-01 1.958e-02 43.390 0.000e+00
------------------------------------------------------------------------
AIC(5) = -2530.136, BIC(5) = -2506.763

Normality Test:
---------------------------------------------------------------
<table>
<thead>
<tr>
<th>Jarque-Bera P-value</th>
<th>Shapiro-Wilk P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>79.85</td>
<td>0</td>
</tr>
<tr>
<td>0.9801</td>
<td>0.009548</td>
</tr>
</tbody>
</table>

Ljung-Box test for standardized residuals:
------------------------------------------------------------------------
<table>
<thead>
<tr>
<th>Statistic</th>
<th>P-value</th>
<th>Chi^2-d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>13.43</td>
<td>0.3385</td>
<td>12</td>
</tr>
</tbody>
</table>

Ljung-Box test for squared standardized residuals:
------------------------------------------------------------------------
<table>
<thead>
<tr>
<th>Statistic</th>
<th>P-value</th>
<th>Chi^2-d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.83</td>
<td>0.4598</td>
<td>12</td>
</tr>
</tbody>
</table>

Lagrange multiplier test:
------------------------------------------------------------------------
<table>
<thead>
<tr>
<th>Lag 1</th>
<th>Lag 2</th>
<th>Lag 3</th>
<th>Lag 4</th>
<th>Lag 5</th>
<th>Lag 6</th>
<th>Lag 7</th>
<th>Lag 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.9655</td>
<td>0.4582</td>
<td>-0.509</td>
<td>-0.7114</td>
<td>0.03989</td>
<td>-0.9966</td>
<td>1.444</td>
<td>1.656</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Lag 9</th>
<th>Lag 10</th>
<th>Lag 11</th>
<th>Lag 12</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4562</td>
<td>1.713</td>
<td>0.2978</td>
<td>0.1433</td>
<td>-0.9991</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TR^2</th>
<th>P-value</th>
<th>F-stat</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.85</td>
<td>0.4575</td>
<td>1.094</td>
<td>0.4738</td>
</tr>
</tbody>
</table>

> predict(spfit,5)
$series.pred:
[1] 0.008621675 0.008629477 0.008637062 0.008644436 0.008651603
$sigma.pred:
[1] 0.05368237 0.05374914 0.05381396 0.05387690 0.05393801
$asymp.sd:
[1] 0.05602398