Stock Options:

- A contract giving its holder the right, but not obligation, to trade shares of a common stock by a certain date for a specified price.
  - Call option: to buy
  - Put option: to sell
  - Specified price: strike price $K$
  - date: expiration $T$

Note: You can also sell a call or put option (underwrite).

Factors affecting the price of an option

- Current stock price: $P_t$
- time to expiration: $T - t$
- Risk-free interest rate: $r$ per annum
- Stock volatility: $\sigma$ annualized

Payoff for European options (exercised at $T$ only)

Call option:

$$V(P_T) = (P_T - K)_+ = \begin{cases} P_T - K & \text{if } P_T > K \\ 0 & \text{if } P_T \leq K \end{cases}$$
The holder only exercises her option if $P_T > K$ (buys the stock via exercising the option and sells the stock on the market).

**Put option:**

$$V(P_T) = (K - P_T)_+ = \begin{cases} K - P_T & \text{if } P_T < K \\ 0 & \text{if } P_T \geq K \end{cases}$$

The holder only exercises her option if $P_T < K$ (buys the stock from the market and sells it via option).

**Mathematical framework**

- Stock (log) price follows a diffusion equation, i.e. a continuous-time continuous stochastic process such as

$$dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dw_t,$$

where $\mu(x_t, t)$ and $\sigma(x_t, t)$ are the drift and diffusion coefficient, respectively, and $w_t$ is a standard Brownian motion (or Wiener process).

- In a complete market, use hedging to derive the price of an option (no arbitrage argument).

- In an incomplete market (e.g. existence of jumps), specify risk and a hedging strategy to minimize the risk.

**Stochastic processes**

- Wiener process (or Standard Brownian motion)

  - notation: $w_t$
– initial value: $w_0 = 0$
– small increments are independent and normal

\[
\text{time points: } 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t
\]
\[
\{ \Delta w_i = w_{t_i} - w_{t_{i-1}} \} \text{ are independent}
\]
\[
\Delta w_t = w_{t+\Delta t} - w_t \sim N(0, \Delta t).
\]

– property: $w_t \sim N(0, t)$
– zero drift and rate of variance change is 1.
– A simple way to understand Wiener processes is to do simulation. In R or S-Plus, this can be achieved by using:

\[
n=5000
\]
\[
at = \text{rnorm}(n)
\]
\[
wt = \text{cumsum}(at)/\sqrt{n}
\]
\[
\text{plot(wt,type='l')}
\]

Repeat the above commands to generate lots of “wt” series.

• Generalized Wiener process

\[
dx_t = \mu dt + \sigma dw_t,
\]

where the drift $\mu$ & rate of volatility change $\sigma$ are constant.

• Ito’s process

\[
dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dw_t,
\]

where both drift and volatility are time-varying.
Figure 1: Time plots of four simulated Wiener processes

- Geometric Brownian motion

\[ dP_t = \mu P_t dt + \sigma P_t dw_t, \]

so that \( \mu(P_t, t) = \mu P_t \) and \( \sigma(P_t, t) = \sigma P_t \) with \( \mu \) and \( \sigma \) being constant.

**Illustration:** Four simulated standard Brownian motions. key feature: variability increases with time.

Assume that the price of a stock follows a geometric Brownian motion. What is the distribution of the log return?
To answer this question, we need Ito’s calculus.

Review of differentiation

$G(x)$: a differentiable function of $x$.

What is $dG(x)$?

Taylor expansion:

$$
\Delta G \equiv G(x + \Delta x) - G(x) = \frac{\partial G}{\partial x} \Delta x \\
+ \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (\Delta x)^2 + \frac{1}{6} \frac{\partial^3 G}{\partial x^3} (\Delta x)^3 + \cdots.
$$

Letting $\Delta x \to 0$, we have

$$
dG = \frac{\partial G}{\partial x} dx.
$$

How about $G(x, y)$?

$$
\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial y} \Delta y \\
+ \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (\Delta x)^2 + \frac{\partial^2 G}{\partial x \partial y} \Delta x \Delta y + \frac{1}{2} \frac{\partial^2 G}{\partial y^2} (\Delta y)^2 + \cdots.
$$

Taking limit as $\Delta x \to 0$ and $\Delta y \to 0$, we have

$$
dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy.
$$

Stochastic differentiation

Now, consider $G(x_t, t)$ with $x_t$ being an Ito’s process.

$$
\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t \\
+ \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (\Delta x)^2 + \frac{\partial^2 G}{\partial x \partial t} \Delta x \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} (\Delta t)^2 + \cdots.
$$
A discretized version of the Ito’s process is

\[ \Delta x = \mu_* \Delta t + \sigma_* \epsilon \sqrt{\Delta t}, \]

where \( \mu_* = \mu(x_t, t) \) and \( \sigma_* = \sigma(x_t, t) \). Therefore,

\[ (\Delta x)^2 = \mu_*^2 (\Delta t)^2 + \sigma_*^2 \epsilon^2 \Delta t + 2 \mu_* \sigma_* \epsilon (\Delta t)^{3/2} \]

\[ = \sigma_*^2 \epsilon^2 \Delta t + H(\Delta t). \]

Thus, \((\Delta x)^2\) contains a term of order \( \Delta t \).

\[ E(\sigma_*^2 \epsilon^2 \Delta t) = \sigma_*^2 \Delta t, \]

\[ \text{Var}(\sigma_*^2 \epsilon^2 \Delta t) = E[\sigma_*^4 (\Delta t)^2] - [E(\sigma_*^2 \epsilon^2 \Delta t)]^2 = 2 \sigma_*^4 (\Delta t)^2, \]

where we use \( E(\epsilon^4) = 3 \). These two properties show that

\[ \sigma_*^2 \epsilon^2 \Delta t \to \sigma_*^2 \Delta t \quad \text{as} \quad \Delta t \to 0. \]

Consequently,

\[ (\Delta x)^2 \to \sigma_*^2 dt \quad \text{as} \quad \Delta t \to 0. \]

Using this result, we have

\[ dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \sigma_*^2 dt \]

\[ = \left( \frac{\partial G}{\partial x} \mu_* + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \sigma_*^2 \right) dt + \frac{\partial G}{\partial x} \sigma_* dw_t. \]

This is the well-known Ito’s lemma.

**Example.** Let \( G(w_t, t) = w_t^2 \). What is \( dG(w_t, t) \)?
Answer: Here $\mu_* = 0$ and $\sigma_* = 1$.

\[
\frac{\partial G}{\partial w_t} = 2w_t, \quad \frac{\partial G}{\partial t} = 0, \quad \frac{\partial^2 G}{\partial w_t^2} = 2.
\]

Therefore,

\[
dw_t^2 = (2w_t \times 0 + 0 + \frac{1}{2} \times 2 \times 1)dt + 2w_t dw_t = dt + 2w_t dw_t.
\]

If $P_t$ follows a geometric Brownian motion, what is the model for $\ln(P_t)$?

Answer: Let $G(P_t, t) = \ln(P_t)$. we have

\[
\frac{\partial G}{\partial P_t} = \frac{1}{P_t}, \quad \frac{\partial G}{\partial t} = 0, \quad \frac{1}{2} \frac{\partial^2 G}{\partial P_t^2} = \frac{1 - 1}{2 P_t^2}.
\]

Consequently, via Ito’s lemma, we obtain

\[
d\ln(P_t) = \left(\frac{1}{P_t} \mu P_t + \frac{1 - 1}{2 P_t^2} \sigma^2 P_t^2\right) dt + \frac{1}{P_t} \sigma P_t dw_t
\]

\[
= \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dw_t.
\]

Thus, $\ln(P_t)$ follows a generalized Wiener Process with drift rate $\mu - \sigma^2/2$ and variance rate $\sigma^2$.

The log return from $t$ to $T$ is normal with mean $(\mu - \sigma^2/2)(T - t)$ and variance $\sigma^2(T - t)$.

Estimation of $\mu$ and $\sigma$

Assume that $n$ log returns are available, say $\{r_t | t = 1, \ldots, n\}$. 


Statistical theory:
Estimate the mean and variance by the sample mean and variance.

\[
\bar{r} = \frac{\sum_{t=1}^{n} r_t}{n},
\]
\[
s_r^2 = \frac{1}{n-1} \sum_{t=1}^{n} (r_t - \bar{r})^2.
\]

Remember the length of time intervals!
Let \( \Delta \) be the length of time intervals measured in years.
Then, the distribution of \( r_t \) is

\[ r_t \sim N((\mu - \sigma^2/2)\Delta, \sigma^2\Delta]. \]

We obtain the estimates

\[
\hat{\sigma} = \frac{s_r}{\sqrt{\Delta}},
\]
\[
\hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = \frac{\bar{r}}{\Delta} + \frac{s_r^2}{2\Delta}.
\]

The data show \( \bar{r} = 0.002276 \) and \( s_r = 0.01915 \).
Since \( \Delta = 1/252 \) year, we obtain that

\[
\hat{\sigma} = \frac{s_r}{\sqrt{\Delta}} = 0.3040, \quad \hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = 0.6198.
\]

Thus, the estimated expected return was 61.98% and the standard deviation was 30.4% per annum for IBM stock in 1998.

Data show \( \bar{r} = 0.00332 \) and \( s_r = 0.026303 \),
Also, \( Q(12) = 10.8 \). Therefore, we have

\[
\hat{\sigma} = \frac{s_r}{\sqrt{\Delta}} = \frac{0.026303}{\sqrt{1.0/252.0}} = 0.418, \quad \hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = 0.924.
\]

Expected return was 92.4% per annum
Estimated s.d. was 41.8% per annum.

Data show \( \bar{r} = -0.00301 \) and \( s_r = 0.05192 \).
Therefore, \( \hat{\sigma} = 0.818, \hat{\mu} = -0.412 \).
Time-varying nature of mean and volatility is clearly shown.

**Distributions of stock prices**
If the price follows

\[
dP_t = \mu P_t dt + \sigma P_t dw_t,
\]
then,

\[
\ln(P_T) - \ln(P_t) \sim N \left[ (\mu - \frac{\sigma^2}{2})(T - t), \sigma^2(T - t) \right].
\]

Consequently, given \( P_t \),

\[
\ln(P_T) \sim N \left[ \ln(P_t) + (\mu - \frac{\sigma^2}{2})(T - t), \sigma^2(T - t) \right],
\]
and we obtain (log-normal dist; ch. 1)

\[
E(P_T) = P_t \exp[\mu(T - t)],
\]
\[
\text{Var}(P_T) = P_t^2 \exp[2\mu(T - t)]\{\exp[\sigma^2(T - t)] - 1\}.
\]
The result can be used to make inference about $P_T$. Simulation is often used to study the behavior of $P_T$.

**Black-Scholes equation**

- Price of stock: $P_t$ is a Geo. B. Motion
- price of derivative: $G_t = G(P_t, t)$ contingent the stock
- Risk neutral world: expected returns are given by the risk-free interest rate (no arbitrage)

From Ito’s lemma:

$$dG_t = \left( \frac{\partial G_t}{\partial P_t} \mu P_t + \frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) dt + \frac{\partial G_t}{\partial P_t} \sigma P_t dw_t.$$ 

A discretized version of the set-up:

$$\Delta P_t = \mu P_t \Delta t + \sigma P_t \Delta w_t,$$

$$\Delta G_t = \left( \frac{\partial G_t}{\partial P_t} \mu P_t + \frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t + \frac{\partial G_t}{\partial P_t} \sigma P_t \Delta w_t,$$

Consider the **Portfolio**:

- short on derivative
- long $\frac{\partial G_t}{\partial P_t}$ shares of the stock.

Value of the portfolio is

$$V_t = -G_t + \frac{\partial G_t}{\partial P_t} P_t.$$
The change in value is
\[ \Delta V_t = -\Delta G_t + \frac{\partial G_t}{\partial P_t} \Delta P_t. \]
by substitution, we have
\[ \Delta V_t = \left( -\frac{\partial G_t}{\partial t} - \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t. \]

**No stochastic** component involved.

The portfolio must be riskless during a small time interval.
\[ \Delta V_t = rV_t \Delta t \]
where \( r \) is the risk-free interest rate. We then have
\[ \left( \frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t = r \left( G_t - \frac{\partial G_t}{\partial P_t} P_t \right) \Delta t. \]
and
\[ \frac{\partial G_t}{\partial t} + rP_t \frac{\partial G_t}{\partial P_t} + \frac{1}{2} \sigma^2 P_t^2 \frac{\partial^2 G_t}{\partial P_t^2} = rG_t, \]
the Black-Scholes differential equ. for derivative pricing.

**Example.** A forward contract on a stock (no dividend). Here
\[ G_t = P_t - K \exp[-r(T - t)] \]
where \( K \) is the delivery price. We have
\[ \frac{\partial G_t}{\partial t} = -rK \exp[-r(T - t)], \quad \frac{\partial G_t}{\partial P_t} = 1, \quad \frac{\partial^2 G_t}{\partial P_t^2} = 0. \]
Substituting these quantities into LHS yields
\[ -rK \exp[-r(T - t)] + rP_t = r \{ P_t - K \exp[-r(T - t)] \}, \]
which equals RHS.

**Black-Scholes formulas**

A European call option: expected payoff

\[ E^*[\max(P_T - K, 0)] \]

Price of the call: (current value)

\[ c_t = \exp[-r(T - t)]E^*[\max(P_T - K, 0)]. \]

In a risk-neutral world, \( \mu = r \) so that

\[ \ln(P_T) \sim N \left[ \ln(P_t) + \left( r - \frac{\sigma^2}{2} \right) (T - t), \sigma^2(T - t) \right]. \]

Let \( g(P_T) \) be the pdf of \( P_T \). Then,

\[ c_t = \exp[-r(T - t)] \int_K^\infty (P_T - K) g(P_T) dP_T. \]

After some algebra (appendix)

\[ c_t = P_t \Phi(h_+) - K \exp[-r(T - t)] \Phi(h_-) \]

where \( \Phi(x) \) is the CDF of \( N(0, 1) \),

\[
\begin{align*}
    h_+ &= \ln(P_t/K) + (r + \sigma^2/2)(T - t) \\
    &= \frac{\ln(P_t/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \\
    &\quad = h_+ - \sigma \sqrt{T - t}.
\end{align*}
\]

See Chapter 6 for some interpretations of the formula.

For put option:

\[ p_t = K \exp[-r(T - t)] \Phi(-h_-) - P_t \Phi(-h_+). \]
Alternatively, use the put-call parity:

\[ p_t - c_t = K \exp[-r(T - t)] - P_t. \]

**Example.** \( P_t = \$80. \sigma = 20\% \text{ per annum.} \ r = 8\% \text{ per annum.} \) What is the price of a European call option with a strike price of $90 that will expire in 3 months? From the assumptions, we have \( P_t = 80, \ K = 90, \ T - t = 0.25, \ \sigma = 0.2 \) and \( r = 0.08. \) Therefore,

\[
\begin{align*}
h_+ &= \frac{\ln(80/90) + (0.08 + 0.04/2) \times 0.25}{0.2 \sqrt{0.25}} = -0.9278 \\
h_- &= h_+ - 0.2 \sqrt{0.25} = -1.0278.
\end{align*}
\]

It can be found

\[
\Phi(-0.9278) = 0.1767, \quad \Phi(-1.0278) = 0.1520.
\]

Therefore,

\[
c_t = 80\Phi(-0.9278) - 90\Phi(-1.0278) \exp(-0.02) = \$0.73.
\]

The stock price has to rise by $10.73 for the purchaser of the call option to break even.

If \( K = \$81, \) then

\[
c_t = 80\Phi(0.125775) - 81 \exp(-0.02)\Phi(0.025775) = \$3.49.
\]

**A note on computer program:** Check the web site: http://www.cse.ucsd.edu/~goguen/courses/130/SayBlackScholes.html
Lower bounds of European options: No dividends.

\[ c_t \geq P_t - K \exp[-r(T - t)]. \]

Why?
Consider two portfolios:

- A: One European call option plus cash \( K \exp[-t(T - t)] \).
- B: One share of the stock.

For A: Invest the cash at risk-free interest rate. At time \( T \), the value is \( K \). If \( P_T > K \), the call option is exercised so that the portfolio is worth \( P_T \). If \( P_T < K \), the call option expires at \( T \) and the portfolio is worth \( K \). Therefore, the value of the portfolio is \( \max(P_T, K) \).

For B: The value at time \( T \) is \( P_T \).

Thus, portfolio A must be worth more than portfolio B today; that is,

\[ c_t + K \exp[-r(T - t)] \geq P_t. \]

See Example 6.7 for an application.

**Stochastic integral**
The formula

\[ \int_0^t dx_s = x_t - x_0 \]

continues hold. In particular,

\[ \int_0^t dw_s = w_t - w_0 = w_t. \]
From
\[ dw_t^2 = dt + 2w_t dw_t \]
we have
\[ w_t^2 = t + 2 \int_0^t w_s dw_s. \]
Therefore,
\[ \int_0^t w_s dw_s = \frac{1}{2}(w_t^2 - t). \]
Different from \( \int_0^t ydy = (y_t^2 - y_0^2)/2 \).
Assume \( x_t \) is a Geo. Brownian motion,
\[ dx_t = \mu x_t dt + \sigma x_t dw_t. \]
Apply Ito’s lemma to \( G(x_t, t) = \ln(x_t) \), we obtain
\[ d\ln(x_t) = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dw_t. \]
Taking integration, we have
\[ \int_0^t d\ln(x_s) = \left( \mu - \frac{\sigma^2}{2} \right) \int_0^t ds + \sigma \int_0^t dw_s. \]
Consequently,
\[ \ln(x_t) = \ln(x_0) + (\mu - \sigma^2/2)t + \sigma w_t, \]
and
\[ x_t = x_0 \exp\left( (\mu - \sigma^2/2)t + \sigma w_t \right). \]
Change \( x_t \) to \( P_t \). The price is
\[ P_t = P_0 \exp\left( (\mu - \sigma^2/2)t + \sigma w_t \right). \]

**Jump diffusion**

Weaknesses of diffusion models:
• no volatility smile (convex function of implied volatility vs strike price)
• fail to capture effects of rare events (tails)

Modification: jump diffusion and stochastic volatility
Jumps are governed by a probability law:
Poisson process: $X_t$ is a Poisson process if
$$Pr(X_t = m) = \frac{\lambda^m t^m}{m!} \exp(-\lambda t), \quad \lambda > 0.$$  
Use a special jump diffusion model by Kou (2002).
$$\frac{dP_t}{P_t} = \mu dt + \sigma dw_t + d\left(\sum_{i=1}^{n_t} (J_i - 1)\right),$$  
• $w_t$: a Wiener process,
• $n_t$: a Poisson process with rate $\lambda$,
• $\{J_i\}$: iid such that $X = \ln(J)$ has a double exp. dist. with pdf
$$f_X(x) = \frac{1}{2\eta} e^{-|x-\kappa|/\eta}, \quad 0 < \eta < 1.$$  
• the above three processes are independent.

$n_t$ = the number of jumps in $[0, t]$ and Poisson($\lambda t$). At the $i$th jump, the proportion of price jump is $J_i - 1$.

For pdf of double exp. dist., see Figure 6.8 of the text.
Stock price under the jump diffusion model:
$$P_t = P_0 \exp[(\mu - \sigma^2/2) t + \sigma w_t] \prod_{i=1}^{n_t} J_i.$$  

This result can be used to obtain the distribution for the return series. Price of an option: Analytical results available, but complicated.

**Example** $P_t = $80. $K = $81. $r = 0.08$ and $T - t = 0.25$. 
Jump: $\lambda = 10$, $\kappa = -0.02$ and $\eta = 0.02$.
We obtain $c_t = $3.92, which is higher than $3.49$ of Example 6.6. 
$p_t = $3.31, which is also higher.