Classification of Financial Risk

1. Credit risk
2. Market risk
3. Operational risk

Some techniques for credit risk measurement

1. Long-term credit rating (High to Low)

<table>
<thead>
<tr>
<th>S&amp;P</th>
<th>Moody</th>
<th>Fitch</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>Aaa</td>
<td>AAA</td>
</tr>
<tr>
<td>AA</td>
<td>Aa</td>
<td>AA</td>
</tr>
<tr>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>BBB</td>
<td>Baa</td>
<td>BBB</td>
</tr>
<tr>
<td>BB</td>
<td>Ba</td>
<td>BB</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>CCC</td>
<td>Caa</td>
<td>CCC</td>
</tr>
<tr>
<td>CC</td>
<td>Ca</td>
<td>CC</td>
</tr>
<tr>
<td>C</td>
<td>C</td>
<td>C</td>
</tr>
<tr>
<td>D</td>
<td>D</td>
<td>D</td>
</tr>
</tbody>
</table>

2. Credit quality over time (transition)
S&P One-year transition matrix
(Source: Standard & Poor’s, Feb. 1997)

<table>
<thead>
<tr>
<th>Ini.</th>
<th>Rating at year-end(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rat.</td>
<td>AAA</td>
</tr>
<tr>
<td>AAA</td>
<td>88.5</td>
</tr>
<tr>
<td>AA</td>
<td>0.76</td>
</tr>
<tr>
<td>A</td>
<td>0.08</td>
</tr>
<tr>
<td>BBB</td>
<td>0.03</td>
</tr>
<tr>
<td>BB</td>
<td>0.02</td>
</tr>
<tr>
<td>B</td>
<td>0.00</td>
</tr>
<tr>
<td>CCC</td>
<td>0.17</td>
</tr>
</tbody>
</table>

3. CreditMetrics (JP Morgan)

4. Altman Z score (Mainly is U.S.)

\[
Z = 3.3 \left( \frac{\text{Earnings before Interest and Taxes [EBIT]}}{\text{Total Assets}} \right) + 1.0 \left( \frac{\text{Sales}}{\text{Total Assets}} \right) + 0.6 \left( \frac{\text{Market Value of Equity}}{\text{Book Value of Debt}} \right) + 1.4 \left( \frac{\text{Retained Earnings}}{\text{Total Assets}} \right) + 1.2 \left( \frac{\text{Working Capital}}{\text{Total Assets}} \right)
\]

5. KMV Corporation’s credit risk model
We focus on the Market Risk What is Value at Risk (VaR)?

- a measure of market risk
- amount a position could decline in a given period
- associated with a given probability

A formal definition:
- time period given: $\Delta t = \ell$
- change in value: $\Delta V(\ell)$
- CDF of the change $F_\ell(x)$
- given probability: $p$
- a long position:
  $$p = Pr[\Delta V(\ell) \leq \text{VaR}] = F_\ell(\text{VaR}).$$
- a short position:
  $$p = Pr[\Delta V(\ell) \geq \text{VaR}]$$
  $$= 1 - Pr[\Delta V(\ell) \leq \text{VaR}]$$
  $$= 1 - F_\ell(\text{VaR}).$$

Quantile: $x_p$ is the $p$th quantile of $F_\ell(x)$ if
$$p = F_\ell(x_p)$$
and $F_\ell(.)$ is continuous.
Factors affect VaR:
1. the probability $p$.
2. the time horizon $\ell$.
3. data frequency.
4. the CDF $F_\ell(x)$.
5. the mark-to-market value of the position.

Why use log returns?
log returns $\approx$ percentage changes.

$\text{VaR} = \text{Value} \times (\text{VaR of log return})$.

**Methods available**

1. RiskMetrics
2. Econometric modeling
3. Empirical quantile
4. Traditional extreme value theory (EVT)
5. EVT based on exceedance over a high threshold

Data used in illustrations:
Daily log returns of IBM stock

- span: July 3, 62 to Dec. 31, 98.
- size: 9190 points
Position: long on $10 millions.

RiskMetrics

• Developed by J.P. Morgan
• \( r_t \) given \( F_{t-1} \): \( N(0, \sigma_t^2) \)
• \( \sigma_t^2 \) follows the special IGARCH(1,1) model
  \[
  \sigma_t^2 = \alpha \sigma_{t-1}^2 + (1 - \alpha) r_{t-1}^2, \quad 1 > \alpha > 0.
  \]
• VaR = 1.65\( \sigma_t \) if \( p = 0.05 \).
• \( k \)-horizon: VaR[\( k \)] = \( \sqrt{k} \)VaR
  The square root of time rule
• Pros: simplicity and transparency
• Cons: model is not adequate

Example: IBM data
Model:
\[
\begin{align*}
  r_t &= a_t, \quad a_t = \sigma_t \epsilon_t, \\
  \sigma_t^2 &= 0.9396 \sigma_{t-1}^2 + (1 - 0.9396) a_{t-1}^2
\end{align*}
\]
Because \( r_{9190} = -0.0128 \) and \( \hat{\sigma}_{9190}^2 = 0.0003472 \),
\( \hat{\sigma}_{9190}^2(1) = 0.000336 \).
For \( p = 0.05 \), VaR of \( r_t \) = \( -1.65 \times \sqrt{0.000336} = -0.03025 \)
\[
\text{VaR} = \$10,000,000 \times 0.03025 = \$302,500.
\]
For $p = 0.01$, VaR of $r_t = -2.3262 \times \sqrt{0.000336} = -0.04265$, and VaR = $426,500$.

**Econometric models**

- $r_t = \mu_t + a_t$ given $F_{t-1}$
- $\mu_t$: a mean equation (Ch.2)
- $\sigma_t^2$: a volatility model (Ch. 3 or 4)
- Pros: sound theory
- Cons: a bit complicated.

IBM data:

**Case 1:** Gaussian

\[
r_t = 0.00066 - 0.0247r_{t-2} + a_t, \quad a_t = \sigma_t \epsilon_t
\]

\[
\sigma_t^2 = 0.00000389 + 0.0799a_{t-1}^2 + 0.9073\sigma_t^2.
\]

From $r_{9189} = -0.00201$, $r_{9190} = -0.0128$ and $\sigma_{9190}^2 = 0.00033455$, we have

\[
\hat{r}_{9190}(1) = 0.00071 \quad \text{and} \quad \hat{\sigma}_{9190}^2(1) = 0.0003211.
\]

If $p = 0.05$, then

\[
0.00071 - 1.6449 \times \sqrt{0.0003211} = -0.02877.
\]

VaR = $10,000,000 \times 0.02877 = $287,700.$
If \( p = 0.01 \), then the quantile is

\[
0.00071 - 2.3262 \times \sqrt{0.0003211} = -0.0409738.
\]

VaR = $409,738.

**Case 2:** Student-\( t_5 \)

\[
r_t = 0.0003 - 0.0335r_{t-2} + a_t, \quad a_t = \sigma_t \epsilon_t
\]

\[
\sigma_t^2 = 0.000003 + 0.0559a_{t-1}^2 + 0.9350\sigma_{t-1}^2.
\]

From the data, \( r_{9189} = -0.00201 \), \( r_{9190} = -0.0128 \) and \( \sigma_{9190}^2 = 0.000349 \), we have

\[
\hat{r}_{9190}(1) = 0.000367 \quad \text{and} \quad \hat{\sigma}_{9190}^2(1) = 0.0003386.
\]

If \( p = 0.05 \), then the quantile is

\[
0.000367 - 1.5608\sqrt{0.0003386} = -0.028354.
\]

VaR = $10,000,000 \times 0.028352 = $283,520.

If \( p = 0.01 \), the quantile is

\[
0.000367 - (3.3649/\sqrt{5/3})\sqrt{0.0003386} = -0.0475943.
\]

VaR = $475,943.

**Discussion:**

- Effects of heavy-tails seen with \( p = 0.01 \).
- Multiple step-ahead forecasts are needed.
Example 7.3 (continued). 15-day horizon.
\[ \hat{r}_{9190}[15] = 0.00998 \text{ and } \sigma_t[15] = 0.0047948. \]
If \( p = 0.05 \), the quantile is \( 0.00998 - 1.6449 \sqrt{0.0047948} = -0.1039191 \).
15-day VaR = $10,000,000 \times 0.1039191 = \$1,039,191$.
RiskMetrics: VaR = $287,700 \times \sqrt{15} = \$1,114,257$.

Empirical quantile
Sample of log returns: \( \{ r_t | t = 1, \cdots, n \} \).
Order statistics:
\[ r_{(1)} \leq r_{(2)} \leq \cdots \leq r_{(n)} \]

\( r_{(i)} \) as the \( i \)th order statistic of the sample.
\( r_{(1)} \) is the sample minimum
\( r_{(n)} \) the sample maximum.
Idea: Use the empirical quantile to estimate the theoretical quantile of \( r_t \).
For a given probability \( p \), what is the empirical quantile?
If \( np = \ell \) is an integer, then it is \( r_{(\ell)} \).
If \( np \) is not an integer, find the two neighboring integers \( \ell_1 < np < \ell_2 \) and use interpolation.
The quantile is
\[ \hat{x}_p = \frac{p - p_1}{p_2 - p_1} r_{(\ell_1)} + \frac{p_2 - p}{p_2 - p_1} r_{(\ell_2)}. \]
IBM data:
\( n = 9190 \). If \( p = 0.05 \), then \( np = 459.5 \).
5% quantile is \((r_{459} + r_{460})/2 = -0.021603\).
VaR = $216,030.

If \(p = 0.01\), then \(np = 91.9\) and the 1% quantile is
\[
\hat{x}_{0.01} = \frac{p_1 - 0.01}{p_2 - p_1} r_{(91)} + \frac{p_2 - 0.01}{p_2 - p_1} r_{(92)}
\]
\[
= \frac{0.0001}{0.00011}(-3.658) + \frac{0.00001}{0.00011}(-3.657)
\]
\[
\approx -3.658.
\]

VaR is $365,800.

**Extreme value theory**: Focus on the tail behavior of \(r_t\).

**Review of extreme value theory**

A properly normalized \(r_{(1)}\) assumes a special distribution:

\[
F_*(x) = \begin{cases} 
1 - \exp[-(1 + kx)^{1/k}] & \text{if } k \neq 0 \\
1 - \exp[- \exp(x)] & \text{if } k = 0 
\end{cases}
\]

for \(x < -1/k\) if \(k < 0\) and for \(x > -1/k\) if \(k > 0\).

\(k\): the *shape parameter*

\(\alpha = -1/k\): tail index of the distribution.

Classification of distributions:

- **Type I**: \(k = 0\), the Gumbel family. The CDF is
  \[
  F_*(x) = 1 - \exp[- \exp(x)], \quad -\infty < x < \infty. \quad (1)
  \]
• Type II: $k < 0$, the Fréchet family. The CDF is
\[
F_*(x) = \begin{cases} 
1 - \exp[-(1 + kx)^{1/k}] & \text{if } x < -1/k \\
1 & \text{otherwise}
\end{cases}
\] (2)

• Type III: $k > 0$, the Weibull family. The CDF here is
\[
F_*(x) = \begin{cases} 
1 - \exp[-(1 + kx)^{1/k}] & \text{if } x > -1/k \\
0 & \text{otherwise}
\end{cases}
\]

The probability density function (pdf) of the normalized minimum is
\[
f_*(x) = \begin{cases} 
(1 + kx)^{1/k-1} \exp[-(1 + kx)^{1/k}] & \text{if } k \neq 0 \\
\exp[x - \exp(x)] & \text{if } k = 0
\end{cases}
\]
where $-\infty < x < \infty$ for $k = 0$, $x < -1/k$ for $k < 0$ and $x > -1/k$ for $k > 0$.

How to use the EVT distribution?
If we know the three parameters, we can compute the quantile!

**Empirical estimation**

Divide the sample into non-overlapping subsamples.
Suppose there are $T$ data points, we devide the data as
\[
\{r_1, \cdots, r_n| r_{n+1}, \cdots, r_{2n}| r_{2n+1}, \cdots, r_{3n}| \cdots | r_{(g-1)n+1}, \cdots, r_{ng}\},
\]
n: size of subgroup
Idea: find the minimum of each subgroup. These minima are the data used to estimate the three parameters.
Several estimation methods available. We use maximum likelihood estimates.

IBM data:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$g$</th>
<th>Scale $\alpha_n$</th>
<th>Location $\beta_n$</th>
<th>Shape Par. $k_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) Minimal returns</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>437</td>
<td>0.823(0.035)</td>
<td>-1.902(0.044)</td>
<td>-0.197(0.036)</td>
</tr>
<tr>
<td>63</td>
<td>145</td>
<td>0.945(0.077)</td>
<td>-2.583(0.090)</td>
<td>-0.335(0.076)</td>
</tr>
<tr>
<td>126</td>
<td>72</td>
<td>1.147(0.131)</td>
<td>-3.141(0.153)</td>
<td>-0.330(0.101)</td>
</tr>
<tr>
<td>252</td>
<td>36</td>
<td>1.542(0.242)</td>
<td>-3.761(0.285)</td>
<td>-0.322(0.127)</td>
</tr>
<tr>
<td>(b) Maximal returns</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>437</td>
<td>0.931(0.039)</td>
<td>2.184(0.050)</td>
<td>-0.168(0.036)</td>
</tr>
<tr>
<td>63</td>
<td>145</td>
<td>1.157(0.087)</td>
<td>3.012(0.108)</td>
<td>-0.217(0.066)</td>
</tr>
<tr>
<td>126</td>
<td>72</td>
<td>1.292(0.158)</td>
<td>3.471(0.181)</td>
<td>-0.349(0.130)</td>
</tr>
<tr>
<td>252</td>
<td>36</td>
<td>1.624(0.271)</td>
<td>4.475(0.325)</td>
<td>-0.264(0.186)</td>
</tr>
</tbody>
</table>

EVT to VaR: Use a two-step procedure, because of the division into subgroup.

VaR for $r_t$:

$$\text{VaR} = \begin{cases} 
\beta_n - \frac{\alpha_n}{k_n} \{1 - [-n \ln(1 - p)]^{k_n}\} & \text{if } k_n \neq 0 \\
\beta_n + \alpha_n \ln[-n \ln(1 - p)] & \text{if } k_n = 0.
\end{cases}$$

For IBM data, if $n = 63$ (quarterly minima), then $\hat{\alpha}_n = 0.945$, $\hat{\beta}_n = -2.583$, and $\hat{k}_n = -0.335$. If $p = 0.01$, the VaR is

$$\text{VaR} = -2.583 - \frac{0.945}{-0.335} \{1 - [-63 \ln(1 - 0.01)]^{-0.335}\}$$

11
= -3.04969

VaR is $304,969.

If $p = 0.05$, then VaR is $166,641$.

For $n = 21$, the results are:

VaR = $340,013$ for $p = 0.01$;
VaR = $184,127$ for $p = 0.05$.

**Discussion:**

- Results depend on the choice of $n$

- VaR seems low, but it might be due to the choice of $p$.
  
  If $p = 0.001$, then
  
  VaR = $546,641$ for the Gaussian AR(2)-GARCH(1,1) model
  VaR = $666,590$ for the extreme value theory with $n = 21$.

**Summary** of IBM data:

Position = $10$ millions.

If $p = 0.05$, then

1. $302,500$ for the RiskMetrics,
2. $287,200$ for an AR(2)-GARCH(1,1) model,
3. $283,520$ for an AR(2)-GARCH(1,1) with $t_5$
4. $216,030$ using the empirical quantile, and
5. $184,127$ for EVT with $n = 21$. 
\( p = 0.01, \) then

1. $426,500 for the RiskMetrics,
2. $409,738 for an AR(2)-GARCH(1,1) model,
3. $475,943 for an AR(2)-GARCH(1,1) model with \( t_5 \)
4. $365,800 for empirical quantile, and
5. $340,013 for EVT with \( n = 21 \).

If \( p = 0.001, \) then

1. $566,443 for the RiskMetrics,
2. $546,641 for an AR(2)-GARCH(1,1) model,
3. $836,341 for an AR(2)-GARCH(1,1) model with \( t_5 \)
4. $780,712 for quantile, and
5. $666,590 for EVT with \( n = 21 \).

**Multi-period VaR** with EVT

\[
\text{VaR}(\ell) = \ell^{1/\alpha}\text{VaR} = \ell^{-k}\text{VaR}
\]

where \( \alpha \) is the tail index and \( k \) is the shape parameter.

For IBM data with \( p = 0.05, \)

\[
\text{VaR}(30) = (30)^{0.335}\text{VaR} = 3.125 \times 184,127 = 575,397.
\]
New approach to VaR
Based on Exceedances over a high threshold
Idea: frequency of big returns and their magnitudes are important.
Statistical theory:
Two-dimensional Poisson process
Two possible cases:
Homogeneous: parameters are fixed over time
Non-homogeneous case: parameters are time-varying, according to some explanatory variables.
IBM data: homogeneous model

<table>
<thead>
<tr>
<th>Thr.</th>
<th>Exc.</th>
<th>Shape Par. $k$</th>
<th>Log(Scale) ln($\alpha$)</th>
<th>Location $\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(a) Original log returns</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.0%</td>
<td>175</td>
<td>$-0.30697(0.09015)$</td>
<td>$0.30699(0.12380)$</td>
<td>$4.69204(0.19058)$</td>
</tr>
<tr>
<td>2.5%</td>
<td>310</td>
<td>$-0.26418(0.06501)$</td>
<td>$0.31529(0.11277)$</td>
<td>$4.74062(0.18041)$</td>
</tr>
<tr>
<td>2.0%</td>
<td>554</td>
<td>$-0.18751(0.04394)$</td>
<td>$0.27655(0.09867)$</td>
<td>$4.81003(0.17209)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(b) Removing the sample mean</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.0%</td>
<td>184</td>
<td>$-0.30516(0.08824)$</td>
<td>$0.30807(0.12395)$</td>
<td>$4.73804(0.19151)$</td>
</tr>
<tr>
<td>2.5%</td>
<td>334</td>
<td>$-0.28179(0.06737)$</td>
<td>$0.31968(0.12065)$</td>
<td>$4.76808(0.18533)$</td>
</tr>
<tr>
<td>2.0%</td>
<td>590</td>
<td>$-0.19260(0.04357)$</td>
<td>$0.27917(0.09913)$</td>
<td>$4.84859(0.17255)$</td>
</tr>
</tbody>
</table>

VaR calculation:
\[
\text{VaR} = \begin{cases} 
\beta + \frac{\alpha}{k} \left\{1 - \left[-T \ln(1 - p)\right]^k\right\} & \text{if } k \neq 0 \\
\beta + \alpha \ln[-T \ln(1 - p)] & \text{if } k = 0
\end{cases}
\]
where $T = 252$, the number trading days in a year.

IBM data: VaR of 5% & 1%

- Case I: original returns
  1. $\eta = 3.0\%$: $228,239 & 359.303$.
  2. $\eta = 2.5\%$: $219,106 & 361,119$.
  3. $\eta = 2.0\%$: $212,981 & 368,552$.

- Case II: remove sample mean
  1. $\eta = 3.0\%$: $232,094 & 363,697$.
  2. $\eta = 2.5\%$: $225,782 & 364,254$.

Non-homogeneous case:

\[
 k_t = \gamma_0 + \gamma_1 x_{1t} + \cdots + \gamma_v x_{vt} \equiv \gamma_0 + \gamma' x_t \\
 \ln(\alpha_t) = \delta_0 + \delta_1 x_{1t} + \cdots + \delta_v x_{vt} \equiv \delta_0 + \delta' x_t \\
 \beta_t = \theta_0 + \theta_1 x_{1t} + \cdots + \theta_v x_{vt} \equiv \theta_0 + \theta' x_t.
\]

For IBM data, explanatory variables include past volatilities, etc.

See Chapter 7 for more details and estimation results.

Illustration:

For December 31, 1998, we have $x_{3,9190} = 0$, $x_{4,9190} = 0.9737$ and $x_{5,9190} = 1.9766$. The parameters become

\[
k_{9190} = -0.01195, \quad \ln(\alpha_{9190}) = 0.19331, \quad \beta_{9190} = 6.105.
\]
If $p = 0.05$, then quantile = 3.03756\% and
\[
\text{VaR} = \$10,000,000 \times 0.0303756 = \$303,756.
\]

If $p = 0.01$, then Var is $497,425$.

For December 30, 1998, we have $x_{3,9189} = 1$, $x_{4,9189} = 0.9737$ and $x_{5,9189} = 1.8757$ and
\[
k_{9189} = -0.2500, \quad \ln(\alpha_{9189}) = 0.52385, \quad \beta_{9189} = 5.8834.
\]
The 5\% VaR becomes
\[
\text{VaR} = \$10,000,000 \times 0.0269139 = \$269,139.
\]

If $p = 0.01$, then VaR becomes $448,323$.

**R and S-Plus Demonstration:**

Both packages use the library: \texttt{evir}. For R, download the library first before using the commands.

*** Please note that the shape parameter "k" of chapter 7 is denoted by "minus xi" in R and S-Plus.

*** Also, the program ‘‘evir’’ uses maxima (right tail) so that one should use ‘‘minus returns’’ for the left tail.

***

(* Command line starts with > *)

(* Output is edited to simplify the handout *)

(* Generate CDF of Weibull, Frechet, & Gumbel dists *)

> z=seq(-5,5,length=200)
just get a sequence of numbers in \([-5,5]\), equally spaced.

\> z[1:10]

\[
\begin{array}{cccccccc}
1 & 5.0000 & -4.94975 & -4.89949 & -4.84925 & -4.79899 & -4.74874 \\
7 & -4.698492 & -4.648241 & -4.597990 & -4.547739 \\
\end{array}
\]

(* Use the command "pgev" to obtain the probability
(CDF) of generalized extreme value dist. *)

\> cdf.f=ifelse((z > -2),pgev(z,xi=0.5),0)
   \quad \leqslant \text{Frechet dist for } z > -2 \text{ only, because } xi=0.5. \\
\> cdf.w=ifelse((z < 2), pgev(z,xi=-0.5),1)
   \quad \leqslant \text{Weibull dist for } z < 2 \text{ only.} \\
\> cdf.g=exp(-exp(-z))

\> plot(z,cdf.w,type='l',xlab='z',ylab='H(z)')
\> lines(z,cdf.f,type='l',lty=2)
\> lines(z,cdf.g,lty=3)
\> legend(-5,1,legend=c("Weibull H(-0.5,0,1)",
   "Frechet H(0.5,0.1)","Gumbel H(0,0,1)"),lty=1:3)

(* Use the command "dgev" to obtain the pdf of
generalized extreme value dist *)

\> pdf.f=ifelse((z > -2),dgev(z,xi=0.5),0)
\> pdf.w=ifelse((z < 2),dgev(z,xi=-0.5),0)
\> pdf.g=exp(-exp(-z))*exp(-z)

\> plot(z,pdf.w,type='l',xlab='z',ylab='density')
\> lines(z,pdf.f,lty=2)
\> lines(z,pdf.g,lty=3)
\> legend(-5.25,0.4,legend=c("Weibull H(-0.5,0,1)",
   "Frechet H(0.5,0,1)","Gumbel H(0,0,1)"),lty=1:3)
(* Other related commands include "qgev" and "rgev" that gives quantiles and generates random variates from generalized extreme value distribution. The parameters must be given in using these two commands. *)

(* For example, to obtain the 95th quantile, use below *)

> qgev(0.95, xi=0.5, mu=0, sigma=1)
[1] 6.830793

**Example**: daily log returns, in percentages, of IBM stock: 1962 to 1998.

> library(evir)
> da=read.table("d-ibmln98.dat")
> ibm=da[,1]
> plot(ibm,type='l')
> qqnorm(ibm) % normal probability plot

> nibm=-ibm % Focus on the left tail.

> m1=gev(nibm,block=21) % fit gen. extreme value dist.
> m1
$n.all
[1] 9190
$n
[1] 438
$data
.....
$block
[1] 21

$par.est
  xi     sigma     mu
0.1956199 0.8239793 1.9031998

$par.ses
  xi     sigma     mu
0.03554473 0.03476737 0.04413629

$varcov
   [,1]        [,2]       [,3]
[1,] 1.263428e-03 -2.782725e-05 -0.0004338483
[2,] -2.782725e-05 1.208770e-03 0.0008475859
[3,] -4.338483e-04 8.475859e-04 0.0019480124

$converged
[1] 0

$nllh.final
[1] 654.3337
attr(,"class")
[1] "gev"

> names(m1)
[1] "n.all" "n" "data" "block" "par.est" [6] "par.ses" "varcov" "converged" "nllh.final"

> m1$n   % numbers of monthly maximum
> ymax=m1$data
> hist(ymax)
> ysort=sort(ymax)
> plot(ysort,-log(-log(ppoints(ysort))),xlab='Monthly maximum')
   # Gumbel qq-plot

> plot(m1) # Model checking plots
Make a plot selection (or 0 to exit):
1: plot: Scatterplot of Residuals
2: plot: QQplot of Residuals
Selection: 1

Make a plot selection (or 0 to exit):
1: plot: Scatterplot of Residuals
2: plot: QQplot of Residuals
Selection: 2

Make a plot selection (or 0 to exit):
1: plot: Scatterplot of Residuals
2: plot: QQplot of Residuals
Selection: 0

> 1-pgev(max(ymax),xi=.196,mu=1.90,sigma=.824)
[1] 5.857486e-05 # Prob. that the drop will exceed the maximum.

> rlevel.gev(m1,k.blocks=36) #return level & its 95% conf. interval.
** Peak Over the Threshold approach (homogeneous case).

> meplot(nibm)  # mean excess plot

> m2=gpd(nibm,threshold=2.5)
> names(m2)
 [1] "n"       "data"    "threshold" "p.less.thresh"
[5] "n.exceed" "method"  "par.ests" "par.ses"
[9] "varcov"   "information" "converged" "nllh.final"

> m2$threshold
[1] 2.5
> m2$n.exceed
[1] 310

> m2  # Obtain all output
$n
[1] 9190
$data

$threshold
[1] 2.5
$p.less.thresh
[1] 0.9662677
$n.exceed
[1] 310
$method
[1] "ml"

$par.est

   xi   beta
0.2641593 0.7786761

$par.se

   xi   beta
0.06659234 0.06714131

$varcov

   [,1]       [,2]
[1,] 0.004434540 -0.002614442
[2,] -0.002614442 0.004507955

$info

[1] "observed"

$converged

[1] 0

$nllh.final

[1] 314.375

attr(,"class")

[1] "gpd"

> plot(m2) # Model checking. Should see all plots.

Make a plot selection (or 0 to exit):
1: plot: Excess Distribution
2: plot: Tail of Underlying Distribution
3: plot: Scatterplot of Residuals
4: plot: QQplot of Residuals
Selection: 1
[1] "threshold = 2.5 xi = 0.264 scale = 0.779 location= 2.5"

Make a plot selection (or 0 to exit):
1: plot: Excess Distribution
2: plot: Tail of Underlying Distribution
3: plot: Scatterplot of Residuals
4: plot: QQplot of Residuals
Selection: 0

> shape(nibm)  % A plot showing the stability of the estimates.

> riskmeasures(m2,c(0.95,0.99))  % Compute VaR and expected shortfall.
   p quantile    sfall
[1,] 0.95  2.208932  3.162654
[2,] 0.99  3.616487  5.075507

Additional information on applying extreme value theory to value at risk calculation.

To traditional approach of EVT

**Return Level**: It is a risk measure based on the idea of subperiods. The $g \ n$-subperiod return level, $L_{n,g}$, is the level that is exceeded in one out of every $g$ subperiods of length $n$.

$$P(r_{n,i} < L_{n,g}) = \frac{1}{g},$$

where $n$ is the length of subperiod used in estimating the GEV model
and \( r_{n,i} \) denotes subperiod minimum. For sufficiently large \( n \),

\[
L_{n,g} = \beta_n + \frac{\alpha_n}{k_n} \{[-\ln(1 - 1/g)]^{k_n} - 1\},
\]

where the shape parameter \( k_n \neq 0 \).

For a short position, the return level is

\[
L_{n,g} = \beta_n + \frac{\alpha_n}{k_n} \{1 - [-\ln(1 - 1/g)]^{1/k_n}\}.
\]

Peaks over Threshold

**Generalized Pareto Distribution**: For simplicity, assume that the shape parameter \( k \neq 0 \). Consider the extreme value distribution of maximum (Eq. (7.29) of the textbook)

\[
F_*(r) = \exp \left[- \left(1 - \frac{k(r - \beta)}{\alpha} \right)^{1/k}\right].
\]

The distribution of \( r \leq x + \eta \) given \( \eta \), where \( x \geq 0 \), is

\[
\text{Pr}(r \leq x + \eta | r > \eta) \approx 1 - \left(1 - \frac{kx}{\psi(\eta)} \right)^{1/k},
\]

where \( \psi(\eta) = \alpha - k(\eta - \beta) \), which depends on \( \eta \).

The distribution with cumulative distribution function

\[
G(x) = 1 - \left[1 - \frac{kx}{\psi(\eta)} \right]^{1/k},
\]

is called a generalized Pareto distribution (GPD).

**Selection of the high threshold**
**Mean Excess:** Given a high threshold \( \eta_o \), suppose the excess \( r - \eta_o \) follows a GPD with parameter \( k \) and \( \psi(\eta_o) \), where \( 0 > k > -1 \). Then the mean excess over the threshold is

\[
E(r - \eta_o | r > \eta_o) = \frac{\psi(\eta_o)}{1 + k}.
\]

For any \( \eta > \eta_o \), the mean excess function is defined as

\[
e(\eta) = E(r - \eta | r > \eta) = \frac{\psi(\eta_o) - k(\eta - \eta_o)}{1 + k}.
\]

The fact that, for a given \( k \), \( e(\eta) \) is a linear function of \( \eta \), where \( \eta > \eta_o \), provides a simple method to infer the threshold \( \eta_o \) for GPD.

Define the empirical mean excess as

\[
e_T(\eta) = \frac{1}{N_\eta} \sum_{i=1}^{N_\eta} (r_{t_i} - \eta),
\]

where \( N_\eta \) is the number of returns that exceed \( \eta \) and \( r_{t_i} \) are the values of the corresponding returns.

The scatterplot \( e_T(\eta) \) versus \( \eta \) is called the mean excess plot, which should be linear for \( \eta > \eta_o \).

In R or S-Plus, the command is **meplot**.

**Use of GPD in VaR**

For a given threshold, estimate GPD to obtain parameters \( k \) and \( \psi(\eta) \). Check the adequacy of the fit; see demonstration. Provided that the model is adequate, the VaR can be computed by

\[
\text{VaR}_q = \eta + \frac{\psi(\eta)}{k} \left\{ 1 - \left[ \frac{T}{N_\eta} (1 - q) \right]^k \right\},
\]
where \( q = 1 - p \) with \( 0 < p < 0.05 \), \( T \) is the sample size and \( N_\eta \) is the number of exceedances.

Alternatively, one can use the formula in Eq. (7.38) of the textbook when one treats the exceedances and exceeding times as a two-dimensional Poisson process. The VaR results obtained are close.

**Expected Shortfall** (ES): the expected loss given that the VaR is exceeded. Specifically,

\[
EES_q = E(r|r > \text{VaR}_q) = \text{VaR}_q + E(r - \text{VaR}_q|r > \text{VaR}_q).
\]

For GPD, it turns out that

\[
ES_q = \frac{\text{VaR}_q}{1 + k} + \frac{\psi(\eta) + k\eta}{1 + k}.
\]

In R or S-Plus, the command is `riskmeasures`.

Let \( r_{n,i} \) be the maximum of a subperiod of length \( n \). Under the traditional EVT, \( r_{n,i} \) follows a generalized extreme value distribution with parameter \((\xi, \sigma, \mu)\).

What is the relationship between quantile of \( r_{n,i} \) and the return \( r_t \)? Let \( Q \) be a real number.

\[
P(r_{n,i} > Q) = 1 - P(r_{n,i} \leq Q)
= 1 - P(\text{all } r_t \text{ in the subperiod } \leq Q)
= 1 - \prod_{t=1}^{n} P(r_t \leq Q) \quad \text{(use independence)}
= 1 - [P(r_t \leq Q)]^n \quad \text{(because of same distribution)}
\]
Consequently, let $p$ be a small upper tail probability of $r_t$ and $Q$ be the corresponding quantile. That is,

$$P(r_t \leq Q) = 1 - p$$

From the above equation, we have

$$P(r_{n,i} > Q) = 1 - (1 - p)^n.$$ 

Therefore,

$$P(r_{n,i} \leq Q) = 1 - P(r_{n,i} > Q) = (1 - p)^n.$$ 

This means that $Q$ is the $(1 - p)^n$-th quantile of the generalized extreme value distribution.

Takeaway: For a small probability $p$, compute $(1 - p)^n$, where $n$ is the length of subperiod, then VaR can be obtained by finding the $(1 - p)^n$-th quantile of the extreme value distribution.