Lecture Note of Bus 41202, Spring 2008:
Stochastic Diffusion & Option Pricing

Key concept: Ito’s lemma

Stock Options:
- A contract giving its holder the right, but not obligation, to trade
  shares of a common stock by a certain date for a specified price.
- Call option: to buy
- Put option: to sell
- Specified price: strike price $K$
- date: expiration $T$ (measured in years)

Note: You can also write a call or put option (underwrite).

Factors affecting the price of an option
- Current stock price: $P_t$
- time to expiration: $T - t$
- Risk-free interest rate: $r$ per annum
- Stock volatility: $\sigma$ annualized

Payoff for European options (exercised at $T$ only)

Call option:

$$V(P_T) = (P_T - K)_+ = \begin{cases} 
P_T - K & \text{if } P_T > K \\
0 & \text{if } P_T \leq K 
\end{cases}$$
The holder only exercises her option if $P_T > K$ (buys the stock via exercising the option and sells the stock on the market).

**Put option:**

$$V(P_T) = (K - P_T)_+ = \begin{cases} 
K - P_T & \text{if } P_T < K \\
0 & \text{if } P_T \geq K
\end{cases}$$

The holder only exercises her option if $P_T < K$ (buys the stock from the market and sells it via option).

**Mathematical framework**

- Stock (log) price follows a diffusion equation, i.e. a continuous-time continuous stochastic process such as

$$dx_t = \mu(x_t, t)dt + \sigma(x_t, t)dw_t,$$

where $\mu(x_t, t)$ and $\sigma(x_t, t)$ are the drift and diffusion coefficient, respectively, and $w_t$ is a standard Brownian motion (or Wiener process).

- In a complete market, use hedging to derive the price of an option (no arbitrage argument).

- In an incomplete market (e.g. existence of jumps), specify risk and a hedging strategy to minimize the risk.

**Stochastic processes**

- Wiener process (or Standard Brownian motion)
  - notation: $w_t$
– initial value: \( w_0 = 0 \)
– small increments are independent and normal

\[
\{ \Delta w_i = w_{t_i} - w_{t_{i-1}} \} \text{ are independent}
\]
\[
\Delta w_t = w_{t+\Delta t} - w_t \sim N(0, \Delta t).
\]
– property: \( w_t \sim N(0, t) \)
– zero drift and rate of variance change is 1.
– A simple way to understand Wiener processes is to do simulation. In R or S-Plus, this can be achieved by using:

\[
\begin{align*}
\text{n=5000} \\
\text{at = rnorm(n)} \\
\text{wt = cumsum(at)/sqrt(n)} \\
\text{plot(wt,type='l')}
\end{align*}
\]

Repeat the above commands to generate lots of “wt” series.

- Generalized Wiener process

\[
dx_t = \mu dt + \sigma dw_t,
\]
where the drift \( \mu \) & rate of volatility change \( \sigma \) are constant.

- Ito’s process

\[
dx_t = \mu(x_t, t) dt + \sigma(x_t, t) dw_t,
\]
where both drift and volatility are time-varying.
Figure 1: Time plots of four simulated Wiener processes

- **Geometric Brownian motion**

\[ dP_t = \mu P_t dt + \sigma P_t dw_t, \]

so that \( \mu(P_t, t) = \mu P_t \) and \( \sigma(P_t, t) = \sigma P_t \) with \( \mu \) and \( \sigma \) being constant.

**Illustration**: Four simulated standard Brownian motions. key feature: variability increases with time.

Assume that the price of a stock follows a geometric Brownian motion. What is the distribution of the log return?
To answer this question, we need Ito’s calculus.

Review of differentiation

$G(x)$: a differentiable function of $x$.

What is $dG(x)$?

Taylor expansion:

$$
\Delta G \equiv G(x + \Delta x) - G(x) = \frac{\partial G}{\partial x} \Delta x
$$

$$
+ \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (\Delta x)^2 + \frac{1}{6} \frac{\partial^3 G}{\partial x^3} (\Delta x)^3 + \cdots.
$$

Letting $\Delta x \to 0$, we have

$$
dG = \frac{\partial G}{\partial x} dx.
$$

How about $G(x, y)$?

$$
\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial y} \Delta y
$$

$$
+ \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (\Delta x)^2 + \frac{\partial^2 G}{\partial x \partial y} \Delta x \Delta y + \frac{1}{2} \frac{\partial^2 G}{\partial y^2} (\Delta y)^2 + \cdots.
$$

Taking limit as $\Delta x \to 0$ and $\Delta y \to 0$, we have

$$
dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial y} dy.
$$

Stochastic differentiation

Now, consider $G(x_t, t)$ with $x_t$ being an Ito’s process.

$$
\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t
$$

$$
+ \frac{1}{2} \frac{\partial^2 G}{\partial x^2} (\Delta x)^2 + \frac{\partial^2 G}{\partial x \partial t} \Delta x \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} (\Delta t)^2 + \cdots.
$$
A discretized version of the Ito’s process is

$$\Delta x = \mu_* \Delta t + \sigma_* \epsilon \sqrt{\Delta t},$$

where $\mu_* = \mu(x_t, t)$ and $\sigma_* = \sigma(x_t, t)$. Therefore,

$$(\Delta x)^2 = \mu_*^2 (\Delta t)^2 + \sigma_*^2 \epsilon^2 \Delta t + 2 \mu_* \sigma_* \epsilon (\Delta t)^{3/2}$$

$$= \sigma_*^2 \epsilon^2 \Delta t + H(\Delta t).$$

Thus, $(\Delta x)^2$ contains a term of order $\Delta t$.

$$E(\sigma_*^2 \epsilon^2 \Delta t) = \sigma_*^2 \Delta t,$$

$$\text{Var}(\sigma_*^2 \epsilon^2 \Delta t) = E[\sigma_*^4 \epsilon^4 (\Delta t)^2] - [E(\sigma_*^2 \epsilon^2 \Delta t)]^2 = 2\sigma_*^4 (\Delta t)^2,$$

where we use $E(\epsilon^4) = 3$. These two properties show that

$$\sigma_*^2 \epsilon^2 \Delta t \to \sigma_*^2 \Delta t \quad \text{as} \quad \Delta t \to 0.$$

Consequently,

$$(\Delta x)^2 \to \sigma_*^2 dt \quad \text{as} \quad \Delta t \to 0.$$

Using this result, we have

$$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \sigma_*^2 dt$$

$$= \left( \frac{\partial G}{\partial x} \mu_* + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \sigma_*^2 \right) dt + \frac{\partial G}{\partial x} \sigma_* dw_t.$$  

This is the well-known Ito’s lemma.

**Example.** Let $G(w_t, t) = w_t^2$. What is $dG(w_t, t)$?
Answer: Here $\mu_* = 0$ and $\sigma_* = 1$.

$$\frac{\partial G}{\partial w_t} = 2w_t, \quad \frac{\partial G}{\partial t} = 0, \quad \frac{\partial^2 G}{\partial w_t^2} = 2.$$ 

Therefore,

$$dw_t^2 = (2w_t \times 0 + 0 + \frac{1}{2} \times 2 \times 1)dt + 2w_t dw_t = dt + 2w_t dw_t.$$ 

If $P_t$ follows a geometric Brownian motion, what is the model for $\ln(P_t)$?

Answer: Let $G(P_t, t) = \ln(P_t)$. we have

$$\frac{\partial G}{\partial P_t} = \frac{1}{P_t}, \quad \frac{\partial G}{\partial t} = 0, \quad \frac{1}{2} \frac{\partial^2 G}{\partial P_t^2} = \frac{1}{2} \frac{1-1}{P_t^2}.$$ 

Consequently, via Ito’s lemma, we obtain

$$d \ln(P_t) = \left( \frac{1}{P_t} \mu P_t + \frac{1}{2} \frac{1-1}{P_t^2} \sigma^2 P_t^2 \right) dt + \frac{1}{P_t} \sigma P_t dw_t$$

$$= \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dw_t.$$ 

Thus, $\ln(P_t)$ follows a generalized Wiener Process with drift rate $\mu - \sigma^2/2$ and variance rate $\sigma^2$.

The log return from $t$ to $T$ is normal with mean $(\mu - \sigma^2/2)(T - t)$ and variance $\sigma^2(T - t)$.

Estimation of $\mu$ and $\sigma$  
Assume that $n$ log returns are available, say $\{r_t | t = 1, \cdots, n\}$.  

Statistical theory:
Estimate the mean and variance by the sample mean and variance.

\[ \bar{r} = \frac{\sum_{t=1}^{n} r_t}{n}, \]
\[ s_r^2 = \frac{1}{n-1} \sum_{t=1}^{n} (r_t - \bar{r})^2. \]

Remember the length of time intervals!
Let \( \Delta \) be the length of time intervals measured in years.
Then, the distribution of \( r_t \) is

\[ r_t \sim N[(\mu - \sigma^2/2)\Delta, \sigma^2\Delta]. \]

We obtain the estimates

\[ \hat{\sigma} = \frac{s_r}{\sqrt{\Delta}}, \]
\[ \hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = \frac{\bar{r}}{\Delta} + \frac{s_r^2}{2\Delta}. \]

**Example.** Daily log returns of IBM stock in 1998.
The data show \( \bar{r} = 0.002276 \) and \( s_r = 0.01915 \).
Since \( \Delta = 1/252 \) year, we obtain that

\[ \hat{\sigma} = \frac{s_r}{\sqrt{\Delta}} = 0.3040, \quad \hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = 0.6198. \]

Thus, the estimated expected return was 61.98% and the standard deviation was 30.4% per annum for IBM stock in 1998.

**Example.** Daily log returns of Cisco stock in 1999.
Data show \( \bar{r} = 0.00332 \) and \( s_r = 0.026303 \),
Also, $Q(12) = 10.8$. Therefore, we have

$$\hat{\sigma} = \frac{s_r}{\sqrt{\Delta}} = \frac{0.026303}{\sqrt{1.0/252.0}} = 0.418, \quad \hat{\mu} = \frac{\bar{r}}{\Delta} + \frac{\hat{\sigma}^2}{2} = 0.924.$$ 

Expected return was 92.4% per annum
Estimated s.d. was 41.8% per annum.

Data show $\bar{r} = -0.00301$ and $s_r = 0.05192$.
Therefore, $\hat{\sigma} = 0.818 \hat{\mu} = -0.412$.
Time-varying nature of mean and volatility is clearly shown.

**Distributions of stock prices**
If the price follows

$$dP_t = \mu P_t dt + \sigma P_t dw_t,$$

then,

$$\ln(P_T) - \ln(P_t) \sim N \left[ (\mu - \frac{\sigma^2}{2})(T - t), \sigma^2(T - t) \right].$$

Consequently, given $P_t$,

$$\ln(P_T) \sim N \left[ \ln(P_t) + (\mu - \frac{\sigma^2}{2})(T - t), \sigma^2(T - t) \right],$$

and we obtain (log-normal dist; ch. 1)

$$E(P_T) = P_t \exp[\mu(T - t)],$$
$$\text{Var}(P_T) = P_t^2 \exp[2\mu(T - t)] \{ \exp[\sigma^2(T - t)] - 1 \}.$$
The result can be used to make inference about $P_T$. Simulation is often used to study the behavior of $P_T$.

**Black-Scholes equation**

- Price of stock: $P_t$ is a Geo. B. Motion
- Price of derivative: $G_t = G(P_t, t)$ contingent the stock
- Risk neutral world: expected returns are given by the risk-free interest rate (no arbitrage)

From Ito’s lemma:

$$dG_t = \left( \frac{\partial G_t}{\partial P_t} \mu P_t + \frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) dt + \frac{\partial G_t}{\partial P_t} \sigma P_t dw_t.$$

A discretized version of the set-up:

$$\Delta P_t = \mu P_t \Delta t + \sigma P_t \Delta w_t,$$

$$\Delta G_t = \left( \frac{\partial G_t}{\partial P_t} \mu P_t + \frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t + \frac{\partial G_t}{\partial P_t} \sigma P_t \Delta w_t,$$

Consider the Portfolio:

- short on derivative
- long $\frac{\partial G_t}{\partial P_t}$ shares of the stock.

Value of the portfolio is

$$V_t = -G_t + \frac{\partial G_t}{\partial P_t} P_t.$$
The change in value is

$$\Delta V_t = -\Delta G_t + \frac{\partial G_t}{\partial P_t} \Delta P_t.$$ 

by substitution, we have

$$\Delta V_t = \left( -\frac{\partial G_t}{\partial t} - \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t.$$ 

**No stochastic** component involved.

The portfolio must be riskless during a small time interval.

$$\Delta V_t = rV_t \Delta t$$

where $r$ is the risk-free interest rate. We then have

$$\left( \frac{\partial G_t}{\partial t} + \frac{1}{2} \frac{\partial^2 G_t}{\partial P_t^2} \sigma^2 P_t^2 \right) \Delta t = r \left( G_t - \frac{\partial G_t}{\partial P_t} P_t \right) \Delta t.$$ 

and

$$\frac{\partial G_t}{\partial t} + r P_t \frac{\partial G_t}{\partial P_t} + \frac{1}{2} \sigma^2 P_t^2 \frac{\partial^2 G_t}{\partial P_t^2} = r G_t,$$

the Black-Scholes differential equ. for derivative pricing.

**Example.** A forward contract on a stock (no dividend). Here

$$G_t = P_t - K \exp\left[-r(T - t)\right]$$

where $K$ is the delivery price. We have

$$\frac{\partial G_t}{\partial t} = -rK \exp[-r(T - t)], \quad \frac{\partial G_t}{\partial P_t} = 1, \quad \frac{\partial^2 G_t}{\partial P_t^2} = 0.$$ 

Substituting these quantities into LHS yields

$$-rK \exp[-r(T - t)] + r P_t = r \{P_t - K \exp[-r(T - t)]\},$$
which equals RHS.

**Black-Scholes formulas**

A European call option: expected payoff

\[
E_*[\max(P_T - K, 0)]
\]

Price of the call: (current value)

\[
c_t = \exp[-r(T - t)]E_*[\max(P_T - K, 0)].
\]

In a risk-neutral world, \(\mu = r\) so that

\[
\ln(P_T) \sim N \left[ \ln(P_t) + \left( r - \frac{\sigma^2}{2} \right)(T - t), \sigma^2(T - t) \right].
\]

Let \(g(P_T)\) be the pdf of \(P_T\). Then,

\[
c_t = \exp[-r(T - t)] \int_K^\infty (P_T - K) g(P_T) dP_T.
\]

After some algebra (appendix)

\[
c_t = P_t \Phi(h_+) - K \exp[-r(T - t)] \Phi(h_-)
\]

where \(\Phi(x)\) is the CDF of \(N(0, 1)\),

\[
h_+ = \frac{\ln(P_t/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}
\]

\[
h_- = \frac{\ln(P_t/K) + (r - \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} = h_+ - \sigma \sqrt{T - t}.
\]

See Chapter 6 for some interpretations of the formula.

For put option:

\[
p_t = K \exp[-r(T - t)] \Phi(-h_-) - P_t \Phi(-h_+).
\]
Alternatively, use the put-call parity:

\[ p_t - c_t = K \exp[-r(T - t)] - P_t. \]

**Put-call parity:** Same underlying stock, same strike price, same time to maturity.

<table>
<thead>
<tr>
<th>current date ( t )</th>
<th>Expiration date ( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Write call} )</td>
<td>( P^* \leq K )</td>
</tr>
<tr>
<td>( \text{Buy put} )</td>
<td>( K &lt; P^* )</td>
</tr>
<tr>
<td>( \text{Buy stock} )</td>
<td>( K - P^* )</td>
</tr>
<tr>
<td>( \text{Borrow} )</td>
<td>( K - P^* )</td>
</tr>
<tr>
<td>( \text{Total} )</td>
<td>( -K )</td>
</tr>
</tbody>
</table>

Current amount: \( c - p - P + K e^{-r(T-t)} \).

On the expiration date:

1. If \( P^* \leq K \): call is worthless, put is \( K - P^* \), stock \( P^* \) and owe bank \( K \). Net worth = 0.

2. If \( K < P^* \): call is \( K - P^* \), put is worthless, stock \( P^* \) and owe \( K \). Net worth = 0.

Under **no arbitrage assumption**: The initial position should be zero. That is, \( c = p + P - K e^{-r(T-t)} \).

**Example.** \( P_t = \$80 \). \( \sigma = 20\% \) per annum. \( r = 8\% \) per annum.
What is the price of a European call option with a strike price of $90 that will expire in 3 months?
From the assumptions, we have \( P_t = 80, \ K = 90, \ T - t = 0.25, \ \sigma = 0.2 \) and \( r = 0.08 \). Therefore,

\[
\begin{align*}
h_+ &= \frac{\ln(80/90) + (0.08 + 0.04/2) \times 0.25}{0.2\sqrt{0.25}} = -0.9278 \\h_- &= h_+ - 0.2\sqrt{0.25} = -1.0278.
\end{align*}
\]

It can be found

\[
\Phi(-0.9278) = 0.1767, \quad \Phi(-1.0278) = 0.1520.
\]

Therefore,

\[
c_t = 80\Phi(-0.9278) - 90\Phi(-1.0278) \exp(-0.02) = 0.73.
\]

The stock price has to rise by $10.73 for the purchaser of the call option to break even.
If \( K = 81 \), then

\[
c_t = 80\Phi(0.125775) - 81 \exp(-0.02)\Phi(0.025775) = 3.49.
\]

**A note on computer program:** Check the web site of Prof. Joseph Goguen of UCSD.
http://www-cse.ucsd.edu/users/goguen/courses/130/SayBlackScholes.html

**Lower bounds of European options:** No dividends.

\[
c_t \geq P_t - K \exp[-r(T - t)].
\]
Why?
Consider two portfolios:

- A: One European call option plus cash $K \exp[-r(T - t)]$.
- B: One share of the stock.

For A: Invest the cash at risk-free interest rate. At time $T$, the value is $K$. If $P_T > K$, the call option is exercised so that the portfolio is worth $P_T$. If $P_T < K$, the call option expires at $T$ and the portfolio is worth $K$. Therefore, the value of the portfolio is $\max(P_T, K)$.

For B: The value at time $T$ is $P_T$.

Thus, portfolio A must be worth more than portfolio B today; that is,

$$c_t + K \exp[-r(T - t)] \geq P_t.$$

See Example 6.7 for an application.

**Stochastic integral**

The formula

$$\int_0^t dx_s = x_t - x_0$$

continues hold. In particular,

$$\int_0^t dw_s = w_t - w_0 = w_t.$$

From

$$dw_t^2 = dt + 2w_t dw_t$$

we have

$$w_t^2 = t + 2 \int_0^t w_s dw_s.$$
Therefore,
\[ \int_0^t w_s dw_s = \frac{1}{2}(w_t^2 - t). \]

Different from \( \int_0^t y dy = \frac{y_t^2 - y_0^2}{2} \).

Assume \( x_t \) is a Geo. Brownian motion,
\[ dx_t = \mu x_t dt + \sigma x_t dw_t. \]

Apply Ito’s lemma to \( G(x_t, t) = \ln(x_t) \), we obtain
\[ d \ln(x_t) = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dw_t. \]

Taking integration, we have
\[ \int_0^t d \ln(x_s) = \left( \mu - \frac{\sigma^2}{2} \right) \int_0^t ds + \sigma \int_0^t dw_s. \]

Consequently,
\[ \ln(x_t) = \ln(x_0) + (\mu - \sigma^2/2)t + \sigma w_t, \]

and
\[ x_t = x_0 \exp[(\mu - \sigma^2/2)t + \sigma w_t]. \]

Change \( x_t \) to \( P_t \). The price is
\[ P_t = P_0 \exp[(\mu - \sigma^2/2)t + \sigma w_t]. \]

**Jump diffusion**

Weaknesses of diffusion models:

- no volatility smile (convex function of implied volatility vs strike price)
• fail to capture effects of rare events (tails)

Modification: jump diffusion and stochastic volatility
Jumps are governed by a probability law:

Poisson process: \( X_t \) is a Poisson process if

\[
Pr(X_t = m) = \frac{\lambda^m t^m}{m!} \exp(-\lambda t), \quad \lambda > 0.
\]

Use a special jump diffusion model by Kou (2002).

\[
\frac{dP_t}{P_t} = \mu dt + \sigma dw_t + d \left( \sum_{i=1}^{n_t} (J_i - 1) \right),
\]

• \( w_t \): a Wiener process,

• \( n_t \): a Poisson process with rate \( \lambda \),

• \( \{J_i\} \): iid such that \( X = \ln(J) \) has a double exp. dist. with pdf

\[
f_X(x) = \frac{1}{2\eta} e^{-|x-\kappa|/\eta}, \quad 0 < \eta < 1.
\]

• the above three processes are independent.

\( n_t \) = the number of jumps in \([0, t]\) and Poisson(\( \lambda t \)). At the \( i \)th jump, the proportion of price jump is \( J_i - 1 \).

For pdf of double exp. dist., see Figure 6.8 of the text.

Stock price under the jump diffusion model:

\[
P_t = P_0 \exp[(\mu - \sigma^2/2)t + \sigma w_t] \prod_{i=1}^{n_t} J_i.
\]

This result can be used to obtain the distribution for the return series.

Price of an option: Analytical results available, but complicated.
Example $P_t = $80. $K = $81. $r = 0.08$ and $T - t = 0.25$.
Jump: $\lambda = 10$, $\kappa = -0.02$ and $\eta = 0.02$.
We obtain $c_t = $3.92, which is higher than $3.49$ of Example 6.6.
$p_t = $3.31, which is also higher.

Some greeks: option value $V$, stock price $P$

1. Delta: $\Delta = \frac{\partial V}{\partial P}$
2. Gamma: $\Gamma = \frac{\partial \Delta}{\partial P}$
3. Theta: $\Theta = -\frac{\partial V}{\partial t}$. 