Problem A: (42 pts) Answer briefly the following questions.

1. Questions 1 to 10. Consider the intraday 1-minute log returns, in percentages, of Caterpillar stock from January 4 to January 8, 2010. The summary statistics and some preliminary analysis of the returns are given in the attached output. Based on the results provided, is the mean of the log return zero? Why?

   **Answer.** No, the mean of the returns is not zero. The 95% confidence interval is $[0.00039,0.0055]$.

2. Is the distribution of the log returns skewed? Perform a statistical test to justify your conclusion.

   **Answer.** The test statistic is $t = -1.19/\sqrt{6/1945} = -2.14$ with a $p$ value 0.032. Thus, the log returns are skewed to the left.

3. Does the distribution of the log returns have heavy tails? Perform a statistical test to justify your conclusion.

   **Answer.** The test statistic is $t = 2.705/\sqrt{24/1945} = 24.35$, which is highly significant. Thus, the log returns have heavy tails.

4. Are there any serial correlations in the log return series? Why? State the null hypothesis you use to justify your answer.

   **Answer.** No, the log returns have no serial correlations. The Ljung-Box statistics give $Q(10) = 12.96$ with a $p$ value 0.23. The null hypothesis is $H_o : \rho_1 = \cdots = \rho_{10} = 0$, where $\rho_i$ is the lag-$i$ ACF of the log returns.

5. Are there any ARCH-effects in the log returns? Why? State the null hypothesis you use to justify your answer.

   **Answer.** Yes, because the Ljung-Box statistics of the squared deviations from the mean give $Q(10) = 338.96$ with a $p$ value close to zero. The null hypothesis is $H_o : \rho_1^* = \cdots = \rho_{10}^* = 0$, where $\rho_i^*$ is the lag-$i$ ACF of $a_t = r_t - \bar{r}$ with $r_t$ denoting the log return and $\bar{r}$ is the mean return.

6. An ARMA(0,0)+GARCH(1,1) model with Student-$t$ innovations is entertained for the log return series. Is the model adequate? Write down the fitted model, including all the parameters.

   **Answer.** Yes, the model is adequate because the model checking statistics all have high $p$ values. The fitted model is

   
   \[
   \begin{align*}
   r_t &= 0.0017 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t_{6.43}^* \\
   \sigma_t^2 &= 0.00012 + 0.0896a_{t-1}^2 + 0.875\sigma_{t-1}^2
   \end{align*}
   \]
where $t^*_v$ denotes a standardized Student-t distribution with $v$ degrees of freedom.

7. Based on the fitted GARCH(1,1) model. Let $v$ be the degrees of freedom of the Student-t innovation. Test $H_0 : v = 5$ vs. $H_a : v \neq 5$. What is the test statistic? Draw your conclusion.

**Answer.** The test statistic is $t = \frac{6.43 - 5}{0.951} = 1.503$ with a $p$ value 0.132. Thus, the null hypothesis cannot be rejected.

8. An ARMA(0,0)+GARCH(1,1) model with skewed Student-t innovations is entertained for the data. Is the model adequate? Write down the fitted model, including all parameters.

**Answer.** Yes, the model is adequate as all model checking statistics have high $p$ values.

The fitted model is

$$
\begin{align*}
\rho_t &= 0.0015 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t^*_v(6.41),
\sigma^2_t &= 0.00012 + 0.0892 a^2_{t-1} + 0.876 \sigma^2_{t-1},
\end{align*}
$$

where $t^*_v,\xi$ denotes a standardized skew Student-t distribution with $v$ degrees of freedom and skew parameter $\xi$.

9. Let $\xi$ be the skew parameter of the fitted model. Test $H_0 : \xi = 1$ vs $H_a : \xi \neq 1$. What is the test statistic employed? Compute the $p$ value of the test and draw your conclusion.

**Answer.** The test statistic is $t = (0.983 - 1)/0.0295 = -0.58$ with a $p$ value 0.56. The null hypothesis of $\xi = 1$ cannot be rejected.

10. Compare the two ARMA(0,0)+GARCH(1,1) models. Which model do you select? Why?

**Answer.** The ARMA(0,0)+GARCH(1,1) model with Student-t innovations. The model has a smaller AIC criterion $-3.0594$, whereas that of the skew innovation is $-3.0586$. This is consistent with the testing result for $\xi$.

11. Suppose that the price $P_t$ of a stock follows the stochastic diffusion model

$$
\frac{dP_t}{P_t} = \mu dt + \sigma P_t dw_t,
$$

where $\mu$ and $\sigma$ are constant and $w_t$ is the standard Brownian motion. What is the distribution of the log return of the stock from time $t$ to $T$?

**Answer.** Let $G_t = \ln(P_t)$. We have $\frac{\partial G_t}{\partial T} = P_t^{-1}, \frac{\partial G_t}{\partial t} = 0$, and $\frac{\partial^2 G_t}{\partial T^2} = -P^2_t$. Applying the Ito’sLemma, we have

$$
\begin{align*}
d[\ln(P_t)] &= \left(\mu - \frac{\sigma^2 P^2_t}{2}\right) dt + \sigma P_t dw_t.
\end{align*}
$$

Therefore, the log return from $t$ to $T$ is

$$
r \sim N\left[\left(\mu - \frac{\sigma^2 P^2_t}{2}\right)(T - t), \sigma^2 P^2_t(T - t)\right].
$$
12. Let $r_{1t}$ and $r_{2t}$ be the monthly simple return of Coca Cola stock and the S&P 500 composite index from January 1960 to December 2009. Let $\mathbf{r}_t = (r_{1t}, r_{2t})'$. Is the bivariate return $\mathbf{r}_t$ serially correlated? Why? State the null hypothesis you use to justify your conclusion.

**Answer.** No, there are no serial correlations in the bivariate return series. The multivariate Ljung-Box statistics give $Q(10) = 43.62$ with a $p$ value 0.32. The null hypothesis is $H_0 : \rho_1 = \cdots = \rho_{10} = 0$, where $\rho_i$ is the lag-$i$ cross correlation matrix of $\mathbf{r}_t$ and $\mathbf{0}$ is a $2 \times 2$ zero matrix.

13. **Questions 13-15.** Consider two time-series models

M1: $x_t = 10.0 + 0.1t + a_t + 0.5a_{t-1}$

M2: $x_t = x_{t-1} + a_t + 0.5a_{t-1}$

where $\{a_t\}$ is a sequence of normal random variables with mean zero and variance 2. Assume that $x_{100} = 10$ and $a_{100} = -2$. For each model, compute a 95% interval forecast of $x_{101}$ at the forecast origin $t = 100$.

**Answer.** For M1, the point prediction is $\hat{x}_{101} = 10 + 0.1 \times 101 + 0.5(-2) = 19.1$. The variance of forecast error is 2. Therefore, the 95% interval forecast is $19.1 \pm 2 \times \sqrt{2}$. That is, $[16.27, 21.93]$. For M2, the point prediction is $\hat{x}_{101} = x_{100} + 0.5a_{100} = 10 + 0.5(-2) = 9$. The variance of forecast error is also 2. Therefore, the 95% interval forecast is $9 \pm 2 \times \sqrt{2}$, i.e., $[6.17, 11.83]$.

14. For each model, compute the variance of 2-step ahead forecast error at the forecast origin 100.

**Answer.** For M1, the 2-step-ahead forecast error is $a_{102} + 0.5a_{101}$ so that the variance of forecast error is $(1+0.5^2)2 = 2.5$. For M2, the 2-step-ahead forecast error is $a_{102} + 1.5a_{101}$ so that the variance of forecast error is 6.5.

15. For each model, compute the 1000-step-ahead prediction of $x_{1100}$ at the forecast origin $t = 100$.

**Answer.** For M1, the prediction is $\hat{x}_{1100} = 1100 = 10 + 0.1 \times 1100 = 120$. For M2, the prediction is $\hat{x}_{1100} = \hat{x}_{101} = x_{100} + 0.5a_{100} = 9$.

16. State two weaknesses in using RiskMetrics to compute value at risk (VaR) of a financial position.

**Answer.** Any two of (a) the assumed model is often rejected by empirical data, (b) the VaR is not a coherent risk measure, and (c) the uncertainty in VaR could be very much.

17. Besides the econometric modeling such as GARCH modeling, describe two additional methods that can be used to estimate stock volatility.

**Answer.** (a) Use high-frequency financial data and (b) use daily open, high, low and close prices.
18. Consider a 3-2-1 feed-forward neural network with skip layer. Suppose that the output variable is binary. Write down the econometric model of this network.

**Answer.** Let the input be $x_1, x_2, x_3$, the hidden nodes be $h_1$ and $h_2$, and the output node is $y$. Then, $y = H(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 h_1 + \beta_5 h_2)$, where $H(\cdot)$ denotes a heaviside function and $h_i$ is defined as $h_i = \alpha_{0i} + \alpha_{1i} x_1 + \alpha_{2i} x_2 + \alpha_{3i} x_3$ for $i = 1, 2$.

19. Describe two difficulties one often faces in assessing credit risk of a loan.

**Answer.** Any two of (a) lack of tradable data available, (b) lack of information on loss given default, (c) accounting data are often delayed, and (d) hard to estimate volatility and correlation between prices of loans.

20. Describe two financial applications of seasonal time series models.

**Answer.** Forecasting quarterly earnings and (b) modeling weather-related derivatives.

21. Describe two effects of overlooking the serial correlations in a linear regression model.

**Answer.** (a) incorrect estimation of the covariance matrix of parameter estimates, (b) may have substantial bias in the estimate in certain situations.

**Problem B.** (18 points) This problem is concerned with forecasting the VIX index of Chicago Board Options Exchange (CBOE). The data span is from January 2, 2004, to December 31, 2009. Two approaches are considered. In the first approach, a pure time series model is used for the VIX index. In the second approach, we employ the lag-1 volatility of the S&P 500 index as an explanatory variable. The latter is estimated by a Gaussian ARMA(1,0)-GARCH(1,1) model using daily log returns, in percentages, of the S&P 500 index. Details are given in the attached output. [Note that the estimated volatility is not annualized. One can multiple the estimate by $\sqrt{252}$, but this will not affect the analysis.]

1. Write down the fitted ARMA(1,0)-GARCH(1,1) model for the daily log returns of the S&P 500 index. Are all estimates statistical significant?

**Answer.** Let $r_t$ be the daily log returns of the S&P 500 index. The fitted model is

$$
\begin{align*}
r_t &= 0.0019 - 0.081 r_{t-1} + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0,1) \\
\sigma_t^2 &= 0.011 + 0.072 a_{t-1}^2 + 0.919 \sigma_{t-1}^2.
\end{align*}
$$

All estimates, except the constant of the mean equation, are significant.

2. Write down the pure time series model for the VIX index. Based on the output provided, is the model adequate? Why?

**Answer.** Let $V_t$ be the VIX index. The fitted model is

$$
(1 - B)V_t = (1 - 0.17B - 0.14B^2 + 0.02B^3 - 0.10B^4 + 0.05B^5)a_t, \quad \sigma_a^2 = 3.35.
$$

Yes, the model is adequate because $Q(5) = 0.56$ with a $p$ value 0.99.
3. Write down the model for VIX index when the lag-1 S&P 500 volatility is used as an explanatory variable. Is the model adequate? Why?

**Answer.** Let $x_{t-1}$ be the lag-1 volatility of the S&P 500 index. The model is

$$(1 - 0.99B)(V_t - 18.55 - 1.85x_{t-1}) = (1 - 0.17B - 0.17B^2 + 0.02B^3 - 0.11B^4 + 0.04B^5)a_t,$$

where $\sigma_a^2 = 3.32$. The model is adequate based on the Q(5) statistic of the residuals.

4. Based on the fitted model, does the lag-1 S&P 500 volatility contribute significantly in modeling the VIX index? Why?

**Answer.** Yes, lagged value of the S&P 500 index volatility contributes significantly to modeling the VIX index. The estimate coefficient 1.85 has a $t$-ratio of 2.61, which is significant at the 5% level.

5. Based on the in-sample fitting, which of the two models of the VIX index is preferred? Why?

**Answer.** The model with lagged S&P 500 index volatility. This model has AIC = 6119.66, which is smaller than 6121.35 of the pure time series model.

6. Based on the backtesting procedure, which model is selected? Why?

**Answer.** Again, the output of sample forecasting also chooses the model with lagged S&P 500 index volatility. The model gives a smaller mean squared error of forecast.

**Problem C.** (10 pts) Consider the quarterly earnings per share of Abbott Laboratories from 1992 to the first quarter of 2010 for 73 observations. Log-transformation of the earnings is taken to stabilize the variability. Two time series models are entertained. The first model is a pure time series model, whereas the second contains a quarterly dummy variable.

1. Write down the fitted pure time series model for the log earning series, including residual variance.

**Answer.** Let $x_t$ be the log earnings. The model is

$$(1 - B)(1 - B^4)x_t = (1 - 0.62B)(1 - 0.34B^4)a_t, \quad \sigma_a^2 = 0.0031.$$

2. Is the fitted model adequate? Is there any ARCH effects in the earnings? Why?

**Answer.** The fitted model is adequate. Its residuals give $Q(12) = 17.66$ with a $p$ value 0.13. Thus, there is no serial correlation in the residuals. Also, the Ljung-Box statistics of the squared residuals give $Q(12) = 12.85$ with a $p$ value of 0.38, indicating that there is no ARCH effect.

3. Write down the fitted model with the dummy variable for the first quarter.

**Answer.** The fitted model is

$$(1 - 0.93B^4)(x_t - 0.17Q_{1t}) = (1 - 0.55B)(1 - 0.30B^4)a_t, \quad \sigma_a^2 = 0.0030,$$

where $Q_{1t}$ is the dummy variable of the first quarter.
4. Is the dummy variable significant at the 5% level? What is the implication of the dummy variable?

Answer. Yes, the coefficient of the dummy variable is significant. Its has a $t$-ratio of $0.174/0.077 = 2.26$, which is greater than the 5% critical value 1.96.

5. Select a model between the two and justify your choice of the model.

Answer. Both models are adequate. The pure seasonal ARMA model has an AIC value $-192.74$, whereas the model with dummy variable has an AIC value $-197.17$. Thus, the model with a quarterly dummy variable is selected.

Problem D. (30 points) Consider the daily log returns of the stocks of Apple and J.P. Morgan Chase from January 3, 2001, to December 31, 2009. The tick symbols are AAPL and JPM, respectively. Suppose that Manager A holds a long position of $1 million on each of the two stocks. Use the attached output to answer the following questions.

1. Manager A decides to use RiskMetrics to calculate VaR of her financial position. To this end, the special Gaussian IGARCH(1,1) model is fitted to the two log-return series. For both stock returns, the $\alpha$ parameter of the IGARCH model is 0.94. Prediction of the conditional variance is given in the output. Calculate the VaR for each stock for the next trading day.

Answer. For Apple stock, we have $\text{VaR}_1 = 2.326\sqrt{0.000243 \times 1000000} = $36,264.
   For JPM, $\text{VaR}_2 = 2.326\sqrt{0.000243 \times 1000000} = $36,264.

2. What are the corresponding expected shortfalls for each financial position?

Answer. For Apple stock, the expected shortfall is $\text{ES} = 2.41580$. For JPM, $\text{ES} = 41,580$.

3. The correlation between the two log returns is 0.327. What is the VaR of Manager A’s financial position for the next 10 trading days?

Answer. VaR = $\sqrt{\text{VaR}_1^2 + \text{VaR}_2^2 + 2\rho\text{VaR}_1\text{VaR}_2} = \sqrt{36264^2 + 36264^2 + 2 \times 0.327 \times 36264^2} = $59,078. For the next 10-trading days, the value of VaR is $186,821.

4. Manager A worries about the heavy tails of the returns so that she adopts an econometric approach with Student-t innovations to calculate VaR for her position. Write down the fitted GARCH(1,1) model for AAPL.

Answer. Let $r_t$ be the negative log returns of the Apple stock. The model is

\[
\begin{align*}
\sigma_t^2 &= 0.071 \times 10^{-4} + 0.035a_{t-1}^2 + 0.956\sigma_{t-1}^2, \\
\epsilon_t &= -0.0018 + a_t, \\
\end{align*}
\]

5. What are the VaR of each financial position based on the econometric approach?

Answer. VaR for the negative Apple log return is $-0.0018 + (3.207/\sqrt{5.66/3.66})\sqrt{0.0003567} = 0.046906$. The VaR for the Apple position is $46,906$. Similarly, the VaR for JPM
negative log return is $-0.00038 + (3.053/\sqrt{6.57/4.57})\sqrt{0.0002057} = 0.036141$. The VaR for the JMP position is $36,141.

6. Manager A also tries the traditional extreme value theory with block size 21 on the Apple stock. What are the estimates of $(x_i, \sigma, \mu)$? What is the VaR for her position on the Apple stock?

**Answer.** The estimates are $(0.263, 0.019, 0.0381)$. The VaR is $74,811$.

7. Finally, Manager A uses the peaks over threshold (POT) approach. She decides to use 4% and 3% as the threshold for the Apple and Chase stock, respectively. Write down the parameter estimates for both stocks?

**Answer.** For AAPL, the estimates are $(x_i, \beta) = (0.3348, 0.0138)$ and for JPM, the estimates are $(x_i, \beta) = (0.2314, 0.0180)$.

8. What are the expected shortfall for each of the two positions Manager A has?

**Answer.** For AAPL, the ES = $121,482$. For JPM, ES = $123,406$.

9. Manager B holds a short position of $1$ million on the J.P. Morgan Chase stock and a long position of $1$ million on the Apple stock. She also decides to use Gaussian GARCH models to estimate VaR. Compute the VaR and expected shortfall for each of the two stock positions for the next trading day.

**Answer.** For AAPL log returns, VaR = $-0.0025 + 2.326 \times 0.0202 = 0.044485$ and expected shortfall is $-0.0025 + f(2.326)/0.01 \times 0.0202 = 0.051337$. Thus, the AAPL position, VaR = $44,485$ and ES = $51,337$. For JPM log returns, VaR = $0.00047 + 2.326 \times 0.01408 = 0.033220$ and ES = $0.00047 + f(2.326)/0.01 \times 0.01408 = 0.037996$. Therefore, for the JPM position, VaR = $33,220$ and ES = $37,996$.

10. What is the VaR and expected shortfall of the next trading day for Manager B?

**Answer.** Recall that the correlation between (long) AAPL and (short) JPM is $-0.327$. Since the portfolio is equally weighted, the mean of the log return is $-0.0025 + 0.00047 = -0.00203$. The standard error of the portfolio log return is $\sqrt{0.0202^2 + 0.01408^2 + 2(-0.327) \times 0.0202 \times 0.01408} = 0.0205$. Therefore, the log portfolio log return VaR = $-0.00203 + 2.326 \times 0.0205 = 0.045653$ and ES = $-0.00203 + f(2.326)/0.01 \times 0.0205 = 0.052607$. Consequently, the VaR is $45,653$ and ES is $52,607$. 
