Problem A: (32 pts) Answer briefly the following questions.

1. Suppose the price $P_t$ of a stock follows the stochastic diffusion equation (SDE) $dP_t = 0.04P_t dt + 0.25P_t dw_t$, where $w_t$ denotes the standard Brownian motion. What are the drift and diffusion for the inverse price $\frac{1}{P_t}$?
   
   A: See Lecture 8 or by Ito’s Lemma, we have
   
   $d\frac{1}{P_t} = (-\mu + \sigma^2)\frac{1}{P_t} dt - \sigma\frac{1}{P_t} dw_t$.
   
   Thus, the drift is $0.0225\frac{1}{P_t}$ and the diffusion $-0.25\frac{1}{P_t}$.

2. Suppose that the price of a stock follows a geometric Brownian motion with drift 5% per annum and volatility 36% per annum. The stock pays no dividends and its current price is $70. Assume further that the risk-free interest rate is 4% per annum. (a) What is the price of a European call option contingent on the stock with a strike price of $71 that will expire in 3 months? (b) What is the corresponding put option price?
   
   A: Apply Black-Scholes formula, $c_t = 4.885$ and $p_t = 5.179$.

3. Give two weaknesses of using value at risk (VaR) to quantify financial risk.
   
   A: (a) Not coherent, (b) it is a point estimate of risk with high variability.

4. Define a coherent risk measure.
   
   A: It must satisfy the following conditions: (a) monotonicity, (b) sub-additivity, (c) positive homogeneity, and (d) translation invariance.

5. Describe two approaches to overcome the market microstructural noises in computing realized volatility.
   
   A: (a) Sub-sampling method and (b) optimal sampling interval.

6. Give two weaknesses of realized volatility as an estimate of daily stock volatility.
   
   A: (a) It does not consider night volatility, (b) sensitive to microstructural noises.

7. Describe two main assumptions used by the RiskMetrics?
   
   A: (a) $r_t | F_{t-1}$ is $N(0, \sigma_t^2)$, (b) $\sigma_t = (1 - \beta)r_{t-1}^2 + \beta\sigma_{t-1}^2$, a special IGARCH(1,1) model.

8. Describe two ways to apply extreme value theory in calculating VaR.
   
   A: (a) Traditional approach using block maximum, (b) Peaks over Thresholds
9. Give two main difficulties in modeling multivariate volatility of asset returns.
   A: (a) High-dimension, $k(k + 1)/2$ series of variances and covariances, (b) volatility matrix must be positive definite for all $t$.

10. (For Questions 10 and 11). The log return $r_t$ of a stock follows the model
    \[
    r_t = -0.016 + 0.1\sigma_t^2 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t_{6.5}^*
    \]
    where $t_v^*$ denotes standardized Student-$t$ distribution ith $v$ degrees of freedom. Suppose further that $r_{100} = 0.02$ and $\sigma_{100} = 0.4$. Calculate the 1-step ahead mean and volatility predictions of $r_t$ at the first origin $T = 100$.
   A: $a_{100} = r_{100} + 0.016 - 0.1\sigma_{100}^2 = 0.02 + 0.016 - 0.016 = 0.02$. Therefore, $\sigma_{101}^2 = 0.05 + 0.05(0.02)^2 + 0.9(0.4)^2 = 0.17402$. The volatility forecast is 0.417. $\hat{r}_{101} = -0.016 + 0.1\sigma_{101}^2 = 0.02$. Therefore, $\sigma_{102}^2 = 0.03 + 0.05 + 0.9\sigma_{101}^2 = 0.00353$.

11. Calculate the 2-step ahead predictions of the mean and volatility of $r_t$ at the forecast origin $T = 100$.
    A: $\sigma_{102}^2 = 0.03 + (0.05 + 0.9)\sigma_{101}^2 = 0.195319$. The volatility forecast is 0.442. $\hat{r}_{102} = -0.016 + 0.1\sigma_{102}^2 = 0.00353$.

12. (For Questions 12 and 13). The quarterly earnings of a company follows the model
    \[
    (1 - B)(1 - B^4)x_t = (1 - 0.4B)(1 - 0.6B^4)a_t, \quad a_t \sim N(0, \sigma^2),
    \]
    where $\sigma^2 = 0.16$. Let $w_t = (1 - B)(1 - B^4)x_t$. Give the lags for which $w_t$ has non-zero ACF.
    A: Lags 1, 3, 4, and 5. (Of course, lag 0 is not zero).

13. Obtain the values of all non-zero ACFs of the $w_t$ series.
    A: $\rho_1 = 0.4/(1 + 0.4^2) = -0.3448$, $\rho_4 = -0.6/(1 + 0.6^2) = -0.4412$, $\rho_3 = \rho_5 = \rho_1 \times \rho_4 = 0.1521$.

14. Let $x_t = (x_{1t}, x_{2t})'$ be a bivariate stationary time series. Denote the mean vector of $x_t$ as $E(x_t) = \mu$. Define the lag-$j$ autocovariance matrix of $x_t$. What is the meaning of the $(1,2)$th element of $\Gamma_j$? Here $(1,2)$th element denotes the upper-right element of the 2-by-2 matrix $\Gamma_j$.
    A: $\Gamma_j = \text{cov}(x_t, x_{t-j}) = E(x_t - \mu)(x_{t-j} - \mu)'$. The $(1,2)$th element denotes the dependence of $x_{1t}$ on $x_{2,t-j}$.

15. Give two reasons for modeling multivariate time series jointly.
    A: Any two of (a) relationship between variables, (b) improve forecasts because of using more information, (c) empirical applications often involve multiple series.
16. Describe two simple approaches discussed in the lecture to model multivariate asset volatility.
A: Any two of (a) exponentially weighted moving average, (b) univariate GARCH models, (c) moving windows.

Problem B. (44 points) Consider the daily adjusted close prices of the stocks of Apple and ExxonMobil from January 3, 2002, to May 31, 2011. The tick symbols are AAPL and XOM, respectively. From the data, we obtain the 2368 daily log returns of the stocks. Consider a long position of $1 million on each of the two stocks. Use the attached output to answer the following questions.

1. (6 points) If RiskMetrics is used, what is the underlying model for the AAPL stock? (Write down the fitted model.) What are the VaR and expected shortfall for the position on AAPL stock?
A: The model is \( r_t = a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0, 1) \) and \( \sigma^2_t = (1 - 0.9672)r^2_{t-1} + 0.9672\sigma^2_{t-1} \). The model gives VaR = $29568 and ES = $33875.

2. (6 points) Again, if RiskMetrics is used, what is the VaR for the position on XOM stock? What is the corresponding VaR for the next 10 trading days? The sample correlation between the two daily log returns is 0.3723. What is the VaR for the joint position of the two stocks?
A: For XOM position, VaR = $23961 and VaR[10] = $75771. The VaR for the portfolio is VaR = \( \sqrt{29568^2 + 23961^2 + 2 \times 0.3723 \times 29568 \times 23961} = $44451. \)

3. (4 points) If ARMA-GARCH models with Gaussian innovations are used to model the daily stock returns, what are the VaR and expected shortfall for AAPL stock?
A: The fitted model seems adequate except for normality assumption. Based on the 1-step ahead prediction, VaR = $34755 and ES = $40148.

4. (4 points) Again, based on GARCH models with Gaussian innovations, what are the VaR for XOM stock position and the point position of the two stocks?
A: Based on the output, VaR = $25835 for the XOM position. The VaR for the portfolio is approximately as VaR = \( \sqrt{34755^2 + 25835^2 + 2 \times 0.327 \times 34755 \times 25835} = $50438. \)

5. (4 points) If GARCH models with Student-\( t \) innovations are used, write down the fitted model for the XOM stock?
A: The model is
\[
\begin{align*}
    r_t &= -9.508 \times 10^{-4} + a_t + 0.0945a_{t-1}, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t^*_{8.378} \\
    \sigma^2_t &= 4.309 \times 10^{-6} + 0.077a^2_{t-1} + 0.904\sigma^2_{t-1}.
\end{align*}
\]
6. (4 points) What are the VaR and expected shortfall for the XOM position based on the GARCH model with Student-\(t\) innovations?
   A: The fitted model appears to be adequate. Based on its prediction, we obtain VaR = $27615 and ES = $34102.

7. (4 points) Next, apply the traditional extreme value theory with block size 21 to the Apple stock. What are the estimates of (\(\xi\), \(\sigma\), \(\mu\))? What is the VaR for the position on the Apple stock?
   A: The estimates are (0.1256, 0.01584, 0.03336). The VaR is $60574.

8. (4 points) Finally, consider the peaks over threshold (POT) approach. The thresholds of 4\% and 2.5\% are used for AAPL and XOM, respectively. Write down the parameter estimates for both stocks? Are these estimates significantly different from zero? Why?
   A: For AAPL stock, the estimates are (0.3211, 0.0112) and for XOM stock, they are (0.2480, 0.0113). Yes, these estimates are significantly different from zero based on their standard errors. The t-ratios of these estimates are all greater than 2.0.

9. (4 points) What are the VaR and expected shortfall for each of the two stock positions based on the POT method?
   A: For AAPL, VaR = $62741 and ES = $90004. For XOM, VaR = $46759 and ES = $68991.

10. (4 points) The exponentially weighted moving average method is used to model the volatility matrix of AAPL and XOM returns. Let \(\Sigma_t\) denotes the volatility matrix. Write down the fitted volatility model. Also, based on the model, the correlation between the two stock on May 31, 2011 is 0.4108. What is the VaR of the joint position based on the RiskMetrics method if this new correlation is used?
    A: The fitted volatility model is
        
        \[
        \Sigma_t = (1 - 0.963) r_{t-1} r_{t-1}' + 0.963 \Sigma_{t-1}.
        \]

    The resulting correlation coefficient is 0.4108. Therefore, the VaR for the portfolio is
    
    \[
    \sqrt{(29568^2 + 23961^2 + 2 \times 0.4108 \times 29568 \times 23961)} = $45061.
    \]

Problem C. (12 points). Consider the intraday trading of the Allstate (ALL) stock on December 01, 2010. There are 14971 trades within the normal trading hours. Thus, we have 14970 price change points. Among those, only 3843 have non-zero price changes. Let \(A_i\) and \(D_i\) be the action and direction of the price change for the \(i\)th trade. That is, \(A_i = 1\) if and
only if the $i$th trade results in a non-zero price change, and $D_i = 1$ if $C_i > 0$, $D_i = -1$ if $C_i < 0$, and $D_i = 0$, otherwise, where $C_i$ denotes the price change of the $i$th trade. Simple logistic regression is used to model $A_i$ and $D_i$. The output is attached. Answer the following questions:

1. (2 points) Write down the fitted model for $A_i$. Are the estimates significantly different from zero? Why?
   
   A: The model is $P(A_i = 1) = \frac{\exp(-1.598 + 1.650 A_{i-1})}{1+\exp(-1.598 + 1.650 A_{i-1})}$. Yes, the two estimates are statistically significant because their t-ratios are large.

2. (2 points) Based on the fitted model, calculate $P(A_i = 1|A_{i-1} = 0)$ and $P(A_i = 1|A_{i-1} = 1)$.
   
   A: Based on the model, $P(A_i = 1|A_{i-1} = 0) = 0.168$ and $P(A_i = 1|A_{i-1} = 1) = 0.513$.

3. (2 points) What is the meaning of the estimate 1.64958?
   
   A: It is the log of odds ratio between $A_{i-1} = 1$ and $A_{i-1} = 0$, where the odd denotes the ratio between $A_{i=1}$ versus $A_i = 0$ given $A_{i-1}$.

4. (2 points) Write down the fitted model for $D_i$ conditional on $A_i = 1$. Are the estimates significantly different from zero? Why?
   
   A: $P(D_i = 1|A_i = 1, D_{i-1} = 1) = \frac{\exp(0.029 - 1.404 D_{i-1})}{1+\exp(0.029 - 1.404 D_{i-1})}$. The constant term is insignificant, but the other coefficient is significant based on their r-ratios.

5. (4 points) Based one the fitted model, calculate $P(D_i = 1|A_i = 1, D_{i-1} = 1)$, $P(D_i = 1|A_i = 1, D_{i-1} = 0)$ and $P(D_i = 1|A_i = 1, D_{i-1} = -1)$.
   
   A: $P(D_i = 1|A_i = 1, D_{i-1} = 0) = 0.507$, $P(D_i = 1|A_i = 1, D_{i-1} = 1) = 0.202$, and $P(D_i = 1|A_i = 1, D_{i-1} = -1) = 0.807$.

Problem D. (12 points). Consider the daily close prices, in log scale, of ExxonMobil (XOM) and Chevron (CVX) stocks from July 31, 2002 to December 31, 2008. We analyze the data to explore trading opportunities. A simple linear regression gives the model

$$xom_t = 0.3004 + 0.9386cvx_t,$$

from which one can construct a spread process $w_t$. Figure 1 shows the $w_t$ series with three horizontal lines at $Y = (0.25, 0.30, 0.35)$, respectively. Based on the plot and the discussions of pairs trading in class, answer the following questions:
1. Describe the basic idea behind pairs trading.
   A: Relative pricing of two stocks having similar risk factors.

2. Write down the linear combination that provides a stationary process of the two log prices.
   A: \( w_t = xom_t - 0.938cvx_t \).

3. If \( w_{t_0} = 0.35 \), which stock is overvalued? Why?
   A: XOM is overvalued, because the spread \( w_t \) is above its mean.

4. (4 points) If the total trading costs (including initiation and closing of a position) for pairs trading are 2%, is there any opportunity to conduct pairs trading between the two stocks? If yes, briefly describe a trading strategy.
   A: Yes, there are trading opportunities. A simple trading strategy is as follows: Short 1 share of XOM stock and long 0.939 shares of CVX when \( w_t \) comes down to 0.35, then unwind the position when \( w_t \) crosses its mean. This gives a long return of 5%, which is greater than the 2% costs. [Other strategies are also possible.]

5. Why is stationarity of \( w_t \) important in pairs trading?
   A: The trading depends on mean-reverting of \( w_t \).