1. Consider the daily price of Caterpillar (CAT) stock from January 3, 2005 to April 21, 2011. You may download the data from Yahoo using the `quantmod` package. Compute the daily log returns using the adjusted closing price. Use a time-series model to remove any serial correlations of the log returns, if necessary.

- Is there any ARCH effect in the daily log returns of CAT stock? Why?
  A: The Ljung-Box statistics indicate some serial correlations in the log returns. PACF of the log returns suggests an AR(3) model. The fitted model is approximately \((1 - 0.012B + 0.037B^2 - 0.085B^3)(r_t - 0.0006) = a_t\). Model checking shows that the AR(3) model is satisfactory. Applying Ljung-Box statistics to the squared residuals of the prior model, we obtain \(Q(10) = 434.48\) with p-value close to zero. Therefore, there are ARCH effects in the daily log returns.

- Fit a GARCH(1,1) model for the log return of CAT stock using Gaussian distribution for the innovations. Write down the fitted model. Is the volatility persistent? Why?
  A: After removing insignificant parameters, we have
  \[
  r_t = 0.00114 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0,1)
  \]
  \[
  \sigma_t^2 = 9.01 \times 10^{-6} + 0.0589a_{t-1}^2 + 0.922\sigma_{t-1}^2.
  \]
  Yes, the volatility is persistent because \(\hat{\alpha}_1 + \hat{\beta}_1 = 0.981\), which is close to 1.

- Fit the GARCH(1,1) model again using a Student-\(t\) distribution for the innovations. Let \(v\) be the degrees of freedom. Test \(H_0 : v = 5\) versus \(H_a : v \neq 5\). Write down the fitted model.
  A: \(t = (5.25 - 5)/(0.683) = 0.366\), which is less than 2. Thus, the hypothesis of 5 degrees of freedom cannot be rejected at the 5% level.

- Fit the GARCH(1,1) model again using a skew Student-\(t\) distribution for the innovations. Write down the fitted model. Is the distribution skew? Why?
  A: The fitted model is
  \[
  r_t = 0.00114 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t_{5.26,97}^*\]
  \[
  \sigma_t^2 = 7.898 \times 10^{-6} + 0.0815a_{t-1}^2 + 0.905\sigma_{t-1}^2.
  \]
  To test for skewness, we have \(H_0 : \xi = 1\) vs \(A_a : \xi \neq 1\). The t-ratio is \(t = (0.9702 - 1)/(0.03352) = -0.889\), the absolute value of which is less than 2. Thus, \(H_0\) cannot be rejected. That is, the distribution seems to be symmetric.
• Compare the three volatility models. Which one do you prefer? Why?
A: Based on AIC or BIC, the GARCH(1,1) model with Student-t innovations is selected.

2. Again, consider the daily log returns of CAT stock of problem 1.
• Use the GARCH(1,1) model with Student-\( t \) innovations of Problem 1 to produce 1-step to 5-step ahead volatility forecasts at the forecast origin 2011-04-21.
A: The forecasts for the return are 0.001295 for all steps. For volatility, the 1-step to 5-step ahead predictions are 0.01753, 0.01764, 0.01775, 0.01785 and 0.01796, respectively.
• Fit an IGARCH(1,1) model with Gaussian innovations to the return series. Write down the fitted model.
A: The model is
\[
\begin{align*}
r_t &= 0.00118 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0, 1) \\
\sigma_t^2 &= 3.429 \times 10^{-6} + (1 - 0.9391) a_{t-1}^2 + 0.9391 \sigma_{t-1}^2.
\end{align*}
\]
• Use the fitted IGARCH(1,1) model to predict the volatility for the next five trading days.
A: The 1-step ahead volatility forecast is 0.01752. For 2-step to 5-step, the predictions are 0.01762, 0.01771, 0.01781, 0.01790, respectively.
• Fit a TGARCH(1,1) model with Student-\( t \) innovations to the return series. Write down the model. Is the leverage effect statistically significant? Why?
A: The fitted model is
\[
\begin{align*}
r_t &= 0.00115 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t^*_{5.32} \\
\sigma_t^2 &= 8.825 \times 10^{-6} + 0.0782 (|a_{t-1}| - 0.21 a_{t-1})^2 + 0.902 \sigma_{t-1}^2.
\end{align*}
\]
The \( t \)-ratio for the leverage effect is 2.279 with p-value 0.02266. Thus, the leverage effect is significant at the 5\% level.

3. Consider the monthly log returns of the CRSP decile 9 portfolio from January 1951 to December 2010. The simple returns are in the file \texttt{m-deciles.txt} under the name \texttt{CAP9RET}.
• Is there any serial correlation in the log returns of decile 9 portfolio? If necessary, find an ARMA model to remove the serial correlations.
A: Yes, the ACF shows a significant lag-1 correlation. The Ljung-Box statistics show \( Q(12) = 24.77 \) with p-value 0.016.
• Is there any ARCH effect in the log return series of decile 9 portfolio?
A: Yes, but it is weak. Using squared residuals of MA(1) model, we have \( Q(10) = 17.38 \) with p-value 0.066, which is slightly greater than 0.05.
• Build a GARCH(1,1) model for the log return series. Write down the model.

A: We started with a GARCH(1,1) model with Gaussian innovations, but the normality assumption is rejected at the 5% level. Therefore, we consider Student-t innovations. The fitted model is

\[ r_t = 0.0123 + a_t - 0.109a_{t-1}, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t_{6.89}^* \]

\[ \sigma_t^2 = 1.25 \times 10^{-4} + 0.110a_{t-1}^2 + 0.842\sigma_{t-1}^2. \]

• Compute 1- to 5-step ahead forecasts for the monthly log return and its volatility based on your fitted model at the forecast origin December 2010.

A: The 1-step to 5-step ahead predictions for the log returns are 0.0185, 0.0123, 0.0123, 0.0123, and 0.0123. Those of the volatility are 0.0564, 0.0565, 0.0562, 0.0560, and 0.0558.

• Fit an IGARCH(1,1) model for the log return series using Gaussian distribution for the innovations. Write down the model.

A: The IGARCH(1,1) model is

\[ r_t = 0.0101 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0, 1) \]

\[ \sigma_t^2 = 6.28 \times 10^{-5} + (1 - 0.860)a_{t-1}^2 + 0.860\sigma_{t-1}^2. \]

If one applies an IGARCH(1,1) model to the MA(1) residuals, the fitted volatility equation remains close as

\[ \sigma_t^2 = 5.11 \times 10^{-5} + (1 - 0.875)a_{t-1}^2 + 0.875\sigma_{t-1}^2. \]

4. The data file m-gesp6110.txt contains the date and monthly simple returns of the General Electric (GE) stock and S&P composite index from January 1961 to December 2010. Transform the simple returns into log returns.

• Is there any serial correlation in the monthly log returns of the GE stock?

A: At the 5% level, Ljung-Box statistics cannot reject the null hypothesis of no serial correlations with Q(10) = 16.31 with p-value 0.091.

• Fit a GARCH(1,1) model to the monthly percentage log returns of GE stock using Gaussian distribution for the innovations. Write down the fitted model and perform model checking.

A: The fitted model is

\[ r_t = 0.0099 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim N(0, 1) \]

\[ \sigma_t^2 = 0.000208 + 0.124a_{t-1}^2 + 0.837\sigma_{t-1}^2. \]

Turn to model checking. The normality is rejected at the 5% level. Other statistics indicate both the mean and volatility equations are adequate.
• Use the fitted model to calculate 1-step to 5-step ahead forecasts for the log returns series and its volatility.
A: The forecasts for the log returns are 0.0099 for all five steps. The volatility predictions are 0.0989, 0.0980, 0.0972, 0.0963, and 0.0955, respectively.

• Fit a GARCH(1,1) model to the monthly log returns of the GE stock using a skewed Student-$t$ distribution. Write down the fitted model. Is the skewness parameter significant?
A: The fitted model is

$$r_t = 0.0104 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t_{10,1.051}^*$$

$$\sigma_t^2 = 0.000240 + 0.129a_{t-1}^2 + 0.828\sigma_{t-1}^2.$$  

Yes, the distribution is skewed. The $t$-ratio for skewness is $t = (1.051 - 1)/(0.0652) = 0.782$, which is less than 2. Thus, the skew parameter is not significantly different from 1.


• Is there any serial correlation in the log return series?
A: No, the Ljung-Box statistics show $Q(10) = 16.11$ with p-value 0.097.

• Is there any ARCH effect in the log return series?
A: Yes, the Ljung-Box statistics of the squared returns show $Q(10) = 145.08$ with p-value close to zero.

• Identify a GARCH model for the log return series, including choosing the innovation distribution. Write down the fitted model.
A: After checking the QQ-plot of the standardized residuals, we selected the GARCH(1,1) model with skew Student-$t$ innovations. The fitted model is

$$r_t = -7.36 \times 10^{-5} + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim t_{6.77,0.941}^*$$

$$\sigma_t^2 = 3.15 \times 10^{-7} + 0.0223a_{t-1}^2 + 0.970\sigma_{t-1}^2.$$  

The $t$-ratio for the skewness is $t = (0.941 - 1)/0.02593 = -2.275$. Thus, the null hypothesis of $\xi = 1$ is rejected at the 5% level. Other model checking statistics show the model is adequate.

• Use the model to produce 1-step to 4-step ahead forecasts for the log return series and its volatility.
A: The forecasts for the log returns are $-7.362 \times 10^{-5}$ whereas those for the volatility are 0.00723, 0.00723, 0.00722, and 0.00722, respectively.