Solutions to Homework Assignment #4

All conclusions are drawn using the type-I error of 5%.


   - Is there any ARCH effect in the daily log returns of AAPL stock? Why?
     Answer: Based on the one-sample t-test and Ljung-Box statistics of the log returns, we see that the null hypothesis of zero mean is rejected, but the null hypothesis of no serial correlations cannot be rejected. Consequently, we can calculate the innovation via
     \[ a_t = r_t - \bar{r}, \]
     where \( \bar{r} \) is the sample mean of the returns. Using Ljung-Box statistics of \( a_t^2 \), we reject the null hypothesis of no ARCH effects in the data.

   - Fit a GARCH(1,1) model for the log return of AAPL stock using Gaussian distribution for the innovations. Write down the fitted model. Is the volatility persistent? Why?
     Answer: The fitted model is
     \[
     r_t = 0.0026 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim iid \ N(0, 1)
     \]
     \[
     \sigma_t^2 = 1.21 \times 10^{-5} + 0.103 a_{t-1}^2 + 0.877 \sigma_{t-1}^2,
     \]
     where all estimates are statistically significant. From the fitted model \( \hat{\alpha}_1 + \hat{\beta}_1 = 0.103+0.877 = 0.98 \), which is close to 1 so that the volatility series is persistent.

   - Fit the GARCH(1,1) model again using a Student-t distribution for the innovations. Let \( v \) be the degrees of freedom. Test \( H_0 : v = 5 \) versus \( H_a : v \neq 5 \). Write down the fitted model.
     Answer: The \( t \)-ratio is \( \frac{6.636-5}{1.159} = 1.41 \), which is less than 1.96. The data do not provide evidence to reject 5 degrees of freedom. The fitted model is
     \[
     r_t = 0.0024 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim iid \ t_{6.636}^*
     \]
     \[
     \sigma_t^2 = 7.49 \times 10^{-6} + 0.088 a_{t-1}^2 + 0.902 \sigma_{t-1}^2,
     \]

   - Fit the GARCH(1,1) model again using a skew Student-t distribution for the innovations. Write down the fitted model. Is the distribution of the log returns skew? Why?

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Answer: The fitted model is

\[
\begin{align*}
    r_t &= 0.0023 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim iid \ t_{6.66,0.99}^* \\
    \sigma_t^2 &= 7.45 \times 10^{-6} + 0.088a_{t-1}^2 + 0.902\sigma_{t-1}^2,
\end{align*}
\]

where the estimated skew parameter is 0.987. To test for skew innovations, we compute the \( t \)-ratio of \( \frac{0.987 - 1}{0.0382} = -0.34 \), which is less than 1.96 in modulus. Therefore, the data provide no support for skew innovations. [You can consider the QQ-plot for the standardized Student-\( t \) model of the prior sub-question. The plot shows a straight line, indicating the symmetric \( t \)-innovations are sufficient.]

- Compare the three volatility models. Which one do you prefer? Why?
  Answer: The model with standardized Student-\( t \) distribution is preferred based on AIC and model checking.

2. Again, consider the daily log returns of AAPL stock of problem 1.

- Use the GARCH(1,1) model with Student-\( t \) innovations of Problem 1 to produce 1-step to 5-step ahead volatility forecasts at the forecast origin 2012-04-30.
  Answer: The volatility forecasts are 0.0312, 0.0312, 0.0311, 0.0311, and 0.0311, respectively.

- Fit an IGARCH(1,1) model with Gaussian innovations to the return series. Write down the fitted model.
  Answer: The fitted IGARCH(1,1) model is

\[
\begin{align*}
    r_t &= 0.0024 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim iid \ N(0,1) \\
    \sigma_t^2 &= 0.059a_{t-1}^2 + 0.941\sigma_{t-1}^2.
\end{align*}
\]

The constant of the volatility equation is set to zero because volatility has no drift.

- Use the fitted IGARCH(1,1) model to predict the volatility for the next five trading days.
  Answer: Based on the fitted IGARCH model, the volatility forecasts are constant and equal to \( \sqrt{0.059 \times (-0.0345)^2 + 0.941 \times (0.0276)^2} = 0.0281 \), where \( a_{1341} = r_{1341} - 0.0024 = -0.0345 \).

- Fit a TGARCH(1,1) model with Student-\( t \) innovations to the return series. Write down the model. Is the leverage effect statistically significant? Why?
  Answer: The fitted model is

\[
\begin{align*}
    r_t &= 0.0022 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim iid \ t_{7.35}^* \\
    \sigma_t^2 &= 1.288 \times 10^{-5} + 0.0914(|a_{t-1}| - 0.349a_{t-1})^2 + 0.875\sigma_{t-1}^2.
\end{align*}
\]

To test the leverage effect, we consider the \( t \)-ratio of \( \gamma_1 \), which is 3.396. Thus, the leverage effect is statistically significant.

- Is there any serial correlation in the log returns, in percentages, of decile 2 portfolio? If necessary, find an ARMA model to remove the serial correlations.
  Answer: The serial correlation is weak. The Ljung-Box statistics with \( Q(12) = 19.28 \) with \( p \)-value 0.082, which is close to 0.05. A closer examination of the sample ACF shows a significant ACF at lag-1. Therefore, we entertained an MA(1) model. The fitted model is
  \[
  r_t = 0.811 + a_t + 0.145a_{t-1}, \quad \sigma_a^2 = 22.39.
  \]

- Is there any ARCH effect in the log return series of decile 2 portfolio?
  Answer: Yes, the Ljung-Box statistics of the residuals of the fitted MA(1) model indicate the existence of ARCH effect, because \( Q(12) \) of the squared residuals is 24.86 with \( p \)-value 0.016.

- Build a GARCH(1,1) model for the log return series. Write down the model. Is the model adequate? Why?
  Answer: This example demonstrates the iterative nature of modeling. We started with an MA(1)+GARCH(1,1) model with Gaussian innovations. The estimated parameters are statistically significant, but the normality assumption is rejected. We then applied an MA(1)+GARCH(1,1) model with Student-\( t \) innovations. The MA(1) coefficient becomes insignificant. Therefore, we apply a GARCH(1,1) model with Student-\( t \) innovations. All fitted models are given in the attached R output. Here I only provide the GARCH(1,1) model with Student-\( t \) innovations:
  \[
  d_t = 1.101 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim iid \ t_{7.34}^*, \quad \sigma_t = 1.47 + 0.129a_{t-1}^2 + 0.814\sigma_{t-1}^2.
  \]
  This model is adequate in describing the first two moments of the returns, as all model checking statistics fail to reject the model. The QQ-plot on the other hand indicates some lack of fit in the tails.

- Fit a GARCH(1,1) with skew Student-\( t \) distributions for the log return series of decile 2 portfolio. Write down the fitted model. Perform test to confirm that the fitted Student-\( t \) distribution is indeed skew.
  Answer: The fitted model is
  \[
  d_t = 0.926 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim iid \ t_{7.34,0.777}^*, \quad \sigma_t = 1.06 + 0.124a_{t-1}^2 + 0.835\sigma_{t-1}^2.
  \]
  The confirm that the distribution is skew, we compute the \( t \)-ratio \( \frac{0.777-0.0447}{0.0522} = -5.03 \), which strongly rejects the null hypothesis of a symmetric distribution.
• Compute 1- to 5-step ahead forecasts for the monthly log return and its volatility based on your fitted model with skew Student-\( t \) distributions at the forecast origin December 2011.

Answer: The forecasts for the return is 0.926 for all steps and the volatility forecasts are 5.431, 5.416, 5.401, 5.387, and 5.373, respectively.

4. Monthly log returns, in percentages, of Coca Cola stock from January 1951 to December 2011.

• Is there any serial correlation in the monthly log returns of the KO stock?
   Answer: No, the Ljung-Box statistics fail to reject the null hypothesis of zero serial correlations.

• Fit a GARCH(1,1) model to the monthly percentage log returns of KO stock using Gaussian distribution for the innovations. Write down the fitted model and perform model checking. Is the model adequate? Why?
   Answer: The fitted model is
   \[
   r_t = 0.011 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim_{iid} N(0, 1)
   \]
   \[
   \sigma_t^2 = 1.90 \times 10^{-4} + 0.101 a_{t-1}^2 + 0.843 \sigma_{t-1}^2.
   \]
   All estimates are statistically significant and the residuals and squared residuals show that the fitted model is adequate in describing the first two moments of the data. However, the normality assumption is clearly rejected.

• Use the fitted model to calculate 1-step to 5-step ahead forecasts for the log returns series and its volatility.
   Answer: The forecasts for the return are 0.011, and the forecasts for volatility are 0.0415, 0.0426, 0.0436, 0.0445, and 0.0454, respectively.

• Fit a GARCH(1,1) model to the monthly log returns of the KO stock using a skew Student-\( t \) distribution. Write down the fitted model. Is the skewness parameter significant?
   Answer: The fitted model is
   \[
   r_t = 0.011 + a_t, \quad a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim_{iid} t_{7.25,.996}^*
   \]
   \[
   \sigma_t^2 = 1.76 \times 10^{-4} + 0.105 a_{t-1}^2 + 0.844 \sigma_{t-1}^2.
   \]
   The estimate of the skew parameter is so close to 1 with standard error 0.053, we obviously cannot reject the null hypothesis of a symmetric innovations.


• Is there any serial correlation in the log return series?
  Answer: Yes, the Ljung-Box statistics show that there exist serial correlations in the data. A careful examination shows a significant lag-5 ACF and significant PACF at lags 5, 10, and 15. These features indicate a seasonal MA model at lag 5.
  The fitted model is
  \[ r_t = a_t - 0.102a_{t-5}, \quad \sigma^2_a = 5.55 \times 10^{-5}. \]

• Is there any ARCH effect in the log return series?
  Answer: Yes, the Ljung-Box statistics of \( a^2_t \) indicate strong ARCH effects.

• Find a simple volatility model (among the models discussed in class) for the log return series, including choosing the innovation distribution. Write down the fitted model.
  Answer: The final model selected is
  \[
  r_t = -9.27 \times 10^{-6} + a_t - 0.012a_{t-1} + 0.0004a_{t-2} + 0.0103a_{t-3} - 0.032a_{t-4} - 0.073a_{t-5}, \\
  a_t = \sigma_t \epsilon_t, \quad \epsilon_t \sim iid t^*_{10, 891}, \\
  \sigma^2_t = 2.26 \times 10^{-7} + 0.046a^2_{t-1} + 0.951\sigma^2_{t-1},
  \]
  where all parameters, but lag-5, of the mean equation are insignificant, and the constant term of the volatility equation is also insignificant. Typically, I would drop insignificant parameters, but the package does not have this capability. Note that the skew parameter is significantly different from 1 so that the innovations are skewed. Also, the estimated degrees of freedom is precisely 10. This indicates that the estimation is not at the global maximum. One possible reason is that Student-t distribution might not be adequate.

• Use the model to produce 1-step to 4-step ahead forecasts for the log return series and its volatility.
  Answer: The forecasts for the return are \( \{7.13, 1.14, -4.30, -5.33\} \times 10^{-4} \). The volatility forecasts are \( \{5.87, 5.88, 5.89, 5.90\} \times 10^{-3} \), respectively.