VaR of a Portfolio

**Basic setup:** Two assets with log returns $r_{1t}$ and $r_{2t}$. The portfolio consists of $w_1$ and $w_2$ amounts invested in asset 1 and asset 2, respectively.

Under RiskMetrics, we have

$$
\begin{align*}
    r_{1t}|F_{t-1} &\sim N(0, \sigma^2_{1t}), \\
    \sigma^2_{1t} &= \beta \sigma^2_{1,t-1} + (1 - \beta) r^2_{1,t-1} \\
    r_{2t}|F_{t-1} &\sim N(0, \sigma^2_{2t}), \\
    \sigma^2_{2t} &= \beta \sigma^2_{2,t-1} + (1 - \beta) r^2_{2,t-1}.
\end{align*}
$$

VaR for the two assets are $w_1 \text{VaR}_1$ and $w_2 \text{VaR}_2$, respectively. For instance, for tail probability 0.05, we have VaR for the two assets as $1.645 w_1 \sigma_{1t}$ and $1.645 w_2 \sigma_{2t}$, respectively.

Let $p_t$ be the log return of the portfolio. Then, we have

$$p_t = wr_{1t} + (1 - w)r_{2t},$$

where $w = \frac{w_1}{w_1 + w_2}$ and $1 - w = \frac{w_2}{w_1 + w_2}$.

**Remark.** $(w_1 + w_2)w = w_1$ and $(w_1 + w_2)(1 - w) = w_2$.

Under RiskMetrics, we have

$$p_t|F_{t-1} \sim N(0, \sigma^2_{pt}),$$

where

$$\sigma^2_{pt} = \text{Var}(r_{pt}|F_{t-1}) = w^2 \sigma^2_{1t} + (1-w)^2 \sigma^2_{2t} + 2w(1-w) \rho_t \sigma_{1t} \sigma_{2t}.$$
The VaR for the portfolio is $(w_1 + w_2)\text{VaR}_p$. For tail probability of 0.05, we have $\text{VaR}_p = 1.645(w_1 + w_2)\sigma_{pt}$. Therefore, the square of VaR for the portfolio with tail probability 0.05 is

$$(\text{VaR}_p)^2 = (1.645)^2(w_1 + w_2)^2\sigma_{pt}^2$$

$$= (1.645)^2(w_1 + w_2)^2$$

$$\times [w^2\sigma_{1t}^2 + (1 - w)^2\sigma_{2t}^2 + 2w(1 - w)\rho_t\sigma_{1t}\sigma_{2t}]$$

$$= (1.645)^2[w_1^2\sigma_{1t}^2 + w_2^2\sigma_{2t}^2 + 2w_1w_2\rho_t\sigma_{1t}\sigma_{2t}]$$

$$= \text{VaR}_1^2 + \text{VaR}_2^2 + 2\rho_t\text{VAR}_1\text{VAR}_2.$$

This is exactly similar to

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\rho \times \text{std}(X)\text{std}(Y).$$

The result can be generalized to more than two assets.